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Static state feedback linearization of nonlinear control systems on homogeneous time scales

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Abstract The paper addresses the problem of static state feedback linearization for nonlinear control systems defined on homogeneous time scales. Necessary and sufficient conditions for generic local linearizability of the considered systems by static state feedback and state transformation are presented in terms of a sequence of subspaces of differential one-forms related to the system.

Keywords Feedback linearization · Nonlinear control systems · Time scale · Differential forms

1 Introduction

There is ongoing need to make the knowledge base of control theory more compact so that the field can continue to grow. Integrating existing results by providing a more abstract structure is, therefore, always necessary, even more so in the fragmented world

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of today. Time scale calculus is a general framework that allows unification of the study of continuous- and discrete-time systems [8]. The term "time scales" refer to the way dynamic systems behave over time. Most engineering applications assume time to be either continuous or (uniformly) discrete, both of which are merged in time scale formalism into a general framework, and follow from the latter as special cases. The main concept of time scale calculus, the delta derivative, is the generalization of both time derivative and the difference operator. Though time scale calculus accommodates more possibilities, regarding the control theory, the most important special cases are continuous- and (uniformly) discrete-time systems that are the examples of systems defined on homogeneous time scale.

The description of the dynamics that relies on the difference operator is often called delta-domain description, see for instance [13,17,18,22,30,37,39]. The delta-domain approach has been promoted as an effective tool for dynamic system modeling and control. Compared to models based on the shift operator, the delta-domain models are less sensitive to round-off errors and do not yield ill-conditioned models when signals are sampled at a high sampling rate, see [23,31,33]. In [1], the authors demonstrate that the numerical properties of structure detection are improved by a delta-domain model and such delta-domain models provide models closely linked to the continuous-time systems. The delta-domain approach allows to address the question of preservation of the system properties under Euler discretization scheme since it treats the discretized systems as a special case of a system defined on a time scale.

Though the literature on time scale calculus is rich (see the references in [3,6,9, 16,19,20,32,34,38]), there is not so many papers addressing the control problems for nonlinear systems defined on time scale [7,12,28]. In the earlier papers [4,5] the algebraic formalism has been developed that allows to address various analysis and control problems for nonlinear systems, defined on homogeneous time scales. This approach has been successfully applied to address the irreducibility, reduction [28] and realization of the input–output equations in the state space form [12].

The theories of continuous- and discrete-time dynamical systems as presented in the literature are different, but the analysis on time scales is nowadays recognized as the right tool to unify the seemingly separate fields of discrete dynamical systems (i.e., difference equations) and continuous dynamical systems (i.e., differential equations), and also to present both continuous- and discrete-time theories in the same language. The goal of this paper is to study the iconic problem of static state feedback linearization for nonlinear systems defined on homogeneous time scales where the state transformation method based on one-forms is used. The presented approach covers the continuous- and discrete-time cases in such a manner that those are the special cases of the formalism. Since delta derivative (used in our paper to describe the dynamical systems) coincides with the time derivative for the continuous-time case, the earlier results for continuous-time systems, see for instance [15,25], follow from our results as a special case, namely the case in which the time scale is the set of real numbers. Note, however, that our results do not recover those given in [2,21,26,29] for the discrete-time systems, defined in terms of the shift operator since the latter is not a delta-derivative. Instead, our formalism includes the description of a discrete-time system based on the difference operator description (delta-domain approach), for which the results shown in the paper are new. Using the difference operator in

the discrete case allows to unify the theories of static state feedback linearization for continuous- and discrete-time nonlinear control systems.

Static state feedback linearization is a much studied research topic since 1980s, both for continuous- and discrete-time cases, see [24,26,27] and the references therein. Many different techniques that speak different languages (including differential geometry and algebra) have been used in linearization studies that make their comparison a difficult task. The paper [35] establishes the explicit relationship between the two sets of necessary and sufficient solvability conditions, given in terms of integrability of certain vector spaces of differential one-forms and involutivity of distributions of vector fields. In particular, it has been demonstrated that the distributions, when they are involutive, are the maximal annihilators of the corresponding codistributions (vector spaces of one-forms). Moreover, in [35], two methods have been compared from the point of view of computational complexity and it was shown that regarding the computation of the state transformation method based on one-forms is simpler. By this reason, we decided to rely on this method in unification. Finally, note that the results of [11] are claimed in [11] to be dual to those of [2] though not shown in detail.

The paper is organized as follows. Section 2 recalls the notions from time scale calculus and algebraic formalism used in the paper, while in Sect. 3 the properties of the sequences of subspaces of differential one-forms are presented, in terms of which our main results are formulated. The basic results given in Sect. 3 together with the facts presented in Sect. 4 contain the key contribution of the paper. Section 4 is devoted to give the main results, i.e., the conditions for the generic local static state feedback linearization of control systems defined on homogeneous time scales. Finally, some illustrative examples that describe our results are presented.

2 Time scale calculus and algebraic formalism

For a general introduction to the calculus on time scales, see [8]. Here, we recall only those notions and facts that will be used later.

A *time scale* $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. We assume that the topology of $\mathbb{T}$ is induced by $\mathbb{R}$. The *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined as $\sigma(t) := \inf\{s \in \mathbb{T} \mid s > t\}$, $\sigma(\max \mathbb{T}) = \max \mathbb{T}$ if there exists a finite max $\mathbb{T}$, the *backward jump operator* $\rho(t) : \mathbb{T} \to \mathbb{T}$ is defined as $\rho(t) := \sup\{s \in \mathbb{T} \mid s < t\}$, $\rho(\min \mathbb{T}) = \min \mathbb{T}$ if there exists a finite min $\mathbb{T}$. The graininess functions $\mu : \mathbb{T} \to [0, \infty)$ and $\nu : \mathbb{T} \to [0, \infty)$ are defined by $\mu(t) := \sigma(t) - t$ and $\nu(t) := t - \rho(t)$, respectively. A time scale is called *homogeneous* if μ and ν are constant functions. Homogeneous time scales are either of the form $t_0 + \mu\mathbb{Z}$, where $t_0 \in \mathbb{R}$ and $\mu > 0$, or are closed intervals, bounded or unbounded, including in particular $\mathbb{R}$.

From now, we assume that $\mathbb{T}$ is a homogeneous time scale.

Remark 1 In [5], we called time scale homogeneous if only μ is a constant function. The reason is that when one only needs the forward shift and delta derivative, it is not necessary to have constant ν. However, if we also need the backward shift and nabla derivative, we have to assume additionally ν to be constant since μ being constant does not yield the latter. For example, in the set of natural numbers $\mu \equiv 1$, but $\nu(1) = 0$ whereas $\nu(k) = 1$ for $k \geq 1$; so ν is not constant.

Let us now recall the definition of the delta derivative f^Δ of a real valued function f.

Definition 1 The *delta derivative of a function* $f : \mathbb{T} \to \mathbb{R}$ at $t \in \mathbb{T}$ is the real number $f^\Delta(t)$ (provided it exists) such that for each $\varepsilon > 0$ there exists a neighborhood $U(\varepsilon)$ of t, $U(\varepsilon) \subset \mathbb{T}$ such that for all $\tau \in U(\varepsilon)$, $|(f(\sigma(t)) - f(\tau)) - f^\Delta(t)(\sigma(t) - \tau)| \leqslant \varepsilon|\sigma(t) - \tau|$. Moreover, we say that f is *delta differentiable* on $\mathbb{T}$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}$.

Remark 2 The delta derivative is a natural extension of time derivative in the continuous-time case and forward difference operator in the discrete-time case. Therefore, for $\mathbb{T} = \mathbb{R}$, $f^\Delta(t) = \lim\limits_{s \to t} \frac{f(t) - f(s)}{t - s} = f'(t)$ and for $\mathbb{T} = \mathbb{Z}$, $f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = f(t + 1) - f(t) =: \Delta f(t)$, where Δ is the usual forward difference operator.

For a function $f : \mathbb{T} \to \mathbb{R}$, we can define the second delta derivative $f^{[2]} := (f^\Delta)^\Delta$ provided that f^Δ is delta differentiable with derivative $f^{[2]} : \mathbb{T} \to \mathbb{R}$. Similarly, we define higher order delta derivatives $f^{[n]} : \mathbb{T} \to \mathbb{R}$, $f^{[n]} := (f^{[n-1]})^\Delta$. Note that for a homogeneous time scale, there is no left-scattered maximal point in $\mathbb{T}$, so $f^{[n]}, n \geqslant 1$ are uniquely defined for all $t \in \mathbb{T}$.

For $f : \mathbb{T} \to \mathbb{R}$, define $f^\sigma := f \circ \sigma$. Denote $f^{\Delta\sigma} := (f^\Delta)^\sigma$ and $f^{\sigma\Delta} := (f^\sigma)^\Delta$.

If f and f^Δ are delta differentiable functions, then for homogeneous time scale one has $f^{\sigma\Delta} = f^{\Delta\sigma}$.

Now, we recall some definitions and facts given in [4,5] that will be used in the paper.

2.1 Differential field and space of one-forms

Consider now the control system, defined on a homogenous time scale $\mathbb{T}$,

$$x^\Delta(t) = f(x(t), u(t)) \tag{1}$$

where $(x(t), u(t)) \in \mathcal{X} \times \mathcal{U}$, $\mathcal{X} \times \mathcal{U}$ is an open subset of $\mathbb{R}^n \times \mathbb{R}^m$, $m \leqslant n$, and $f : \mathcal{X} \times \mathcal{U} \to \mathbb{R}^n$ is analytic. We assume that the input applied to system (1) is infinitely many times delta differentiable, i.e., $u^{[k]}$ exists for all $k \geqslant 0$. Let us define $\widetilde{f}(x, u) := x + \mu f(x, u)$ and assume that there exists a map $\varphi : \mathcal{X} \times \mathcal{U} \to \mathbb{R}^m$ such that $\Phi = (\widetilde{f}, \varphi)^T$ is an analytic diffeomorphism[1] from the set $\mathcal{X} \times \mathcal{U}$ onto $\Phi(\mathcal{X} \times \mathcal{U})$. This means that from $(\bar{x}, z) = (\widetilde{f}(x, u), \varphi(x, u)) = \Phi(x, u)$, we can uniquely compute (x, u) as an analytic function of $(\bar{x}, z)$. For $\mu = 0$, this condition is always satisfied with $\varphi(x, u) = u$.

Note that for $\mathbb{T} = \mathbb{R}$, the Eq. (1) becomes an ordinary differential equation and when $\mathbb{T} = h\mathbb{Z}$, for $h > 0$, it becomes the difference equation. Both $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = h\mathbb{Z}$ are homogeneous time scales.

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[1] This assumption guarantees that the system $x^\sigma = \widetilde{f}(x, u)$ is submersive, that is generically rank $\frac{\partial \widetilde{f}(x,u)}{\partial(x,u)} = n$.

For notational convenience, $(x_1, \ldots, x_n)$ will simply be written as x, and $(u_1^{[k]}, \ldots, u_m^{[k]})$ as $u^{[k]}$, for $k \geqslant 0$. For $i \leqslant k$, let $u^{[i..k]} := (u^{[i]}, \ldots, u^{[k]})$. Consider the infinite set of real (independent) indeterminates

$$\mathcal{C} = \{x_i, \; i = 1, \ldots, n, \; u_j^{[k]}, \; j = 1, \ldots, m, \; k \geqslant 0\}$$

and let $\mathcal{K}$ be the (commutative) field of meromorphic functions in a finite number of the variables from the set $\mathcal{C}$. Let $\sigma_f : \mathcal{K} \to \mathcal{K}$ be an operator defined by

$$\sigma_f(F)(x, u^{[0..k+1]}) := F(\sigma_f(x), \sigma_f(u^{[0..k]})), \tag{2}$$

where $F \in \mathcal{K}$, $\sigma_f(u^{[0..k]}) := u^{[0..k]} + \mu u^{[1..k+1]}$, for $k \geqslant 0$ and by (1) $\sigma_f(x) := x + \mu x^\Delta = x + \mu f(x, u)$. Under the assumption about the existence of φ such that Φ is an analytic diffeomorphism, σ_f is an injective endomorphism.

The field $\mathcal{K}$ can be equipped with an operator $\Delta_f : \mathcal{K} \to \mathcal{K}$ defined by

$$
\begin{aligned}
&\Delta_f(F)(x, u^{[0..k+1]}) \\
&:= \begin{cases}
\frac{1}{\mu}[F(x + \mu f(x, u), u^{[0..k]} + \mu u^{[1..k+1]}) - F(x, u^{[0..k]})], & \text{if } \mu \neq 0 \\
\frac{\partial F}{\partial x}(x, u^{[0..k]}) f(x, u) + \sum_{k \geqslant 0} \frac{\partial F}{\partial u^{[0..k]}}(x, u^{[0..k]}) u^{[1..k+1]}, & \text{if } \mu = 0.
\end{cases}
\end{aligned}
\tag{3}
$$

Remark 3 Note that in both continuous-time and discrete-time case, the operator Δ_f corresponds to the total delta derivative of a function with respect to the dynamic of the system, i.e., for control $u(\cdot)$ applied to system (1), for solution $x(\cdot)$ of (1) corresponding to control u and $F \in \mathcal{K}$ we get

$$\frac{\Delta}{\Delta t}(F(x(t), u^{[0..k]}(t))) = \Delta_f(F)(x(t), u^{[0..k+1]}(t)). \tag{4}$$

On the right hand side of (4), the operator Δ_f is applied to function F which depends on indeterminates from the set $\mathcal{C}$ and substituting $x(\cdot)$ and $u^{[l]}(\cdot)$ instead of x and $u^{[l]}$, respectively, we get relation (4) where the time appears.

