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Multiple positive solutions for a fourth-order integral boundary value problem on time scales

Yongkun Li* and Yanshou Dong

* Correspondence: ykxie@ynu.edu.cnDepartment of Mathematics,
Yunnan University Kunming,
Yunnan 650091 People's Republic
of China**Abstract**

In this article, we investigate the multiplicity of positive solutions for a fourth-order system of integral boundary value problem on time scales. The existence of multiple positive solutions for the system is obtained by using the fixed point theorem of cone expansion and compression type due to Krasnosel'skill. To demonstrate the applications of our results, an example is also given in the article.

Keywords: positive solutions, fixed points, integral boundary conditions, time scales

1 Introduction

Boundary value problem (BVP) for ordinary differential equations arise in different areas of applied mathematics and physics and so on, the existence and multiplicity of positive solutions for such problems have become an important area of investigation in recent years, lots of significant results have been established by using upper and lower solution arguments, fixed point indexes, fixed point theorems and so on (see [1-8] and the references therein). Especially, the existence of positive solutions of nonlinear BVP with integral boundary conditions has been extensively studied by many authors (see [9-18] and the references therein).

However, the corresponding results for BVP with integral boundary conditions on time scales are still very few [19-21]. In this article, we discuss the multiple positive solutions for the following fourth-order system of integral BVP with a parameter on time scales

$$\left\{ \begin{array}{l} x^{(4\Delta)}(t) + \lambda f(t, x(t), x^\Delta(t), x^{\Delta\Delta}(t), \gamma(t), \gamma^\Delta(t), \gamma^{\Delta\Delta}(t)) = 0, \quad t \in (0, \sigma(T))_{\mathbb{T}}, \\ \gamma^{(4\Delta)}(t) + \mu g(t, x(t), x^\Delta(t), x^{\Delta\Delta}(t), \gamma(t), \gamma^\Delta(t), \gamma^{\Delta\Delta}(t)) = 0, \quad t \in (0, \sigma(T))_{\mathbb{T}}, \\ x(0) = x^\Delta(0) = 0, \\ \gamma(0) = \gamma^\Delta(0) = 0, \\ a_1 x^{\Delta\Delta}(0) - b_1 x^{\Delta\Delta\Delta}(0) = \int_0^{\sigma(T)} x^{\Delta\Delta}(s) A_1(s) \Delta s, \\ c_1 x^{\Delta\Delta}(\sigma(T)) + d_1 x^{\Delta\Delta\Delta}(\sigma(T)) = \int_0^{\sigma(T)} x^{\Delta\Delta}(s) B_1(s) \Delta s, \\ a_2 \gamma^{\Delta\Delta}(0) - b_2 \gamma^{\Delta\Delta\Delta}(0) = \int_0^{\sigma(T)} \gamma^{\Delta\Delta}(s) A_2(s) \Delta s, \\ c_2 \gamma^{\Delta\Delta}(\sigma(T)) + d_2 \gamma^{\Delta\Delta\Delta}(\sigma(T)) = \int_0^{\sigma(T)} \gamma^{\Delta\Delta}(s) B_2(s) \Delta s, \end{array} \right. \quad (1.1)$$

where $a_i, b_i, c_i, d_i \geq 0$, and $\rho_i = a_i c_i \sigma(T) + a_i d_i + b_i c_i > 0 (i = 1, 2)$, $0 < \lambda, \mu < +\infty$, $f, g \in C((0, \sigma(T))_{\mathbb{T}} \times (\mathbb{R}^+)^6, \mathbb{R}^+)$, $\mathbb{R}^+ = [0, +\infty)$, A_i and B_i are nonnegative and rd-continuous on $[0, \sigma(T)]_{\mathbb{T}} (i = 1, 2)$.

The main purpose of this article is to establish some sufficient conditions for the existence of at least two positive solutions for system (1.1) by using the fixed point theorem of cone expansion and compression type. This article is organized as follows. In Section 2, some useful lemmas are established. In Section 3, by using the fixed point theorem of cone expansion and compression type, we establish sufficient conditions for the existence of at least two positive solutions for system (1.1). An illustrative example is given in Section 4.

2 Preliminaries

In this section, we will provide several foundational definitions and results from the calculus on time scales and give some lemmas which are used in the proof of our main results.

A time scale \mathbb{T} is a nonempty closed subset of the real numbers \mathbb{R} .

Definition 2.1. [22] For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ by $\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\}$, while the backward jump operator $\rho : \mathbb{T} \rightarrow \mathbb{T}$ by $\rho(t) = \sup\{\tau \in \mathbb{T} : \tau < t\}$.

In this definition, we put $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$, where \emptyset , denotes the empty set. If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. We also need, below, the set \mathbb{T}^k , which is derived from the time scale \mathbb{T} as follows: if \mathbb{T} has a left-scattered maximum m , then $\mathbb{T}^k = \mathbb{T} - m$. Otherwise, $\mathbb{T}^k = \mathbb{T}$.

Definition 2.2. [22] Assume that $x : \mathbb{T} \rightarrow \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then x is called differentiable at $t \in \mathbb{T}$ if there exists a $\theta \in \mathbb{R}$ such that for any given $\varepsilon > 0$, there is an open neighborhood U of t such that

$$|x(\sigma(t)) - x(t) - x^\Delta(t)| |\sigma(t) - t| \leq \varepsilon |\sigma(t) - t|, \quad s \in U.$$

In this case, $x^\Delta(t)$ is called the delta derivative of x at t . The second derivative of $x(t)$ is defined by $x^{\Delta\Delta}(t) = (x^\Delta)^\Delta(t)$.

In a similar way, we can obtain the fourth-order derivative of $x(t)$ is defined by $x^{(4\Delta)}(t) = (((x^\Delta)^\Delta)^\Delta)^\Delta(t)$.

Definition 2.3. [22] A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in \mathbb{T} and its left-sided limits exist at left-dense points in \mathbb{T} . The set of rd-continuous functions $f : \mathbb{T} \rightarrow \mathbb{R}$ will be denoted by $C_{rd}(\mathbb{T})$.

Definition 2.4. [22] A function $F : \mathbb{T} \rightarrow \mathbb{R}$ is called a delta-antiderivative of $f : \mathbb{T} \rightarrow \mathbb{R}$ provide $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^k$. In this case we define the integral of f by

$$\int_a^t f(s) \Delta s = F(t) - F(a).$$

For convenience, we denote $I = [0, \sigma(T)]_{\mathbb{T}}$, $I' = (0, \sigma(T))_{\mathbb{T}}$ and for $i = 1, 2$, we set

$$D_{1i} = \frac{Q_{1i}}{1 - P_{1i}}, \quad D_{2i} = \frac{P_{2i}}{1 - Q_{2i}}, \quad K_{1i} = \frac{1}{1 - P_{1i}}, \quad K_{2i} = \frac{1}{1 - Q_{2i}},$$

where

$$\begin{aligned}
 P_{1i} &= \int_0^{\sigma(T)} B_i(s) \frac{a_i s + b_i}{\rho_i} \Delta s, & P_{2i} &= \int_0^{\sigma(T)} A_i(s) \frac{a_i s + b_i}{\rho_i} \Delta s, \\
 Q_{1i} &= \int_0^{\sigma(T)} B_i(s) \frac{d_i + c_i(\sigma(T) - s)}{\rho_i} \Delta s, & Q_{2i} &= \int_0^{\sigma(T)} A_i(s) \frac{d_i + c_i(\sigma(T) - s)}{\rho_i} \Delta s.
 \end{aligned}$$

