

An Interpretation of Completeness and Basu's Theorem

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In order to obtain a statistical interpretation of completeness of a sufficient statistic T , an attempt is made to characterize this property as equivalent to the condition that all ancillary statistics are independent of T . For technical reasons, this characterization is not correct, but two correct versions are obtained in Section 3 by modifying either the definition of ancillarity or of completeness. The light this throws on the meaning of completeness is discussed in Section 4. Finally, Section 5 reviews some of the difficulties encountered in models that lack completeness.

KEY WORDS: Ancillary statistic; Basu's theorem; Completeness; Conditioning; Unbiased estimation.

1. INTRODUCTION

Let X be a random observable distributed according to a distribution from the family $\mathcal{P} = \{P_\theta, \theta \in \Omega\}$. Recall the following definitions.

A statistic $V = V(X)$ is *ancillary* if its distribution does not depend on θ .

A statistic $T = T(X)$ is *sufficient* if the conditional distribution of X given T does not depend on θ .

These two concepts are complementary. If the rest of the data is discarded, a sufficient statistic by itself retains all the information about θ contained in the data; an ancillary statistic by itself contains no information about θ .

Since sufficiency reduces the data without loss of information, a sufficient statistic is of greatest interest when it is minimal, that is, when it provides the greatest reduction of the data. Formally, a sufficient statistic is *minimal* if it is a function of every other sufficient statistic. Although exceptions are possible (Landers and Rogge 1972; Pitcher 1975), minimal sufficient statistics typically exist and are easy to construct (Bahadur 1954; Lehmann and Scheffé 1950).

A sufficient statistic T is *complete* if

$$E_\theta f(T) = 0 \quad \text{for all } \theta \Rightarrow f(t) = 0 \quad (\text{a.e. } \mathcal{P}), \quad (1.1)$$

and is *boundedly complete* if (1.1) holds for all bounded

f . A necessary condition for T to be complete or boundedly complete is that it is minimal.

The properties of minimality and completeness are of a rather different nature. Under mild assumptions a minimal sufficient statistic exists and whether or not T is minimal then depends on the choice of T . On the other hand, existence of a complete sufficient statistic is equivalent to the completeness of the minimal sufficient statistic, and hence is a property of the model \mathcal{P} . We shall say that a model is complete if its minimal sufficient statistic is complete.

It has been pointed out in the literature that definition (1.1) is purely technical and does not suggest a direct statistical interpretation. Since various aspects of statistical inference become particularly simple for complete models, one would expect this concept to carry statistical meaning. It is the principal aim of this article to attempt a statistical characterization of models which are complete or boundedly complete. In the last section, we shall briefly survey some of the difficulties presented by incomplete models.

2. AN EXAMPLE AND BASU'S THEOREM

In order to get some insight into the problem, consider the following example.

Example 1. Let X_1, \dots, X_n be iid with density $f(x_i - \theta)$, and contrast what happens when f is normal, exponential, or uniform on the one hand, and in most other cases, for example, when f is logistic, Cauchy, or double exponential on the other. In the first type of situation the data can be reduced without loss of information to a small number of sufficient statistics: one for the normal (\bar{X}) and exponential ($X_{(1)}$), and two for the uniform ($X_{(1)}, X_{(n)}$), where $X_{(1)} < \dots < X_{(n)}$ denote the order statistics. In the second type of situation, the set of n order statistics is minimal sufficient. (Intermediate cases are of course possible, as is seen, for example, by considering $f(x - \theta) = Ce^{-(x-\theta)^{2k}}$.)

As will be seen later, this difference is related to the following property of the ancillary information. In any location problem, the $n - 1$ differences $Y_i = X_{(n)} - X_{(i)}$ ($i = 1, \dots, n - 1$) constitute an $(n - 1)$ -dimensional ancillary statistic. In the normal and exponential cases, the Y 's are independent of the minimal sufficient statistic

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and hence carry no information about θ either by themselves or in conjunction with the minimal sufficient statistic. On the other hand, when the n order statistics are minimal, the differences Y_i —although still ancillary—are functions of the minimal sufficient statistic. Thus, sufficiency has not been successful in squeezing out the ancillary material. By themselves, the Y 's carry no information but in conjunction with the rest of the data they do.

