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Schur-convexity of dual form of some symmetric functions

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Abstract

By the properties of a Schur-convex function, Schur-convexity of the dual form of some symmetric functions is simply proved.

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1 Introduction

Throughout the article, $\mathbb{R}$ denotes the set of real numbers, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ denotes n-tuple (n-dimensional real vectors), the set of vectors can be written as

$$\mathbb{R}^n = \{\boldsymbol{x} = (x_1, \ldots, x_n) : x_i \in \mathbb{R}, i = 1, \ldots, n\},$$

$$\mathbb{R}^n_+ = \{\boldsymbol{x} = (x_1, \ldots, x_n) : x_i > 0, i = 1, \ldots, n\}.$$

In particular, the notations $\mathbb{R}$ and $\mathbb{R}_+$ denote $\mathbb{R}^1$ and $\mathbb{R}^1_+$, respectively.

For convenience, we introduce some definitions as follows.

Definition 1 [1, 2] Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$.

(i) $\boldsymbol{x} \ge \boldsymbol{y}$ means $x_i \ge y_i$ for all $i = 1, 2, \ldots, n$.

(ii) Let $\Omega \subset \mathbb{R}^n$, $\varphi : \Omega \to \mathbb{R}$ is said to be increasing if $\boldsymbol{x} \ge \boldsymbol{y}$ implies $\varphi(\boldsymbol{x}) \ge \varphi(\boldsymbol{y})$. φ is said to be decreasing if and only if $-\varphi$ is increasing.

Definition 2 [1, 2] Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$.

(i) $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x} \prec \boldsymbol{y}$) if $\sum_{i=1}^k x_{[i]} \le \sum_{i=1}^k y_{[i]}$ for $k = 1, 2, \ldots, n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, where $x_{[1]} \ge \cdots \ge x_{[n]}$ and $y_{[1]} \ge \cdots \ge y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order.

(ii) Let $\Omega \subset \mathbb{R}^n$, $\varphi : \Omega \to \mathbb{R}$ is said to be a Schur-convex function on Ω if $\boldsymbol{x} \prec \boldsymbol{y}$ on Ω implies $\varphi(\boldsymbol{x}) \le \varphi(\boldsymbol{y})$. φ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is Schur-convex function on Ω.

Definition 3 [1, 2] Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$.

(i) $\Omega \subset \mathbb{R}^n$ is said to be a convex set if $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, $0 \le \alpha \le 1$ implies

$$\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} = (\alpha x_1 + (1-\alpha)y_1, \ldots, \alpha x_n + (1-\alpha)y_n) \in \Omega.$$

(ii) Let $\Omega \subset \mathbb{R}^n$ be a convex set. A function $\varphi : \Omega \to \mathbb{R}$ is said to be a convex function on Ω if

$$\varphi(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \le \alpha\varphi(\boldsymbol{x}) + (1-\alpha)\varphi(\boldsymbol{y})$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ and all $\alpha \in [0,1]$. φ is said to be a concave function on Ω if and only if $-\varphi$ is a convex function on Ω.

(iii) Let $\Omega \subset \mathbb{R}^n$. A function $\varphi : \Omega \to \mathbb{R}$ is said to be a log-convex function on Ω if the function $\ln \varphi$ is convex.

Definition 4 [1]

(i) $\Omega \subset \mathbb{R}^n$ is called a symmetric set, if $x \in \Omega$ implies $Px \in \Omega$ for every $n \times n$ permutation matrix P.

(ii) The function $\varphi : \Omega \to \mathbb{R}$ is called symmetric if for every permutation matrix P, $\varphi(Px) = \varphi(x)$ for all $x \in \Omega$.

Theorem A (Schur-convex function decision theorem [1, p.84]) *Let $\Omega \subset \mathbb{R}^n$ be symmetric and have a nonempty interior convex set. Ω^0 is the interior of Ω. $\varphi : \Omega \to \mathbb{R}$ is continuous on Ω and differentiable in Ω^0. Then φ is the Schur-convex* (*Schur-concave*) *function if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial\varphi}{\partial x_1} - \frac{\partial\varphi}{\partial x_2}\right) \ge 0 \ (\le 0) \tag{1}$$

holds for any $\boldsymbol{x} \in \Omega^0$.

The Schur-convex functions were introduced by Schur in 1923 and have important applications in analytic inequalities, elementary quantum mechanics and quantum information theory. See [1].