To establish the existence of multiple positive solutions of system (1.1), let us list the following assumptions:

$$(H_1) P_{ji}, Q_{ji} \in [0, 1), D_{11} D_{21} \in [0, 1), D_{21} D_{22} \in [0, 1), \quad j, i = 1, 2.$$

In order to overcome the difficulty due to the dependence of f, g on derivatives, we first consider the following second-order nonlinear system

$$\left\{ \begin{aligned}
 &u^{\Delta\Delta}(t) + \lambda f(t, A_2 u, A_1 u, A_0 u, A_2 v, A_1 v, A_0 v) = 0, \quad t \in (0, \sigma(T))_{\mathbb{T}}, \\
 &v^{\Delta\Delta}(t) + \mu g(t, A_2 v, A_1 v, A_0 v, A_2 u, A_1 u, A_0 u) = 0, \quad t \in (0, \sigma(T))_{\mathbb{T}}, \\
 &a_1 u(0) - b_1 u^{\Delta}(0) = \int_0^{\sigma(T)} u(s) A_1(s) \Delta s, \\
 &c_1 u(\sigma(T)) + d_1 u^{\Delta}(\sigma(T)) = \int_0^{\sigma(T)} u(s) B_1(s) \Delta s, \\
 &a_2 v(0) - b_2 v^{\Delta}(0) = \int_0^{\sigma(T)} v(s) A_2(s) \Delta s, \\
 &c_2 v(\sigma(T)) + d_2 v^{\Delta}(\sigma(T)) = \int_0^{\sigma(T)} v(s) B_2(s) \Delta s,
 \end{aligned} \right. \quad (2.1)$$

where A_0 is the identity operator, and

$$A_i u(t) = \int_0^t (t - \sigma(s))^{i-1} u(s) \Delta s, \quad A_i v(t) = \int_0^t (t - \sigma(s))^{i-1} v(s) \Delta s, \quad i = 1, 2. \quad (2.2)$$

For the proof of our main results, we will make use of the following lemmas.

Lemma 2.1. *The fourth-order system (1.1) has a solution (x, y) if and only if the nonlinear system (2.1) has a solution (u, v) .*

Proof. If (x, y) is a solution of the fourth-order system (1.1), let $u(t) = x^{\Delta\Delta}(t)$, $v(t) = y^{\Delta\Delta}(t)$, then it follows from the boundary conditions of system (1.1) that

$$A_1 u(t) = x^{\Delta}(t), \quad A_2 u(t) = x(t), \quad A_1 v(t) = y^{\Delta}(t), \quad A_2 v(t) = y(t).$$

Thus $(u, v) = (x^{\Delta\Delta}(t), y^{\Delta\Delta}(t))$ is a solution of the nonlinear system (2.1).

Conversely, if (u, v) is a solution of the nonlinear system (2.1), let $x(t) = A_2 u(t)$, $y(t) = A_2 v(t)$, then we have

$$x^{\Delta}(t) = A_1 u(t), \quad x^{\Delta\Delta}(t) = u(t), \quad y^{\Delta}(t) = A_1 v(t), \quad y^{\Delta\Delta}(t) = v(t),$$

which imply that

$$x(0) = 0, x^\Delta(0) = 0, \gamma(0) = 0, \gamma^\Delta(0) = 0.$$

Consequently, $(x, y) = (A_2u(t), A_2v(t))$ is a solution of the fourth-order system (1.1). This completes the proof.

Lemma 2.2. *Assume that $D_{11}D_{21} \neq 1$ holds. Then for any $h_1 \in C(I, \mathbb{R}^+)$, the following BVP*

$$\begin{cases} u^{\Delta\Delta}(t) + h_1(t) = 0, & t \in (0, \sigma(T))_{\mathbb{T}}, \\ a_1u(0) - b_1u^\Delta(0) = \int_0^{\sigma(T)} u(s)A_1(s)\Delta s, \\ c_1u(\sigma(T)) + d_1u^\Delta(\sigma(T)) = \int_0^{\sigma(T)} u(s)B_1(s)\Delta s \end{cases} \quad (2.3)$$

has a solution

$$u(t) = \int_0^{\sigma(T)} H_1(t, s)h_1(s)\Delta s,$$

where

$$\begin{aligned} H_1(t, s) &= G_1(t, s) + r_1(t) \int_0^{\sigma(T)} B_1(\tau)G_1(\tau, s)\Delta\tau + r_2(t) \int_0^{\sigma(T)} A_1(\tau)G_1(\tau, s)\Delta\tau, \\ G_1(t, s) &= \frac{1}{\rho_1} \begin{cases} (a_1\sigma(s) + b_1)[d_1 + c_1(\sigma(T) - t)], & \sigma(s) < t, \\ (a_1t + b_1)[d_1 + c_1(\sigma(T) - \sigma(s))], & t \leq \sigma(s), \end{cases} \\ r_{11}(t) &= \frac{K_{11}(a_1t + b_1) + K_{11}D_{21}[d_1 + c_1(\sigma(T) - t)]}{\rho_1(1 - D_{11}D_{21})}, \\ r_{21}(t) &= \frac{K_{21}D_{11}(a_1t + b_1) + K_{21}[d_1 + c_1(\sigma(T) - t)]}{\rho_1(1 - D_{11}D_{21})}. \end{aligned}$$

Proof. First suppose that u is a solution of system (2.3). It is easy to see by integration of BVP(2.3) that

$$u^\Delta(t) = u^\Delta(0) - \int_0^t h_1(s)\Delta s. \quad (2.4)$$

Integrating again, we can obtain

$$u(t) = u(0) + tu^\Delta(0) - \int_0^t (t - \sigma(s))h_1(s)\Delta s. \quad (2.5)$$

Let $t = \sigma(T)$ in (2.4) and (2.5), we obtain

$$u^\Delta(\sigma(T)) = u^\Delta(0) - \int_0^{\sigma(T)} h_1(s)\Delta s, \quad (2.6)$$

$$u(\sigma(T)) = u(0) + \sigma(T)u^\Delta(0) - \int_0^{\sigma(T)} (\sigma(T) - \sigma(s))h_1(s)\Delta s. \quad (2.7)$$

Substituting (2.6) and (2.7) into the second boundary value condition of system (2.3), we obtain

$$c_1 u(0) + (c_1 \sigma(T) + d_1) u^\Delta(0) = \int_0^{\sigma(T)} [d_1 + c_1(\sigma(T) - \sigma(s))] h_1(s) \Delta s + \int_0^{\sigma(T)} u(s) B_1(s) \Delta s. \tag{2.8}$$

From (2.8) and the first boundary value condition of system (2.3), we have

$$u^\Delta(0) = \frac{a_1}{\rho_1} \left(\int_0^{\sigma(T)} [d_1 + c_1(\sigma(T) - \sigma(s))] h_1(s) \Delta s + \int_0^{\sigma(T)} u(s) B_1(s) \Delta s - \frac{c_1}{a_1} \int_0^{\sigma(T)} u(s) A_1(s) \Delta s \right), \tag{2.9}$$

$$u(0) = \frac{b_1}{\rho_1} \left(\int_0^{\sigma(T)} [d_1 + c_1(\sigma(T) - \sigma(s))] h_1(s) \Delta s + \int_0^{\sigma(T)} u(s) B_1(s) \Delta s - \frac{c_1}{a_1} \int_0^{\sigma(T)} u(s) A_1(s) \Delta s \right) + \frac{1}{a_1} \int_0^{\sigma(T)} u(s) A_1(s) \Delta s. \tag{2.10}$$