The uniform case is intermediate. Here the Y 's are not independent of the minimal sufficient statistic, but the ratios $Z_i = Y_i/Y_1$ ($i = 2, \dots, n - 1$) are. If we represent the ordered sample by

$$\{X_{(1)} \times X_{(n)}, Y_1 = X_{(n)} - X_{(1)}, Z_2, \dots, Z_{n-1}\}$$

we see that the minimal sufficient statistic $T = (X_{(1)}, X_{(n)})$ has pushed out the $n - 2$ Z 's, which carry no information by themselves or in conjunction with the rest of the data, but not Y_1 .

Note that the minimal sufficient statistic is complete in the normal and exponential cases, but not in the other four cases. This suggests that completeness of a minimal sufficient statistic T may be associated with the ability of T completely to rid itself of the ancillary part of the data by making it independent of T . As a further illustration of this idea, consider a sample from a uniform distribution on (θ_1, θ_2) where now both θ 's are unknown. The minimal sufficient statistic continues to be $T = (X_{(1)}, X_{(n)})$. However, Y_1 is no longer ancillary, although the Z 's continue to be so. In the present case, T is more successful in discarding ancillary material than it was in the one-parameter case, and T is now complete.

These comments are closely related to a theorem of Basu (1955, 1958), which states that when a sufficient statistic T is boundedly complete, then all ancillary statistics are independent of T . This would provide the desired characterization of complete models as those in which the minimal sufficient statistic is completely successful in discarding all ancillary material, if the converse were also true, that is, if the independence of all ancillary statistics from a sufficient statistic implied that T were boundedly complete. Unfortunately this is not correct (see Sec. 3, Ex. 2) but it contains a germ of truth.

3. SOME ADAPTATIONS OF BASU'S THEOREM

The converse of Basu's theorem fails because ancillarity is concerned with the whole distribution of a statistic, while completeness is a property dealing only with expectation. We shall now show that correct versions of the converse can be obtained by replacing either ancillarity with the corresponding first-order property or completeness with a condition reflecting the whole distribution.

Let us call a real-valued statistic $V = V(X)$ *first-order ancillary* if $E_\theta(V)$ is independent of θ . (A first-order ancillary is simply an unbiased estimator of zero plus a constant.) Then we have the following first-order version of Basu's theorem.

Theorem 1. A necessary and sufficient condition for a sufficient statistic T to be boundedly complete is that every bounded first-order ancillary V is uncorrelated (for all θ) with every bounded real-valued function of T .

The proof is based on the following characterization of all bounded, unbiased estimators of zero.

Lemma 1. If a sufficient statistic T is boundedly complete, then a bounded statistic $S = S(X)$ is an unbiased estimator of zero if and only if (iff)

$$\eta(t) = E[S | t] = 0 \text{ (a.e. } \mathcal{P}^T\text{)}. \quad (3.1)$$

(Here (a.e. \mathcal{P}^T) means: except on a set N of values of T , which has probability zero for all θ .)

Proof of Lemma 1. If S is an unbiased estimator of zero, then $E(S) = E[\eta(T)] = 0$ for all θ , and (3.1) follows from completeness. Conversely, if (3.1) holds, taking the expectation of both sides shows S to be an unbiased estimator of zero.

Proof of Theorem 1. (a) *Necessity.* Suppose that T is boundedly complete and that V is a bounded first-order ancillary. Without loss of generality suppose that $E(V) = 0$. Then for any bounded $f(t)$ we have

$$\text{cov}_\theta(f(T), V) = E_\theta[f(T)V] = E_\theta\psi(T)$$

where $\psi(t) = f(t)E[V | t]$. By the lemma, $E[V | t]$ and therefore $\psi(t)$ is zero (a.e. \mathcal{P}^T) and hence $\text{cov}_\theta(f(T), V) = 0$ for all θ .

(b) *Sufficiency.* If T is not boundedly complete, there exists a bounded function f such that $E_\theta f(T) = 0$ for all θ but $f(T) \neq 0$ with positive probability for some θ_0 . Put $V(X) = f(T)$. Then $\text{cov}_{\theta_0}(V, f(T)) = E_{\theta_0}[f^2(T)] > 0$, and this proves the condition to be sufficient.

