Eur. Phys. J. C (2015) 75:167
DOI 10.1140/epjc/s10052-015-3389-6

THE EUROPEAN PHYSICAL JOURNAL C

CrossMark

Regular Article - Theoretical Physics

Polarization operator in the 2+1 dimensional quantum electrodynamics with a nonzero fermion density in a constant uniform magnetic field

V. R. Khalilov[a], **I. V. Mamsurov**

Faculty of Physics, M.V. Lomonosov Moscow State University, 119991 Moscow, Russia

Received: 19 February 2015 / Accepted: 31 March 2015 / Published online: 25 April 2015

Abstract The polarization operator (tensor) for planar charged fermions in a constant uniform magnetic field is calculated in the one-loop approximation of $2+1$-dimensional quantum electrodynamics (QED_{2+1}) with a nonzero fermion density. We construct the Green function of the Dirac equation with a constant uniform external magnetic field in QED_{2+1} at a finite chemical potential, find the imaginary part of this Green function, and then obtain the polarization tensor related to the combined contribution from real particles occupying the finite number of energy levels and magnetic field. We expect that some physical effects under consideration seem likely to be revealed in a monolayer graphene sample in the presence of an external constant uniform magnetic field B perpendicular to it.

1 Introduction

Planar charged fermions governed by the Dirac equation with external electromagnetic fields attract considerable interest in connection with problems of the quantum Hall effect [1], high-temperature superconductivity [2] as well as graphene (see, e.g., [3–6]). In graphene, the electron dynamics at low energies is described by the two-dimensional Dirac equation for massless fermions [4,7–9], though the case of massive charged fermions is also of interest [10].

It is well known [11] that every energy level of a planar electron in an uniform magnetic field is degenerate and the number of degenerate states per unit area is $|eB|/(2\pi\hbar c)$. So, the kinetic energy of the electrons can completely be quenched at strong magnetic fields. Moreover, the kinetic energy per Dirac electron is of order $\varepsilon \approx v_{\mathrm{F}}\sqrt{2|eB|\hbar/c}$ (where v_{F} is the Fermi–Dirac velocity) and is comparable with the Coulomb energy per electron $E_C = e^2/(\epsilon_0 l_B)$, where $l_B = \sqrt{\hbar c/|eB|}$ is the so-called magnetic length and ϵ_0 is the dielectric constant of the medium [6]. In the two-dimensional electron gas, this enhancing of the Coulomb interactions between electrons in the presence of strong magnetic fields, probably leads to the fractional quantum Hall effect (FQHE) [12,13]. The features of the fractional quantum Hall effect in graphene were predicted, for example, in [14–16] and the observation of the fractional quantum Hall effect in suspended graphene was reported in [17–19].

Important physical quantities related to the vacuum polarization are the vacuum charge and current densities induced by the background field. Polarization effects in the massive QED_{2+1} with a constant uniform magnetic field and with a nonzero fermion density were studied in [20,21]. In particular, the contribution of the induced Chern–Simons term to the polarization tensor and the effective Lagrangian with the electron density corresponding to the occupation of n Landau levels in an uniform magnetic field were calculated in [21].

Since the effective fine structure constant in graphene is large, QED_{2+1} effects can be significant already in the one-loop approximation. Important quantum relativistic effects were discussed in [22] (Klein paradox) and [23–25] (Casimir effect). The polarization operator in a strong magnetic field perpendicular to the graphene membrane has been calculated in [26–28]. The problem of light absorption in graphene was investigated in [29] and the Faraday effect in a monolayer graphene sample in a strong constant uniform magnetic field perpendicular to it was considered in [30].

The induced vacuum current in the field of a solenoid perpendicular to the graphene sample was investigated in [31], and vacuum polarization in QED_{2+1} with an Aharonov–Bohm (AB) potential for massive and massless fermions was studied in [32]. The vacuum electric current due to vacuum polarization in the AB potential for massive case was observed in [33] in “a quantum-tunneling system using

[a] e-mail: khalilov@phys.msu.ru

two-dimensional ionic structures in a linear Paul trap”. A very important phenomenon—charged impurity screening in graphene due to the vacuum polarization by a Coulomb field—was investigated in [9,34–37]. The effect of spin on the dynamics of the two-dimensional Dirac oscillator in a magnetic cosmic string background was considered in [38].

In this work, we have calculated the polarization tensor of planar charged fermions in the presence of an external constant uniform magnetic field in the one-loop approximation of QED_{2+1} at the finite chemical potential. We have shown that the one-loop polarization tensor induces physical effects, which seem to be likely to be revealed in a monolayer graphene sample in a strong constant uniform magnetic field aligned perpendicularly to the sample.

We shall adopt the units where $c = \hbar = 1$.

2 Vacuum polarization by a constant uniform magnetic field in QED_{2+1}

The polarization operator (PO) in a constant uniform magnetic field in QED_{2+1} is diagonal with respect to the photon three-momentum and in the momentum representation is determined by

$$\Pi^{\mu\nu}(p) = -ie^2 \int \frac{d^3k}{(2\pi)^3} \mathrm{tr}[\gamma^\mu S^c(k, B)\gamma^\nu S^c(k-p, B)], \tag{1}$$

where $S^c(k, B)$ is the causal Green function of the Dirac equation with a constant uniform magnetic field B in the momentum representation. In the coordinate representation the Green function $S^c(x^\mu, x'^\mu, B)$ of the Dirac equation for a fermion of the mass m and charge e in an external constant uniform magnetic field in $2 + 1$ dimensions satisfies the equation

$$(\gamma^\mu P_\mu - m)S(x^\mu, x'^\mu, B) = \delta^3(x^\mu - x'^\mu), \tag{2}$$

where $x^\mu = x^0, x^1, x^2 \equiv t, x, y$ is the three-vector, $P_\mu = -i\partial_\mu - eA_\mu$ is the generalized fermion momentum operator. It is well known [39] that the Green's function in a constant magnetic field is not translation invariant in the coordinate representation; it is of a product of the non-translation invariant phase and a translation invariant function.

