CHY representations for gauge theory and gravity amplitudes with up to three massive particles

Stephen G. Naculich[1]

Department of Physics, Bowdoin College,
Brunswick, ME 04011, U.S.A.

E-mail: naculich@bowdoin.edu

ABSTRACT: We show that a wide class of tree-level scattering amplitudes involving scalars, gauge bosons, and gravitons, up to three of which may be massive, can be expressed in terms of a Cachazo-He-Yuan representation as a sum over solutions of the scattering equations. These amplitudes, when expressed in terms of the appropriate kinematic invariants, are independent of the masses and therefore identical to the corresponding massless amplitudes.

Contents

1 Introduction

The new representations for tree-level gauge and gravitational amplitudes in arbitrary spacetime dimensions discovered by Cachazo, He, and Yuan [1, 2], while as yet lacking a quantum-field-theoretic foundation, have illuminated and unified many previous results. The tree-level n-gluon amplitude for pure Yang-Mills theory (up to an overall factor proportional to g_{YM}^{n-2}) is given by

$$\mathcal{A}_{\mathrm{gauge}} = (-1)^{n-1} \sum_{\{\sigma\} \in \mathrm{solutions}} \frac{C(\{\sigma\}) \, E(\{\epsilon, k, \sigma\})}{\mathrm{det}' \, \Phi(\{k, \sigma\})} \tag{1.1}$$

where $\{\sigma\}$ is a set of n points on $\mathbb{CP}^1$, $C(\{\sigma\})$ depends on group theory structure constants, $E(\{\epsilon, k, \sigma\})$ depends on the momenta and polarizations of the gluons, and $\mathrm{det}' \, \Phi(\{k, \sigma\})$ is a Jacobian factor defined in section 4. The sum is over the $(n-3)!$ solutions of the scattering equations [3]

$$\sum_{b \neq a} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0, \qquad \sigma_a \in \mathbb{CP}^1, \qquad a = 1, \cdots, n \,. \tag{1.2}$$

The structure of eq. (1.1) as a sum over products of color- and kinematic-dependent factors is closely related to BCJ color-kinematic duality [4]. For $n = 4$, the scattering equations have a single solution, so that eq. (1.1) implies that the four-gluon amplitude factors into a color-dependent term and a polarization-dependent term. The factorization of a wide class of four-point gauge theory amplitudes was first recognized over thirty years ago [5, 6] following the discovery that radiative gauge theory amplitudes vanish in certain kinematic regions [7–12].

The tree-level gravity amplitude (up to an overall factor proportional to κ^{n-2}) is

$$\mathcal{A}_{\text{grav}} = (-1)^{n-1} \sum_{\{\sigma\} \in \text{solutions}} \frac{E(\{\epsilon, k, \sigma\})\, E(\{\tilde{\epsilon}, k, \sigma\})}{\det' \Phi(\{k, \sigma\})} \tag{1.3}$$

and its structure clearly echoes the double-copy procedure of refs. [4, 13, 14]. The Kawai-Lewellen-Tye (KLT) relations of tree-level string theory [15] in the field theory limit [16, 17] also follow from eq. (1.3). In particular, eq. (1.3) implies that the four-graviton amplitude factors into a pair of gauge-theory partial amplitudes. This has long been known from the field-theory limit of the four-point closed string amplitude [18] as well as from an explicit field theory calculation [19]. Many other four-point amplitudes involving gravitons have also been shown to factor into a pair of gauge theory amplitudes in an arbitrary number of spacetime dimensions [20–23].

In refs. [24, 25], Cachazo, He, and Yuan extended their results to amplitudes for massless particles in a variety of theories using dimensional reduction and other techniques. Other work inspired by these developments includes refs. [26–53].