Substituting (2.9) and (2.10) into (2.5), we have

$$u(t) = \int_0^{\sigma(T)} G_1(t, s) h_1(s) \Delta s + \frac{a_1 t + b_1}{\rho_1} \int_0^{\sigma(T)} u(s) B_1(s) \Delta s + \frac{d_1 + c_1(\sigma(T) - t)}{\rho_1} \int_0^{\sigma(T)} u(s) A_1(s) \Delta s. \tag{2.11}$$

By (2.11), we get

$$\int_0^{\sigma(T)} u(s) B_1(s) \Delta s = \frac{1}{1 - P_{11}} \int_0^{\sigma(T)} B_1(s) \int_0^{\sigma(T)} G_1(s, \tau) h_1(\tau) \Delta \tau \Delta s + \frac{Q_{11}}{1 - P_{11}} \int_0^{\sigma(T)} u(s) A_1(s) \Delta s, \tag{2.12}$$

$$\int_0^{\sigma(T)} u(s) A_1(s) \Delta s = \frac{1}{1 - Q_{21}} \int_0^{\sigma(T)} B_1(s) \int_0^{\sigma(T)} G_1(s, \tau) h_1(\tau) \Delta \tau \Delta s + \frac{P_{21}}{1 - Q_{21}} \int_0^{\sigma(T)} u(s) B_1(s) \Delta s. \tag{2.13}$$

By (2.12) and (2.13), we get

$$\int_0^{\sigma(T)} u(s)A_1(s)\Delta s = \frac{K_{11}D_{21}}{1 - D_{11}D_{21}} \int_0^{\sigma(T)} B_1(s) \int_0^{\sigma(T)} G_1(s, \tau)h_1(\tau)\Delta\tau\Delta s$$

$$+ \frac{K_{21}}{1 - D_{11}D_{21}} \int_0^{\sigma(T)} A_1(s) \int_0^{\sigma(T)} G_1(s, \tau)h_1(\tau)\Delta\tau\Delta s, \tag{2.14}$$

$$\int_0^{\sigma(T)} u(s)B_1(s)\Delta s = \frac{K_{11}}{1 - D_{11}D_{21}} \int_0^T B_1(s) \int_0^T G_1(s, \tau)h_1(\tau)\Delta\tau\Delta s$$

$$+ \frac{K_{21}D_{11}}{1 - D_{11}D_{21}} \int_0^{\sigma(T)} A_1(s) \int_0^{\sigma(T)} G_1(s, \tau)h_1(\tau)\Delta\tau\Delta s. \tag{2.15}$$

Substituting (2.14) and (2.15) into (2.11), we have

$$u(t) = \int_0^{\sigma(T)} G_1(t, s)h_1(s)\Delta s$$

$$+ \frac{K_{11}(a_1t + b_1) + K_{11}D_{21}[d_1 + c_1(\sigma(T) - t)]}{\rho_1(1 - D_{11}D_{21})} \int_0^{\sigma(T)} B_1(s) \int_0^{\sigma(T)} G_1(s, \tau)h_1(\tau)\Delta\tau\Delta s$$

$$+ \frac{K_{21}D_{11}(a_1t + b_1) + K_{21}[d_1 + c_1(\sigma(T) - t)]}{\rho_1(1 - D_{11}D_{21})} \int_0^{\sigma(T)} A_1(s) \int_0^{\sigma(T)} G_1(s, \tau)h_1(\tau)\Delta\tau\Delta s$$

$$= \int_0^{\sigma(T)} G_1(t, s)h_1(s)\Delta s + r_{11}(t) \int_0^{\sigma(T)} B_1(s) \int_0^{\sigma(T)} G_1(s, \tau)h_1(\tau)\Delta\tau\Delta s$$

$$+ r_{21}(t) \int_0^{\sigma(T)} A_1(s) \int_0^{\sigma(T)} G_1(s, \tau)h_1(\tau)\Delta\tau\Delta s$$

$$= \int_0^{\sigma(T)} H_1(t, s)h_1(s)\Delta s. \tag{2.16}$$

Conversely, suppose $u(t) = \int_0^{\sigma(T)} H_1(t, s)h_1(s)\Delta s$, then

$$u(t) = \int_0^{\sigma(T)} G_1(t, s)h_1(s)\Delta s + r_{11}(t) \int_0^{\sigma(T)} B_1(s) \int_0^{\sigma(T)} G_1(s, \tau)h_1(\tau)\Delta\tau\Delta s$$

$$+ r_{21}(t) \int_0^{\sigma(T)} A_1(s) \int_0^{\sigma(T)} G_1(s, \tau)h_1(\tau)\Delta\tau\Delta s. \tag{2.17}$$

Direct differentiation of (2.17) implies

$$\begin{aligned}
 u^\Delta(t) = & \frac{1}{\rho_1} \left(a_1 \int_t^{\sigma(T)} [d_1 + c_1(\sigma(T) - \sigma(s))]h_1(s)\Delta s - c_1 \int_0^t (a_1\sigma(s) + b_1)h_1(s) \Delta s \right) \\
 & + \frac{a_1K_{11} - c_1K_{11}D_{21}}{\rho_1(1 - D_{11}D_{21})} \int_0^{\sigma(T)} B_1(s) \int_0^{\sigma(T)} G_1(s, \tau)h_1(\tau)\Delta\tau\Delta s \\
 & + \frac{a_1K_{21}D_{11} - c_1K_{21}}{\rho_1(1 - D_{11}D_{21})} \int_0^{\sigma(T)} A_1(s) \int_0^{\sigma(T)} G_1(s, \tau)h_1(\tau)\Delta\tau\Delta s
 \end{aligned}$$

and

$$u^{\Delta\Delta}(t) = -h_1(t),$$

and it is easy to verify that

$$\begin{aligned}
 a_1u(0) - b_1u^\Delta(0) &= \int_0^{\sigma(T)} u(s)A_1(s)\Delta s, \\
 c_1u(\sigma(T)) + d_1u^\Delta(\sigma(T)) &= \int_0^{\sigma(T)} u(s)B_1(s)\Delta s.
 \end{aligned}$$

This completes the proof.

Lemma 2.3. Assume that $D_{12}D_{22} \neq 1$ holds. Then for any $h_2 \in C(I, \mathbb{R}^+)$, the following BVP

$$\begin{cases}
 v^{\Delta\Delta}(t) + h_2(t) = 0, & t \in (0, \sigma(T))_{\mathbb{T}}, \\
 a_2v(0) - b_2v^\Delta(0) = \int_0^{\sigma(T)} v(s)A_2(s)\Delta s, \\
 c_2v(\sigma(T)) + d_2v^\Delta(\sigma(T)) = \int_0^{\sigma(T)} v(s)B_2(s)\Delta s
 \end{cases}$$

has a solution

$$v(t) = \int_0^{\sigma(T)} H_2(t, s)h_2(s)\Delta s,$$

where

$$\begin{aligned}
 H_2(t, s) &= G_2(t, s) + r_{12}(t) \int_0^{\sigma(T)} B_2(\tau)G_2(\tau, s)\Delta\tau + r_{22}(t) \int_0^{\sigma(T)} A_2(\tau)G_2(\tau, s)\Delta\tau, \\
 G_2(t, s) &= \frac{1}{\rho_2} \begin{cases} (a_2\sigma(s) + b_2)[d_2 + c_2(\sigma(T) - t)], & \sigma(s) < t, \\ (a_2t + b_2)[d_2 + c_2(\sigma(T) - \sigma(s))], & t \leq \sigma(s), \end{cases} \\
 r_{12}(t) &= \frac{K_{12}(a_2t + b_2) + K_{12}D_{22}[d_2 + c_2(\sigma(T) - t)]}{\rho_2(1 - D_{12}D_{22})}, \\
 r_{22}(t) &= \frac{K_{22}D_{12}(a_2t + b_2) + K_{22}[d_2 + c_2(\sigma(T) - t)]}{\rho_2(1 - D_{12}D_{22})}.
 \end{aligned}$$

Proof. The proof is similar to that of Lemma 2.2 and will omit it here.