The operator Δ_f satisfies, for all $F, G \in \mathcal{K}$, the conditions

(i) $\Delta_f(F + G) = \Delta_f(F) + \Delta_f(G)$,
(ii) $\Delta_f(FG) = \Delta_f(F)G + \sigma_f(F)\Delta_f(G)$ (generalized Leibniz rule).

An operator satisfying the generalized Leibniz rule is called a "σ_f-derivation" and a commutative field endowed with a σ_f-derivation is called a σ_f-*differential field* [14]. Therefore, under the assumption about the existence of φ such that Φ is an analytic diffeomorphism, $\mathcal{K}$ endowed with the delta derivative Δ_f is a σ_f-differential field. The σ_f-differential field $\mathcal{K}$ is called *inversive*, if every element of $\mathcal{K}$ has a pre-image in $\mathcal{K}$ with respect to σ_f, i.e., $\sigma_f^{-1}(F)$ is defined for all $F \in \mathcal{K}$, see [14]. For $\mu = 0$, $\sigma_f = \sigma_f^{-1} = \text{id}$ and $\mathcal{K}$ is inversive. Though $\mathcal{K}$ is not inversive in general, it is always possible to embed $\mathcal{K}$ into an inversive σ_f-differential overfield $\mathcal{K}^*$, called

the *inversive closure* of $\mathcal{K}$ [14]. Since σ_f is an injective endomorphism, it can be extended to $\mathcal{K}^*$ such that $\sigma_f : \mathcal{K}^* \to \mathcal{K}^*$ is an automorphism. For $\mu \neq 0$, the overfield $\mathcal{K}^*$ consists of meromorphic functions in a finite number of the independent variables $\mathcal{C}^* = \mathcal{C} \cup \{z_s^{\langle -\ell \rangle}, \ s = 1, \ldots, m, \ \ell \geqslant 1\}$, where $z_i^{\langle -k \rangle} = \sigma_f(z_i^{\langle -k-1 \rangle})$ for $k \geqslant 1$, and $z_i = \varphi_i(x, u) = \sigma_f(z_i^{\langle -1 \rangle})$, see [4,5]. Let $z := (z_1, \ldots, z_m)$. Then, $\sigma_f^{-1}(x, u) = \psi(x, z^{\langle -1 \rangle})$, where ψ is a certain vector valued function, determined by f in (1) and the extension $z = \varphi(x, u)$. Although the choice of variables z is not unique, all possible choices yield isomorphic field extensions. We extend the operator Δ_f to new variables by

$$\Delta_f(z^{\langle -\ell \rangle}) := \frac{z^{\langle -\ell+1 \rangle} - z^{\langle -\ell \rangle}}{\mu}, \quad l \geqslant 1.$$

The extension of operator Δ_f to $\mathcal{K}^*$ can be made in analogy to (3). Such operator Δ_f is now a σ_f-derivation of $\mathcal{K}^*$. A practical procedure for construction of $\mathcal{K}^*$ (for $\mu \neq 0$) is given in [5], where the explicit construction of an inversive σ_f-differential overfield $\mathcal{K}^*$ is unique up to an isomorphism, see [5].

Note that $\mathcal{K}^* = \mathcal{K}$ for $\mu = 0$.

We assume that for system (1) the assumption of independence of controls

$$\text{rank}_{\mathcal{K}} \frac{\partial f}{\partial u} = m \tag{5}$$

holds.[2] We will consider regular state feedbacks, which are defined as follows:

Definition 2 *A regular static state feedback* is an analytic mapping $\phi : \mathcal{X} \times \mathcal{V} \to \mathcal{U}$,

$$(x, v) \mapsto u = \phi(x, v) \tag{6}$$

satisfying rank $\frac{\partial \phi}{\partial v}(x, v) = m$ for all $(x, v) \in \mathcal{X} \times \mathcal{V}$ and $\phi(\mathcal{X} \times \mathcal{V}) = \mathcal{U}$, where $\mathcal{V} \subset \mathbb{R}^m$.

Remark 4 The regular static state feedback defined above is not globally invertible, but the condition rank $\frac{\partial \phi}{\partial v} = m$ guarantees local invertibility with respect to v for each fixed point $x \in \mathcal{X}$. Moreover, the map $\vartheta : (\mathcal{X}, \mathcal{V}) \ni (x, v) \mapsto (x, \phi(x, v)) \in (\mathcal{X}, \mathcal{U})$ is a local analytic diffeomorphism.

From now on

$$\mathcal{C}^* = \begin{cases} \mathcal{C}, & \text{if } \mu = 0 \\ \mathcal{C} \cup \{z^{\langle -\ell \rangle} | \ell \geqslant 1\}, & \text{if } \mu \neq 0. \end{cases}$$

Consider the infinite set of differentials of indeterminates $d\mathcal{C}^* = \{d\zeta_i, \ \zeta_i \in \mathcal{C}^*\}$. Then, $dx_i, i = 1, \ldots, n, du_j^{[k]}, j = 1, \ldots, m, k \geqslant 0$, can be treated as differentials of

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[2] This assumption, though natural, is not necessary for construction of $\mathcal{K}^*$.

coordinate functions which select x_i or $u_j^{[k]}$ from the set of all variables, respectively. Define $\mathcal{E} := \mathrm{span}_{\mathcal{K}^*} d\mathcal{C}^*$. Any element of $\mathcal{E}$ is a vector of the form

$$\omega = \sum_{i=1}^{n} A_i dx_i + \sum_{k \geqslant 0} \sum_{j=1}^{m} B_{jk} du_j^{[k]} + \sum_{\ell \geqslant 1} \sum_{s=1}^{m} C_{s\ell} dz_s^{\langle -\ell \rangle}$$

where only a finite number of coefficients B_{jk} and $C_{s\ell}$ are nonzero elements of $\mathcal{K}^*$. Since F is a meromorphic function in a finite number of variables from the set $\mathcal{C}^*$, then an operator $d : \mathcal{K}^* \to \mathcal{E}$ can be defined in the standard manner:

$$dF := \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} dx_i + \sum_{k \geqslant 0} \sum_{j=1}^{m} \frac{\partial F}{\partial u_j^{[k]}} du_j^{[k]} + \sum_{\ell \geqslant 1} \sum_{s=1}^{m} \frac{\partial F}{\partial z_s^{\langle -\ell \rangle}} dz_s^{\langle -\ell \rangle}. \qquad (7)$$

The elements of $\mathcal{E}$ are called differential *one-forms*. One says that $\omega \in \mathcal{E}$ is an *exact one-form* if $\omega = dF$ for some $F \in \mathcal{K}^*$. We will refer to dF as to the *total differential* (or simply the *differential*) of F.

If $\omega = \sum_i A_i d\zeta_i$ is a one-form, where $A_i \in \mathcal{K}^*$ and $\zeta_i \in \mathcal{C}^*$, one can define the operators $\Delta_f : \mathcal{E} \to \mathcal{E}$ and $\sigma_f : \mathcal{E} \to \mathcal{E}$ by

$$\Delta_f(\omega) := \sum_i \{ \Delta_f(A_i) d\zeta_i + \sigma_f(A_i) d[\Delta_f(\zeta_i)] \},$$

and

$$\sigma_f(\omega) := \sum_i \sigma_f(A_i) d[\sigma_f(\zeta_i)].$$

The operator $\sigma_f : \mathcal{E} \to \mathcal{E}$ is invertible and the inverse operator $\sigma_f^{-1} : \mathcal{E} \to \mathcal{E}$ is given by

$$\sigma_f^{-1} \left(\sum_i A_i d\zeta_i \right) = \sum_i \sigma_f^{-1}(A_i) d[\sigma_f^{-1}(\zeta_i)],$$

for $A_i \in \mathcal{K}^*$ and $\zeta_i \in \mathcal{C}^*$.

Since $\sigma_f(A_i) = A_i + \mu \Delta_f(A_i)$ (see [5]),

$$\Delta_f(\omega) = \sum_i \{ \Delta_f(A_i) d\zeta_i + (A_i + \mu \Delta_f(A_i)) d[\Delta_f(\zeta_i)] \}.$$

For the homogeneous time scale $\mathbb{T}$, we have for $F \in \mathcal{K}^*$

$$[dF]^{\Delta_f} = d[F^{\Delta_f}] \quad \text{and} \quad [dF]^{\sigma_f} = d[F^{\sigma_f}].$$

Note that the following relation between operators Δ_f and σ_f holds

$$\sigma_f = \mathrm{id} + \mu \cdot \sigma_f, \tag{8}$$

where id denotes the identity operator.

We will use sometimes the more compact notations $F^{\Delta_f}, \omega^{\Delta_f}, F^{\sigma_f}$ and ω^{σ_f} instead of $\Delta_f(F), \Delta_f(\omega), \sigma_f(F)$ and $\sigma_f(\omega)$.

2.2 Vector fields and p-forms

Let $\mathcal{E}'$ be the dual vector space of $\mathcal{E}$, i.e., the space of linear mappings from $\mathcal{E}$ to $\mathcal{K}^*$. The elements of $\mathcal{E}'$ are of the form

$$X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + \sum_{k \geqslant 0} \sum_{j=1}^{m} b_{jk} \frac{\partial}{\partial u_j^{[k]}} + \sum_{\ell \geqslant 1} \sum_{s=1}^{m} c_{s\ell} \frac{\partial}{\partial z_s^{\langle -\ell \rangle}}, \tag{9}$$

where $a_i, b_{jk}, c_{s\ell} \in \mathcal{K}^*$ and are called vector fields. Taking $\omega = \sum_{i=1}^{n} A_i \mathrm{d}x_i + \sum_{k=0}^{p} \sum_{j=1}^{m} B_{jk} \mathrm{d}u_j^{[k]} + \sum_{\ell=1}^{q} \sum_{s=1}^{m} C_{s\ell} \mathrm{d}z_s^{\langle -\ell \rangle} \in \mathcal{E}$ and the vector field $X \in \mathcal{E}'$ of the form (9), we get

$$X(\omega) =: \langle X, \omega \rangle = \sum_{i=1}^{n} a_i A_i + \sum_{k=0}^{p} \sum_{j=1}^{m} b_{jk} B_{jk} + \sum_{\ell=1}^{q} \sum_{s=1}^{m} c_{s\ell} C_{s\ell}.$$

The delta-derivative X^{Δ_f} and the forward-shift X^{σ_f} of $X \in \mathcal{E}'$ may be defined uniquely by the equations

$$\langle X^{\Delta_f}, \omega \rangle = \langle X, \sigma_f^{-1}(\omega) \rangle^{\Delta_f} - \langle X, [\sigma_f^{-1}(\omega)]^{\Delta_f} \rangle \tag{10}$$

and

$$\langle X^{\sigma_f}, \omega \rangle = \langle X, \sigma_f^{-1}(\omega) \rangle^{\sigma_f},$$

respectively, where ω is an arbitrary one-form. Note that $\langle X, \sigma_f^{-1}(\omega) \rangle \in \mathcal{K}^*$, so $\langle X, \sigma_f^{-1}(\omega) \rangle^{\sigma_f}$ and $\langle X, \sigma_f^{-1}(\omega) \rangle^{\Delta_f}$ are well defined.

Proposition 1 [4] *Let $X \in \mathcal{E}'$. Then for arbitrary $\omega \in \mathcal{E}$*

$$X^{\sigma_f} = X + \mu X^{\Delta_f}, \tag{11}$$

$$\langle X, \omega \rangle^{\Delta_f} = \langle X^{\Delta_f}, \omega \rangle + \langle X^{\sigma_f}, \omega^{\Delta_f} \rangle. \tag{12}$$

Let $\mathcal{E}^p$ be the set of differential p-forms, see [4]. Every p-form $\alpha \in \mathcal{E}^p$ has a unique representative of the form

$$\alpha = \sum_{i_1 < \cdots < i_p} A_{i_1 \ldots i_p} \mathrm{d}\zeta_{i_1} \wedge \cdots \wedge \mathrm{d}\zeta_{i_p},$$

where $A_{i_1 \cdots i_p} \in \mathcal{K}^*$.