Most of the attention so far has been focused on amplitudes for massless particles. The factorization of four-point gauge theory amplitudes found in ref. [5, 6], however, required that only one of the gauge bosons involved be massless, and the four-point gravitational amplitudes considered in refs. [20–23] involved massive particles as well. This suggests that CHY representations may be possible for various amplitudes containing massive particles. Dolan and Goddard proposed a modification of the scattering equations for massive particles with equal mass and derived a CHY representation for ϕ^3 theory [32, 39]. In ref. [50], the scattering equations were generalized to arbitrary masses, and CHY representations were proposed for gauge theory amplitudes involving a pair of massive scalar particles in the fundamental representation and an arbitrary number of gluons, as well as for gravitational amplitudes involving a pair of massive scalar particles and an arbitrary number of gravitons.

In this paper, we derive CHY representations for a large class of tree-level gauge theory and gravitational amplitudes, with up to three massive particles. We require only that the remaining (massless) particles be flavor-preserving (as is generally the case). Furthermore, we show that these amplitudes, when written in terms of the appropriate kinematic invariants, are independent of the masses of the particles, and thus identical to purely massless amplitudes. Our approach is first to show that all the diagrams contributing to an amplitude with up to three massive particles (with the remaining particles being flavor-preserving) can be kinematically regarded as the dimensional reduction of diagrams of purely massless amplitudes, with non-zero momenta in the extra dimensions. Then, by different choices of the polarization vectors, we can use eqs. (1.1) and (1.3) in higher dimensions to derive CHY representations for mixed amplitudes in lower dimensions, as was done in ref. [25]. In this way, we derive a variety of gauge theory amplitudes involving massive gauge bosons and scalar particles, and gravitational amplitudes involving gravitons, massive gauge particles, and massive scalars. We recover the results of ref. [50] as a special case.

This paper is structured as follows. In section 2, we show that the propagator matrix for an amplitude with up to three massive particles has the same rank as that for a purely

massless amplitude, motivating the search for a CHY-type representation for such ampli-
tudes. In section 3, we derive the scattering equations for amplitudes with up to three
massive particles from massless scattering equations in higher dimensions. In section 4, we
derive a CHY representation for the massive propagator matrix. In section 5, we dimension-
ally reduce massless gauge boson amplitudes to obtain CHY representations for amplitudes
with massive and massless gauge bosons, and mixed amplitudes containing gauge bosons
and adjoint or fundamental scalars. In section 6, we dimensionally reduce pure gravi-
ton amplitudes to obtain CHY representations for mixed amplitudes containing gravitons,
massive or massless gauge bosons, and massive scalars. Section 7 contains our conclusions.

2 Propagator matrix

In this section, we briefly review the propagator matrix that appears in the amplitude
for massless gauge bosons [54]. We then show that the propagator matrix for an n-point
amplitude with up to three massive particles (assuming that the remaining $n - 3$ particles
are flavor-preserving) is identical to that for an amplitude with all massless particles. Like
the latter, therefore, it has rank $(n - 3)!$. This in turn suggests the possibility of finding a
CHY representation in terms of a sum over the $(n-3)!$ solutions of the scattering equations
for these amplitudes.

We begin with the tree-level n-gluon amplitude written as a sum over cubic diagrams [4]

$$\mathcal{A}_{\text{gauge}} = \sum_i \frac{c_i n_i}{d_i} \tag{2.1}$$

where c_i, n_i, and d_i are the color factors, kinematic numerators, and denominators (arising
from the propagators) associated with the diagram. The color factors c_i of each of the
cubic diagrams appearing in eq. (2.1) can be rewritten, using Jacobi identities, in terms of
a basis of $(n - 2)!$ color factors [55, 56]

$$c_i = \sum_{\gamma \in S_{n-2}} M_{i,1\gamma n} \mathbf{c}_{1\gamma n} \tag{2.2}$$

where γ denotes a permutation of $\{2, \cdots, n - 1\}$ and

$$\mathbf{c}_{1\gamma n} \equiv \mathbf{c}_{1\gamma(2)\cdots\gamma(n-1)n} \equiv \sum_{\mathsf{b}_1,\ldots,\mathsf{b}_{n-3}} f^{\mathsf{a}_1 \mathsf{a}_{\gamma(2)} \mathsf{b}_1} f^{\mathsf{b}_1 \mathsf{a}_{\gamma(3)} \mathsf{b}_2} \ldots f^{\mathsf{b}_{n-3} \mathsf{a}_{\gamma(n-1)} \mathsf{a}_n} . \tag{2.3}$$