In recent years, many scholars use the Schur-convex function decision theorem to determine the Schur-convexity of many symmetric functions.

Xia *et al.* [3] proved that the symmetric function

$$E_k\left(\frac{\boldsymbol{x}}{1+\boldsymbol{x}}\right) = \sum_{1\le i_1<\cdots<i_k\le n} \prod_{j=1}^{k} \frac{x_{i_j}}{1+x_{i_j}}, \quad k = 1,\ldots,n, \tag{2}$$

is Schur-convex on $\mathbb{R}^n_+$.

Chu *et al.* [4] proved that the symmetric function

$$E_k\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}\right) = \sum_{1\le i_1<\cdots<i_k\le n} \prod_{j=1}^{k} \frac{x_{i_j}}{1-x_{i_j}}, \quad k = 1,\ldots,n, \tag{3}$$

is Schur-convex on $[\frac{k-1}{2(n-1)}, 1)^n$ and Schur-concave on $[0, \frac{k-1}{2(n-1)}]^n$.

Xia and Chu [5] proved that the symmetric function

$$E_k\left(\frac{1-\boldsymbol{x}}{\boldsymbol{x}}\right) = \sum_{1\le i_1<\cdots<i_k\le n} \prod_{j=1}^{k} \frac{1-x_{i_j}}{x_{i_j}}, \quad k = 1,\ldots,n, \tag{4}$$

is Schur-convex on $(0, \frac{2n-k-1}{2(n-1)}]^n$ and Schur-concave on $[\frac{2n-k-1}{2(n-1)}, 1]^n$.

Xia and Chu [6] also proved that the symmetric function

$$E_k\left(\frac{1+\boldsymbol{x}}{1-\boldsymbol{x}}\right)=\sum_{1\le i_1<\cdots<i_k\le n}\prod_{j=1}^{k}\frac{1+x_{i_j}}{1-x_{i_j}},\quad k=1,\ldots,n, \tag{5}$$

is Schur-convex on $(0,1)^n$.

Mei *et al.* [7] proved that the symmetric function

$$E_k\left(\frac{1}{\boldsymbol{x}}-\boldsymbol{x}\right)=\sum_{1\le i_1<\cdots<i_k\le n}\prod_{j=1}^{k}\left(\frac{1}{x_{i_j}}-x_{i_j}\right),\quad k=1,\ldots,n, \tag{6}$$

is Schur-convex on $(0,1)^n$. More results for Schur convexity of the symmetric functions, we refer the reader to [8].

In this paper, by the properties of a Schur-convex function, we study Schur-convexity of the dual form of the above symmetric functions, and we obtained the following results.

Theorem 1 *The symmetric function*

$$E_k^*\left(\frac{\boldsymbol{x}}{1+\boldsymbol{x}}\right)=\prod_{1\le i_1<\cdots<i_k\le n}\sum_{j=1}^{k}\frac{x_{i_j}}{1+x_{i_j}},\quad k=1,\ldots,n, \tag{7}$$

is a Schur-concave function on $\mathbb{R}_+^n$.

Theorem 2 *The symmetric function*

$$E_k^*\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}\right)=\prod_{1\le i_1<\cdots<i_k\le n}\sum_{j=1}^{k}\frac{x_{i_j}}{1-x_{i_j}},\quad k=1,\ldots,n, \tag{8}$$

is a Schur-convex function on $[\frac{1}{2},1)^n$.

Theorem 3 *The symmetric function*

$$E_k^*\left(\frac{1-\boldsymbol{x}}{\boldsymbol{x}}\right)=\prod_{1\le i_1<\cdots<i_k\le n}\sum_{j=1}^{k}\frac{1-x_{i_j}}{x_{i_j}},\quad k=1,\ldots,n, \tag{9}$$

is a Schur-convex function on $(0,\frac{1}{2}]^n$.

Theorem 4 *The symmetric function*

$$E_k^*\left(\frac{1+\boldsymbol{x}}{1-\boldsymbol{x}}\right)=\prod_{1\le i_1<\cdots<i_k\le n}\sum_{j=1}^{k}\frac{1+x_{i_j}}{1-x_{i_j}},\quad k=1,\ldots,n, \tag{10}$$

is a Schur-convex function on $(0,1)^n$.