The analogous result holds for completeness instead of bounded completeness if attention is restricted to statistics with finite variance.

Instead of modifying the definition of ancillarity to obtain a converse of Basu's theorem, one can instead modify the definition of completeness. Quite generally, call a sufficient statistic \mathcal{F} -complete if $E_\theta f(T) = 0$ for all θ and $f \in \mathcal{F}$ implies $f(t) = 0$ (a.e. \mathcal{P}^T) where \mathcal{F} is some given class of real-valued functions. Completeness and bounded completeness correspond respectively to the case that F is the class of all integrable or all bounded functions. The case that it is the class of all square integrable functions is relevant for the analog of Theorem 1 mentioned following the proof of that theorem.

If F_0 is the class of all two-valued functions, we have the following result, which provides a partial converse of Basu's theorem.

Theorem 2. If

$$\text{Every ancillary statistic is independent of } T, \quad (3.2)$$

then T is F_0 -complete.

Proof. Suppose T is not F_0 -complete. Then there exists a two-valued function f such that $E_\theta f(T) = 0$ for all θ ,

but f is not zero with probability one. Let $f(t) = -a$ when $t \in A$ and $f(t) = b$ otherwise. Then $P_\theta(T \in A) = b/(a + b)$ is independent of θ so that the indicator of A , $I_A(T)$, is an ancillary. This provides a contradiction.

Basu's theorem, together with Theorem 2, shows that

$$\text{Bounded completeness} \Rightarrow (3.2) \Rightarrow F_0\text{-completeness.} \tag{3.3}$$

That neither of these implications holds in the other direction is shown by the following example.

Example 2. Let X be distributed as

-5	-4	-3	-2	-1	1	2	3	4	5
$\alpha' p^2 q$	$\alpha' p q^2$	$\frac{1}{2} p^3$	$\frac{1}{2} q^3$	$\gamma' p q$	$\gamma p q$	$\frac{1}{2} q^3$	$\frac{1}{2} p^3$	$\alpha p q^2$	$\alpha p^2 q$

where the first row lists the values X takes on and the second row their probabilities and where $\alpha, \gamma, \alpha', \gamma'$ are positive constants satisfying

$$\alpha + \gamma = \alpha' + \gamma' = \frac{3}{2}.$$

It is easy to check the following facts.

- I If $0 < p < 1, q = 1 - p$, the probabilities add up to 1.
- II $T = |X|$ is minimal sufficient.
- III $P(X > 0) = \frac{1}{2}$ so that the indicator V of the set $\{X > 0\}$ is an ancillary statistic.
- IV If $\alpha \neq \alpha', V$ is not independent of T , so that (3.2) does not hold.
- V To see that T is F_0 -complete, note that $Ef(T) = 0$ iff

$$f(1)[\gamma + \gamma']pq + f(2)q^3 + f(3)p^3 + (\alpha + \alpha')[f(4)pq^2 + f(5)p^2q] = 0.$$

This holds for all $0 < p < 1$ iff

$$\begin{aligned} f(2) = f(3) = 0, \quad f(1) = a, \\ f(4) = f(5) = -\frac{\gamma + \gamma'}{\alpha + \alpha'} a \end{aligned} \tag{3.4}$$

for some a . Since no two-valued f can satisfy (3.4), T is F_0 -complete.

It now follows from (IV) that F_0 -completeness does not imply (3.2). Also, since (3.4) shows T not to be boundedly complete, F_0 -completeness does not imply bounded completeness.

That (3.2) does not imply bounded completeness can be seen from any example without bounded completeness in which no ancillary statistic exists and (3.2) therefore holds vacuously. A less trivial example is obtained by putting $\alpha = \alpha', \gamma = \gamma'$ in Example 2. The totality of ancillary statistics is then obtained by selecting any of the 5 sign combinations $\pm 1, \pm 2, \pm 3, \pm 4, \pm 5$, for example $+1, -2, -3, +4, -5$ and setting

$$\begin{aligned} V(+1) = V(-2) = V(-3) = V(4) = V(-5) = 1 \\ V(-1) = V(+2) = V(+3) = V(-4) = V(5) = 0. \end{aligned} \tag{3.5}$$

For any such $V, P(V = 1) = P(V = 0) = \frac{1}{2}$ and V is independent of T , so that (3.2) holds. On the other hand, T has been seen not to be boundedly complete.