The color factors $\mathbf{c}_{1\gamma n}$ are associated with half-ladder diagrams, i.e., diagrams in which
there is a line connecting particles 1 and n to which each of the other particles is directly
connected. The assumption of color-kinematic duality [4] is that the kinematic numerators
n_i obey the same Jacobi identities as c_i, and therefore can be written in terms of $(n - 2)!$
half-ladder numerators

$$n_i = \sum_{\gamma \in S_{n-2}} M_{i,1\gamma n} \mathbf{n}_{1\gamma n} . \tag{2.4}$$

Eqs. (2.2) and (2.4) can be used to rewrite the amplitude (2.1) as

$$\mathcal{A}_{\text{gauge}} = \sum_{\gamma \in S_{n-2}} \sum_{\delta \in S_{n-2}} \mathbf{n}_{1\gamma n} \, m(1\gamma n | 1\delta n) \, \mathbf{c}_{1\delta n} \tag{2.5}$$
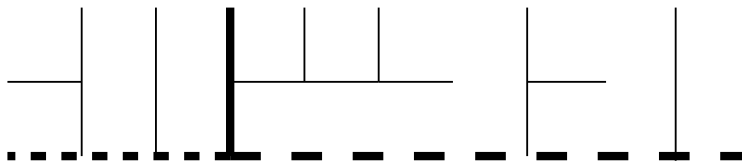
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Figure 1. A cubic diagram in which thin lines denote massless propagators, while the thick lines denote massive propagators (with possibly different masses).

where we define the entries of the $(n-2)! \times (n-2)!$ propagator matrix [54]

$$m(1\gamma n|1\delta n) = \sum_i \frac{M_{i,1\gamma n} M_{i,1\delta n}}{d_i} \tag{2.6}$$

as the sum (weighted by the denominators $1/d_i$ of the diagrams) over those cubic diagrams that contribute to both $\mathbf{c}_{1\gamma n}$ and $\mathbf{c}_{1\delta n}$.

More generally, we can define the propagator matrix for a gauge theory amplitude with particles other than gluons by using group theory relations to rewrite the color factors in terms of a half-ladder basis (for example, see ref. [50] for amplitudes with a pair of particles in the fundamental representation). Now consider an n-point amplitude with up to three massive particles, with masses m_1, m_2, and m_n, where the remaining $(n-3)$ particles are flavor-preserving. Each of the individual cubic diagrams contributing to the sum (2.1) can be viewed as a skeleton with three massive lines meeting at a vertex fleshed out with branches of massless lines emanating from each of the massive lines and from other massless lines; an example is shown in figure 1. The denominator factors associated with the side branches are of the form

$$\left(\sum_{a \subset S} k_a \right)^2 = \sum_{\substack{a,b \subset S \\ a \neq b}} 2 k_a \cdot k_b \tag{2.7}$$

where S is some subset of the (massless) momenta $\{3, \cdots, n-1\}$. The denominator factors associated with the three massive lines are of the form

$$\left(k_c + \sum_{a \subset S} k_a \right)^2 - m_c^2 = \left(\sum_{a \subset S} k_a \right)^2 + \sum_{a \subset S} 2 k_c \cdot k_a, \qquad c \in \{1, 2, n\}. \tag{2.8}$$

Thus, all of the inverse propagators (2.7) and (2.8) are given by linear combinations of the kinematic invariants

$$k_1 \cdot k_a, \qquad k_2 \cdot k_a, \qquad k_n \cdot k_a, \qquad k_a \cdot k_b, \qquad a, b \in \{3, \cdots, n-1\} \tag{2.9}$$

with no explicit dependence on the masses m_1, m_2, and m_n. (Of course, momentum conservation can be used to eliminate $k_n \cdot k_a$, leaving $n(n-3)/2$ independent invariants.)