Lemma 2.4. *Suppose that (H_1) is satisfied, for all $t, s \in I$ and $i = 1, 2$, we have*

- (i) $G_i(t, s) > 0, H_i(t, s) > 0,$
- (ii) $\text{Lim}_i G_i(\sigma(s), s) \leq H_i(t, s) \leq M_i G_i(\sigma(s), s),$
- (iii) $m G_i(\sigma(s), s) \leq H_i(t, s) \leq M G_i(\sigma(s), s),$

where

$$M_i = 1 + \overline{r_{1i}} \int_0^{\sigma(T)} B_i(\tau) \Delta \tau + \overline{r_{2i}} \int_0^{\sigma(T)} A_i(\tau) \Delta \tau, \quad \overline{r_{ji}} = \max_{0 \leq t \leq 1} r_i(t),$$

$$m_i = 1 + \underline{r_{1i}}(t) \int_0^{\sigma(T)} B_i(\tau) \Delta \tau + \underline{r_{2i}}(t) \int_0^{\sigma(T)} A_i(\tau) \Delta \tau, \quad \underline{r_{ji}} = \min_{0 \leq t \leq 1} r_i(t),$$

$$M = \max\{M_1, M_2\}, \quad m = \min\{L_1 m_1, L_2 m_2\}, \quad L_i = \min \left\{ \frac{d_i}{d_i + c_i}, \frac{b_i}{a_i + b_i} \right\}, \quad i, j = 1, 2.$$

Proof. It is easy to verify that $G_i(t, s) > 0, H_i(t, s) > 0$ and $G_i(t, s) \leq G_i(\sigma(s), s)$, for all $t, s \in I$. Since

$$\frac{G_i(t, s)}{G_i(\sigma(s), s)} = \begin{cases} \frac{d_i + c_i(\sigma(T) - t)}{d_i + c_i(\sigma(T) - \sigma(s))}, & \sigma(s) < t, \\ \frac{a_i t + b_i}{a_i \sigma(s) + b_i}, & \sigma(s) \geq t. \end{cases}$$

Thus $G_i(t, s)/G_i(\sigma(s), s) \geq L_i$ and we have

$$G_i(t, s) \geq L_i G_i(\sigma(s), s).$$

On the one hand, from the definition of L_i and m_i , for all $t, s \in I$, we have

$$H_i(t, s) = G_i(t, s) + r_{1i}(t) \int_0^{\sigma(T)} B_i(\tau) G_i(\tau, s) \Delta \tau + r_{2i}(t) \int_0^{\sigma(T)} A_i(\tau) G_i(\tau, s) \Delta \tau$$

$$\geq L_i G_i(\sigma(s), s) \left(1 + r_{1i}(t) \int_0^{\sigma(T)} B_i(\tau) \Delta \tau + r_{2i}(t) \int_0^{\sigma(T)} A_i(\tau) \Delta \tau \right)$$

$$\geq L_i m_i G_i(\sigma(s), s),$$

and on the other hand, we obtain easily that from the definition of M_i , for all $t, s \in I$,

$$H_i(t, s) \leq G_i(\sigma(s), s) + r_{1i}(t) \int_0^{\sigma(T)} B_i(\tau) G_i(\sigma(s), s) \Delta \tau + r_{2i}(t) \int_0^{\sigma(T)} A_i(\tau) G_i(\sigma(s), s) \Delta \tau$$

$$\leq M_i G_i(\sigma(s), s).$$

Finally, it is easy to verify that $m G_i(\sigma(s), s) \leq H_i(t, s) \leq M G_i(\sigma(s), s)$. This completes the proof.

Lemma 2.5. [23] *Let E be a Banach space and P be a cone in E . Assume that Ω_1 and Ω_2 are bounded open subsets of E , such that $0 \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$, and let $T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow P$ be a completely continuous operator such that either*

- (i) $\|Tu\| \leq \|u\|, \forall u \in P \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|, \forall u \in P \cap \partial\Omega_2$, or
- (ii) $\|Tu\| \geq \|u\|, \forall u \in P \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|, \forall u \in P \cap \partial\Omega_2$

holds. Then T has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

To obtain the existence of positive solutions for system (2.1), we construct a cone P in the Banach space $Q = C(I, \mathbb{R}^+) \times C(I, \mathbb{R}^+)$ equipped with the norm $\|(u, v)\| = \|u\| + \|v\| = \max_{t \in I} |u| + \max_{t \in I} |v|$ by

$$P = \left\{ (u, v) \in Q \mid u(t) \geq 0, v(t) \geq 0, \min_{t \in I} (u(t) + v(t)) \geq \frac{m}{M} \|(u, v)\| \right\}.$$

It is easy to see that P is a cone in Q .

Define two operators $T_\lambda, T_\mu : P \rightarrow C(I, \mathbb{R}^+)$ by

$$T_\lambda(u, v)(t) = \lambda \int_0^T H_1(t, s) f(t, A_2 u, A_1 u, A_0 u, A_2 v, A_1 v, A_0 v) \Delta s, \quad t \in I,$$

$$T_\mu(u, v)(t) = \mu \int_0^T H_2(t, s) g(t, A_2 u, A_1 u, A_0 u, A_2 v, A_1 v, A_0 v) \Delta s, \quad t \in I.$$

Then we can define an operator $T : P \rightarrow C(I, \mathbb{R}^+)$ by

$$T(u, v) = (T_\lambda(u, v), T_\mu(u, v)), \quad \forall (u, v) \in P.$$

Lemma 2.6. *Let (H_1) hold. Then $T : P \rightarrow P$ is completely continuous.*

Proof. Firstly, we prove that $T : P \rightarrow P$. In fact, for all $(u, v) \in P$ and $t \in I$, by Lemma 2.4(i) and (H_1) , it is obvious that $T_\lambda(u, v)(t) > 0, T_\mu(u, v)(t) > 0$. In addition, we have

$$T_\lambda(u, v)(t) = \lambda \int_0^{\sigma(T)} H_1(t, s) f(t, A_2 u, A_1 u, A_0 u, A_2 v, A_1 v, A_0 v) \Delta s$$

$$\leq \lambda M \int_0^{\sigma(T)} G_1(\sigma(s), s) f(t, A_2 u, A_1 u, A_0 u, A_2 v, A_1 v, A_0 v) \Delta s, \tag{2.18}$$

which implies $\|T_\lambda(u, v)\| \leq \lambda M \int_0^{\sigma(T)} G_1(\sigma(s), s) f(t, A_2 u, A_1 u, A_0 u, A_2 v, A_1 v, A_0 v) \Delta s$. And we have

$$T_\lambda(u, v)(t) \geq \lambda L_1 m_1 \int_0^{\sigma(T)} G_1(\sigma(s), s) f(t, A_2 u, A_1 u, A_0 u, A_2 v, A_1 v, A_0 v) \Delta s$$

$$\geq \frac{m}{M} \|T_\lambda(u, v)\|.$$

In a similar way,

$$T_\mu(u, v)(t) \geq \frac{m}{M} \|T_\mu(u, v)\|.$$

Therefore,

$$\min_{t \in I} (T_\lambda(u, v)(t) + T_\mu(u, v)(t)) \geq \frac{m}{M} \|T_\lambda(u, v)\| + \frac{m}{M} \|T_\mu(u, v)\|$$

$$= \frac{m}{M} \|T_\lambda(u, v), T_\mu(u, v)\|.$$

This shows that $T : P \rightarrow P$.