Theorem 5 *The symmetric function*

$$E_k^*\left(\frac{1}{\boldsymbol{x}}-\boldsymbol{x}\right)=\prod_{1\le i_1<\cdots<i_k\le n}\sum_{j=1}^{k}\left(\frac{1}{x_{i_j}}-x_{i_j}\right),\quad k=1,\ldots,n, \tag{11}$$

is a Schur-convex function on $(0,\sqrt{\sqrt{5}-2})^n$.

2 Lemmas

To prove the above three theorems, we need the following lemmas.

Lemma 1 ([1, p.97], [2]) *If φ is symmetric and convex (concave) on a symmetric convex set Ω, then φ is Schur-convex (Schur-concave) on Ω.*

Lemma 2 [2, p.64] *Let $\Omega \subset \mathbb{R}^n$, $\varphi : \Omega \to \mathbb{R}_+$. Then $\log \varphi$ is Schur-convex (Schur-concave) if and only if φ is Schur-convex (Schur-concave).*

Lemma 3 ([1, p.642], [2]) *Let $\Omega \subset \mathbb{R}^n$ be an open convex set, $\varphi : \Omega \to \mathbb{R}$. For $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, define one variable function $g(t) = \varphi(t\boldsymbol{x} + (1-t)\boldsymbol{y})$ on the interval $(0,1)$. Then φ is convex (concave) on Ω if and only if g is convex (concave) on $[0,1]$ for all $\boldsymbol{x}, \boldsymbol{y} \in \Omega$.*

Lemma 4 *Let $\boldsymbol{x} = (x_1, \dots, x_m)$ and $\boldsymbol{y} = (y_1, \dots, y_m) \in \mathbb{R}_+^m$. Then the function $p(t) = \log g(t)$ is concave on $[0,1]$, where*

$$g(t) = \sum_{j=1}^{m} \frac{tx_j + (1-t)y_j}{1 + tx_j + (1-t)y_j}.$$

Proof

$$p'(t) = \frac{g'(t)}{g(t)},$$

where

$$g'(t) = \sum_{j=1}^{m} \frac{x_j - y_j}{(1 + tx_j + (1-t)y_j)^2},$$

$$p''(t) = \frac{g''(t)g(t) - (g'(t))^2}{g^2(t)},$$

where

$$g''(t) = -\sum_{j=1}^{m} \frac{2(x_j - y_j)^2}{(1 + tx_j + (1-t)y_j)^3}.$$

Thus,

$$\begin{aligned} &g''(t)g(t) - \bigl(g'(t)\bigr)^2 \\ &\quad = \left(-\sum_{j=1}^{m} \frac{2(x_j - y_j)^2}{(1 + tx_j + (1-t)y_j)^3}\right)\left(\sum_{j=1}^{m} \frac{tx_j + (1-t)y_j}{1 + tx_j + (1-t)y_j}\right) \\ &\qquad - \left(\sum_{j=1}^{m} \frac{x_j - y_j}{(1 + tx_j + (1-t)y_j)^2}\right)^2 \\ &\quad \le 0, \end{aligned}$$

and then $p''(t) \le 0$, that is, $p(t)$ is concave on $[0,1]$.

The proof of Lemma 4 is completed. □

Lemma 5 *Let* $\mathbf{x} = (x_1, \ldots, x_m)$ *and* $\mathbf{y} = (y_1, \ldots, y_m) \in [\frac{1}{2}, 1)^m$. *Then the function* $q(t) = \log \psi(t)$ *is convex on* $[0,1]$, *where*