Because the inverse propagators d_i, when written in terms of the invariants (2.9), do not depend on the masses, neither does the propagator matrix (2.6). And since the $(n-2)! \times (n-2)!$ propagator matrix for amplitudes with massless particles has rank $(n-3)!$ as a consequence of momentum conservation [54], the same is true for the propagator matrix

for amplitudes with up to three massive particles. This suggests the possibility of expressing it as a sum over the $(n-3)!$ solutions of the scattering equations, which we will do in the next two sections.

On the other hand, if there are four or more massive particles in the amplitude, the rank of the propagator matrix will generally be larger than $(n-3)!$. For example, the amplitude for four massive particles, one of which is flavor-preserving, has a 2×2 propagator matrix with non-vanishing determinant, and thus rank two.

3 Scattering equations

In order to obtain CHY representations for amplitudes with massive particles, we first need the appropriate scattering equations. In this section we show that the scattering equations for an n-point amplitude with up to three massive particles are identical to, and therefore have the same set of solutions as, the scattering equations for massless particles. We also show that the massive scattering equations for amplitudes with up to three massive particles can be derived from the massless scattering equations in higher dimensions.

The scattering equations for massless particles are given in eq. (1.2). In refs. [32, 39], Dolan and Goddard proposed a modification of these equations for massive particles with equal mass. The scattering equations were subsequently generalized [50] for particles with distinct masses m_a:

$$f_a \equiv \sum_{b \neq a} \frac{k_a \cdot k_b + \Delta_{ab}}{\sigma_a - \sigma_b} = 0, \qquad \sigma_a \in \mathbb{CP}^1, \qquad a = 1, \cdots, n \,. \tag{3.1}$$

This set of equations is invariant under $\mathrm{SL}(2, \mathbb{C})$ transformations

$$\sigma \longrightarrow \frac{A\sigma + B}{C\sigma + D}, \qquad AD - BC = 1 \tag{3.2}$$

provided $\sum_{a=1}^n k_a = 0$ and that Δ_{ab} satisfy the constraints

$$\Delta_{ab} = \Delta_{ba}, \qquad \sum_{b \neq a} \Delta_{ab} = m_a^2 \,. \tag{3.3}$$

Furthermore, when the constraints (3.3) are satisfied, the three sums

$$\sum_{a=1}^n f_a, \qquad \sum_{a=1}^n \sigma_a f_a, \qquad \sum_{a=1}^n \sigma_a^2 f_a \tag{3.4}$$

vanish identically (i.e., before imposing $f_a = 0$), which implies that only $n-3$ of the massive scattering equations (3.1) are independent.

Now consider an amplitude with three massive particles, m_1, m_2, and m_n. The constraints (3.3) can be satisfied by choosing

$$\Delta_{12} = \Delta_{21} = \frac{1}{2} \left(m_1^2 + m_2^2 - m_n^2 \right) \,,$$
$$\Delta_{2n} = \Delta_{n2} = \frac{1}{2} \left(m_2^2 + m_n^2 - m_1^2 \right) \,,$$

$$\Delta_{n1} \;=\; \Delta_{1n} = \frac{1}{2}\left(m_n^2 + m_1^2 - m_2^2\right) \tag{3.5}$$

with the remaining Δ_{ab} vanishing. Since only $n-3$ of the n scattering equations (3.1) are independent, we can eliminate those for $a = 1$, 2, and n. Since Δ_{ab} vanishes in the remaining equations (for $a = 3, \cdots, n-1$), they are identical to the scattering equations for massless particles, and hence have the same set of $(n-3)!$ solutions, when expressed in terms of the kinematic invariants (2.9). This will be crucial in the remainder of this paper.