The *exterior product* (alternatively called the *wedge product*) of a p-form $\omega_1 = \sum_{i_1 < \cdots < i_p} F_{i_1 \cdots i_p} \mathrm{d}\zeta_{i_1} \wedge \cdots \wedge \mathrm{d}\zeta_{i_p}$ and q-form $\omega_2 = \sum_{j_1 < \cdots < j_q} G_{j_1 \cdots j_q} \mathrm{d}\zeta_{j_1} \wedge \cdots \wedge \mathrm{d}\zeta_{j_q}$, denoted as $\omega_1 \wedge \omega_2$, is $(p+q)$-form defined in the following way

$$\omega_1 \wedge \omega_2 = \sum_{i_1 < \cdots < i_p} \sum_{j_1 < \cdots < j_q} F_{i_1 \cdots i_p} G_{j_1 \cdots j_q} \mathrm{d}\zeta_{i_1} \wedge \cdots \wedge \mathrm{d}\zeta_{i_p} \wedge \mathrm{d}\zeta_{j_1} \wedge \cdots \wedge \mathrm{d}\zeta_{j_q},$$

where $F_{i_1 \cdots i_p}, G_{j_1 \cdots j_q} \in \mathcal{K}^*$ and $\zeta_{i_l}, \zeta_{j_s} \in \mathcal{C}^*, l = 1, \ldots, p, s = 1, \ldots, q$, see [4].

Let $\mathbb{E}$ be the space of all forms, i.e., $\mathbb{E} = \biguplus_{p \geq 0} \mathcal{E}^p$, where $\mathcal{E}^0 := \mathcal{K}^*, \mathcal{E}^1 := \mathcal{E}$.

Exterior differential d is an operator $\mathrm{d} : \mathbb{E} \to \mathbb{E}$ that satisfies the following properties:

(i) $\mathrm{d}(\mathcal{E}^p) \in \mathcal{E}^{p+1}$ and $\mathrm{d} : \mathcal{E}^p \to \mathcal{E}^{p+1}$ is an $\mathbb{R}$-linear operator.
(ii) $\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^s \alpha \wedge \mathrm{d}\beta$, where $\alpha \in \mathcal{E}^s$ and $\beta \in \mathcal{E}^{p-s}$,
(iii) if $F \in \mathcal{K}^*$, then $\mathrm{d}F$ coincides with the ordinary differential, see (7).
(iv) $\mathrm{d}^2 = 0$, where $\mathrm{d}^2 = \mathrm{d} \circ \mathrm{d} : \mathbb{E} \to \mathbb{E}$.

The properties (i)–(iii) define uniquely the operator d.

The operators $\Delta_f : \mathcal{K}^* \to \mathcal{K}^*$ and $\sigma_f : \mathcal{K}^* \to \mathcal{K}^*$, related to system (1), induce the operators $\Delta_f : \mathcal{E}^p \to \mathcal{E}^p$ and $\sigma_f : \mathcal{E}^p \to \mathcal{E}^p$ by

$$
\begin{aligned}
\Delta_f &\left(\sum_{i_1 < \ldots < i_p} A_{i_1 \ldots i_p} \mathrm{d}\zeta_{i_1} \wedge \cdots \wedge \mathrm{d}\zeta_{i_p} \right) \\
&:= \sum_{i_1 < \cdots < i_p} \left[A_{i_1 \cdots i_p}^{\Delta_f} \mathrm{d}\zeta_{i_1} \wedge \cdots \wedge \mathrm{d}\zeta_{i_p} + A_{i_1 \cdots i_p}^{\sigma_f} \mathrm{d}\zeta_{i_1}^{\Delta_f} \wedge \mathrm{d}\zeta_{i_2} \wedge \cdots \wedge \mathrm{d}\zeta_{i_p} \right. \\
&\quad + A_{i_1 \cdots i_p}^{\sigma_f} \mathrm{d}\zeta_{i_1}^{\sigma_f} \wedge \mathrm{d}\zeta_{i_2}^{\Delta_f} \wedge \mathrm{d}\zeta_{i_3} \wedge \cdots \wedge \mathrm{d}\zeta_{i_p} \\
&\quad \left. + \cdots + A_{i_1 \cdots i_p}^{\sigma_f} \mathrm{d}\zeta_{i_1}^{\sigma_f} \wedge \cdots \wedge \mathrm{d}\zeta_{i_{p-1}}^{\sigma_f} \wedge \mathrm{d}\zeta_{i_p}^{\Delta_f} \right]
\end{aligned}
\tag{13}
$$

and

$$\sigma_f \left(\sum_{i_1 < \cdots < i_p} A_{i_1 \cdots i_p} \mathrm{d}\zeta_{i_1} \wedge \cdots \wedge \mathrm{d}\zeta_{i_p} \right) := \sum_{i_1 < \cdots < i_p} \left[A_{i_1 \cdots i_p}^{\sigma_f} \mathrm{d}\zeta_{i_1}^{\sigma_f} \wedge \cdots \wedge \mathrm{d}\zeta_{i_p}^{\sigma_f} \right],$$

$$\tag{14}$$

where $\zeta_{i_1}, \ldots, \zeta_{i_p} \in \mathcal{C}^*$ and $A_{i_1 \cdots i_p} \in \mathcal{K}^*$.

Proposition 2 [4] *Let $\omega \in \mathcal{E}^p$, $p \geqslant 1$. Then, for a homogeneous time scale $\mathbb{T}$:*

$$d[\omega^{\Delta_f}] = [d\omega]^{\Delta_f} \quad and \quad d[\omega^{\sigma_f}] = [d\omega]^{\sigma_f}.$$

Now let us consider an operator $i_X : \mathbb{E} \to \mathbb{E}$ associated with a vector field $X \in \mathcal{E}'$ that satisfies the following properties:

(i) $i_X(\mathcal{E}^{p+1}) \in \mathcal{E}^p$, for $p \geqslant 0$ and $i_X : \mathcal{E}^{p+1} \to \mathcal{E}^p$ is a $\mathcal{K}^*$-linear operator.
(ii) $i_X(\omega_1 \wedge \omega_2) = i_X(\omega_1) \wedge \omega_2 + (-1)^p \omega_1 \wedge i_X(\omega_2)$, where p is the degree of ω_1;
(iii) $i_X(F) = 0$, for all $F \in \mathcal{K}^*$;
(iv) $i_X(dx_i) = X_i$, where by X_i is denoted the ith component of X.

The operator i_X is called *interior product* with respect to X. Note that $i_X(d\zeta_i) = \langle X, d\zeta_i \rangle$ so that for a differential 2-form $\theta = \sum_{i,j} a_{ij} d\zeta_i \wedge d\zeta_j$ we have

$$i_X\theta = \sum_{i,j} a_{ij}(\langle X, d\zeta_i \rangle d\zeta_j - \langle X, d\zeta_j \rangle d\zeta_i).$$

3 Sequence of subspaces of one-forms

In this section, we introduce the sequence of subspaces of $\mathcal{E}$ being the main tool for characterizing solvability/solution of numerous problems in control theory, for example accessibility (irreducibility), realization and feedback linearization.

Introduce, in analogy with [15], the sequence of subspaces $(\mathcal{H}_k)$ of $\mathcal{E}$ defined by

$$\mathcal{H}_0 := \text{span}_{\mathcal{K}^*}\{dx, du\}$$
$$\mathcal{H}_{k+1} := \text{span}_{\mathcal{K}^*}\{\omega \in \mathcal{H}_k \mid \omega^{\Delta_f} \in \mathcal{H}_k\}, \quad k \geqslant 0 \tag{15}$$

Then, $\mathcal{H}_1 := \text{span}_{\mathcal{K}^*}\{dx\}$. Note that $\dim_{\mathcal{K}^*} \mathcal{H}_0 = n + m$ and $\dim_{\mathcal{K}^*} \mathcal{H}_1 = n$.

Let $\Delta_f^k := \underbrace{\Delta_f \circ \Delta_f \circ \cdots \circ \Delta_f}_{k\text{-times}}$. The *relative degree* of a one-form $\omega \in \mathcal{H}_0$ is

defined as:

$$r := \min\left\{k \geqslant 0 \mid \Delta_f^k(\omega) = \omega^{\Delta_f^k} \notin \mathcal{H}_1\right\}.$$

If such integer does not exist, set $r = \infty$. The *relative degree* of a meromorphic function $A(x, u)$ is defined to be the relative degree of the one-form $dA(x, u)$.

The following proposition is a simple consequence of the construction of $\mathcal{H}_k$.

Proposition 3 (i) *For $k \geqslant 0$, $\mathcal{H}_k$ is the space of one-forms that have relative degree greater than or equal to k.*
(ii) *There exists an integer $0 < k^* \leqslant n$ such that, for $0 \leqslant k \leqslant k^*$, $\mathcal{H}_{k+1} \subsetneq \mathcal{H}_k$ and $\mathcal{H}_{k^*+1} = \mathcal{H}_\ell, \ell \geqslant k^* + 1$.*

Proof Point (i) is a simple consequence of definition (15). Concerning (ii), the existence of the integer k^* comes from the fact that each $\mathcal{H}_k$ is a finite-dimensional $\mathcal{K}^*$-vector space so that, at each step, either its dimension decreases or $\mathcal{H}_{k+1} = \mathcal{H}_k$. Moreover, $k^* \leqslant n = \dim_{\mathcal{K}^*} \mathcal{H}_1$.

Define $\mathcal{H}_\infty := \mathcal{H}_{k^*+1}$. Then from the construction of $\mathcal{H}_k$, we get

Proposition 4 $\mathcal{H}_\infty$ *is the largest subspace of* $\mathcal{H}_1$ *that is invariant under* σ_f-*derivation* Δ_f.

Now, we will show that the coordinate transformation yields an inversive closure isomorphic to the original one. Consequently, we have some correspondence between the constructed subspaces of one-forms.

Let $\hat{x} = \xi(x)$, $\xi = (\xi_1, \ldots, \xi_n)$ and $\xi : \mathcal{X} \to \widehat{\mathcal{X}} \subset \mathbb{R}^n$ be an analytic diffeomorphism. Then,

$$\hat{x}^\Delta(t) = \hat{f}(\hat{x}(t), u(t)),$$

where $t \in \mathbb{T}$ and $\hat{f}(\hat{x}, u) := \xi^{\Delta_f}(x, u)\big|_{x=\xi^{-1}(\hat{x})}$.

Note that x and $z^{\langle-\ell\rangle}$, $\ell \geqslant 1$, belong to some open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $\widehat{\mathcal{K}}^*$ be the field of functions in variables from the set $\widehat{\mathcal{C}} := \{\hat{x}, u^{[k]}, z^{\langle-\ell\rangle} \mid k \geqslant 0, \ell \geqslant 1\}$, where $z = \varphi(x, u) = \varphi(\xi^{-1}(\hat{x}), u) =: \hat{\varphi}(\hat{x}, u)$ and $\hat{x} \in \xi(U)$. Moreover, $\widehat{\mathcal{K}}^*$ is equipped with operators $\widehat{\Delta}_{\hat{f}}$ and $\widehat{\sigma}_{\hat{f}}$ defined in a similar manner as operators Δ_f and σ_f (see definitions (3) and (2), respectively). Observe that if $\mathcal{U} = \mathbb{R}^n$, then $\mathcal{K}^* = \widehat{\mathcal{K}}^*$. It is easy to see that the relation similar to (8) holds also for the operators $\widehat{\Delta}_{\hat{f}}$ and $\widehat{\sigma}_{\hat{f}}$, i.e.,

$$\widehat{\Delta}_{\hat{f}} = \mathrm{id} + \mu \cdot \widehat{\sigma}_{\hat{f}}. \tag{16}$$

Let $\widehat{\mathcal{E}} := \mathrm{span}_{\widehat{\mathcal{K}}^*}\{d\hat{x}, du^{[k]}, dz^{\langle-\ell\rangle}, \ k \geqslant 0, \ell \geqslant 1\}$. Then, ξ induces the isomorphisms $\xi^* : \widehat{\mathcal{K}}^* \to \mathcal{K}^*$ and $\xi^* : \widehat{\mathcal{E}} \to \mathcal{E}$ by

$$\xi^*(\widehat{A}_i)(x, u^{[0..k]}, z^{\langle-1\rangle}, \ldots, z^{\langle-\ell\rangle}) := \widehat{A}_i(\xi(x), u^{[0..k]}, z^{\langle-1\rangle}, \ldots, z^{\langle-\ell\rangle}) \tag{17}$$

and

$$\xi^*\left(\sum_{i=1}^n \widehat{A}_i d\hat{x}_i + \sum_{k \geqslant 0}\sum_{j=1}^m \widehat{B}_{jk} du_j^{[k]} + \sum_{\ell \geqslant 1}\sum_{s=1}^m \widehat{C}_{s\ell} dz_s^{\langle-\ell\rangle}\right)$$
$$:= \sum_{i=1}^n \sum_{j=1}^n \xi^*(\widehat{A}_i)\frac{\partial \xi_i}{\partial x_j} dx_j$$
$$+ \sum_{k \geqslant 0}\sum_{j=1}^m \xi^*(\widehat{B}_{jk}) du_j^{[k]} + \sum_{\ell \geqslant 1}\sum_{s=1}^m \xi^*(\widehat{C}_{s\ell}) dz_s^{\langle-\ell\rangle}, \tag{18}$$

respectively, where $\hat{x}_i = \xi_i(x)$, $i = 1, \ldots, n$.