Secondly, we prove that T is continuous and compact, respectively. Let $\{(u_k, v_k)\} \in P$ be any sequence of functions with $\lim_{k \rightarrow \infty} (u_k, v_k) = (u, v) \in P$,

$$|T_\lambda(u_k, v_k)(t) - T_\lambda(u, v)(t)| \leq \lambda M_1 \sup_{t \in I} |f(t, A_2 u_k, A_1 u_k, A_0 u_k, A_2 v_k, A_1 v_k, A_0 v_k) - f(t, A_2 u, A_1 u, A_0 u, A_2 v, A_1 v, A_0 v)| \int_0^{\sigma(T)} G_1(\sigma(s), s) \Delta s,$$

from the continuity of f , we know that $\|T_\lambda(u_k, v_k) - T_\lambda(u, v)\| \rightarrow 0$ as $k \rightarrow \infty$. Hence T_λ is continuous.

T_λ is compact provided that it maps bounded sets into relatively compact sets. Let $\bar{f} = \sup_{t \in I'} |f(t, A_2 u, A_1 u, A_0 u, A_2 v, A_1 v, A_0 v)|$, and let Ω be any bounded subset of P , then there exists $r > 0$ such that $\|(u, v)\| \leq r$ for all $(u, v) \in \Omega$. Obviously, from (2.16), we know that

$$T_\lambda(u, v)(t) \leq \lambda M \bar{f} \int_0^{\sigma(T)} G_1(\sigma(s), s) \Delta s,$$

so, $T_\lambda \Omega$ is bounded for all $(u, v) \in \Omega$. Moreover, let

$$\begin{aligned} L'_1 = & \frac{\lambda \bar{f}}{\rho_1} \left(a_1 \int_0^{\sigma(T)} [d_1 + c_1(\sigma(T) - \sigma(s))] \Delta s + c_1 \int_0^{\sigma(T)} (a_1 \sigma(s) + b_1) \Delta s \right) \\ & + \frac{\lambda \bar{f} |a_1 K_{11} - c_1 K_{11} D_{21}|}{\rho_1 (1 - D_{11} D_{21})} \int_0^{\sigma(T)} B_1(s) \int_0^{\sigma(T)} G_1(s, \tau) \Delta \tau \Delta s \\ & + \frac{\lambda \bar{f} |a_1 K_{21} D_{11} - c_1 K_{21}|}{\rho_1 (1 - D_{11} D_{21})} \int_0^{\sigma(T)} A_1(s) \int_0^{\sigma(T)} G_1(s, \tau) \Delta \tau \Delta s. \end{aligned}$$

We have

$$\begin{aligned} & |T_\lambda(u, v)^\Delta(t)| \\ & \leq \frac{\lambda}{\rho_1} \left| a_1 \int_t^{\sigma(T)} [d_1 + c_1(\sigma(T) - \sigma(s))] f(s, A_2 u, A_1 u, A_0 u, A_2 v, A_1 v, A_0 v) \Delta s \right. \\ & \quad \left. - c_1 \int_0^t (a_1 \sigma(s) + b_1) f(s, A_2 u, A_1 u, A_0 u, A_2 v, A_1 v, A_0 v) \Delta s \right| \\ & \quad + \frac{\lambda |a_1 K_{11} - c_1 K_{11} D_{21}|}{\rho_1 (1 - D_{11} D_{21})} \int_0^{\sigma(T)} B_1(s) \int_0^{\sigma(T)} G_1(s, \tau) f(s, A_2 u, A_1 u, A_0 u, A_2 v, A_1 v, A_0 v) \Delta \tau \Delta s \\ & \quad + \frac{\lambda |a_1 K_{21} D_{11} - c_1 K_{21}|}{\rho_1 (1 - D_{11} D_{21})} \int_0^{\sigma(T)} A_1(s) \int_0^{\sigma(T)} G_1(s, \tau) f(s, A_2 u, A_1 u, A_0 u, A_2 v, A_1 v, A_0 v) \Delta \tau \Delta s \\ & \leq L'_1. \end{aligned}$$

Thus, for any $(u, v) \in \Omega$ and $\forall \varepsilon > 0$, let $\delta = \frac{\varepsilon}{L'_1}$, then for $t_1, t_2 \in I, |t_1 - t_2| < \delta$, we have

$$|T_\lambda(u, v)(t_1) - T_\lambda(u, v)(t_2)| \leq L'_1 |t_1 - t_2| < \varepsilon.$$

So, for all $(u, v) \in \Omega$, $T_\lambda \Omega$ is equicontinuous. By Ascoli-Arzela theorem, we obtain that $T_\lambda : P \rightarrow P$ is completely continuous. In a similar way, we can prove that $T_\mu : P \rightarrow P$ is completely continuous. Therefore, $T : P \rightarrow P$ is completely continuous. This completes the proof.

3 Main results

In this section, we will give our main results on multiplicity of positive solutions of system (1.1). In the following, for convenience, we set

$$f_\beta = \lim_{\sum_{i=1}^6 |\varphi_i| \rightarrow \beta} \inf_{t \in I'} \frac{f(t, \varphi_1, \dots, \varphi_6)}{q_1(t) \sum_{i=1}^6 |\varphi_i|}, \quad f^\alpha = \lim_{\sum_{i=1}^6 |\varphi_i| \rightarrow \alpha} \sup_{t \in I'} \frac{f(t, \varphi_1, \dots, \varphi_6)}{q_2(t) \sum_{i=1}^6 |\varphi_i|},$$

$$g_\beta = \lim_{\sum_{i=1}^6 |\varphi_i| \rightarrow \beta} \inf_{t \in I'} \frac{f(t, \varphi_1, \dots, \varphi_6)}{q_3(t) \sum_{i=1}^6 |\varphi_i|}, \quad g^\alpha = \lim_{\sum_{i=1}^6 |\varphi_i| \rightarrow \alpha} \sup_{t \in I'} \frac{f(t, \varphi_1, \dots, \varphi_6)}{q_4(t) \sum_{i=1}^6 |\varphi_i|},$$

where $q_i(t), q_j(t) \in C_{rd}(I', \mathbb{R}^+)$ satisfy

$$0 < \int_0^{\sigma(T)} G_1(\sigma(s), s) q_i(s) \Delta s < +\infty \quad (i = 1, 2),$$

$$0 < \int_0^{\sigma(T)} G_2(\sigma(s), s) q_j(s) \Delta s < +\infty \quad (j = 3, 4).$$