Finally, we show that the massive scattering equations (3.1) are precisely the equations that arise if one regards the massive particles as massless particles in a higher-dimensional space. In particular, the momentum K_a for a massless particle in $(d + M)$ dimensions can be written

$$K_a \;=\; (k_a | \kappa_a) \tag{3.6}$$

where k_a and κ_a are the components of momentum in d- and M-dimensional subspaces respectively.[1] We will refer to κ_a as the *internal momentum* of a particle; the mass of the particle in d dimensions is given by $m_a^2 = \kappa_a^2$. The massless scattering equations in $(d + M)$ dimensions

$$\sum_{b \neq a} \frac{K_a \cdot K_b}{\sigma_a - \sigma_b} \;=\; 0, \qquad a = 1, \cdots, n \tag{3.7}$$

imply that the scattering equations (3.1) hold in d dimensions with

$$\Delta_{ab} \;=\; -\,\kappa_a \cdot \kappa_b \tag{3.8}$$

and the constraints (3.3) hold automatically by virtue of momentum conservation in the internal dimensions.

If $M = 1$, then momentum conservation in the internal space requires $\sum_{a=1}^{n}(\pm m_a) = 0$ (for some choice of signs), which is too restrictive for our purposes. If $M \geq 2$, however, there is no constraint on the masses of the particles.

It must be stressed, however, that by no means can every massive amplitude be obtained from the dimensional reduction of a massless amplitude because momentum conservation (including conservation of internal momentum) must hold at each vertex of a diagram, placing severe constraints on the masses of the intermediate particles in any given diagram. A simple example will suffice to explain the difficulty. Consider the four-point amplitude involving $\bar{u}$, d, W^+, and Z^0. Since the Z^0 is flavor- and therefore mass-preserving, conservation of momentum at any vertex involving the Z^0 implies that $(\kappa_i + \kappa_Z)^2 = \kappa_i^2$, where κ_i is the internal momentum for $\bar{u}$, d, or W^+. Together with overall internal momentum conservation ($\kappa_Z = -\sum_i \kappa_i$), this implies that $\kappa_Z^2 = 0$, which implies that the Z_0 must be massless.

For the amplitudes considered in this paper, however, with (up to) three massive particles, and the remaining (massless) particles flavor-preserving, internal momentum conservation at each vertex involving a massless ($\kappa = 0$) flavor-conserving particle holds automatically, and internal momentum conservation at the vertex involving the three massive particles, $\kappa_1 + \kappa_2 + \kappa_n = 0$, is easily satisfied for appropriate choices of κ_i.

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[1]We will use a mostly-minus metric for K_a and k_a, but an all-plus metric for the internal components κ_a.

4 Double-partial amplitudes

In this section, we show that the propagator matrix (2.6) for an amplitude with up to three massive particles can be written as a sum over the solutions of the scattering equations (3.1).

First we recall from section 3 that kinematically we can regard the scattering of n particles (up to three of them massive and the rest flavor-preserving) in d dimensions as the scattering of n massless particles in $d+M$ dimensions, with the same propagator matrix.

Next, recall that Cachazo, He, and Yuan [2] demonstrated that the entries of the propagator matrix for massless amplitudes can be interpreted as double-partial amplitudes of a theory of massless scalar particles transforming in the adjoint of the color group $\mathrm{U}(N) \times \mathrm{U}(\tilde{N})$ and presented a compact formula for computing them. First they defined an $n \times n$ matrix $\Phi(\{K, \sigma\})$ with entries

$$\Phi_{ab} \;=\; \frac{2K_a \cdot K_b}{(\sigma_a - \sigma_b)^2}, \quad a \neq b; \qquad \Phi_{aa} \;=\; -\sum_{c \neq a} \frac{2K_a \cdot K_c}{(\sigma_a - \sigma_c)^2} \tag{4.1}$$