The Dirac γ^μ-matrix algebra in $2 + 1$ dimensions is well known to be representable in terms of the two-dimensional Pauli matrices σ_j,

$$\gamma^0 = \tau\sigma_3, \quad \gamma^1 = i\sigma_1, \quad \gamma^2 = i\sigma_2, \tag{3}$$

where the parameter $\tau = \pm 1$ can label two types of fermions in accordance with the signature of the two-dimensional Dirac matrices [40]; it can be applied to characterize two states of the fermion spin (spin “up” and “down”) [41]. We take the magnetic field vector potential in the Cartesian coordinates in the Landau gauge $A_0 = 0$, $A_1 = 0$, $A_2 = Bx$; then the magnetic field is defined as $B = \partial_1 A_2 - \partial_2 A_1 \equiv F_{21}$, where $F_{\mu\nu}$ is the electromagnetic field tensor.

The positive-frequency Dirac equation solutions (the particle states) in the considered field corresponding to the energy eigenvalues (the Landau levels),

$$E_n^+ \equiv E_n = \sqrt{m^2 + 2n|eB|}, \quad n = 0, 1, \ldots, \tag{4}$$

are given by [42]

$$\Psi^+(t, \mathbf{r}) = \frac{1}{\sqrt{2E_n}} \begin{pmatrix} \sqrt{E_n + \tau m}\, U_n(z) \\ -\mathrm{sign}(eB)\sqrt{E_n - \tau m}\, U_{n-1}(z) \end{pmatrix} \times \exp(-iE^+ t + ip_2 y), \tag{5}$$

where the normalized functions $U_n(z)$ are expressed through the Hermite polynomials $H_n(z)$ as

$$U_n(z) = \frac{|eB|^{1/4}}{(2^n n! \pi^{1/2})^{1/2}} e^{-z^2/2} H_n(z), \; z = \sqrt{|eB|}(x - p_2/eB),$$

and p_2 is the eigenvalue because $-i\partial_y \Psi^+(t, \mathbf{r}) = p_2 \Psi^+(t, \mathbf{r})$. All the energy levels except the lowest level ($n = 0$) with $\tau = 1$ for $eB > 0$ and $\tau = -1$ for $eB < 0$ are doubly degenerate on spin $\tau = \pm 1$. This means that the eigenvalues of the fermion energy except the lowest level are actually spin-independent in the configuration under investigation. For definiteness, we consider the case where $eB < 0$. The negative-frequency Dirac equation solutions (the antiparticle states) corresponding to negative energies $E_n^- = -E_n$ can be constructed from (5) by means of the charge-conjugation operation.

The exact expression for the free electron propagator in an external magnetic field in $3 + 1$ dimensions was found for the first time by Schwinger [39]. The Green function in a constant uniform magnetic field in the momentum representation in 2+1 dimensions was obtained in [42] in the form

$$S^c(p, B) = -\frac{i}{|eB|} \int_0^\infty \frac{dz}{\cos z} \exp\left[\frac{iz}{|eB|}\left(p_0^2 - m^2 - \mathbf{p}^2 \frac{\tan z}{z} + i\epsilon\right)\right] \times \left[(\gamma^0 p^0 + m)\exp(i\tau\sigma_3 z) - \frac{(\gamma^1 p^1 + \gamma^2 p^2)}{\cos z}\right], \tag{6}$$

where $z = |eB|s$ and s is the “proper time”. It is well to note that $S^c(p, B)$ *is the Fourier transform only of the translation invariant part of* $S(x^\mu, x'^\mu, B)$ [42]. The main properties and tensor structure of the PO can be obtained from the requirements of relativistic and gauge invariance and also from the symmetry of the external field. In the considered external field the PO must be diagonal with respect to the “photon” three-momentum and depend only on three independent scalars, which can be constructed from the three-momentum p^μ and the tensor of external magnetic field $F^{\mu\nu}$:

$$p^2=(p^0)^2-\mathbf{p}^2,\quad p_\mu F^{\mu\nu}F_{\nu\rho}p^\rho\equiv B^2\mathbf{p}^2,\quad F^{\mu\nu}F_{\mu\nu}\equiv 2B^2. \tag{7}$$

We introduce the orthonormalized system of the three-vectors l_i^μ,

$$l_1^\mu=\frac{1}{\sqrt{\mathbf{p}^2}}(0,-p_2,p_1),\quad l_2^\mu=\frac{1}{\sqrt{p^2\mathbf{p}^2}}(\mathbf{p}^2,p_0p_1,p_0p_2),$$
$$l_3^\mu=\frac{1}{\sqrt{p^2}}(p_0,p_1,p_2), \tag{8}$$

which satisfy the relations

$$g_{\mu\nu}l_i^\mu l_k^\nu=g_{ik},\quad \sum_{i,k}g^{ik}l_i^\mu l_k^\nu=g^{\mu\nu},$$
$$-(g^{\mu\nu}-p^\mu p^\nu/p^2)=\sum_{j=1,2}l_j^\mu l_j^\nu, \tag{9}$$

where $g^{\mu\nu}$ is the Minkowski tensor $g^{11}=g^{22}=-g^{00}=-1$, and the nonzero diagonal components of g^{ik} are $g^{11}=g^{22}=-g^{33}=-1$.