Theorem 3.1. Assume that (H_1) holds. Assume further that

(H_2) there exist a constant $R > 0$, and two functions $p_i(t) \in C_{rd}(I, \mathbb{R}_+)$ satisfying

$$0 < \int_0^{\sigma(T)} G_i(\sigma(s), s) p_i(s) \Delta s < +\infty \quad (i = 1, 2) \text{ such that}$$

$$f(t, \varphi_1, \dots, \varphi_6) \leq R p_1(t), \quad \forall t \in I, \quad 0 < \sum_{i=1}^6 |\varphi_i| \leq R,$$

$$g(t, \varphi_1, \dots, \varphi_6) \leq R p_2(t), \quad \forall t \in I, \quad 0 < \sum_{i=1}^6 |\varphi_i| \leq R,$$

and one of the following conditions is satisfied

$$(E_1) \lambda \in \left(\frac{M_3}{f_0}, M_4\right), \mu \in \left(\frac{N_3}{g_\infty}, N_4\right),$$

$$(E_2) \lambda \in \left(\frac{M_3}{f_\infty}, M_4\right), \mu \in \left(\frac{N_3}{g_0}, N_4\right),$$

$$(E_3) \lambda \in \left(\frac{M_3}{F_\alpha}, M_4\right), \mu \in (0, N_4),$$

$$(E_4) \lambda \in (0, M_4), \mu \in \left(\frac{N_3}{G_\alpha}, N_4\right),$$

where

$$M_3 = \left(\frac{m^2}{M} \int_0^{\sigma(T)} G_1(\sigma(s), s) q_1(s) \Delta s\right)^{-1}, \quad M_4 = \left(O_1 M N \int_0^{\sigma(T)} G_1(\sigma(s), s) p_1(s) \Delta s\right)^{-1},$$

$$N_3 = \left(\frac{m^2}{M} \int_0^{\sigma(T)} G_2(\sigma(s), s) q_3(s) \Delta s\right)^{-1}, \quad N_4 = \left(o_2 M N \int_0^{\sigma(T)} G_2(\sigma(s), s) p_2(s) \Delta s\right)^{-1},$$

$$F_\alpha = \min\{f_0, f_\infty\}, \quad G_\alpha = \{g_0, g_\infty\},$$

O_1, O_2 satisfy $\frac{1}{O_1} + \frac{1}{O_2} \leq 1, N = 1 + \sigma(T) + (\sigma(T))^2$. Then system (1.1) has at least two positive solutions.

Proof. We only prove the case in which (H_2) and (E_1) hold, the other case can be proved similarly. Firstly, from (2.2), we have

$$\sum_{i=0}^2 A_i u(t) + \sum_{i=0}^2 A_i v(t) \leq \|(u, v)\| + \sigma(T)\|(u, v)\| + (\sigma(T))^2\|(u, v)\| = N\|(u, v)\|. \quad (3.1)$$

Take $R_1 = \frac{R}{N}$, and let $\Omega_1 = \{(u, v) \in Q; \|(u, v)\| < R_1\}$. For any $t \in I, (u, v) \in \partial\Omega_1 \cap P$, it follows from $\lambda < M_4, \mu < N_4$ and (H_2) that

$$\begin{aligned} T_\lambda(u, v)(t) &= \lambda \int_0^{\sigma(T)} H_1(t, s) f(s, A_2 u, A_1 u, A_0 u, A_2 v, A_1 v, A_0 v) \Delta s \\ &\leq M_4 M \int_0^{\sigma(T)} G_1(\sigma(s), s) f(s, A_2 u, A_1 u, A_0 u, A_2 v, A_1 v, A_0 v) \Delta s \\ &\leq M_4 M R \int_0^{\sigma(T)} G_1(\sigma(s), s) p_1(s) \Delta s \\ &= N M_4 M R_1 \int_0^{\sigma(T)} G_1(\sigma(s), s) p_1(s) \Delta s \leq \frac{1}{O_1} R_1 \end{aligned}$$

and

$$\begin{aligned} T_\mu(u, v)(t) &= \mu \int_0^{\sigma(T)} H_2(t, s) g(s, A_2 u, A_1 u, A_0 u, A_2 v, A_1 v, A_0 v) \Delta s \\ &\leq N_4 M \int_0^{\sigma(T)} G_2(\sigma(s), s) g(s, A_2 u, A_1 u, A_0 u, A_2 v, A_1 v, A_0 v) \Delta s \\ &\leq N_4 M R \int_0^{\sigma(T)} G_2(\sigma(s), s) p_2(s) \Delta s \\ &= N_4 M N R_1 \int_0^{\sigma(T)} G_2(\sigma(s), s) p_2(s) \Delta s \leq \frac{1}{O_2} R_1. \end{aligned}$$

Consequently, for any $(u, v) \in \partial\Omega_1 \cap P$, we have

$$\|T(u, v)\| = \|T_\lambda(u, v)\| + \|T_\mu(u, v)\| < \frac{1}{O_1} R_1 + \frac{1}{O_2} R_1 \leq R_1. \quad (3.2)$$

Second, from $\lambda > \frac{M_3}{f_0}$, we can choose $\varepsilon_1 > 0$ such that $\lambda f_0 > M_3 + \varepsilon_1$, then there exists $0 < l_1 < N R_1$ such that for any $\sum_{i=1}^6 |\varphi_i| < l_1$ and $t \in I$,

$$f(t, \varphi_1, \dots, \varphi_6) \geq \frac{M_3 + \varepsilon_1}{\lambda} q_1(t) \sum_{i=1}^6 |\varphi_i|.$$

And since

$$\sum_{i=0}^2 A_i u(t) + \sum_{i=0}^2 A_i v(t) \geq u(t) + v(t) \geq \frac{m}{M} \|(u, v)\|. \tag{3.3}$$

Take

$$R_2 = \frac{l_1}{N} < R_1.$$

For all $(u, v) \in \Omega_2 \cap P$, where $\Omega_2 = \{(u, v) \in Q; \|(u, v)\| < R_2\}$, we have

$$\sum_{i=0}^2 A_i u(t) + \sum_{i=0}^2 A_i v(t) \geq u(t) + v(t) \geq \frac{m}{M} R_2.$$

Thus, for all $(u, v) \in \Omega_2 \cap P$, we have

$$\begin{aligned} T_\lambda(u, v)(t) &\geq \lambda m \int_0^{\sigma(T)} G_1(\sigma(s), s) f(s, A_2 u, A_1 u, A_0 u, A_2 v, A_1 v, A_0 v) \Delta s \\ &\geq m(M_3 + \varepsilon_1) \left(\sum_{i=0}^2 |A_i u| + \sum_{i=0}^2 |A_i v| \right) \int_0^{\sigma(T)} G_1(\sigma(s), s) q_1(s) \Delta s \\ &\geq M_3 \frac{m^2}{M} R_2 \int_0^{\sigma(T)} G_1(\sigma(s), s) q_1(s) \Delta s = R_2. \end{aligned}$$

Consequently, for all $(u, v) \in \Omega_2 \cap P$, we have

$$\|T(u, v)\| \geq \|T_\lambda(u, v)\| \geq \|(u, v)\|. \tag{3.4}$$

Finally, from $\mu > N_3/g_\infty$, we can choose $\varepsilon_2 > 0$ such that $\mu g_\infty > N_3 + \varepsilon_2$. then, there exists $l_2 > \frac{m}{M} R$ such that for any $\sum_{i=1}^6 |\varphi_i| < l_2$ and $t \in I$,

$$g(t, \varphi_1, \dots, \varphi_6) \geq \frac{N_3 + \varepsilon_2}{\mu} q_3(t) \sum_{i=1}^6 |\varphi_i|.$$

Take

$$R_3 = \left(\frac{m}{M}\right)^{-1} l_2 > R_1.$$

For all $(u, v) \in \Omega_3 \cap P$, where $\Omega_3 = \{(u, v) \in Q; \|(u, v)\| < R_3\}$, from (3.3), we have

$$\sum_{i=0}^2 A_i u(t) + \sum_{i=0}^2 A_i v(t) \geq \frac{m}{M} R_3 = l_2. \tag{3.5}$$