Remark 5 Note that if $\mu = 0$, then the variables $z_s^{\langle -\ell \rangle}$, $s = 1, \ldots, m$, $\ell \geqslant 1$ do not appear and $\sigma_f = \widehat{\sigma}_{\hat{f}} = \mathrm{id}$.

If $\widehat{G} \in \widehat{\mathcal{K}}^*$ and $\widehat{G}$ depends only on $u^{[0..k]}$, $z^{\langle -1 \rangle}, \ldots, z^{\langle -\ell \rangle}$ and does not depend on x, then $\xi^*(\widehat{G}) = \widehat{G}$.

In general, we have

$$
\begin{aligned}
(\xi^* \circ \widehat{\sigma}_{\hat{f}})(\hat{x})(x, u) &= \xi^*(\hat{x} + \mu \hat{f}(\hat{x}, u))(x, u) = \xi^*(\hat{x})(x, u) + \mu \xi^*(\hat{f})(x, u) \\
&= \xi(x) + \mu \hat{f}(\xi(x), u) = \xi(x) + \mu \Delta_f(\xi)(x, u) \\
&= \sigma_f(\xi(x))(x, u) = \sigma_f(\xi^*(\hat{x}))(x, u) = (\sigma_f \circ \xi^*)(\hat{x})(x, u)
\end{aligned}
$$

and

$$
\begin{aligned}
(\xi^* \circ \widehat{\sigma}_{\hat{f}})(u^{[k]})(u^{[k]}, u^{[k+1]}) &= \xi^*(u^{[k]} + \mu u^{[k+1]}) = \xi^*(u^{[k]}) + \mu \xi^*(u^{[k+1]}) \\
&= u^{[k]} + \mu \Delta_f(u^{[k]})(u^{[k]}, u^{[k+1]}) = \sigma_f(u^{[k]})(u^{[k]}, u^{[k+1]}) \\
&= \sigma_f(\xi^*(u^{[k]}))(u^{[k]}, u^{[k+1]}) = (\sigma_f \circ \xi^*)(u^{[k]})(u^{[k]}, u^{[k+1]}).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
(\xi^* \circ \widehat{\sigma}_{\hat{f}})(z^{\langle -1 \rangle})(x, u) &= (\xi^* \circ \widehat{\sigma}_{\hat{f}})(\widehat{\sigma}_{\hat{f}}^{-1}(\hat{\varphi}(\hat{x}, u)))(x, u) = \xi^*(\hat{\varphi})(x, u) \\
&= \hat{\varphi}(\xi(x), u) = \varphi((\xi^{-1} \circ \xi)(x), u) = \varphi(x, u) \\
&= \sigma_f(\sigma_f^{-1}(\varphi(x, u))) = \sigma_f(z^{\langle -1 \rangle})(x, u) = (\sigma_f \circ \xi^*)(z^{\langle -1 \rangle})(x, u)
\end{aligned}
$$

and for $\ell \geqslant 2$

$$
\begin{aligned}
(\xi^* \circ \widehat{\sigma}_{\hat{f}})(z^{\langle -\ell \rangle})(z^{\langle -\ell+1 \rangle}) &= \xi^*(z^{\langle -\ell+1 \rangle}) = z^{\langle -\ell+1 \rangle} \\
&= \sigma_f(z^{\langle -\ell \rangle})(z^{\langle -\ell+1 \rangle}) = (\sigma_f \circ \xi^*)(z^{\langle -\ell \rangle})(z^{\langle -\ell+1 \rangle}).
\end{aligned}
$$

Therefore, we get

$$
\xi^* \circ \widehat{\sigma}_{\hat{f}} = \sigma_f \circ \xi^*.
$$

Note that $u^{[k]}$, $k \geqslant 0$ and $z^{\langle -\ell \rangle}$, $\ell \geqslant 1$ are the same for $\mathcal{K}^*$ and $\widehat{\mathcal{K}}^*$. Consequently, $\widehat{\sigma}_{\hat{f}}(z^{\langle -\ell \rangle}) = z^{\langle -\ell+1 \rangle} = \sigma_f(z^{\langle -\ell \rangle})$.

Proposition 5 *For $\widehat{A} \in \widehat{\mathcal{K}}^*$*

$$
\xi^* \left(\widehat{A}^{\widehat{\Delta}_{\hat{f}}} \right) = [\xi^* (\widehat{A})]^{\Delta_f} \tag{19}
$$

and

$$
\xi^* \left(\widehat{A}^{\widehat{\sigma}_{\hat{f}}} \right) = [\xi^* (\widehat{A})]^{\sigma_f} . \tag{20}
$$

Proof Let $\widehat{A} \in \widehat{\mathcal{K}}^*$. Then, $\xi^*(\widehat{A}) \in \mathcal{K}^*$. Since, by (3), for $\mu \neq 0$

$$\widehat{A}^{\widehat{\Delta}_{\hat{f}}}(\hat{x}, u^{[0..k+1]}, z^{\langle -1 \rangle}, \ldots, z^{\langle -\ell+1 \rangle})$$
$$= \frac{1}{\mu} \cdot \left[\widehat{A}(\hat{x} + \mu \hat{f}(\hat{x}, u), u^{[0..k]} + \mu u^{[1..k+1]}, \hat{\varphi}(\hat{x}, u), z^{\langle -1 \rangle}, \ldots, z^{\langle -\ell+1 \rangle}) \right.$$
$$\left. - \widehat{A}(\hat{x}, u^{[0..k]}, z^{\langle -1 \rangle}, z^{\langle -2 \rangle}, \ldots, z^{\langle -\ell \rangle}) \right],$$

and for $\mu = 0$

$$\widehat{A}^{\widehat{\Delta}_{\hat{f}}}(\hat{x}, u^{[0..k+1]}) = \frac{\partial \widehat{A}}{\partial \hat{x}}\left(\hat{x}, u^{[0..k]}\right) \hat{f}(\hat{x}, u) + \sum_{j=0}^{k} \frac{\partial \widehat{A}}{\partial u^{[j]}}\left(\hat{x}, u^{[0..k]}\right) u^{[j+1]},$$

we get

$$\xi^*(\widehat{A}^{\widehat{\Delta}_{\hat{f}}})(x, u^{[0..k+1]}, z^{\langle -1 \rangle}, \ldots, z^{\langle -\ell+1 \rangle})$$
$$= \begin{cases} \frac{1}{\mu} \cdot \left[\widehat{A}(\xi(x) + \mu \xi^{\Delta_f}(x, u), u^{[0..k]} + \mu u^{[1..k+1]}, \varphi(x, u), z^{\langle -1 \rangle}, \ldots, \right. \\ \quad \left. z^{\langle -\ell+1 \rangle}) - \widehat{A}(\xi(x), u^{[0..k]}, z^{\langle -1 \rangle}, z^{\langle -2 \rangle}, \ldots, z^{\langle -\ell \rangle}) \right], \qquad \mu \neq 0 \\[2mm] \frac{\partial \widehat{A}}{\partial \hat{x}}(\xi(x), u^{[0..k]}) \frac{\partial \xi}{\partial x} f(x, u) + \sum_{j=0}^{k} \frac{\partial \widehat{A}}{\partial u^{[j]}}(\xi(x), u^{[0..k]}) u^{[j+1]}, \qquad \mu = 0 \end{cases}$$
$$= [\widehat{A}(\xi(x), u^{[0..k]}, z^{\langle -1 \rangle}, z^{\langle -2 \rangle}, \ldots, z^{\langle -\ell \rangle})]^{\Delta_f}$$
$$= [\xi^*(\widehat{A})(x, u^{[0..k]}, z^{\langle -1 \rangle}, z^{\langle -2 \rangle}, \ldots, z^{\langle -\ell \rangle})]^{\Delta_f}$$
$$= [\xi^*(\widehat{A})]^{\Delta_f}(x, u^{[0..k+1]}, z^{\langle -1 \rangle}, z^{\langle -2 \rangle}, \ldots, z^{\langle -\ell+1 \rangle}).$$

Using the relations (8), (16) and (19), one gets

$$\xi^*(\widehat{A}^{\widehat{\sigma}_{\hat{f}}}) = \xi^*(\widehat{A} + \mu \cdot \widehat{A}^{\widehat{\Delta}_{\hat{f}}}) = \xi^*(\widehat{A}) + \mu \cdot [\xi^*(\widehat{A})]^{\Delta_f} = [\xi^*(\widehat{A})]^{\sigma_f}.$$

Hence the proposition holds.

Proposition 6 *Let $\widehat{\omega} \in \widehat{\mathcal{E}}$. Then,*

$$\xi^*(\widehat{\omega}^{\widehat{\Delta}_{\hat{f}}}) = [\xi^*(\widehat{\omega})]^{\Delta_f}. \tag{21}$$

The proof is given in the Appendix.

In $\widehat{\mathcal{E}}$, we have the following sequence of subspaces $(\widehat{\mathcal{H}}_k)$:

$$\begin{aligned} \widehat{\mathcal{H}}_0 &= \mathrm{span}_{\widehat{\mathcal{K}}^*}\{d\hat{x}_i, du_j, \; i = 1, \ldots, n, \; j = 1, \ldots, m\} \\ \widehat{\mathcal{H}}_1 &= \mathrm{span}_{\widehat{\mathcal{K}}^*}\{d\hat{x}_i, \; i = 1, \ldots, n\} \\ \widehat{\mathcal{H}}_k &= \mathrm{span}_{\widehat{\mathcal{K}}^*}\{\widehat{\omega} \in \widehat{\mathcal{H}}_{k-1} \mid \widehat{\omega}^{\widehat{\Delta}_{\hat{f}}} \in \widehat{\mathcal{H}}_{k-1}\}, \quad k \geqslant 2. \end{aligned} \tag{22}$$

Proposition 7 *For subspaces* $\mathcal{H}_k$, $\widehat{\mathcal{H}}_k$ *defined by* (15) *and* (22), *respectively, we have, for* $k \geqslant 0$

$$\xi^*(\widehat{\mathcal{H}}_k) = \mathcal{H}_k. \tag{23}$$

Proof The proof is by the induction principle. Since the diffeomorphism ξ induces isomorphism $\xi^* : \widehat{\mathcal{E}} \to \mathcal{E}$ and for arbitrary $\widehat{\omega}_x = \sum_{i=1}^n \widehat{A}_i d\hat{x}_i \in \widehat{\mathcal{H}}_1 \subset \widehat{\mathcal{H}}_0$, $\widehat{\omega}_u = \sum_{k \geqslant 0} \sum_{j=1}^m \widehat{B}_{jk} d\hat{u}_j^{[k]} \in \widehat{\mathcal{H}}_0$ by (18) we have

$$\xi^*(\widehat{\omega}_x + \widehat{\omega}_u) = \xi^* \left(\sum_{i=1}^n \widehat{A}_i d\hat{x}_i + \sum_{k \geqslant 0} \sum_{j=1}^m \widehat{B}_{jk} d\hat{u}_j^{[k]} \right)$$

$$= \sum_{i=1}^n \sum_{j=1}^n \xi^*(\widehat{A}_i) \frac{\partial \xi_i}{\partial x_j} dx_j + \sum_{k \geqslant 0} \sum_{j=1}^m \xi^*(\widehat{B}_{jk}) d\hat{u}_j^{[k]} \in \mathcal{H}_0$$

and $\xi^*(\widehat{\omega}_x) = \sum_{i=1}^n \sum_{j=1}^n \xi^*(\widehat{A}_i) \frac{\partial \xi_i}{\partial x_j} dx_j \in \mathcal{H}_1$, so (23) holds for $k = 0, 1$. Assume that (23) is true for $k = n$, and let $\widehat{\omega} \in \widehat{\mathcal{H}}_{n+1}$. From definition (15), we get $\widehat{\omega} \in \widehat{\mathcal{H}}_n$ and $\widehat{\omega}^{\Delta_{\hat{f}}} \in \widehat{\mathcal{H}}_n$. Since $\xi^*(\widehat{\mathcal{H}}_n) = \mathcal{H}_n$,

$$\xi^*(\widehat{\omega}) \in \mathcal{H}_n \quad \text{and} \quad \xi^*(\widehat{\omega}^{\Delta_{\hat{f}}}) \in \mathcal{H}_n.$$

By (21), we get

$$[\xi^*(\widehat{\omega})]^{\Delta_f} \in \mathcal{H}_n$$

and it is equivalent to $\xi^*(\widehat{\omega}) \in \mathcal{H}_{n+1}$. Hence using the induction principle, we get that (23) holds for all $k \geqslant 0$. $\qquad \blacksquare$

Remark 6 The dimension of the subspace $\mathcal{H}_k$, $k \geqslant 0$, is invariant under diffeomorphism ξ, i.e., $\dim_{\mathcal{K}^*} \mathcal{H}_k = \dim_{\mathcal{K}^*} \widehat{\mathcal{H}}_k$, for $\mathcal{H}_k = \xi^*(\widehat{\mathcal{H}}_k)$.