The vectors l_i^μ, $i=1,2$ are not eigenvectors of the PO; the PO eigenvalue corresponding to the eigenvector l_3^μ is equal to zero due to the gauge invariance,

$$p_\mu\Pi^{\mu\nu}=\Pi^{\mu\nu}p_\nu=0. \tag{10}$$

As a result of the calculations, we find the PO in the fully transversal form [21]:

$$\Pi^{\mu\nu}(p,b)=\frac{i^{1/2}e^2}{8\pi^{3/2}b^{1/2}}\int\limits_0^\infty\frac{x^{1/2}\mathrm{d}x}{\sin x}\int\limits_{-1}^1\mathrm{d}u[\Pi_1 l_1^\mu l_1^\nu$$
$$+\Pi_2 l_2^\mu l_2^\nu+C\times i\tau e^{\mu\nu\rho}p_\rho]$$
$$\times\exp\left[\frac{ix}{b}\left(\frac{p_0^2(1-u^2)}{4}\right.\right.$$
$$-\frac{\mathbf{p}^2\sin[x(1+u)/2]\sin[x(1-u)/2]}{x\sin x}$$
$$\left.\left.-m^2+i\epsilon\right)\right],\quad b=|eB|, \tag{11}$$

where

$$\Pi_1=p_0^2(\cos ux-u\cot x\sin ux)$$
$$-\mathbf{p}^2\frac{u\sin x\sin 2ux-2\cos ux+\cos x(1+\cos 2ux)}{\sin^2 x},$$
$$\Pi_2=p^2(\cos ux-u\cot x\sin ux),\quad C=2m\cos ux \tag{12}$$

and $e^{\mu\nu\rho}$ is a fully antisymmetric unit tensor. We note that $\Pi^{\mu\nu}(p)$ is not symmetric tensor in $2+1$ dimensions. In (11) the last term is the so-called induced Chern–Simons term; using the relation $e^{\mu\nu\rho}p_\rho=-\sqrt{p^2}(l_1^\mu l_2^\nu-l_2^\mu l_1^\nu)$ it can be written in another form. It should be noted that the mass term in the considered QED_{2+1} model is not invariant with respect to the operations of spatial (and time) inversion, therefore the induced Chern–Simons term must be generated dynamically by the external magnetic field; it contributes to the vacuum polarization only in the one-loop QED_{2+1} approximation [43,44].

In the limit $eB=0$, we obtain from (11)

$$\Pi^{\mu\nu}(p)=\frac{i^{1/2}e^2}{8\pi^{3/2}}\int\limits_0^\infty\frac{\mathrm{d}s}{\sqrt{s}}\int\limits_{-1}^1\mathrm{d}u[(1-u^2)(g^{\mu\nu}p^2-p^\mu p^\nu)$$
$$-2\mathrm{i}m\tau e^{\mu\nu\rho}p_\rho]\exp\left[is\left(\frac{p^2(1-u^2)}{4}-m^2+i\epsilon\right)\right]. \tag{13}$$

The PO (13) is a function of only one scalar p^2 and its analytic properties can be studied in the complex p^2 plane. It is important that $\Pi^{\mu\nu}(p)$ is a real function on the negative real half axis $p^2<0$, which allows us to perform integrations in (13) for the domain $p^2<0$ and to obtain [21]

$$\Pi^{\mu\nu}(p)=\frac{e^2}{4\pi}\left[(g^{\mu\nu}p^2-p^\mu p^\nu)\left(\frac{4m^2+p^2}{p^2\sqrt{-p^2}}\arctan\sqrt{\frac{-p^2}{4m^2}}\right.\right.$$
$$\left.\left.-\frac{2m}{p^2}\right)--im\tau e^{\mu\nu\rho}\frac{4p_\rho}{\sqrt{-p^2}}\arctan\sqrt{\frac{-p^2}{4m^2}}\right]. \tag{14}$$

It should be noted that the free polarization operator in 2+1 dimensions was obtained in another form and without the Chern–Simons term in [45].

The singularities of $\Pi^{\mu\nu}(p)$ lie on the positive real half axis of p^2 and the point $p^2=4m^2$ is the branch point (the threshold for the creation of fermion pairs), so $\Pi^{\mu\nu}(p)$ is an analytic function in the complex p^2 plane with a cut $[4m^2,\infty)$; the domains $p^2<0$ and $p^2>4m^2$ are physical domains, and the domain $0\le p^2\le 4m^2$ is nonphysical (the definition of the physical/nonphysical domain is given, for example, in [46]). The function $\Pi^{\mu\nu}(p)$ in the whole domain of p^2 can be obtained by the analytic continuation of (14). In the domain $p^2>4m^2$ the free polarization operator gets the imaginary part, which is on the upper edge of the cut:

$$\mathrm{Im}\,\Pi^{\mu\nu}(p)=-\frac{e^2}{4\pi}(g^{\mu\nu}p^2-p^\mu p^\nu)\frac{4m^2+p^2}{p^2\sqrt{p^2}}. \tag{15}$$

The imaginary part of the PO has a discontinuity $2\mathrm{Im}\,\Pi^{\mu\nu}(p)$ in going across the cut.