Thus, for all $(u, v) \in \Omega_3 \cap P$, we have

$$\begin{aligned} T_\mu(u, v)(t) &\geq \mu m \int_0^{\sigma(T)} G_2(\sigma(s), s) g(s, A_2 u, A_1 u, A_0 u, A_2 v, A_1 v, A_0 v) \Delta s \\ &\geq m(N_3 + \varepsilon_2) \left(\sum_{i=0}^2 |A_i u| + \sum_{i=0}^2 |A_i v| \right) \int_0^{\sigma(T)} G_2(\sigma(s), s) q_3(s) \Delta s \\ &\geq N_3 \frac{m^2}{M} R_3 \int_0^{\sigma(T)} G_2(\sigma(s), s) q_3(s) \Delta s = R_3. \end{aligned}$$

Consequently, for all $(u, v) \in \Omega_3 \cap P$, we have

$$\|T(u, v)\| \geq \|T_\mu(u, v)\| \geq \|(u, v)\|. \tag{3.6}$$

From (3.2), (3.4), and (ii) of Lemma 2.5, it follows that system (2.1) has one positive solution $(u_1, v_1) \in P$ with $R_2 \leq \|(u_1, v_1)\| \leq R_1$. Therefore, from Lemma 2.1, it follows that system (1.1) has one positive solution (x_1, y_1) . In the same way, from (3.2), (3.6), and (i) of Lemma 2.5, it follows that system (2.1) has one positive solution $(u_2, v_2) \in P$ with $R_1 \leq \|(u_2, v_2)\| \leq R_3$. Therefore, from Lemma 2.1, it follows that system (1.1) has one positive solution (x_2, y_2) . Above all, system (1.1) has at least two positive solutions. This completes the proof.

Theorem 3.2. *Assume that (H_1) holds. Suppose further that*

(H_3) there exist a constant $R_0 > 0$, and two functions $w_i(t) \in C_{rd}(I, R_+)$ ($i = 1, 2$) satisfying $0 < \int_0^{\sigma(T)} G_i(\sigma(s), s)w_i(s)\Delta s < +\infty$ such that

$$f(t, \varphi_1, \dots, \varphi_6) \geq R_0 w_1(t), \quad \forall t \in I, \quad \sum_{i=1}^6 |\varphi_i| > R_0, \tag{3.7}$$

or

$$g(t, \varphi_1, \dots, \varphi_6) \geq R_0 w_2(t), \quad \forall t \in I, \quad \sum_{i=1}^6 |\varphi_i| > R_0. \tag{3.8}$$

Then system (1.1) has at least two positive solutions for each $\lambda \in (M_5, \frac{M_6}{F_0})$ and $\mu \in (N_5, \frac{N_6}{G_0})$, where

$$M_5 = \left(\frac{m^2}{M} \int_0^{\sigma(T)} G_1(\sigma(s), s)w_1(s)\Delta s \right)^{-1}, \quad N_5 = \left(\frac{m^2}{M} \int_0^{\sigma(T)} G_2(\sigma(s), s)w_2(s)\Delta s \right)^{-1},$$

$$M_6 = \left(O_1 MN \int_0^{\sigma(T)} G_1(\sigma(s), s)q_2(s)\Delta s \right)^{-1}, \quad N_6 = \left(o_2 MN \int_0^{\sigma(T)} G_2(\sigma(s), s)q_4(s)\Delta s \right)^{-1},$$

$$F^\alpha = \max\{f^0, f^\infty\} < \infty, \quad G^\alpha = \max\{g^0, g^\infty\} < \infty.$$

Proof. We only prove the case in which (3.7) holds. The other case in which (3.8) holds can be proved similarly.

Take

$$R'_1 = \left(\frac{m}{M} \right)^{-1} R_0$$

and let $\Omega_4 = \{(u, v) \in Q; \|(u, v)\| < R'_1\}$. For any $t \in I$, $(u, v) \in \partial\Omega_4 \cap P$, it follows from $\lambda > M_5$ and (H_3) that

$$\begin{aligned} T_\lambda(u, v)(t) &= \lambda \int_0^{\sigma(T)} H_1(t, s)f(s, A_2u, A_1u, A_0u, A_2v, A_1v, A_0v)\Delta s \\ &\geq M_5 m \int_0^{\sigma(T)} G_1(\sigma(s), s)f(s, A_2u, A_1u, A_0u, A_2v, A_1v, A_0v)\Delta s \\ &\geq m M_5 R_0 \int_0^{\sigma(T)} G_1(\sigma(s), s)w_1(s)\Delta s \\ &= M_5 \frac{m^2}{M} R'_1 \int_0^{\sigma(T)} G_1(\sigma(s), s)w_1(s)\Delta s = R'_1. \end{aligned}$$

Consequently, for any $(u, v) \in \partial\Omega_4 \cap P$, we have

$$\|T(u, v)\| \geq \|T_\lambda(u, v)\| \geq \|(u, v)\|. \tag{3.9}$$

From $\lambda < \frac{M_6}{f^0}$, $\mu < \frac{N_6}{g^0}$, we know that $\lambda < \frac{M_6}{f^0}$, $\mu < \frac{N_6}{g^0}$, we can choose $\varepsilon_3 > 0$ such that $M_6 - \varepsilon_3 > 0$, $N_6 - \varepsilon_3 > 0$ and $\lambda f^0 < M_6 - \varepsilon_3$, $\mu g^0 < N_6 - \varepsilon_3$. Then there exists $0 < l_3 < NR_0 < NR'_1$ such that for any $\sum_{i=1}^6 |\varphi_i| < l_3$ and $t \in I$,

$$f(t, \varphi_1, \dots, \varphi_6) \leq \frac{M_6 - \varepsilon_3}{\lambda} q_2(t) \sum_{i=1}^6 |\varphi_i|,$$

$$g(t, \varphi_1, \dots, \varphi_6) \leq \frac{N_6 - \varepsilon_3}{\mu} q_4(t) \sum_{i=1}^6 |\varphi_i|.$$

Take $R'_2 = \frac{l_3}{N} < R'_1$ and $\Omega_5 = \{(u, v) \in Q; \|(u, v)\| < R'_2\}$. Then, for any $(u, v) \in \Omega_5 \cap P$, from(3.1), we have

$$\begin{aligned} T_\lambda(u, v)(t) &= \lambda \int_0^{\sigma(T)} H_1(t, s) f(s, A_2u, A_1u, A_0u, A_2v, A_1v, A_0v) \Delta s \\ &\leq \lambda M \int_0^{\sigma(T)} G_1(\sigma(s), s) f(s, A_2u, A_1u, A_0u, A_2v, A_1v, A_0v) \Delta s \\ &\leq M(M_6 - \varepsilon_3) \left(\sum_{i=0}^2 |A_i u| + \sum_{i=0}^2 |A_i v| \right) \int_0^{\sigma(T)} G_1(\sigma(s), s) q_2(s) \Delta s \\ &\leq MNM_6 R'_2 \int_0^{\sigma(T)} G_1(\sigma(s), s) q_2(s) \Delta s = \frac{1}{O_1} R'_2 \end{aligned}$$

and

$$\begin{aligned} T_\mu(u, v)(t) &= \mu \int_0^{\sigma(T)} H_2(t, s) g(s, A_2u, A_1u, A_0u, A_2v, A_1v, A_0v) \Delta s \\ &\leq \mu M \int_0^{\sigma(T)} G_2(\sigma(s), s) g(s, A_2u, A_1u, A_0u, A_2v, A_1v, A_0v) \Delta s \\ &\leq M(N_6 - \varepsilon_3) \left(\sum_{i=0}^2 |A_i u| + \sum_{i=0}^2 |A_i v| \right) \int_0^{\sigma(T)} G_2(\sigma(s), s) q_4(s) \Delta s \\ &\leq MNN_6 R'_2 \int_0^{\sigma(T)} G_2(\sigma(s), s) q_4(s) \Delta s = \frac{1}{O_2} R'_2. \end{aligned}$$