Let $u = \phi(x, v)$ be a static state feedback such that the map $\vartheta : (\mathcal{X}, \mathcal{V}) \ni (x, v) \mapsto (x, \phi(x, v)) \in (\mathcal{X}, \mathcal{U})$ is an analytic diffeomorphism. Then, the field $\widetilde{\mathcal{K}}^*$ of meromorphic functions in a finite number of variables from the set $\widetilde{\mathcal{C}} := \{x, v^{[k]}, z^{\langle -\ell \rangle} \mid k \geqslant 0, \ell \geqslant 1\}$ where $z = \varphi(x, \phi(x, v))$, and $\widetilde{\mathcal{E}}$ can be constructed similarly as $\widehat{\mathcal{K}}^*$ and $\widehat{\mathcal{E}}$, respectively. Then, ϑ induces the isomorphisms $\vartheta^* : \widetilde{\mathcal{K}}^* \to \mathcal{K}^*$ and $\vartheta^* : \widetilde{\mathcal{E}} \to \mathcal{E}$ by the formulas similar to (17) and (18). Moreover, the sequence $(\widetilde{\mathcal{H}}_k)$ can be defined by analogy with (22). Subspaces $\mathcal{H}_k$ and $\widetilde{\mathcal{H}}_k$, for $k \geqslant 0$, are isomorphic similarly as in Proposition 7. In particular,

$$\mathcal{H}_0 = \text{span}_{\mathcal{K}^*} \{dx, du\} = \text{span}_{\widetilde{\mathcal{K}}^*} \left\{ dx, \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial v} dv \right\} = \text{span}_{\widetilde{\mathcal{K}}^*} \left\{ dx, \frac{\partial \phi}{\partial v} dv \right\}.$$

Since ϑ is a diffeomorphism, one gets

$$\widetilde{\mathcal{H}}_0 = \mathrm{span}_{\widetilde{\mathcal{K}}^*}\{\mathrm{d}x, \mathrm{d}v\}.$$

Note that the relative degrees of the one-forms are invariant under regular static state feedback. This fact is well known for continuous-time systems, see Proposition 8.8 in [36], and carries easily over for systems, defined on homogeneous time scales. Then, the following property characterizes the subspaces $\mathcal{H}_k$:

Corollary 1 *Let $k \geqslant 0$. The subspaces $\mathcal{H}_k$ and $\widetilde{\mathcal{H}}_k$ are isomorphic, i.e., $\dim \mathcal{H}_k = \dim \widetilde{\mathcal{H}}_k$, for $\mathcal{H}_k = \vartheta^*(\widetilde{\mathcal{H}}_k)$. Moreover, for $k \geqslant 1$ the subspaces $\mathcal{H}_k$ are invariant under regular static state feedback, so $\mathcal{H}_k = \widetilde{\mathcal{H}}_k$. Feedback invariance comes from the fact that the relative degrees are invariant under regular static state feedback and from the definition of the subspaces $\mathcal{H}_k$ and $\widetilde{\mathcal{H}}_k$, $k \geqslant 1$.*

Theorem 1 *Suppose $\mathcal{H}_\infty = \{0\}$. Then, there exists a list of integers $r_1, \ldots, r_m$ and one-forms $\omega_1, \ldots, \omega_m \in \mathcal{H}_1$ whose relative degrees are, respectively, $r_1, \ldots, r_m$ such that*

(i) $\mathrm{span}_{\mathcal{K}^*}\{\omega_i^{\Delta_f^j}, \ 1 \leqslant i \leqslant m, 0 \leqslant j \leqslant r_i - k\} = \mathcal{H}_k, k \geqslant 0$, *in particular*

$\mathrm{span}_{\mathcal{K}^*}\{\omega_i^{\Delta_f^j}, \ 1 \leqslant i \leqslant m, 0 \leqslant j \leqslant r_i - 1\} = \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}x\} = \mathcal{H}_1,$

$\mathrm{span}_{\mathcal{K}^*}\{\omega_i^{\Delta_f^j}, \ 1 \leqslant i \leqslant m, 0 \leqslant j \leqslant r_i\} = \mathrm{span}_{\mathcal{K}^*}\{\mathrm{d}x, \mathrm{d}u\} = \mathcal{H}_0,$

(ii) *the one-forms $\{\omega_i^{\Delta_f^j}, \ 1 \leqslant i \leqslant m, j \geqslant 0\}$ are linearly independent over the field $\mathcal{K}^*$; in particular $\sum_{i=1}^m r_i = n$.*

Proof Let $\mathcal{W}_{k^*} = \{\eta_1, \ldots, \eta_{r_{k^*}}\}$ be a basis for $\mathcal{H}_{k^*}$.[3] Then by definition (15), the elements of $\mathcal{W}_{k^*}$ and $\mathcal{W}_{k^*}^{\Delta_f} = \{\eta_1^{\Delta_f}, \ldots, \eta_{r_{k^*}}^{\Delta_f}\}$ belong to $\mathcal{H}_{k^*-1}$. Now, we want to prove that the vectors in $\mathcal{W}_{k^*} \cup \mathcal{W}_{k^*}^{\Delta_f}$ are independent and we prove it by contradiction. Suppose that $\mathcal{W}_{k^*} \cup \mathcal{W}_{k^*}^{\Delta_f}$ are linearly dependent. Then, there exist some coefficients $a_i, b_i, 1 \leqslant i \leqslant r_{k^*}$, some of them nonzero, such that $\sum_i (a_i \eta_i + b_i \eta_i^{\Delta_f}) = 0$. The linear independence of the η_i's implies that not all the b_i's vanish. Denote $\sigma_f^{-1}(b_i)$ by $b_i^{\rho_f}$ and consider the one-form $\omega = \sum_i b_i^{\rho_f} \eta_i \in \mathcal{H}_{k^*}$ whose delta derivative is $\omega^{\Delta_f} = \sum_i ((b_i^{\rho_f})^{\Delta_f} \eta_i + b_i \eta_i^{\Delta_f}) = \sum_i (b_i^{\rho_f})^{\Delta_f} \eta_i - \sum_i a_i \eta_i$. Hence $\omega \in \mathcal{H}_{k^*+1} = \mathcal{H}_\infty$, which contradicts the assumption $\mathcal{H}_\infty = \{0\}$. Therefore, $\mathcal{W}_{k^*} \cup \mathcal{W}_{k^*}^{\Delta}$ are linearly independent. Hence it is always possible to choose a set (possibly empty) $\mathcal{W}_{k^*-1}$ such that $\mathcal{W}_{k^*} \cup \mathcal{W}_{k^*}^{\Delta} \cup \mathcal{W}_{k^*-1}$ is a basis for $\mathcal{H}_{k^*-1}$. Repeating this procedure $k^* - 1$ times, we obtain

---

[3] Note that k^* is defined by Proposition 3.

$$\mathcal{H}_k = \text{span}_{\mathcal{K}^*} \left\{ \mathcal{W}_i^{\Delta_f^j}, \ k \leqslant i \leqslant k^*, 0 \leqslant j \leqslant i - k \right\}, \quad 0 \leqslant k \leqslant k^*. \quad (24)$$

It can be proved by induction that, for $0 \leqslant k \leqslant k^*$, the set

$$\left\{ \mathcal{W}_{k^*}, \ldots, \mathcal{W}_{k^*}^{\Delta_f^{k^*-k}}, \ldots, \mathcal{W}_{k+1}, \mathcal{W}_{k+1}^{\Delta_f}, \mathcal{W}_k \right\}$$

is linearly independent. $\mathcal{H}_1 = \text{span}_{\mathcal{K}^*}\{dx\}$ and assumption (5) imply $\mathcal{W}_0 = \emptyset$. Note that $\mathcal{H}_0 = \mathcal{H}_1 \oplus \text{span}_{\mathcal{K}^*}\{du\}$. By (24), we get

$$\text{span}_{\mathcal{K}^*}\{du\} = \text{span}_{\mathcal{K}^*} \left\{ \mathcal{W}_1^{\Delta_f}, \mathcal{W}_2^{\Delta_f^2}, \ldots, \mathcal{W}_{k^*}^{\Delta_f^{k^*}} \right\}.$$

Since $\dim \text{span}_{\mathcal{K}^*}\{du\} = m$, there exist $\omega_1, \ldots, \omega_m$ such that

$$\{\omega_1, \ldots, \omega_m\} = \mathcal{W}_{k^*} \cup \cdots \cup \mathcal{W}_1,$$

r_i is the relative degree of the one-form ω_i for $i = 1, \ldots, m$, and

$$\text{span}_{\mathcal{K}^*}\{du\} = \text{span}_{\mathcal{K}^*} \left\{ \omega_1^{\Delta_f^{r_1}}, \ldots, \omega_m^{\Delta_f^{r_m}} \right\}.$$

Consequently, $\mathcal{H}_1 = \text{span}_{\mathcal{K}^*}\{\omega_i, \ldots, \omega_i^{\Delta_f^{r_i-1}}, i = 1, \ldots, m\}$ and by $\dim \mathcal{H}_1 = n$, we get $\sum_{i=1}^{m} r_i = n$.

Corollary 2 *Suppose* $\mathcal{H}_\infty = \{0\}$. *Then, there exists a basis* $\{\omega_{i,j}, \ 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant r_i\}$ *of* $\mathcal{H}_1$ *such that*

$$
\begin{aligned}
\omega_{i,1}^{\Delta_f} &= \omega_{i,2} \\
\omega_{i,2}^{\Delta_f} &= \omega_{i,3} \\
&\vdots \\
\omega_{i,r_i-1}^{\Delta_f} &= \omega_{i,r_i} \\
\omega_{i,r_i}^{\Delta_f} &= \sum_{s=1}^{m} \sum_{j=1}^{r_s} a_{s,j}^i \omega_{s,j} + \sum_{j=1}^{m} b_j^i du_j, \quad i = 1, 2, \ldots, m,
\end{aligned}
\quad (25)
$$

where $a_{s,j}^i, b_j^i \in \mathcal{K}^*$ *and* $[b_j^i]$ *has the inverse in the ring of* $m \times m$ *matrices with entries in* $\mathcal{K}^*$.

Proof For $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant r_i$, we take $\omega_{i,j} = \omega_i^{\Delta_f^{j-1}}$.

Proposition 8 *For $1 \leqslant k \leqslant k^* + 1$, there exist n_k one-forms $\alpha_1, \ldots, \alpha_{n_k}$ that depend on variables $\{x, u^{[i]} \mid i \geqslant 0\}$ for $k = 1$ and variables $\{x, u^{[i]}, z^{\langle -j \rangle} \mid i \geqslant 0, 1 \leqslant j \leqslant k - 1\}$ for $k \geqslant 2$ and constitute a basis for $\mathcal{H}_k$.*

Proof The proof is by induction. Proposition 8 is evidently true for $k = 1$ and $\dim \mathcal{H}_1 = n_1 = n$. Suppose it is true for some integer $k \geqslant 1$. Let $\{\eta_1, \ldots, \eta_{n_k}, \vartheta_1, \ldots, \vartheta_{n_{k-1} - n_k}\}$ and $\{\eta_1, \ldots, \eta_{n_k}\}$ be, respectively, the bases of $\mathcal{H}_{k-1}$ and $\mathcal{H}_k$. An arbitrary element $\alpha = \sum_i a_i \eta_i \in \mathcal{H}_k$ belongs to $\mathcal{H}_{k+1}$ if and only if $\alpha^{\Delta_f} = \sum_i a_i^{\Delta_f} \eta_i + a_i^{\sigma_f} \eta_i^{\Delta_f} \in \mathcal{H}_k$. Since $\eta_i \in \mathcal{H}_k$, it follows that $\eta_i, \eta_i^{\Delta_f} \in \mathcal{H}_{k-1}$. Hence

$$\alpha^{\Delta_f} = \sum_i a_i^{\Delta_f} \eta_i + a_i^{\sigma_f} \left(\sum_j b_{ij} \eta_j + \sum_\ell c_{i\ell} \vartheta_\ell \right).$$

Thus, $\alpha \in \mathcal{H}_{k+1}$ if and only if the coefficients $a_i^{\sigma_f}$ satisfy the following system of linear equations:

$$\sum_i a_i^{\sigma_f} c_{i\ell} = 0, \quad 1 \leqslant \ell \leqslant n_{k-1} - n_k. \tag{26}$$

System (26) has $n_k - \operatorname{rank}_{\mathcal{K}^*}[c_{i\ell}] =: s$ linearly independent solutions and $s = \dim_{\mathcal{K}^*} \mathcal{H}_{k+1}$. Observe also that $s = n_{k+1}$ from definition of $\mathcal{H}_{k+1}$.

Note that from the induction assumption, the coefficients $a_i^{\sigma_f}$ may be chosen to depend only on variables $\{x, u^{[i]}, z^{\langle -j \rangle} \mid j \leqslant k - 1, i \geqslant 0\}$. Since $a_i = \sigma_f^{-1}(a_i^{\sigma_f})$, so a_i depends only on the variables $\{x, u^{[i]}, z^{\langle -j \rangle} \mid j \leqslant k, i \geqslant 0\}$.