We can find the polarization operator for charged massless fermions putting in the above formulas $m=0$. In particular, the free polarization operator for the case $m=0$ is a real function on the negative real half axis $p^2<0$ and has the extremely simple form:

$$\Pi^{\mu\nu}(p,m=0)=\frac{e^2}{8}\frac{(g^{\mu\nu}p^2-p^\mu p^\nu)}{\sqrt{-p^2}}. \tag{16}$$

The free polarization operator (16) is transverse. Now the point $p^2 = 0$ is the branch point (the threshold for the creation of massless fermion pairs), so $\Pi^{\mu\nu}(p, m = 0)$ is an analytic function in the complex p^2 plane with a cut $[0, \infty)$. In the domain $p^2 > 0$ $\Pi^{\mu\nu}(p, m = 0)$ is pure imaginary and on the upper edge of the cut it has the form

$$\Pi^{\mu\nu}(p, m = 0) = -i\frac{e^2}{8}\frac{(g^{\mu\nu}p^2 - p^\mu p^\nu)}{\sqrt{p^2}}. \quad (17)$$

In this form the polarization operator has been calculated in [45]. In condensed matter problems, the $\Pi^{00}(p, m = 0)$ component of the polarization tensor is actual one that, for example, at $p^2 < 0$ has the form

$$\Pi^{00}(p, m = 0) = -\frac{e^2}{8}\frac{\mathbf{p}^2}{\sqrt{\mathbf{p}^2 - p_0^2}}. \quad (18)$$

This formula is in agreement with that obtained for graphene in the so-called random phase approximation in [47] (see also [5,48]).

It is convenient to represent the polarization operator for the case $m = 0$ in a weak constant uniform magnetic field as follows:

$$\Pi^{\mu\nu}(p, eB, m = 0) = \Pi^{\mu\nu}(p, m = 0) + e^2(eB)^2\left[\pi_1(p)\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) + \pi_2(p)l_1^\mu l_1^\nu\right], \quad (19)$$

where the first term is the PO $\Pi^{\mu\nu}(p, m = 0)$ and $\pi_1(p), \pi_2(p)$ are functions of only p. In particular, in the domain $p^2 < 0$ they can be estimated up to numerical constants as

$$\pi_1(p) \approx (a\mathbf{p}^2 + cp^2)/(-p^2)^{5/2}, \quad \pi_2(p) \approx d\mathbf{p}^2/(-p^2)^{5/2}, \quad (20)$$

where $1 > a > c > d > 0.1$. These formulas are in agreement with results obtained in [49]. Since fermions are massless, dimensionless factors eB/p^2 are built with $|p|$ in place of m.

To calculate the main contribution in (11) in a strong magnetic field let us rotate the contour of integration in x on $-\pi/2$ to obtain

$$\Pi^{\mu\nu}(p, b) = \frac{e^2}{8\pi^{3/2}\sqrt{b}}e^{-\mathbf{p}^2/2b}(g^{\mu\nu}p^2 - p^\mu p^\nu) \times \int_0^\infty \frac{x^{1/2}\mathrm{d}x}{\sinh x}\int_{-1}^1 \mathrm{d}u(u\coth x\sinh x - \cosh ux) \times \exp\left[-\frac{x}{b}\left(\frac{p_0^2(1-u^2)}{4} - m^2\right)\right]. \quad (21)$$

Neglecting the term $\sim p_0^2/b$ in the exponent we integrate (21) in u and obtain

$$\Pi^{\mu\nu}(p, b) = -\frac{e^2}{4\pi^{3/2}\sqrt{b}}e^{-\mathbf{p}^2/2b}(g^{\mu\nu}p^2 - p^\mu p^\nu) \times \int_0^\infty \frac{\mathrm{d}x}{\sqrt{x}}\left(\frac{\coth x}{x} - \frac{1}{\sinh^2 x}\right)e^{-xm^2/b}. \quad (22)$$

We see that the integral is maximum at $m = 0$, so we can estimate it putting $m = 0$ in the exponent. But the integrand does not contain b at $m = 0$. Therefore, the leading contribution in b in (22) is proportional to $b^{-1/2}$

$$\Pi^{\mu\nu}(p, b) = \frac{e^2 C}{4\pi^{3/2}\sqrt{b}}(g^{\mu\nu}p^2 - p^\mu p^\nu), \quad C \sim 2. \quad (23)$$

This result is in agreement with that obtained in [50]. Thus, the polarization tensor in QED_{2+1} is very small ($\sim |eB|^{-1/2}$) in a large constant uniform magnetic field. This feature of the PO in QED_{2+1} essentially differs from that in the massive QED_{3+1}, where the Π^{33} component plays a major role and increases in a large magnetic field as [51] $\Pi^{33} \sim |eB|/m^2$. Physically, this is because only electrons from the lowest Landau level couple to the longitudinal components of the photon at $|eB| \gg m^2$ [50–52]. It is seen that the vectors l_j^μ, $j = 1, 2$ are spacelike if $p^2 > 0$, but if $p^2 = 0$, then the only vector l_1^μ is still spacelike, whereas $l_2^\mu \to p^\mu/\sqrt{p^2}$. It means that the "first" photon mode ($\sim l_1^\mu$) becomes almost free in a very strong magnetic field, and the "second" photon mode ($\sim l_2^\mu$) therefore does not exist.