$$\psi(t) = \sum_{j=1}^{m} \frac{tx_j + (1-t)y_j}{1 - tx_j - (1-t)y_j}.$$

Proof

$$q'(t) = \frac{\psi'(t)}{\psi(t)},$$

where

$$\psi'(t) = \sum_{j=1}^{m} \frac{x_j - y_j}{(1 - tx_j - (1-t)y_j)^2},$$

$$q''(t) = \frac{\psi''(t)\psi(t) - (\psi'(t))^2}{\psi^2(t)},$$

where

$$\psi''(t) = \sum_{j=1}^{m} \frac{2(x_j - y_j)^2}{(1 - tx_j - (1-t)y_j)^3}.$$

By the Cauchy inequality, we have

$$\begin{aligned}
&\psi''(t)\psi(t) - \left(\psi'(t)\right)^2 \\
&\quad = \left(\sum_{j=1}^{m} \frac{2(x_j - y_j)^2}{(1 - tx_j - (1-t)y_j)^3}\right)\left(\sum_{j=1}^{m} \frac{tx_j + (1-t)y_j}{1 - tx_j - (1-t)y_j}\right) - \left(\sum_{j=1}^{m} \frac{x_j - y_j}{(1 - tx_j - (1-t)y_j)^2}\right)^2 \\
&\quad \geq \left(\sum_{j=1}^{m} \frac{\sqrt{2}|x_j - y_j|}{(1 - tx_j - (1-t)y_j)^{\frac{3}{2}}} \frac{\sqrt{tx_j + (1-t)y_j}}{\sqrt{1 - tx_j - (1-t)y_j}}\right)^2 - \left(\sum_{j=1}^{m} \frac{x_j - y_j}{(1 - tx_j - (1-t)y_j)^2}\right)^2 \\
&\quad = \left(\sum_{j=1}^{m} \frac{\sqrt{2}|x_j - y_j|\sqrt{tx_j + (1-t)y_j}}{(1 - tx_j - (1-t)y_j)^2}\right)^2 - \left(\sum_{j=1}^{m} \frac{x_j - y_j}{(1 - tx_j - (1-t)y_j)^2}\right)^2.
\end{aligned}$$

From $x_j, y_j \in [\frac{1}{2}, 1)$ it follows that $\sqrt{2}\sqrt{tx_j + (1-t)y_j} \geq 1$, hence $\psi''(t)\psi(t) - (\psi'(t))^2 \geq 0$, and then $q''(t) \geq 0$, that is, $q(t)$ is convex on $[0,1]$.

The proof of Lemma 5 is completed. □

Lemma 6 *Let* $\mathbf{x} = (x_1, \ldots, x_m)$ *and* $\mathbf{y} = (y_1, \ldots, y_m) \in (0, \frac{1}{2}]^m$. *Then the function* $r(t) = \log \varphi(t)$ *is convex on* $[0,1]$, *where*

$$\varphi(t) = \sum_{j=1}^{m} \frac{1 - tx_j - (1-t)y_j}{tx_j + (1-t)y_j}.$$

Proof

$$r'(t) = \frac{\varphi'(t)}{\varphi(t)},$$

where

$$\varphi'(t) = -\sum_{j=1}^{m} \frac{x_j - y_j}{(tx_j + (1-t)y_j)^2},$$

$$r''(t) = \frac{\varphi''(t)\varphi(t) - (\varphi'(t))^2}{\varphi^2(t)},$$

where

$$\varphi''(t) = \sum_{j=1}^{m} \frac{2(x_j - y_j)^2}{(tx_j + (1-t)y_j)^3}.$$

By the Cauchy inequality, we have

$$\begin{aligned}
&\varphi''(t)\varphi(t) - \bigl(\varphi'(t)\bigr)^2 \\
&= \left(\sum_{j=1}^{m} \frac{2(x_j - y_j)^2}{(tx_j + (1-t)y_j)^3}\right)\left(\sum_{j=1}^{m} \frac{1 - tx_j - (1-t)y_j}{tx_j + (1-t)y_j}\right) - \left(-\sum_{j=1}^{m} \frac{x_j - y_j}{(tx_j + (1-t)y_j)^2}\right)^2 \\
&\geq \left(\sum_{j=1}^{m} \frac{\sqrt{2}|x_j - y_j|}{(tx_j + (1-t)y_j)^{\frac{3}{2}}} \frac{\sqrt{1 - tx_j - (1-t)y_j}}{\sqrt{tx_j + (1-t)y_j}}\right)^2 - \left(\sum_{j=1}^{m} \frac{x_j - y_j}{(tx_j + (1-t)y_j)^2}\right)^2 \\
&= \left(\sum_{j=1}^{m} \frac{\sqrt{2}|x_j - y_j|\sqrt{1 - tx_j - (1-t)y_j}}{(tx_j + (1-t)y_j)^2}\right)^2 - \left(\sum_{j=1}^{m} \frac{x_j - y_j}{(tx_j + (1-t)y_j)^2}\right)^2.
\end{aligned}$$

From $x_j, y_j \in (0, \frac{1}{2}]$ it follows that $\sqrt{2}\sqrt{1 - tx_j - (1-t)y_j} \geq 1$, hence $\varphi''(t)\varphi(t) - (\varphi'(t))^2 \geq 0$, and then $r''(t) \geq 0$, that is, $r(t)$ is convex on $[0,1]$.