That F_0 -completeness is a very weak (and somewhat unnatural) condition is indicated by the fact that a sufficient statistic can be F_0 -complete with being minimal. As a simple example, suppose that X and Y are independent Poisson variables with $E(X) = E(Y) = \lambda$. Then $T = X + Y$ is minimal sufficient and complete, but (X, Y) is F_0 -complete. This follows from the facts that the conditional distribution of X given $X + Y = t$ is the binomial distribution $b(\frac{1}{2}, t)$ and that for no constant c does there exist a nontrivial two-valued function $f(x, y)$ such that $E[f(X, Y) | t] = c$ for all t .

Example 2, together with (3.3), shows that bounded completeness is a stronger condition than (3.2) while F_0 -completeness is not strong enough. It is interesting that there exists an intermediate completeness condition which is exactly equivalent to (3.2). Although the condition is rather intractable and not likely to be of much use, we give it here for the sake of completeness.

Let F_1 be the class of all functions $f(t)$ which are the conditional expectation of a two-valued function of X given t . If $X = T$, then F_1 reduces to F_0 .

Theorem 3. Condition (3.2) is necessary and sufficient for T to be F_1 -complete in the model \mathcal{P} .

Proof. (a) Suppose T is F_1 -complete with respect to \mathcal{P} and that V is an ancillary statistic. Then for any set A , the probability $P(V \in A) = p$ is independent of θ , and $g(V) = I_A(V) - p$ is both a two-valued function of X and an unbiased estimator of zero. If $f(t) = E[g(V) | t]$, it follows as in the proof of Theorem 1(a) that $E_\theta f(T) = 0$ for all θ . Since $f \in F_1$, this implies

$$f(t) = P[V \in A | t] - p = 0 \text{ a.e.}$$

and hence the independence of V and T .

(b) Conversely, suppose that T is not F_1 -complete, so that there exists $f \in F_1$ which is not 0 a.e. but such that $E_\theta f(T) = 0$ for all θ . Let g be a nontrivial two-valued function of X for which $E[g(X) | t] = f(t)$, say $g(x) = a$ if $x \in A$ and $g(x) = b$ otherwise. Then $E_\theta g(X) = 0$, the indicator V of A is an ancillary statistic as in the proof of Theorem 2, but by the assumption about f, V is not independent of T .

4. INTERPRETATION OF COMPLETENESS

Ancillary statistics by themselves have no value for estimating θ and the same is likely to be true of first-order ancillaries (since their expectation is independent of θ), although statistics with either property may be valuable when used in conjunction with other statistics. The results of the preceding section can be summarized roughly by saying that the various forms of completeness of a sufficient statistics T characterize the success of T in separating the informative part of the data from that part which by itself carries no (or little) information.

Basu's theorem explains how a complete sufficient statistic achieves this separation. It does this by making the ancillary part of the data independent of (or uncorrelated with) T . For the technical reasons mentioned in Section 2, this interpretation of completeness is not as clear-cut as one might like. Roughly speaking, however, a sufficient statistic T is complete if it contains no ancillary information or equivalently if all ancillary information is independent of (or at least uncorrelated with) T .

The idea that it is the lack of correlation between first-order ancillaries and functions of T that makes complete sufficient statistics so effective as a noise-free distillation of the data finds some support in a theorem of Lehmann and Scheffé (1950) and Rao (1952). Suppose T is not complete, and consider the functions $f(T)$ which are uncorrelated with all first-order ancillaries (all functions being assumed to be square integrable). One might then expect these f 's (though not sufficient) to be particularly effective estimators. The theorem in question states in part that these functions are the only unbiased estimators with uniformly minimum variance (UMVU).