3 Green function and polarization operator at a nonzero fermion density

Now we construct the Green function to the Dirac equation for the case of a nonzero fermion density (the finite chemical potential). For graphene, only the case with a fixed sign of τ is interesting, and we hence assume that $\tau = 1$ in this section, without restricting the generality. The Green function with the finite chemical potential can be obtained from the Green function in the momentum representation (6) by shifting the variable $p^0 \to p^0 + \mu + i\delta\mathrm{sign}(p^0)$, where μ is the chemical potential. We have

$$S^c(p, m, \mu) = -\frac{i}{|eB|}\int_0^\infty \frac{\mathrm{d}z}{\cos z}\exp\left[\frac{iz}{|eB|}\left(\left[p^0 + \mu + i\delta\mathrm{sign}p^0\right]^2 - m^2 - \mathbf{p}^2\frac{\tan z}{z} + i\epsilon\right)\right] \times \left[(\gamma^0(p^0 + \mu) + m)\exp(i\sigma_3 z) - \frac{(\gamma^1 p^1 + \gamma^2 p^2)}{\cos z}\right], \quad \delta \to +0. \quad (24)$$

If $\mu > m$, there are real particles occupying the Landau levels, if $\mu < m$, then there are no real particles. We also assume $\mu > 0$ without loss of generality. The integration path passes below the singularities in the integrand in (24) and the imaginary term $i\delta\,\mathrm{sign}\,p^0$ is essential near the poles [53]; the Green function (24) has the poles at the points $p^0 = \pm E_n - \mu$ as well as an imaginary part $\mathrm{Im}S^c(p,\mu)$ related to the presence of real charged fermions. Rotating the integration path over z in (24) into the lower half-plane we can extend (24) onto the whole complex plane of p^0 with the cuts $[p^0_+,\infty)$, $[-p^0_-,-\infty)$ at the real axis. Then, as a function of p^0, the Green function (24) is a limit of some analytic function $S^c(p^0)$. Denoting the integrand in (24) as $S(z)$, we represent $\mathrm{Im}S^c(p,\mu)$ via the discontinuities ΔS at the edges of the cuts in the form $\int_{-\infty}^{\infty} S(z)\mathrm{d}z$. For this, we apply the method for calculation of the PO discontinuities in the presence of various external electromagnetic fields in vacuum [51,54], which was extended to the case of a nonzero fermion density in [21]. Finally, we obtain $S^c(p,\mu)$ in the form

$$
\begin{aligned}
S^c(p,m,\mu) = &-\frac{i}{|eB|}\int_0^\infty \frac{\mathrm{d}z}{\cos z}\exp\left[\frac{iz}{|eB|}\left(\left[p^0+\mu\right.\right.\right.\\
&\left.\left.\left.+i\delta\,\mathrm{sign}\,p^0\right]^2 - m^2 - \mathbf{p}^2\frac{\tan z}{z} + i\epsilon\right)\right](\gamma P + M)\\
&-2i\pi(\gamma\bar{P}+\bar{M}),\quad \delta\to +0,
\end{aligned}
\tag{25}
$$

where

$$
\begin{aligned}
P^0 &= (p_0+\mu)\cos z + im\sin z,\quad M = m\cos z + i(p_0+\mu)\sin z,\\
&\quad \mathbf{P} = -\mathbf{p}/\cos z,\\
\bar{P} &= [m_0(p_0+\mu+m)/2m]\delta[(p_0+\mu-m)/m_0]e^{-\mathbf{p}^2/|eB|}\\
&\quad + mm_0\sum_{n=1}^{\infty}\frac{(-1)^n}{E_n}I_{nn}(\mathbf{p}^2/|eB|)\,[\delta[(p_0+\mu-E_n)/m_0]\\
&\quad + \delta[(p_0+\mu+E_n)/m_0]]\,,\\
\bar{M} &= [m_0(p_0+\mu+m)/2m]\delta[(p_0+\mu-m)/m_0]e^{-\mathbf{p}^2/|eB|}\\
&\quad + (p^0+\mu)m_0\sum_{n=1}^{\infty}\frac{(-1)^n}{E_n}I_{nn}(\mathbf{p}^2/|eB|)\,[\delta[(p_0+\mu-E_n)/m_0]\\
&\quad + \delta[(p_0+\mu+E_n)/m_0]]\,.
\end{aligned}
\tag{26}
$$

Here m_0 is a parameter of the dimension of mass, we have the Laguerre function $I_{nn}(x) = (1/n!)e^{-x/2}L_n(x)$, $L_n(x)$ is the Laguerre polynomial, and all the differences must satisfy the inequalities $m+\mu > 0$, $E_n^+ - \mu > 0$, and $-m-\mu < 0$, $E_n^- - \mu < 0$.