where K_a are the $(d+M)$-dimensional momenta of massless particles. When the scattering equations (3.7) are satisfied, this matrix satisfies $\sum_{a=1}^{n} \Phi_{ab} = \sum_{a=1}^{n} \sigma_a \Phi_{ab} = \sum_{a=1}^{n} \sigma_a^2 \Phi_{ab} = 0$, and therefore has rank $(n-3)$. Define the nonsingular matrix Φ_{pqr}^{ijk} by removing rows i, j, and k, and columns p, q, and r, and let $|\Phi|_{pqr}^{ijk}$ be its signed determinant. The combination

$$\det{}'\Phi(\{K, \sigma\}) \equiv \frac{|\Phi|_{pqr}^{ijk}}{(\sigma_{p,q}\sigma_{q,r}\sigma_{r,p})(\sigma_{i,j}\sigma_{j,k}\sigma_{k,i})} \tag{4.2}$$

where $\sigma_{a,b} \equiv \sigma_a - \sigma_b$, was then shown to be independent of the choices of removed rows and columns. CHY then demonstrated that the entries of the propagator matrix can be computed using

$$m(1\gamma n|1\delta n) = \sum_{\{\sigma\}\in\text{solutions}} \frac{(-1)^{n-1}}{\det{}'\Phi(\{K,\sigma\})} \frac{1}{(\sigma_{1,\gamma(2)}\cdots\sigma_{\gamma(n-1),n}\sigma_{n,1})(\sigma_{1,\delta(2)}\cdots\sigma_{\delta(n-1),n}\sigma_{n,1})} \tag{4.3}$$

where the sum is over the $(n-3)!$ solutions of the scattering equations (3.7).

Now we need to rewrite eq. (4.3) in terms of d-dimensional quantities. Since $\det{}'\Phi$ is independent of the choices of removed rows and columns, we can choose to remove the rows and columns associated with the massive particles (1, 2, and n). Since the remaining rows and columns correspond to massless particles (with $\kappa_a = 0$), the numerators in Φ_{12n}^{12n} are given by $K_a \cdot K_c = k_a \cdot k_c$ with $a \in \{3, \cdots, n-3\}$ and $c \in \{1, \cdots, n\}$, which are precisely the set of kinematic invariants (2.9). This implies that[2]

$$\det{}'\Phi(\{K, \sigma\}) \;=\; \det{}'\Phi(\{k, \sigma\}). \tag{4.4}$$

provided that the determinant is expressed in terms of the invariants (2.9). Furthermore, as we saw in section 3, the massless scattering equations in $d + M$ dimensions (3.7) are

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[2]Clearly this would not be the case if four or more of the particles were massive.

precisely equivalent to the massive scattering equations in d dimensions (3.1), and therefore have the same solutions. Thus, the propagator matrix for the massive amplitude is given by

$$m(1\gamma n|1\delta n) \;=\; \sum_{\{\sigma\}\in\text{solutions}} \frac{(-1)^{n-1}}{\det'\Phi(\{k,\sigma\})} \frac{1}{(\sigma_{1,\gamma(2)}\cdots\sigma_{\gamma(n-1),n}\sigma_{n,1})(\sigma_{1,\delta(2)}\cdots\sigma_{\delta(n-1),n}\sigma_{n,1})}$$

$$(4.5)$$

where the sum is over the solutions of the scattering equations (3.1).

Finally, the expression (4.5) has no explicit dependence on the masses, nor does it have any implicit dependence on the masses since the solutions $\{\sigma\}$ of the scattering equations, when expressed in terms of the kinematic invariants (2.9), are independent of the masses. Hence, eq. (4.5) for an amplitude with up to three massive particles is the same as that for massless amplitudes, as we already saw in section 2.