Consequently, for any $(u, v) \in \partial\Omega_5 \cap P$, we have

$$\|T(u, v)\| = \|T_\lambda(u, v)\| + \|T_\mu(u, v)\| < \frac{1}{O_1} R'_2 + \frac{1}{O_2} R'_2 \leq R'_2. \tag{3.10}$$

From $\lambda < \frac{M_6}{F^\alpha}$, $\mu < \frac{N_6}{G^\alpha}$ we know that $\lambda < \frac{M_6}{f^\infty}$, $\mu < \frac{N_6}{g^\infty}$, we can choose $\varepsilon_4 > 0$ such that $M_6 - \varepsilon_4 > 0$, $N_6 - \varepsilon_4 > 0$ and $\lambda f^\infty < M_6 - \varepsilon_4$, $\mu g^\infty < N_6 - \varepsilon_4$. Then there exists $l_4 > \frac{m}{M}R'_1$ such that for any $\sum_{i=1}^6 |\varphi_i| > l_4$ and $t \in I$,

$$f(t, \varphi_1, \dots, \varphi_6) \leq \frac{M_6 - \varepsilon_3}{\lambda} q_2(t) \sum_{i=1}^6 |\varphi_i|,$$

$$g(t, \varphi_1, \dots, \varphi_6) \leq \frac{N_6 - \varepsilon_3}{\mu} q_4(t) \sum_{i=1}^6 |\varphi_i|.$$

Take $R'_3 = (\frac{m}{M})^{-1}l_4 > R'_1$ and let $\Omega_6 = \{(u, v) \in Q; \|(u, v)\| < R'_3\}$. Then, for any $(u, v) \in \Omega_6 \cap P$, we have

$$\begin{aligned} T_\lambda(u, v)(t) &= \lambda \int_0^{\sigma(T)} H_1(t, s) f(s, A_2 u, A_1 u, 3A_0 u, A_2 v, A_1 v, A_0 v) \Delta s \\ &\leq \lambda M \int_0^{\sigma(T)} G_1(\sigma(s), s) f(s, A_2 u, A_1 u, A_0 u, A_2 v, A_1 v, A_0 v) \Delta s \\ &\leq M(M_6 - \varepsilon_3) \left(\sum_{i=0}^2 |A_i u| + \sum_{i=0}^2 |A_i v| \right) \int_0^{\sigma(T)} G_1(\sigma(s), s) q_2(s) \Delta s \\ &\leq M N M_6 R'_3 \int_0^{\sigma(T)} G_1(\sigma(s), s) q_2(s) \Delta s = \frac{1}{O_1} R'_3 \end{aligned}$$

and

$$\begin{aligned} T_\mu(u, v)(t) &= \mu \int_0^{\sigma(T)} H_2(t, s) g(s, A_2 u, A_1 u, A_0 u, A_2 v, A_1 v, A_0 v) \Delta s \\ &\leq \mu M \int_0^{\sigma(T)} G_2(\sigma(s), s) g(s, A_2 u, A_1 u, A_0 u, A_2 v, A_1 v, A_0 v) \Delta s \\ &\leq M(N_6 - \varepsilon_3) \left(\sum_{i=0}^2 |A_i u| + \sum_{i=0}^2 |A_i v| \right) \int_0^{\sigma(T)} G_2(\sigma(s), s) q_4(s) \Delta s \\ &\leq M N N_6 R'_3 \int_0^{\sigma(T)} G_2(\sigma(s), s) q_4(s) \Delta s = \frac{1}{O_2} R'_3. \end{aligned}$$

Consequently, for any $(u, v) \in \partial\Omega_6 \cap P$, we have

$$\|T(u, v)\| = \|T_\lambda(u, v)\| + \|T_\mu(u, v)\| < \frac{1}{O_1} R'_3 + \frac{1}{O_2} R'_3 \leq R'_3. \tag{3.11}$$

From (3.9), (3.10) and (i) of Lemma 2.5, it follows that system (2.1) has one positive solution $(u_1, v_1) \in P$ with $R'_1 \leq \|(u_1, v_1)\| \leq R'_2$. Therefore, from Lemma 2.1, it follows that system (1.1) has one positive solution (x_1, y_1) . In the same way, from (3.9), (3.11) and (ii) of Lemma 2.5, it follows that system (2.1) has one positive solution $(u_2, v_2) \in$

P with $R'_1 \leq \|(u_2, v_2)\| \leq R'_3$. Therefore, from Lemma 2.1, it follows that system (1.1) has one positive solution (x_2, y_2) . Above all, system (1.1) has at least two positive solutions. This completes the proof.

4 An example

Consider the following BVP with integral boundary conditions:

$$\left\{ \begin{array}{l} x^{(4\Delta)}(t) + \lambda f(t, x(t), x^\Delta(t), x^{\Delta\Delta}(t), \gamma(t), \gamma^\Delta(t), \gamma^{\Delta\Delta}(t)) = 0, \quad t \in (0, \sigma(T))_{\mathbb{T}}, \\ \gamma^{(4\Delta)}(t) + \mu g(t, x(t), x^\Delta(t), x^{\Delta\Delta}(t), \gamma(t), \gamma^\Delta(t), \gamma^{\Delta\Delta}(t)) = 0, \quad t \in (0, \sigma(T))_{\mathbb{T}}, \\ x(0) = x^\Delta(0) = 0, \\ \gamma(0) = \gamma^\Delta(0) = 0, \\ x^{\Delta\Delta}(0) - x^{\Delta\Delta\Delta}(0) = \int_0^{\sigma(T)} x^{\Delta\Delta}(s)A_1(s)\Delta s, \\ x^{\Delta\Delta}(\sigma(T)) + x^{\Delta\Delta\Delta}(\sigma(T)) = \int_0^{\sigma(T)} x^{\Delta\Delta}(s)B_1(s)\Delta s, \\ \gamma^{\Delta\Delta}(0) - \gamma^{\Delta\Delta\Delta}(0) = \int_0^{\sigma(T)} \gamma^{\Delta\Delta}(s)A_2(s)\Delta s, \\ \gamma^{\Delta\Delta}(\sigma(T)) + \gamma^{\Delta\Delta\Delta}(\sigma(T)) = \int_0^{\sigma(T)} \gamma^{\Delta\Delta}(s)B_2(s)\Delta s, \end{array} \right. \quad (4.1)$$

where $A_1(t) = B_1(t) = t$, $A_2(t) = B_2(t) = t/2$ and

$$f(t, \phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6) = 2t \left(\sum_{i=1}^6 \phi_i \right)^{\frac{1}{2}}, \quad t \in (0, \sigma(T))_{\mathbb{T}}, \quad \phi_i \geq 0, \quad i = 1, \dots, 6,$$

$$g(t, \phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6) = \frac{t}{2} \left(\sum_{i=1}^6 \phi_i \right)^3, \quad t \in (0, \sigma(T))_{\mathbb{T}}, \quad \phi_i \geq 0, \quad i = 1, \dots, 6.$$

we choose $O_1 = 2$, $O_2 = 4$, $R = 1$, $p_1(t) = 2t$, $p_2(t) = \frac{t}{2}$, $q_1(t) = q_3(t) = 1$. It is easy to check that $f_0 = g_\infty = \infty$, (H_1) , (H_2) and (E_1) are satisfied. Therefore, by Theorem 3.1, system (4.1) has at least two positive solutions for each $\lambda \in (0, M_4)$, $\mu \in (0, N_4)$.

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Authors' contributions

All authors contributed equally to the manuscript and typed, read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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