The Green function for charged massless fermions is easily derived from Eqs. (25) and (26) to read

$$
\begin{aligned}
S^c(p,\mu,m=0) = &-\frac{i}{|eB|}\int_0^\infty \frac{\mathrm{d}z}{\cos z}\exp\left[\frac{iz}{|eB|}\left([p^0+\mu\right.\right.\\
&\left.\left.+i\delta\,\mathrm{sign}\,p^0]^2 - \mathbf{p}^2\frac{\tan z}{z} + i\epsilon\right)\right]\\
&\times(\gamma P + M) - 2i\pi(\gamma\bar{P}+\bar{M}),\quad \delta\to +0,
\end{aligned}
\tag{27}
$$

where

$$
\begin{aligned}
P^0 &= (p_0+\mu)\cos z,\quad M = i(p_0+\mu)\sin z,\\
\mathbf{P} &= -\mathbf{p}/\cos z,\\
\bar{P} &= [m_0(p_0+\mu)/2m]\delta[(p_0+\mu)/m_0]|_{m\to 0}e^{-\mathbf{p}^2/|eB|},\\
\bar{M} &= [m_0(p_0+\mu)/2m]\delta[(p_0+\mu)/m_0]|_{m\to 0}e^{-\mathbf{p}^2/|eB|}\\
&\quad + m_0(p^0+\mu)\sum_{n=1}^{\infty}\frac{(-1)^n}{\varepsilon_n}I_{nn}(\mathbf{p}^2/|eB|)\,[\delta[(p_0+\mu\\
&\quad -\varepsilon_n)/m_0] + \delta[(p_0+\mu+\varepsilon_n)/m_0]]\,,\quad \varepsilon_n^{\pm} = \pm\sqrt{2|eB|n}.
\end{aligned}
\tag{28}
$$

A massless fermion does not have a spin degree of freedom in 2 + 1 dimensions [55] but the Dirac equation for charged massless fermions in an external magnetic field in 2 + 1 dimensions keeps the spin parameter. Therefore, all the energy levels except the lowest level, $n = 0$, are doubly degenerate; the levels with $n = n_r, \tau = 1$ and $n = n_r + 1, \tau = -1$ (where $n_r = 0, 1, 2\ldots$ is the radial quantum number) coincide.

In graphene, the set of Landau levels in an uniform magnetic field aligned perpendicularly to the monolayer sample is given by (see, for example, [6,13])

$$
\varepsilon_n = \pm v_{\mathrm{F}}\sqrt{2|eB|n},\quad n = 0,1,2,\ldots,
\tag{29}
$$

where v_{F} is the Fermi–Dirac velocity, and the $\pm$ signs label the states of positive (electron) and negative (hole) energy, respectively. They play the same role as the band index for the conduction (+) and the valence (−) band. In addition, the states of positive and negative energy for charged massless fermions has the same energy $\varepsilon_0 = 0$ but opposite spins in the ground states.

We now discuss briefly the polarization tensor (PT) related to contributions coming from real particles. We note that the PT contains terms with $\mu \neq 0$, $B = 0$ and $\mu \neq 0$, $B \neq 0$. Terms with $\mu \neq 0$, $B = 0$ are very cumbersome and we do not give them. In the absence of a magnetic field the $\Pi^{00}(p,\mu)$ component of the PT in monolayer graphene has been studied in one-loop approximation in [56–58]. The $\Pi^{00}(p,\mu)$ component mainly contributes to the PT in the presence of a weak magnetic field and so it should be taken into account in this case.

Here we give only the expression for the PT related to the combined contribution from real particles and the magnetic field. It is natural to consider that the medium contains particles, which implies $\mu > m$ and real antiparticles are absent. We also assume that $\mu \neq m, E_n$, which means that all the n Landau levels are fully filled and no levels are partly filled.

As a result of long calculations (see [21]), one can obtain the PT related to the above combined contribution in the form

$$\Pi^{\mu\nu}(p,\mu b) = \frac{e^2}{2\pi|m|}\int\limits_0^\infty \mathrm{d}x[p^2\Pi_r l_2^\mu l_2^\nu + m\tau C_r \times i e^{\mu\nu\rho} p_\rho] \times \exp\left[ix\left(\frac{(p_0+\mu)^2}{b} - 1\right) - \frac{\mathbf{p}^2(1+i\sin 2x - \cos 2x)}{2b} - \epsilon x\right], \qquad (30)$$

where

$$\Pi_r = \frac{1}{2}\sin\frac{2m(p_0+\mu)x}{b} + \frac{m}{E_n}L_n\left(\frac{\mathbf{p}^2\sin^2 x}{2b}\right) \times \sin\frac{2E_n(p_0+\mu)x}{b}\left[\frac{\mu^2-m^2}{2|eB|}\right]$$
$$C_r = \frac{i}{2}\cos\frac{2m(p_0+\mu)x}{b} + iL_n\left(\frac{\mathbf{p}^2\sin^2 x}{2b}\right) \times \cos\frac{2E_n(p_0+\mu)x}{b}\left[\frac{\mu^2-m^2}{2|eB|}\right]. \qquad (31)$$

Here the first terms give the contribution from the $n = 0$ Landau level and $[(\mu^2-m^2)/2|eB|] \equiv N$ denotes the integer part of the function $u = (\mu^2 - m^2)/2|eB|$, i.e. the largest integer $\leq u$. We assume that the argument of u is not equal to an integer; otherwise the function would become ambiguous.