The proof of Proposition 8 provides a procedure to compute bases of the subspaces $\mathcal{H}_k$, $k \geqslant 2$. Let $\omega_{i,j}$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant r_i$ be a basis of $\mathcal{H}_1$ satisfying (25). Observe that there exists an integer number M such that one-forms $\omega_{i,j}$ depend on $(x, u, u^{[1]}, \ldots, u^{[\iota]}, z^{\langle -1 \rangle}, \ldots, z^{\langle -\kappa \rangle}) \in \mathcal{X} \times \mathcal{U} \times \mathbb{R}^{\iota + \kappa} \subset \mathbb{R}^M$. Let $\mathcal{S}$ be an open and dense subset of $\mathcal{X} \times \mathcal{U} \times \mathbb{R}^{\iota + \kappa}$ such that the forms $\omega_i^{\Delta_f^j}$, $1 \leqslant i \leqslant k^*$, $0 \leqslant j \leqslant r_i - 1$ evaluated at $(x, u, u^{[1]}, \ldots, u^{[\iota]}, z^{\langle -1 \rangle}, \ldots, z^{\langle -\kappa \rangle}) \in \mathcal{S}$ are linearly independent over $\mathbb{R}$.

Let us recall that a codistribution on $V \subset \mathbb{R}^M$ is a map $H : V \ni p \mapsto H_p$, where H_p is a linear subspace of $T_p^* \mathbb{R}^M$. The codistribution H is *locally integrable* if for each point $p \in V$ there exist some neighbourhood $\mathcal{V}$ of p and exact one-forms defined on $\mathcal{V}$ such that for every $q \in \mathcal{V}$ the one-forms evaluated at q form the basis of H_q. If $p \mapsto \dim H_p$ is constant and H is generated by one-forms $\omega_1, \ldots, \omega_r$ (i.e., at each $p \in V$, H_p is spanned by $\omega_1(p), \ldots, \omega_r(p)$), then local integrability of H is equivalent to the Frobenius condition, i.e., $d\omega_k \wedge \omega_1 \wedge \cdots \wedge \omega_r = 0$, for $1 \leq k \leq r$, see for instance [10].

Observe that the spaces $\mathcal{H}_k$ are not codistributions in the sense defined above. To associate with $\mathcal{H}_k$ a codistribution on $\mathcal{S}$, for $p \in \mathcal{S}$, we set H_p to be the space of $\omega(p)$,

where $\omega \in \mathcal{H}_k$ is well defined at p. This codistribution is generated by the one-forms given in Proposition 8 and $\mathcal{S}$ is the subset of $\mathcal{X} \times \mathcal{U} \times \mathbb{R}^{l+\kappa}$ on which these one-forms are well defined and linearly independent at every point. We say that $\mathcal{H}_k$ is *locally integrable* if the codistribution associated with $\mathcal{H}_k$ is locally integrable.

Assumption 1 Assume that $\mathcal{H}_\infty = \{0\}$ and for each $(x, u, u^{[1]}, \ldots, u, u^{[1]}, \ldots, u^{[l]},$
$z^{\langle -1 \rangle}, \ldots, z^{\langle -\kappa \rangle}) \in \mathcal{S}$ the one-forms $\omega_i^{\Delta_f^j}$, $1 \leqslant i \leqslant k^*$, $0 \leqslant j \leqslant r_i - 1$ evaluated at $(x, u, u^{[1]}, \ldots, u, u^{[1]}, \ldots, u^{[l]}, z^{\langle -1 \rangle}, \ldots, z^{\langle -\kappa \rangle})$ are linearly independent over $\mathbb{R}$.

Observe that under Assumption 1, all $\mathcal{H}_k$, $k \geqslant 2$, define constant-dimensional codistributions on $\mathcal{S}$.

Remark 7 Later, we shall restrict variables x and u to a neighbourhood of some $(\bar{x}, \bar{u}) \in \mathcal{X} \times \mathcal{U}$. This will result in $\mathcal{H}_k$ and $\mathcal{H}_\infty$ restricted to such a neighborhood. Let us notice that $\mathcal{H}_\infty = \{0\}$ if and only if $\mathcal{H}_\infty$ restricted to all such neighbourhoods is equal 0.

4 Feedback linearization

Definition 3 System Σ of the form (1) is said to be *linearizable* by static state feedback if there exist a state analytic diffeomorphism $\xi : \mathcal{X} \to \widehat{\mathcal{X}}$

$$\hat{x} = \xi(x) \tag{27}$$

and a static state feedback (6) such that $\vartheta : (\mathcal{X}, \mathcal{V}) \ni (x, v) \mapsto (x, \phi(x, v)) \in (\mathcal{X}, \mathcal{U})$ is an analytic diffeomorphism and, in new coordinates, we have

$$\hat{x}^\Delta(t) = A \cdot \hat{x}(t) + B \cdot v(t), \tag{28}$$

where the pair (A, B) is controllable, i.e., rank $[B \; AB \; \ldots \; A^{n-1}B] = n$.

The variables $(\hat{x}, v)$ of system (28) belong to some open subset of $\mathbb{R}^n \times \mathbb{R}^m$.

Definition 4 System Σ of the form (1) is said to be *generically locally linearizable* by static state feedback if there is an open and dense subset T of $\mathcal{X} \times \mathcal{U}$ such that for every $(\bar{x}, \bar{u}) \in T$ there is a neighborhood V of $(\bar{x}, \bar{u})$ contained in $\mathcal{X} \times \mathcal{U}$ such that Σ restricted to V is linearizable by static state feedback.

Observe that the static state feedback (6) used in Definitions 3 and 4 is regular.

Proposition 9 *If system (1) is generically locally linearizable by static state feedback, then $\mathcal{H}_\infty = \{0\}$.*

Proof From the controllability of the pair (A, B) we get $\widehat{\mathcal{H}}_\infty = \{0\}$. Since $\widehat{\mathcal{H}}_\infty = \{0\}$, then $\mathcal{H}_\infty = \{0\}$ as the image of 0 with respect to a linear map.

Remark 8 The property $\mathcal{H}_\infty = \{0\}$ corresponds to accessibility of system (1). See [2,15] for more details.

Theorem 2 *Assume that Assumption* 1 *holds. Then, system* (1) *is generically locally linearizable by static state feedback if and only if* $\mathcal{H}_k$ *is locally integrable for* $1 \leqslant k \leqslant k^*$.

Proof $\Leftarrow$: Since from Assumption 1 $\mathcal{H}_\infty = \{0\}$, then by Corollary 2 there exists a basis $\{\omega_{i,j}, \ 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant r_i\}$ of $\mathcal{H}_1 = \operatorname{span}_{\mathcal{K}^*}\{dx\}$ such that in this basis the first-order approximation of (1), i.e.,

$$dx^{\Delta}(t) = \frac{\partial f}{\partial x}(x(t), u(t))dx(t) + \frac{\partial f}{\partial u}(x(t), u(t))du(t),$$

locally takes form (25). Note that "locally" means in some neighborhood of arbitrary point $(\bar{x}, \bar{u}) \in T$. By Frobenius's Theorem, there is no loss of generality if we assume that the basis $\{\omega_{i,j}, \ 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant r_i\}$ contains exact one-forms. Thus, every $\omega_{i,j}$ can be integrated, i.e., there exist $\xi_{i,j}$ such that $\omega_{i,j} = d\xi_{i,j}(x)$. Since $d\hat{x}_{i,j}^{\Delta} = [d\xi_{i,j}(x)]^{\Delta_f} = \omega_{i,j}^{\Delta_f} = \omega_{i,j+1} = d\xi_{i,j+1}(x) = d\hat{x}_{i,j}$ for $j = 1, \ldots, r_i - 1$, in coordinates $\hat{x}_{i,j} = \xi_{i,j}(x), 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant r_i$, system (1) can be written as:

$$\begin{aligned}
\hat{x}_{i,1}^{\Delta}(t) &= \hat{x}_{i,2}(t) \\
\hat{x}_{i,2}^{\Delta}(t) &= \hat{x}_{i,3}(t) \\
&\vdots \\
\hat{x}_{i,r_i-1}^{\Delta}(t) &= \hat{x}_{i,r_i}(t) \\
\hat{x}_{i,r_i}^{\Delta}(t) &= \hat{f}(\hat{x}(t), u(t)), \quad i = 1, 2, \ldots, m,
\end{aligned} \tag{29}$$

where $\frac{\partial \hat{f}_i}{\partial \hat{x}_{s,j}} = a_{s,j}^i$ and $\frac{\partial \hat{f}_i}{\partial u_j} = b_j^i$. Under the feedback $u(t) = \phi(\hat{x}(t), v(t))$, where

$$\hat{f}_i(\hat{x}(t), \phi(\hat{x}(t), v(t))) = v_i(t), \quad i = 1, 2, \ldots, m,$$

the system has the Brunovsky canonical form (28) with controllability indices $r_1, \ldots, r_m$, where the pair (A, B) is controllable.

$\Rightarrow$: For a linear system, the $\widehat{\mathcal{H}}_k$'s are integrable and this property is invariant under both regular static state feedback and state diffeomorphism, so for $1 \leqslant k \leqslant k^*$ the $\mathcal{H}_k$'s are locally integrable.

Remark 9 For continuous-time case, Theorem 2 yields Theorem 9.1 in [15] which is a differential-form version of a vector-field characterization presented in [24,27]. For discrete-time systems, the results of Theorem 2 are new because they are described in terms of difference operator while the shift operator is used for discrete time in the literature, see for instance [2].

Now let us present examples that illustrate our results. The first example has been suggested by one of the reviewers.

Example 1 Consider the following nonlinear control system defined on homogeneous time scale $\mathbb{T}$ with $\mu \equiv \text{const} \geqslant 0$:

$$x_1^\Delta = x_2^2$$
$$x_2^\Delta = u, \tag{30}$$

where $(x_1, x_2) \in \mathcal{X} = \mathbb{R}^2$ and $u \in \mathcal{U} = \mathbb{R}$.

By formula (24), $\mathcal{H}_0 = \text{span}_{\mathcal{K}^*}\{dx_1, dx_2, du\}$. From Proposition 3 (i), it easily follows that $\mathcal{H}_1 = \text{span}_{\mathcal{K}^*}\{dx_1, dx_2\}$, because relative degrees of dx_1 and dx_2 are obviously greater or equal to 1, and relative degree of du equals to 0. Since $(dx_1)^{\Delta_f} = 2x_2 dx_2 \in \mathcal{H}_1$, $(dx_1)^{\Delta_f^2} = 2u dx_2 + 2(x_2 + \mu u)du \notin \mathcal{H}_1$ and $(dx_2)^{\Delta_f} = du \notin \mathcal{H}_1$, the relative degrees of dx_1 and dx_2 are equal to 2 and 1, respectively. Then, $\mathcal{H}_2 = \text{span}_{\mathcal{K}^*}\{dx_1\}$ and $\mathcal{H}_3 = \mathcal{H}_\infty = \{0\}$.

From Theorem 1, there exist the one-form $\omega_1 = dx_1$ whose relative degree is equal to 2 and

$$\mathcal{H}_0 = \text{span}_{\mathcal{K}^*}\left\{dx_1, (dx_1)^{\Delta_f}, (dx_1)^{\Delta_f^2}\right\},$$
$$= \text{span}_{\mathcal{K}^*}\{dx_1, 2x_2 dx_2, 2u dx_2 + 2(x_2 + \mu u)du\},$$
$$\mathcal{H}_1 = \text{span}_{\mathcal{K}^*}\left\{dx_1, (dx_1)^{\Delta_f}\right\} = \text{span}_{\mathcal{K}^*}\{dx_1, 2x_2 dx_2\},$$
$$\mathcal{H}_2 = \text{span}_{\mathcal{K}^*}\{dx_1\},$$
$$\mathcal{H}_3 = \mathcal{H}_\infty = \{0\}.$$

Note that all $\mathcal{H}_k, k \geqslant 0$, are integrable, but they do not satisfy Assumption 1 at $x_2 = 0$, $x_2 + \mu u = 0$. But for $x_2 \neq 0$ and $x_2 + \mu u \neq 0$, we have (in terms of Corollary 2) the one-forms $\omega_{1,1} = dx_1$, $\omega_{1,2} = 2x_2 dx_2$ and

$$\omega_{1,1}^{\Delta_f} = \omega_{1,2}$$
$$\omega_{1,2}^{\Delta_f} = \frac{u}{x_2}\omega_{1,2} + (2x_2 + 2\mu u)du.$$

Since for $\mathcal{S} = \{(x_1, x_2, u) : x_2 \neq 0 \ \wedge \ x_2 + \mu u \neq 0\} \subset \mathbb{R}^3$ the assumptions of Theorem 2 are satisfied, the new local state variables can be defined as follows: $\hat{x}_1 := x_1$ and $\hat{x}_2 := x_2^2$. In these coordinates, the system takes the linearized form:

$$\hat{x}_1^\Delta = \hat{x}_2$$
$$\hat{x}_2^\Delta = v,$$

where $v = 2u\sqrt{\hat{x}_2} + \mu u^2$.