The proof of Lemma 6 is completed. □

Lemma 7 *Let* $\boldsymbol{x} = (x_1, \ldots, x_m)$ *and* $\boldsymbol{y} = (y_1, \ldots, y_m) \in (0,1)^m$. *Then the function* $h(t) = \log f(t)$ *is convex on* $[0,1]$, *where*

$$f(t) = \sum_{j=1}^{m} \frac{1 + tx_j + (1-t)y_j}{1 - tx_j - (1-t)y_j}.$$

Proof

$$h'(t) = \frac{f'(t)}{f(t)},$$

where

$$f'(t) = \sum_{j=1}^{m} \frac{2(x_j - y_j)}{(1 - tx_j - (1-t)y_j)^2},$$

$$h''(t) = \frac{f''(t)f(t) - (f'(t))^2}{f^2(t)},$$

where

$$f''(t) = \sum_{j=1}^{m} \frac{4(x_j - y_j)^2}{(1 - tx_j - (1-t)y_j)^3}.$$

By the Cauchy inequality, we have

$$\begin{aligned}
&f''(t)f(t) - \bigl(f'(t)\bigr)^2 \\
&= \left(\sum_{j=1}^{m} \frac{4(x_j - y_j)^2}{(1 - tx_j - (1-t)y_j)^3}\right)\left(\sum_{j=1}^{m} \frac{1 + tx_j + (1-t)y_j}{1 - tx_j - (1-t)y_j}\right) \\
&\quad - \left(\sum_{j=1}^{m} \frac{2(x_j - y_j)}{(1 - tx_j - (1-t)y_j)^2}\right)^2 \\
&\geq \left(\sum_{j=1}^{m} \frac{2|x_j - y_j|}{(1 - tx_j - (1-t)y_j)^{\frac{3}{2}}} \frac{\sqrt{1 + tx_j + (1-t)y_j}}{\sqrt{1 - tx_j - (1-t)y_j}}\right)^2 - \left(\sum_{j=1}^{m} \frac{2(x_j - y_j)}{(1 - tx_j - (1-t)y_j)^2}\right)^2 \\
&= \left(\sum_{j=1}^{m} \frac{2|x_j - y_j|\sqrt{1 + tx_j + (1-t)y_j}}{(1 - tx_j - (1-t)y_j)^2}\right)^2 - \left(\sum_{j=1}^{m} \frac{2(x_j - y_j)}{(1 - tx_j - (1-t)y_j)^2}\right)^2.
\end{aligned}$$

From $x_j, y_j \in (0,1)$ it follows that $\sqrt{2}\sqrt{1 + tx_j + (1-t)y_j} \geq 1$, hence $f''(t)f(t) - (f'(t))^2 \geq 0$, and then $h''(t) \geq 0$, that is, $h(t)$ is convex on $[0,1]$.

The proof of Lemma 7 is completed. □

Lemma 8 *Let* $\boldsymbol{x} = (x_1, \ldots, x_m)$ *and* $\boldsymbol{y} = (y_1, \ldots, y_m) \in (0, \sqrt{\sqrt{5} - 2})^m$. *Then the function* $s(t) = \log w(t)$ *is convex on* $[0,1]$, *where*

$$w(t) = \sum_{j=1}^{m} \left(\frac{1}{tx_j + (1-t)y_j} - \bigl(tx_j + (1-t)y_j\bigr)\right).$$

Proof

$$s'(t) = \frac{w'(t)}{w(t)},$$

where

$$w'(t) = -\sum_{j=1}^{m} (x_j - y_j)\left(\frac{1}{(tx_j + (1-t)y_j)^2} + 1\right),$$