The part of the polarization tensor determined by the function Π_r is of main interest. Calculating it for the case of a weak constant uniform magnetic field $(p_0+\mu)^2 \gg |eB|$, we obtain

$$\Pi^{\mu\nu}(p,\mu b) = \frac{e^2}{2\pi}p^2\Pi_r l_2^\mu l_2^\nu. \qquad (32)$$

Here

$$\Pi_r = \frac{|eB|}{(p_0+\mu)[(p_0+\mu)^2-4m^2]}\theta(\mu-|m|) + \frac{2|eB|}{(p_0+\mu)[(p_0+\mu)^2-4E_N^2]} \times N\theta(\mu - E_N^+), \qquad (33)$$

where $\theta(z)$ is the Heaviside function; therefore the first term gives the contribution if the ground Landau level is occupied and the second one contributes if the N Landau levels are occupied. If the magnetic field is strong, $(p_0+\mu)^2 \ll |eB|$, then only the ground Landau level is occupied in the massive case and we obtain

$$\Pi_r = \frac{p_0+\mu}{|eB|}, \quad \mu > m, \quad |eB| \gg m^2. \qquad (34)$$

For the case of massless charged fermions we must put $m = 0$ and replace E_N^+ by ε_N^+ in Eq. (33). In a strong magnetic field $(p_0+\mu)^2/|eB| \ll 1$, we also obtain

$$\Pi_r = \frac{p_0+\mu}{|eB|}\theta(\mu) + \frac{2(p_0+\mu)}{|eB|} \times N\theta(\mu-\varepsilon_N^+), \qquad (35)$$

where the first and second terms, respectively, give the contributions if the ground Landau level is occupied and the N Landau levels are occupied. The main feature is that the polarization tensor (30) with a nonzero fermion density is extremely small ($\sim|eB|^{-1}$) in a strong constant uniform magnetic field.

It is well to note that the chemical potential (the Fermi energy) of noninteracting electrons is chosen to be situated between a completely filled N and a completely empty $N+1$ Landau level, which corresponds to the integer quantum Hall effect regime. The correlations between interacting electrons in the strong-correlation limit of partially filled Landau levels lead to the formation of incompressible quantum-liquid phases, which display the fractional quantum Hall effect [12,59].

4 Discussion

We have calculated the polarization tensor in the one-loop approximation of 2 + 1-dimensional quantum electrodynamics with a nonzero fermion density in a constant uniform magnetic field. The polarization tensor contains the contributions from virtual (vacuum) and real charged particles occupying a finite number of the Landau levels. In particular, we have found that the polarization tensor in QED_{2+1} in a strong constant uniform magnetic field is proportional to $\sim e^2|p^2|/\sqrt{|eB|}$, $|eB| \gg |p^2|$ (contribution of virtual particles) and $\sim e^2|\mathbf{p}^2|(p_0+\mu)/|eB|$, $|eB| \gg (p_0+\mu)^2$ (contribution of real particles). It means that photons become almost free in a very strong magnetic field.

It should be noted that if one, for instance, needs to investigate the propagation of electromagnetic waves in a real graphene strip we must consider it as the three-dimensional object of extremely small but nonzero thickness (see also [60]).

In addition, we emphasize that the polarization tensors (11) and (30) at nonzero fermion mass are finite in the limit $p_\rho \to 0$ because they contain antisymmetric terms that do not vanish in this limit. The coefficient multiplying $ie^{\mu\nu\rho}p_\rho$ in (11) and (30) with $p_\nu = 0$ is called the induced Chern–Simons coefficient, and its appearance in the effective Lagrangian of QED_{2+1} with an external magnetic field means that photons dynamically gain "masses". The induced Chern–Simons coefficient is calculated exactly and has the form

$$C_{CS} = -\mathrm{sign}(\mathrm{m}\tau)\frac{e^2}{4\pi}\theta(|m|-\mu) + \mathrm{sign}(\mathrm{eB}) \times \frac{e^2}{2\pi}\left(\frac{1}{2}\delta_{\mathrm{sign(eB)},\mathrm{sign}(\tau)}\theta(\mu-|m|) + N\theta(\mu-E_N^+)\right),$$

where the first term gives the contribution of virtual fermions.

Acknowledgments The work was supported in part by the Ministry of Education and Science of the Russian Federation Grant (Agreement No 14.576.21.0025 of 27.07.2014).