Unfortunately, it turns out that in many common problems there are no functions f for which the above property holds. For example, when X_1, \dots, X_n are iid according to the uniform distribution on $(\theta - \frac{1}{2}, \theta + \frac{1}{2})$, it was shown by Lehmann and Scheffé (1950, 1956) that no function of θ has a UMVU estimator. One might expect this property to hold fairly generally in location problems with incomplete minimal sufficient statistic. (For some partial results in this direction see Bondesson 1975.) The corresponding result was proved by Unni (1978) for the case that X_1, \dots, X_n is a sample from the normal distribution $N(r\sigma, \sigma^2)$, $r = \text{known}$, and for the Behrens-Fisher problem (i.e., the case of samples from two normal distributions $N(\xi, \sigma^2)$, $N(\xi, \tau^2)$ with unknown variances and common unknown mean.) Examples in which the minimal sufficient statistic is not complete but in which some parametric functions have UMVU estimators can of course be constructed (see for instance Ex. 5.2 of Lehmann and Scheffé 1950), but even then the dimension of the space of such functions is typically quite low.

5. ANCILLARY STATISTICS AND THE PRINCIPLE OF CONDITIONALITY

When a model is not complete, the minimal sufficient statistic will contain ancillary or at least first-order ancillary statistics, and as a result its dimension will typically be larger than that of the parameter space. The question then arises as to how to use the ancillary information to effect a further reduction of the data in order to facilitate the inference problem. Little is known about ways of utilizing first-order ancillaries (the theorems by Lehmann and Scheffé and by Rao mentioned in Section 4 can be viewed as a contribution in this direction) but there is considerable literature on the use of ancillary statistics. The ideas were first put forward by R.A. Fisher (1934, 1935) and an excellent discussion of the issues can

be found in Cox and Hinkley (1974). Many additional results are given by Barndorff-Nielsen (1978). As a brief introduction to the subject, we shall here discuss a few examples.

Example 3. (a) In the standard introductory example, which is implicit in the writing of Fisher, the natural ancillary statistic is the sample size. To be specific, let N be a random variable taking on values $1, 2, \dots$ with known probabilities π_1, π_2, \dots . Having observed $N = n$, we perform n binomial trials with success probability p and let S_n be the number of successes in these trials. For the observation $X = (N, S_N)$, X is minimal sufficient and N is ancillary.

The natural estimator S_N/N of p is unbiased and has variance $pq E(1/N)$. It is, however, not clear that this is the appropriate measure of accuracy for the following reason. The most striking feature of the example is that N , while not providing any information about p , tells us how much information about p is contained in S_N . If the observed value n of N is large, S_N is highly informative; for small n it carries little information. For this reason it may be better to cite as measure of accuracy the conditional variance pq/n as appropriate in those cases in which n has been obtained.

If this argument is accepted, not only the variance but the whole problem should be viewed conditionally. This includes in particular the choice of estimate which in the present case would remain unchanged but in others might not. From this point of view, conditioning on N has the advantage of reducing the random part of the data from (N, S_N) to the simple one-dimensional statistic S_n . Since S_n is complete, the conditional model is complete, there exist UMVU estimators for p and pq/n , and so on.

(b) Example (a) may seem artificial (although very similar situations arise in sampling from finite populations) but some of its principal features are in fact quite typical. Consider for instance once more the location problem of Example 1. Here the differences Y_i provide the ancillary information which is analogous to N , and typically give an indication of the dispersion of the conditional distribution of an estimator of θ such as the mean or median. This is particularly clear in the case of the uniform distribution. The minimal sufficient statistic can then be represented as (Z, Y) with $Z = \frac{1}{2}[X_{(1)} + X_{(n)}]$ and $Y = X_{(n)} - X_{(1)}$. The conditional distribution of Z , given $Y = y$, is the uniform distribution on $(\theta - \frac{1}{2}(1 - y), \theta + \frac{1}{2}(1 - y))$. Thus Y again measures the amount of information Z contains about θ and it is natural to restrict attention to the conditional distribution of Z given Y . In this case even the conditional model is not complete. (Additional ancillaries exist but are rather unsatisfactory. See Basu 1964, Sec. 4.)