$$s''(t) = \frac{w''(t)w(t) - (w'(t))^2}{w^2(t)},$$

where

$$w''(t) = \sum_{j=1}^{m} \frac{2(x_j - y_j)^2}{(tx_j + (1-t)y_j)^3}.$$

By the Cauchy inequality, we have

$$\begin{aligned}
&w''(t)w(t) - \left(w'(t)\right)^2 \\
&\quad = \left(\sum_{j=1}^m \frac{2(x_j - y_j)^2}{(tx_j + (1-t)y_j)^3}\right)\left(\sum_{j=1}^m \left(\frac{1}{tx_j + (1-t)y_j} - \left(tx_j + (1-t)y_j\right)\right)\right) \\
&\qquad - \left(-\sum_{j=1}^m (x_j - y_j)\left(\frac{1}{(tx_j + (1-t)y_j)^2} + 1\right)\right)^2 \\
&\quad \geq \left(\sum_{j=1}^m \frac{\sqrt{2}|x_j - y_j|}{(tx_j + (1-t)y_j)^{\frac{3}{2}}}\sqrt{\frac{1}{tx_j + (1-t)y_j} - \left(tx_j + (1-t)y_j\right)}\right)^2 \\
&\qquad - \left(\sum_{j=1}^m (x_j - y_j)\left(\frac{1}{(tx_j + (1-t)y_j)^2} + 1\right)\right)^2 \\
&\quad = \left(\sum_{j=1}^m \frac{\sqrt{2}|x_j - y_j|\sqrt{1 - (tx_j + (1-t)y_j)^2}}{(tx_j + (1-t)y_j)^2}\right)^2 - \left(\sum_{j=1}^m (x_j - y_j)\frac{1 + (tx_j + (1-t)y_j)^2}{(tx_j + (1-t)y_j)^2}\right)^2.
\end{aligned}$$

Let $u_j := tx_j + (1-t)y_j$. From $x_j, y_j \in (0, \sqrt{\sqrt{5} - 2})$ it follows that $u_j^2 \leq \sqrt{5} - 2$. Since

$$\begin{aligned}
&u_j^2 \leq \sqrt{5} - 2 \quad \Leftrightarrow \quad \left(u_j^2 + 2\right)^2 \leq 5 \quad \Leftrightarrow \quad u_j^4 + 4u_j^2 - 1 \leq 0 \\
&\quad \Leftrightarrow \quad 2\left(1 - u_j^2\right) \geq \left(1 + u_j^2\right)^2 \quad \Leftrightarrow \quad \sqrt{2}\sqrt{1 - u_j^2} \geq 1 + u_j^2,
\end{aligned}$$

so $w''(t)w(t) - (w'(t))^2 \geq 0$, and then $s''(t) \geq 0$, that is, $s(t)$ is convex on $[0,1]$.

The proof of Lemma 8 is completed. □

3 Proof of main results

Proof of Theorem 4 For any $1 \leq i_1 < \cdots < i_k \leq n$, by Lemma 3 and Lemma 7, it follows that $\log\sum_{j=1}^k \frac{1+x_{i_j}}{1-x_{i_j}}$ is convex on $(0,1)^k$. Obviously, $\log\sum_{j=1}^k \frac{1+x_{i_j}}{1-x_{i_j}}$ is also convex on $(0,1)^n$, and then $\log E_k^*(\frac{1+\boldsymbol{x}}{1-\boldsymbol{x}}) = \sum_{1\leq i_1<\cdots<i_k\leq n}\log\sum_{j=1}^k \frac{1+x_{i_j}}{1-x_{i_j}}$ is convex on $(0,1)^n$. Furthermore, it is clear that $\log E_k^*(\frac{1+\boldsymbol{x}}{1-\boldsymbol{x}})$ is symmetric on $(0,1)^n$. By Lemma 1, it follows that $\log E_k^*(\frac{1+\boldsymbol{x}}{1-\boldsymbol{x}})$ is Schur-convex on $(0,1)^n$, and then from Lemma 2 we conclude that $E_k^*(\frac{1+\boldsymbol{x}}{1-\boldsymbol{x}})$ is also Schur-convex on $(0,1)^n$.

The proof of Theorem 4 is completed. □

Similar to the proof of Theorem 4, we can use Lemma 4, Lemma 5, Lemma 6 and Lemma 8 respectively to prove Theorem 1, Theorem 2, Theorem 3 and Theorem 5; therefore we omit the details of the proof.

Remark 1 Using the Schur-convex function decision theorem, Liu *et al.* [9] have proved Theorem 3. Xia and Chu [10] have proved that the symmetric function

$$E_k^*\left(\frac{1+\boldsymbol{x}}{\boldsymbol{x}}\right) = \prod_{1\leq i_1<\cdots<i_k\leq n}\sum_{j=1}^k \frac{1+x_{i_j}}{x_{i_j}}, \quad k = 1,\ldots,n, \tag{12}$$

is a Schur-convex function on $\mathbb{R}_+^n$.

The reader may wish to prove the inequality (12) by the properties of a Schur-convex function.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors co-authored this paper together. All authors read and approved the final manuscript.

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