References

1. The Quantum Hall Effect, 4th edn, ed. by R.E. Prange, S.M. Girvin (Springer, New York, 1990)
2. F. Wilczek, *Fractional Statistics and Anyon Superconductivity* (World Scientific, Singapore, 1990)
3. K.S. Novoselov et al., Science **306**, 666 (2004)
4. A.H. Castro Neto et al., Rev. Mod. Phys. **81**, 109 (2009)
5. N.M.R. Peres, Rev. Mod. Phys. **82**, 2673 (2010)
6. V.N. Kotov et al., Rev. Mod. Phys. **84**, 1067 (2012)
7. K.S. Novoselov et al., Nature **438**, 197 (2005)
8. Z. Jiang, Y. Zhang, H.L. Stormer, P. Kim, Phys. Rev. Lett. **99**, 106802 (2007)
9. I.S. Terekhov, A.I. Milstein, V.N. Kotov, O.P. Sushkov, Phys. Rev. Lett. **100**, 076803 (2008)
10. F. Guinea, M.I. Katsnelson, A.K. Geim, Nat. Phys. **6**, 30 (2009)
11. L.D. Landau, E.M. Lifshitz, *Quantum Mechanics*, 3rd edn. (Pergamon, New York, 1977)
12. R.B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983)
13. M.O. Goerbig, Rev. Mod. Phys. **83**, 1193 (2011)
14. N.M.R. Peres, F. Guinea, A.H. Castro, Neto. Phys. Rev. B **73**, 125411 (2006)
15. C. Toke, P.E. Lammert, J.K. Jain, V.H. Crespi, Phys. Rev. B **74**, 235417 (2006)
16. D.V. Khveshcenko, Phys. Rev. B **75**, 159405 (2007)
17. X. Du, I. Skachko, F. Duerr, A. Luican, E.Y. Andrei, Nature **462**, 192 (2009)
18. K.I. Bolotin, F. Ghahari, M.D. Shulman, H.L. Stormer, P. Kim, Nature **462**, 196 (2009)
19. I. Skachko, X. Du, F. Duerr, D.A. Abanin, L.S. Levitov, E.Y. Andrei, Phil. Trans. R. Soc. **A368**, 5403 (2010)
20. A.J. Niemi, G.W. Semenoff, Phys. Rev. Lett. **51**, 2077 (1983)
21. V.R. Khalilov, Theor. Math. Phys. **125**, 1413 (2000)
22. M.I. Katsnelson, A.K. Geim, K.S. Novoselov, Nature Phys. **2**, 620 (2006)
23. J.F. Dobson, A. White, A. Rubio, Phys. Rev. Lett. **96**, 073201 (2006)
24. G. Gómez-Santos, Phys. Rev. B **80**, 245424 (2009)
25. J. Sarabadani, A. Naji, R. Asgari, R. Podgornik, Phys. Rev. B **84**, 155407 (2011)
26. E.V. Gorbar, V.P. Gusynin, V.A. Miransky, I.A. Shovkovy, Phys. Rev. B **66**, 045108 (2002)
27. V.P. Gusynin, S.G. Sharapov, Phys. Rev. B **73**, 245411 (2006)
28. P.K. Pyatkovskiy, V.P. Gusynin, Phys. Rev. B **83**, 075422 (2011)
29. I. Fialkovsky, D.V. Vassilevich, Int. J. Mod. Phys. A **27**, 1260007 (2012)
30. I. Fialkovsky, D.V. Vassilevich, Eur. Phys. J. B **85**, 384 (2012)
31. R. Jackiw, A.I. Milstein, S.-Y. Pi, I.S. Terekhov, Phys. Rev. B **80**, 033413 (2009)
32. V.R. Khalilov, Eur. Phys. J. C **74**, 2708 (2014)
33. A. Noguchi, Y. Shikano, K. Toyoda, S. Urabe, Nature Commun. **5**, 3868 (2014)
34. V.M. Pereira, J. Nilsson, A.H. Castro, Neto. Phys. Rev. Lett. **99**, 166802 (2007)
35. A.V. Shytov, M.I. Katsnelson, L.S. Levitov, Phys. Rev. Lett. **99**, 236801 (2007)
36. K. Nomura, A.H. MacDonald, Phys. Rev. Lett. **98**, 076602 (2007)
37. I.F. Herbut, Phys. Rev. Lett. **104**, 066404 (2010)
38. F.M. Andrade, E.O. Silva, Eur. Phys. J. C **74**, 3182 (2014)
39. J. Schwinger, Phys. Rev. **82**, 664 (1951)
40. Y. Hosotani, Phys. Lett. B **319**, 332 (1993)
41. C.R. Hagen, Phys. Rev. Lett. **64**, 503 (1990)
42. V.P. Gusynin, V.A. Miransky, I.A. Shovkovy, Phys. Rev. D **52**, 4718 (1995)
43. T. Bernstein, A. Lee, Phys. Rev. D **32**, 1020 (1985)
44. S. Coleman, B. Hill, Phys. Lett. B **159**, 184 (1985)
45. T.W. Appelquist, M. Bowick, D. Karabali, L.C.R. Wijewardhana, Phys. Rev. D **33**, 3704 (1986)
46. V.B. Berestetzkii, E.M. Lifshitz, L.P. Pitaevskii, Quantum Electrodynamics, 2nd ed (Pergamon, New York, 1982)
47. J. González, F. Guinea, M.A.H. Vozmediano, Phys. Rev. B **59**, 2474 (1999). arXiv:cond-mat/9807130
48. C. Popovici, C.S. Fischer, L. von Smekal, Phys. Rev. B **88**, 205429 (2013). arXiv:1308.6199 [hep-ph]
49. D. Valenzuela, S. Hernándes-Ortiz, M. Loewe, A. Raya, J. Phys. Math. Theor. **A48**, 065402 (2015)
50. A.V. Shpagin, Dynamical mass generation in (2+1) dimensional electrodynamics in an external magnetic field (1996, unpublished, Preprint). arXiv:hep-ph/9611412
51. A.E. Lobanov, V.R. Khalilov, ZhETF **77**, 548 (1979)
52. V.P. Gusynin, V.A. Miransky, I.A. Shovkovy, Nucl. Phys. B **563**, 361 (1999)
53. E.M. Lifshitz, L.P. Pitaevskii, Statistical Physics, 2nd part (Nauka, Moscow, 1978)
54. A.E. Shabad, FIAN Report No. 60, (FIAN, Moscow, 1974)
55. R. Jackiw, V.P. Nair, Phys. Rev. D **43**, 1933 (1991)
56. E.V. Gorbar, V.P. Gusynin, V.A. Miransky, I.A. Shovkovy, Phys. Rev. B **66**, 045108 (2002)
57. P.K. Pyatkovskiy, J. Phys. Condens. Matter **21**, 025506 (2009)
58. A. Qaiumzadeh, R. Asgari, Phys. Rev. B **79**, 075414 (2009)
59. D.C. Tsui, H. Störmer, A.C. Gossard, Phys. Rev. Lett. **48**, 1559 (1982)
60. O. Coquand, B. Machet, Refractive properties of graphene in a medium-strong external magnetic field (2014). arXiv:1410.6585v1 [hep-ph]