Example 2 Consider the following nonlinear control system defined on $\mathbb{T} = \mathbb{Z}$ (with $\mu \equiv 1$):

$$x_1^\Delta = -x_1 + u x_2^2$$
$$x_2^\Delta = -x_2 + u x_2, \tag{31}$$

where $(x_1, x_2) \in \mathcal{X} = \{x \mid x_i \neq 0, i = 1, 2\} \subset \mathbb{R}^2$ and $(u_1, u_2) \in \mathcal{U} = \mathbb{R}\backslash\{0\}$.

By formula (24), $\mathcal{H}_0 = \text{span}_{\mathcal{K}^*}\{dx_1, dx_2, du\}$. From Proposition 3 (i), it easily follows that $\mathcal{H}_1 = \text{span}_{\mathcal{K}^*}\{dx_1, dx_2\}$, since relative degrees of dx_1 and dx_2 are obviously greater or equal to 1, and relative degree of du equals to 0. To compute the next elements of the sequence $\mathcal{H}_k$, the algorithm in the proof of Proposition 8 has to be applied repeatedly.

To find $\mathcal{H}_2$, we assume $k = 1$, $r_k = 2$ and $r_{k-1} - r_k = 1$. The basis vectors may be chosen as $\eta_1 = dx_1$, $\eta_2 = dx_2$ and $\vartheta = du$. The next step is to find $\eta_i^{\Delta_f}$ in terms of η_1, η_2 and ϑ for $i = 1, 2$, i.e., $\eta_1^{\Delta_f} = -dx_1 + 2x_2u\,dx_2 + x_2^2\,du = -\eta_1 + 2x_2u\eta_2 + x_2^2\vartheta$ and $\eta_2^{\Delta_f} = -dx_2 + u\,dx_2 + x_2\,du = (u-1)\eta_2 + x_2\vartheta$. The coefficients of ϑ are $(c_{11}, c_{21})^T = (x_2^2, x_2)^T$ and thus one possible solution of (33) is $(a_1^{\sigma_f}, a_2^{\sigma_f})^T = (-1, x_2)^T$. To find a_2, we have to express $x_2^{\sigma_f^{-1}}$ from (31). By (31), $x_1^{\sigma_f} = x_1 + x_1^{\Delta_f} = ux_2^2$ and $x_1^{\sigma_f} = ux_2$. Dividing the first equation by the second gives us $(x_1 \cdot x_2^{-1})^{\sigma_f} = x_2$, which yields after application of the backward jump operator $x_2^{\sigma_f^{-1}} = x_1 \cdot x_2^{-1}$. Hence $(a_1, a_2)^T = (-1, x_1 \cdot x_2^{-1})^T$ and $\mathcal{H}_2 = \text{span}_{\mathcal{K}^*}\{a_1\eta_1 + a_2\eta_2\} = \text{span}_{\mathcal{K}^*}\{-dx_1 + x_1 \cdot x_2^{-1}dx_2\} = \text{span}_{\mathcal{K}^*}\{x_2^{-1}dx_1 - x_1 \cdot x_2^{-2}dx_2\} = \text{span}_{\mathcal{K}^*}\{d(x_1 \cdot x_2^{-1})\}$.

For computing $\mathcal{H}_3$, one may choose $\eta_1 = -x_2dx_1 + x_1dx_2$ and $\vartheta = dx_1$. Then, $\eta_1^{\Delta_f} = x_2dx_1 - (x_1u^2x_2^2)dx_2 = -[1 + (u^2x_2^2x_1^{-1})]\eta_1 - (u^2x_2^3x_1^{-1})\vartheta$. From here $c_{11} = -u^2x_2^3x_1^{-1} \neq 0$. Since the system $a_1^{\sigma_f}c_{11} = 0$ does not admit a non-zero solution, the subspace $\mathcal{H}_3 = \{0\}$.

According to Proposition 3, $k^* = 3$. Since Assumption 1 is satisfied and all the subspaces $\mathcal{H}_k$, $1 \leqslant k \leqslant 3$, are locally integrable, the conditions of Theorem 2 are fulfilled and thus, system (31) is generically locally linearizable by static state feedback. One may choose[4] new state variables as $\hat{x}_1 := x_1x_2^{-1}$, $\hat{x}_2 := x_2 - x_1x_2^{-1}$. In these variables, the system takes linearized form:

$$\hat{x}_1^{\Delta} = \hat{x}_2$$
$$\hat{x}_2^{\Delta} = v,$$

where $v = (u-1)\hat{x}_1 + (u-2)\hat{x}_2$. We could consider the system on $\mathbb{R}^2$, but in order to guarantee the constant dimensions of codistributions $\mathcal{H}_k$, $k \geqslant 2$ the state space has to be reduced to $\{(x_1, x_2) : x_1 \neq 0, x_2 \neq 0\}$.

Remark 10 Note that for the systems defined on the homogeneous time scale with $\mu > 0$, by (8) and (15), we get $\omega \in \mathcal{H}_{k+1}$ if and only if $\omega \in \mathcal{H}_k$ and $\sigma_f(\omega) \in \mathcal{H}_k$. Hence

$$\text{span}_{\mathcal{K}^*}\{\omega \in \mathcal{H}_k \mid \Delta_f(\omega) \in \mathcal{H}_k\} = \text{span}_{\mathcal{K}^*}\{\omega \in \mathcal{H}_k \mid \sigma_f(\omega) \in \mathcal{H}_k\}$$

and, consequently, $\mathcal{H}_{k+1}$ in (15) can be alternatively defined as:

$$\mathcal{H}_{k+1} = \text{span}_{\mathcal{K}^*}\{\omega \in \mathcal{H}_k \mid \sigma_f(\omega) \in \mathcal{H}_k\}$$

---

[4] The next example describes this process in details.

like in [2] where the shift operator is used in the description of the subspaces of differential one-forms. Therefore, for the discrete-time systems the subspaces $\mathcal{H}_k$, $k \geqslant 0$, defined by (15), are the same as in [2], but the formalism used in the description of the subspaces is different. We base on difference operator that is a special case of delta derivative while in [2] the shift operator formalism is used. Additionally, using the delta-domain approach we have one description that works for both the continuous and discrete systems. Though the computation of the delta-derivative is different in the continuous- and discrete-time cases, the results obtained by means of it are the same for both time domains.

Example 3 Consider a nonlinear control system defined on homogeneous time scale $\mathbb{T}$ with $\mu \geqslant 0$:

$$
\begin{aligned}
x_1^\Delta &= \frac{x_4 x_6}{x_3} \\
x_2^\Delta &= u_2 + (1 + u_1)x_6 \\
x_3^\Delta &= \frac{1}{x_6^2}(x_3^2 + \mu x_3 x_6(u_2 + u_1 x_6) + x_6^3(u_2 + u_1 x_6)) \\
x_4^\Delta &= \mu u_2^2 + \frac{u_1 x_4 x_6^2}{x_3} + u_2 \left(\frac{x_3}{x_6} + \mu u_1 x_6 + \frac{x_4 x_6}{x_3} \right) \\
x_5^\Delta &= x_2 - \frac{x_3}{x_6} \\
x_6^\Delta &= \frac{x_3}{x_6},
\end{aligned}
\tag{32}
$$

where $(x_1, \ldots, x_6) \in \mathcal{X} = \{x \mid x_3 \neq 0 \text{ and } x_6 \neq 0\} \subset \mathbb{R}^6$ and $(u_1, u_2) \in \mathcal{U} = \mathbb{R}^2$.

The algorithm given in the proof of Proposition 8 allows to find the subspaces $\mathcal{H}_k$:

$$
\begin{aligned}
\mathcal{H}_0 &= \mathrm{span}_{\mathcal{K}^*} \{dx_1, \ldots, dx_6, du_1, du_2\} \\
\mathcal{H}_1 &= \mathrm{span}_{\mathcal{K}^*} \{dx_1, \ldots, dx_6\} \\
\mathcal{H}_2 &= \mathrm{span}_{\mathcal{K}^*} \left\{ dx_1, dx_5, dx_6, d\left(x_2 - \frac{x_3}{x_6}\right) \right\} \\
\mathcal{H}_3 &= \mathrm{span}_{\mathcal{K}^*} \left\{ dx_5, d\left(x_2 - \frac{x_3}{x_6}\right) \right\} \\
\mathcal{H}_4 &= \mathrm{span}_{\mathcal{K}^*} \{dx_5\} \\
\mathcal{H}_5 &= \{0\}.
\end{aligned}
$$

The conditions of Theorem 2 are satisfied for (32), thus (32) is generically locally linearizable by static state feedback and can be represented in the form (29). Note that the computations below can be done even if some of $\mathcal{H}_k$'s are non-integrable. Namely, one may always find the linearized system equations in terms of one-forms, not necessarily exact, as in (25).

Following the proof of Theorem 1, we construct the sets $\mathcal{W}_k$, $k = k^*, \ldots, 1$, necessary to find the state transformations to linearize the equations. Obviously, $\mathcal{W}_{k^*} =$

$\mathcal{W}_5 = 0$. Then, one has to choose $\mathcal{W}_{k^*-1} = \mathcal{W}_4$ in a such manner that the elements of $\mathcal{W}_5 \cup \mathcal{W}_5^{\Delta} \cup \mathcal{W}_4$ form a basis for $\mathcal{H}_4$. Obviously, $\mathcal{W}_4 = \{dx_5\}$. As a next step, one has to find $\mathcal{W}_3$ such that $\mathcal{W}_5 \cup \mathcal{W}_5^{\Delta_f} \cup \mathcal{W}_5^{\Delta_f^2} \cup \mathcal{W}_4 \cup \mathcal{W}_4^{\Delta_f} \cup \mathcal{W}_3$ is the basis for $\mathcal{H}_3$. The sets $\mathcal{W}_4$ and $\mathcal{W}_4^{\Delta_f} = \{dx_5^{\Delta_f}\} = \{d(x_2 - \frac{x_3}{x_6})\}$ span $\mathcal{H}_3$, hence $\mathcal{W}_3 = \emptyset$. Next, we find $\mathcal{W}_2$ such that $\mathcal{W}_5 \cup \cdots \cup \mathcal{W}_5^{\Delta_f^3} \cup \mathcal{W}_4 \cup \mathcal{W}_4^{\Delta_f} \cup \mathcal{W}_4^{\Delta_f^2} \cup \mathcal{W}_3 \cup \mathcal{W}_3^{\Delta_f} \cup \mathcal{W}_2$ is the basis for $\mathcal{H}_2$. The set $\mathcal{W}_4^{\Delta_f^2} = \{[d(x_2 - \frac{x_3}{x_6})]^{\Delta_f}\} = \{dx_6\}$, thus we may choose $\mathcal{W}_2 = \{dx_1\}$. On the last step, we have to find $\mathcal{W}_1$; regarding that $\mathcal{W}_4^{\Delta_f^3} = \{dx_6^{\Delta_f}\} = \{d(\frac{x_3}{x_6})\}$ and $\mathcal{W}_2^{\Delta_f} = \{dx_1^{\Delta_f}\} = \{d(\frac{x_4 x_6}{x_3})\}$ it follows that $\mathcal{W}_1 = \emptyset$. Finally, we may take $\omega_1 = dx_5$ and $\omega_2 = dx_1$, and their relative degrees are $r_1 = 4$ and $r_2 = 2$, respectively.

In terms of Corollary 2, the one-forms $\omega_{1,1} = dx_5$, $\omega_{1,2} = d(x_2 - \frac{x_3}{x_6})$, $\omega_{1,3} = dx_6$, $\omega_{1,4} = d(\frac{x_3}{x_6})$, $\omega_{2,1} = dx_1$ and $\omega_{2,2} = d(\frac{x_4 x_6}{x_3})$, and

$$\omega_{1,1}^{\Delta_f} = \omega_{1,2}$$
$$\omega_{1,2}^{\Delta_f} = \omega_{1,3}$$
$$\omega_{1,3}^{\Delta_f} = \omega_{1,4}$$
$$\omega_{1,4}^{\Delta_f} = u_1\omega_{1,3} + x_6 du_1 + du_2$$
$$\omega_{2,1}^{\Delta_f} = \omega_{2,2}$$
$$\omega_{2,2}^{\Delta_f} = du_2.$$

Since the assumptions of Theorem 2 are satisfied, the new state variables can be defined as follows: $\hat{x}_1 := x_5$, $\hat{x}_2 := x_2 - \frac{x_3}{x_6}$, $\hat{x}_3 := x_6$, $\hat{x}_4 := \frac{x_3}{x_6}$, $\hat{x}_5 := x_1$ and $\hat{x}_6 := \frac{x_4 x_6}{x_3}$. In these coordinates, the system takes the linearized form:

$$\hat{x}_1^{\Delta} = \hat{x}_2$$
$$\hat{x}_2^{\Delta} = \hat{x}_3$$
$$\hat{x}_3^{\Delta} = \hat{x}_4$$
$$\hat{x}_4^{\Delta} = v_1$$
$$\hat{x}_5^{\Delta} = \hat{x}_6$$
$$\hat{x}_6^{\Delta} = v_2,$$

where $v_1 = u_2 + u_1\hat{x}_3$ and $v_2 = u_2$.