(c) As a last illustration consider the ancillary statistic $V = I\{x > 0\}$ of Ex. 2. The probabilities of the conditional distribution of X given $V = 1$ are

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ (3 - 2\alpha)pq, & q^3, & p^3, & 2\alpha pq^2, & 2\alpha pq^2. \end{matrix} \quad (5.1)$$

Given $V = 0$, they are the corresponding values with α replaced by α' . It is easy to see that an experiment with outcome probabilities (5.1) is more informative than the corresponding experiment with α' when $\alpha > \alpha'$. (The reason is that the events with probabilities $2\alpha pq^2$ and $2\alpha p^2 q$ can be split at random into four events with probabilities $2\alpha' pq^2$, $2\alpha' p^2 q$, $2(\alpha - \alpha') pq^2$, $2(\alpha - \alpha') p^2 q$. The latter two have a combined probability of $2(\alpha - \alpha') pq$, which can be combined with $(3 - 2\alpha) pq$ to yield an event with probability $(3 - 2\alpha') pq$. Thus, from the α -experiment and a table of random numbers one can generate the α' -experiment but not vice versa.)

The above considerations suggest the conditionality principle, first proposed by Fisher (1935, 1936, 1956) and investigated as a fundamental principle by Birnbaum (1962): in the presence of an ancillary statistic V , statistical inference should be carried out not in the overall model but in the conditional model given V .

Apart from possible reservations which one may have about the principle itself, difficulties in implementing it arise from the fact that ancillary statistics are not unique. In Example 3(b), for instance, $\sum (X_i - \bar{X})^2$ is ancillary, as is $\sum |X_i - \bar{X}|$ or any other function of the differences. To avoid this difficulty it is tempting, in analogy with the definition of a minimal sufficient statistic, to call an ancillary statistic V maximal if every other ancillary statistic is a function of V . One would then condition on this maximal V , thereby carrying the conditioning process as far as possible.

Unfortunately, a maximal V in this strong sense often does not exist. Basu (1964) has therefore introduced a weaker concept and has defined V as maximal if there exists no ancillary statistic V' of which V is a function. Such maximal ancillaries, however, frequently are not unique. In searching for a way out of the difficulty, it is helpful to recall the motivation for conditioning which was given in the comments after Example 3, and which was stated by Fisher (1935, p. 48) as follows: "Ancillary statistics are only useful when different samples of the same size can supply different amounts of information, and serve to distinguish those which supply more from those which supply less."

Fisher's statement leads to the idea that the better an ancillary statistic is, in distinguishing the more informative samples from those that are less so, the more useful it is as a conditioning variable. An implementation of this criterion for choosing an ancillary has been provided by Cox (1971), who compares different ancillaries V in terms of the variance of the Fisher information of the conditional distribution of X given V . An ancillary V is preferred to V' if it has the larger such variance for all parameter values. Such a principle will resolve the choice of ancillary in many cases although it may of course turn out that it leads to different choices for different parameter values.

If $X = (Y, V)$, where V is ancillary, conditioning on V can be thought of as data reduction obtained by replacing the random quantity V by a known constant v ,

so that part of the randomness of the data is eliminated. It frequently happens that the conditional model for Y given v is complete, in which case conditioning is particularly rewarding. It is interesting to note that in this latter case the ancillary V is essentially maximal in the sense of Basu (1959). This result (to which we hope to return in a later paper) presents a considerable strengthening of Theorem 7 of Basu (1959).

In summary it is seen that models involving ancillaries contained in the minimal sufficient statistic are more difficult to analyze than complete models. Even in so relatively simple a situation as that of Example 1, no general agreement has been reached whether to assess the accuracy of the best estimator of θ (the Pitman estimator) conditionally or unconditionally. The situation becomes even more complicated when nuisance parameters are present, a possibility we have not considered here.

Commenting on his efforts to develop a satisfactory small-sample theory of estimation, Fisher wrote in 1956 (pp. 157 and 158): "The most important step which has been taken so far to complete the structure of the theory of estimation is the recognition of Ancillary statistics." Despite some important recent work on conditional inference by Efron and Hinkley (1978) and Hinkley (1977), we still seem to be far from a full understanding of this difficult topic and a satisfactory small-sample theory of models that lack completeness.

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