Let us now give the example where the integrability of $\mathcal{H}_k$ depends on the time scale.

Example 4 Consider a nonlinear control system defined on homogeneous time scale $\mathbb{T}$ with $\mu \geqslant 0$:

$$x_1^\Delta = (x_1 + \mu x_3)u_2 x_2$$
$$x_2^\Delta = u_1 + x_4$$
$$x_3^\Delta = u_2 x_2 + x_3$$
$$x_4^\Delta = u_1 + x_1 x_3,$$

(33)

where $(x_1, \ldots, x_4) \in \mathcal{X} \subset \mathbb{R}^4$ and $(u_1, u_2) \in \mathbb{R}^2$.

Let $\mathcal{S} := \{(x, u_1, u_2, x_3^{\langle-1\rangle}) \mid x_1 \neq 0, \ x_2 \neq 0, \ x_3 \neq 0 \text{ and } 1 + x_3 - (1+\mu)x_3^{\langle-1\rangle} \neq 0\}$ be the subset of $\mathcal{X} \times \mathcal{U} \times \mathbb{R}$. Then, one gets the following subspaces:

$$\mathcal{H}_1 = \text{span}_{\mathcal{K}^*}\{dx_1, dx_2, dx_3, dx_4\},$$
$$\mathcal{H}_2 = \text{span}_{\mathcal{K}^*}\{dx_2 - dx_4, ((1+\mu)x_3^{\langle-1\rangle} - 1 - x_3)dx_1 + (x_1 + \mu x_3^{\langle-1\rangle})dx_3\},$$

which define constant-dimensional codistributions on $\mathcal{S}$. Note that if $\mu \neq 0$, then in general $\mathcal{H}_2$ is not integrable. For $\mu = 0$, $x_3^{\langle-1\rangle} = \sigma_f^{-1}(x_3) = x_3$, $1 + x_3 - (1 + \mu)x_3^{\langle-1\rangle} = 1 \neq 0$ and $\mathcal{H}_2 = \text{span}_{\mathcal{K}^*}\{dx_2 - dx_4, -dx_1 + x_1 dx_3\}$ is integrable. Therefore, for $\mu = 0$, the conditions of Theorem 2 are satisfied and one may define new state variables as:

$$\hat{x}_1 := e^{-x_3}x_1$$
$$\hat{x}_2 := -e^{-x_3}x_1 x_3$$
$$\hat{x}_3 := x_2 - x_4$$
$$\hat{x}_4 := x_4 - x_1 x_3.$$

Hence the coordinates x_i can be expressed in terms of $\hat{x}_i$ as follows:

$$x_1 = e^{-\hat{x}_2/\hat{x}_1}\hat{x}_1$$
$$x_2 = -e^{-\hat{x}_2/\hat{x}_1}\hat{x}_2 + \hat{x}_3 + \hat{x}_4$$
$$x_3 = -\frac{\hat{x}_2}{\hat{x}_1}$$
$$x_4 = -e^{-\hat{x}_2/\hat{x}_1}\hat{x}_2 + \hat{x}_4.$$

For $\mu = 0$, in the new coordinates system (33) takes the linearized form:

$$\hat{x}_1^\Delta = \hat{x}_2$$
$$\hat{x}_2^\Delta = v_1$$
$$\hat{x}_3^\Delta = \hat{x}_4$$
$$\hat{x}_4^\Delta = v_2,$$

where $v_1 = (1 + e^{-\hat{x}_2/\hat{x}_1}u_2\hat{x}_1)\hat{x}_2 + \hat{x}_2^2/\hat{x}_1 - u_2\hat{x}_1(\hat{x}_3 + \hat{x}_4)$ and $v_2 = u_1 - e^{-\hat{x}_2/\hat{x}_1}u_2(\hat{x}_1 - \hat{x}_2)(e^{-\hat{x}_2/\hat{x}_1}\hat{x}_2 - \hat{x}_3 - \hat{x}_4)$.

5 Conclusions

In the paper, the necessary and sufficient conditions for the generic local static state feedback linearizability of nonlinear control systems defined on homogeneous time scales are given. Our main contribution has been to show the properties of subspaces of differential one-forms that contain considerable structural information about the system. Then, one of the main results, i.e., necessary and sufficient conditions for generic local linearizability by static state feedback, are formulated in terms of these subspaces. Our future goal is devoted to extend the results to non-homogeneous but regular time scales.

Appendix: The proof of Proposition 6

Proof Let $\widehat{\omega} = \widehat{\omega}_x + \widehat{\omega}_u + \widehat{\omega}_z \in \widehat{\mathcal{E}}$, where $\widehat{\omega}_x = \sum_i \widehat{A}_i d\hat{x}_i$, $\widehat{\omega}_u = \sum_{k \geq 0} \sum_{j=1}^m \widehat{B}_{jk} du_j^{[k]}$ and $\widehat{\omega}_z = \sum_{\ell \geq 1} \sum_{s=1}^m \widehat{C}_{s\ell} dz_s^{\langle -\ell \rangle}$. Then, $\xi^*(\widehat{\omega}) \in \mathcal{E}$ and by (19) and (20) we get

$$
\begin{aligned}
\xi^* \left(\widehat{\omega}_x^{\widehat{\Delta}_{\hat{f}}} \right) &= \xi^* \left(\sum_i \widehat{A}_i^{\widehat{\Delta}_{\hat{f}}} d\hat{x}_i + \sum_i \widehat{A}_i^{\widehat{\sigma}_{\hat{f}}} \left[d\hat{x}_i \right]^{\widehat{\Delta}_{\hat{f}}} \right) \\
&= \sum_i \left[\xi^* \left(\widehat{A}_i^{\widehat{\Delta}_{\hat{f}}} \right) d\xi_i(x) + \xi^* \left(\widehat{A}_i^{\widehat{\sigma}_{\hat{f}}} \right) \left[d\xi_i(x) \right]^{\Delta_f} \right] \\
&= \sum_i \left\{ \left[\xi^* \left(\widehat{A}_i \right) \right]^{\Delta_f} \sum_j \frac{\partial \xi_i}{\partial x_j} dx_j + \left[\xi^* \left(\widehat{A}_i \right) \right]^{\sigma_f} \left[\sum_j \frac{\partial \xi_i}{\partial x_j} dx_j \right]^{\Delta_f} \right\} \\
&= \sum_i \sum_j \left\{ \left[\xi^* \left(\widehat{A}_i \right) \right]^{\Delta_f} \frac{\partial \xi_i}{\partial x_j} dx_j + \left[\xi^* \left(\widehat{A}_i \right) \right]^{\sigma_f} \left[\left(\frac{\partial \xi_i}{\partial x_j} \right)^{\Delta_f} dx_j \right. \right. \\
&\quad \left. \left. + \left(\frac{\partial \xi_i}{\partial x_j} \right)^{\sigma_f} \left[dx_j \right]^{\Delta_f} \right] \right\} \\
&= \sum_i \sum_j \left\{ \left[\xi^* \left(\widehat{A}_i \right) \frac{\partial \xi_i}{\partial x_j} \right]^{\Delta_f} dx_j + \left[\xi^* \left(\widehat{A}_i \right) \frac{\partial \xi_i}{\partial x_j} \right]^{\sigma_f} \left[dx_j \right]^{\Delta_f} \right\}
\end{aligned}
$$

$$= \left[\sum_i \sum_j \xi^* \left(\widehat{A}_i \right) \frac{\partial \xi_i}{\partial x_j} dx_j \right]^{\Delta_f} = \left[\sum_i \sum_j \xi^* \left(\widehat{A}_i \right) \frac{\partial \xi_i}{\partial x_j} dx_j \right]^{\Delta_f}$$

$$= \left[\xi^* \left(\widehat{\omega}_x \right) \right]^{\Delta_f},$$

$$\xi^* \left(\widehat{\omega}_u^{\widehat{\Delta}_f} \right) = \xi^* \left(\sum_{k \geqslant 0} \sum_{j=1}^m \left[\widehat{B}_{jk}^{\widehat{\Delta}_f} du_j^{[k]} + \widehat{B}_{jk}^{\widehat{\sigma}_f} du_j^{[k+1]} \right] \right)$$

$$= \sum_{k \geqslant 0} \sum_{j=1}^m \left[\xi^* \left(\widehat{B}_{jk}^{\widehat{\Delta}_f} \right) du_j^{[k]} + \xi^* \left(\widehat{B}_{jk}^{\widehat{\sigma}_f} \right) du_j^{[k+1]} \right]$$

$$= \sum_{k \geqslant 0} \sum_{j=1}^m \left[\left(\xi^* \left(\widehat{B}_{jk} \right) \right)^{\Delta_f} du_j^{[k]} + \left(\xi^* \left(\widehat{B}_{jk} \right) \right)^{\sigma_f} du_j^{[k+1]} \right]$$

$$= \left(\sum_{k \geqslant 0} \sum_{j=1}^m \xi^* \left(\widehat{B}_{jk} \right) du_j^{[k]} \right)^{\Delta_f} = \left[\xi^* \left(\sum_{k \geqslant 0} \sum_{j=1}^m \widehat{B}_{jk} du_j^{[k]} \right) \right]^{\Delta_f}$$

$$= \left[\xi^* \left(\widehat{\omega}_u \right) \right]^{\Delta_f}$$

and similarly

$$\xi^* \left(\widehat{\omega}_z^{\widehat{\Delta}_f} \right) = \xi^* \left(\sum_{\ell \geqslant 1} \sum_{s=1}^m \left[\widehat{C}_{s\ell}^{\widehat{\Delta}_f} dz_s^{\langle -\ell \rangle} + \widehat{C}_{s\ell}^{\widehat{\sigma}_f} \left[dz_s^{\langle -\ell \rangle} \right]^{\widehat{\Delta}_f} \right] \right)$$

$$= \sum_{\ell \geqslant 1} \sum_{s=1}^m \left[\xi^* \left(\widehat{C}_{s\ell}^{\widehat{\Delta}_f} \right) dz_s^{\langle -\ell \rangle} + \xi^* \left(\widehat{C}_{s\ell}^{\widehat{\sigma}_f} \right) \left[dz_s^{\langle -\ell \rangle} \right]^{\Delta_f} \right]$$

$$= \sum_{\ell \geqslant 1} \sum_{s=1}^m \left[\left(\xi^* \left(\widehat{C}_{s\ell} \right) \right)^{\Delta_f} dz_s^{\langle -\ell \rangle} + \left(\xi^* \left(\widehat{C}_{s\ell} \right) \right)^{\sigma_f} \left[dz_s^{\langle -\ell \rangle} \right]^{\Delta_f} \right]$$

$$= \sum_{\ell \geqslant 1} \sum_{s=1}^m \left(\xi^* \left(\widehat{C}_{s\ell} \right) dz_s^{\langle -\ell \rangle} \right)^{\Delta_f} = \left[\xi^* \left(\sum_{\ell \geqslant 1} \sum_{s=1}^m \left(\widehat{C}_{s\ell} \right) dz_s^{\langle -\ell \rangle} \right) \right]^{\Delta_f}$$

$$= \left[\xi^* \left(\widehat{\omega}_z \right) \right]^{\Delta_f}.$$

Hence

$$\xi^* \left(\widehat{\omega}^{\widehat{\Delta}_f} \right) = \xi^* \left(\widehat{\omega}_x^{\widehat{\Delta}_f} + \widehat{\omega}_u^{\widehat{\Delta}_f} + \widehat{\omega}_z^{\widehat{\Delta}_f} \right) = \xi^* \left(\widehat{\omega}_x^{\widehat{\Delta}_f} \right) + \xi^* \left(\widehat{\omega}_u^{\widehat{\Delta}_f} \right) + \xi^* \left(\widehat{\omega}_z^{\widehat{\Delta}_f} \right)$$

$$= \left[\xi^* \left(\widehat{\omega}_x \right) \right]^{\Delta_f} + \left[\xi^* \left(\widehat{\omega}_u \right) \right]^{\Delta_f} + \left[\xi^* \left(\widehat{\omega}_z \right) \right]^{\Delta_f}$$

$$= \left[\xi^* \left(\widehat{\omega}_x \right) + \xi^* \left(\widehat{\omega}_u \right) + \xi^* \left(\widehat{\omega}_z \right) \right]^{\Delta_f}$$

$$= \left[\xi^* \left(\widehat{\omega}_x + \widehat{\omega}_u + \widehat{\omega}_z \right) \right]^{\Delta_f} = \left[\xi^* \left(\widehat{\omega} \right) \right]^{\Delta_f}.$$

Therefore (21) holds.

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