5 Gauge theory amplitudes

In this section, we demonstrate that various gauge theory amplitudes involving up to three massive particles can be expressed, like those for massless particles, as a sum over the $(n-3)!$ solutions of the scattering equations. Moreover, the expressions for these amplitudes, when written in terms of the kinematic invariants (2.9), are independent of the masses of the particles.

We begin in $d+M$ dimensions with the CHY representation for the tree-level scattering amplitude of n massless gauge bosons [1, 2]

$$\mathcal{A}_{\text{gauge}} \;=\; (-1)^{n-1} \sum_{\{\sigma\}\in\text{solutions}} \frac{C(\{\sigma\})\, E(\{\mathcal{E},K,\sigma\})}{\det'\Phi(\{K,\sigma\})}. \tag{5.1}$$

Here $\mathcal{E}_a$ and K_a are the $(d+M)$-dimensional polarizations and momenta of the gauge bosons, and the sum is over the solutions of the scattering equations (3.7). The first factor in the numerator is

$$C(\{\sigma\}) \;=\; \sum_{\gamma\in S_{n-2}} \frac{\mathbf{c}_{1\gamma n}}{\sigma_{1,\gamma(2)}\cdots\sigma_{\gamma(n-1),n}\sigma_{n,1}} \tag{5.2}$$

where $\mathbf{c}_{1\gamma n}$ was defined in eq. (2.3), and $\det'\Phi$ was defined in eq. (4.2). It is apparent that color-kinematic dual numerators $\mathbf{n}_{1\gamma n}$ can be used to construct

$$E(\{\mathcal{E},K,\sigma\}) \;=\; \sum_{\delta\in S_{n-2}} \frac{\mathbf{n}_{1\delta n}}{\sigma_{1,\delta(2)}\cdots\sigma_{\delta(n-1),n}\sigma_{n,1}} \tag{5.3}$$

so that eq. (4.3) implies that eqs. (5.1) and (2.5) are equivalent. The novelty of the CHY approach, however, is in providing an alternative definition for $E(\{\mathcal{E},K,\sigma\})$ in terms of the Pfaffian of a matrix, so that eq. (5.1) yields the gauge theory amplitudes directly without resort to Feynman diagrams.

One first defines a $2n\times 2n$ matrix

$$\Psi(\{\mathcal{E},K,\sigma\}) \;=\; \begin{pmatrix} A & -C^{\mathrm{T}} \\ C & B \end{pmatrix} \tag{5.4}$$

where A, B, and C are the $n \times n$ submatrices

$$
A_{ab} = \begin{cases} \dfrac{K_a \cdot K_b}{\sigma_a - \sigma_b}, & a \neq b; \\[2ex] 0, & a = b; \end{cases} \qquad B_{ab} = \begin{cases} \dfrac{\mathcal{E}_a \cdot \mathcal{E}_b}{\sigma_a - \sigma_b}, & a \neq b; \\[2ex] 0, & a = b; \end{cases} \qquad C_{ab} = \begin{cases} \dfrac{\mathcal{E}_a \cdot K_b}{\sigma_a - \sigma_b}, & a \neq b; \\[2ex] -\displaystyle\sum_{c \neq a} \dfrac{\mathcal{E}_a \cdot K_c}{\sigma_a - \sigma_c}, & a = b. \end{cases}
$$

(5.5)

The matrix Ψ has two null vectors, so to obtain a nonvanishing Pfaffian, it is necessary to remove two of the first n rows and columns. It was shown in ref. [1] that the resulting "reduced Pfaffian" is independent (when the scattering equations are satisfied) of the choice of rows and columns deleted, but for our purposes (in order to make manifest the mass independence of the result) it will prove advantageous to choose ones corresponding to two of the massive particles. We therefore define $\Psi_{1,n}$ as the matrix obtained by deleting the 1st and nth rows and columns from Ψ, in which case the CHY prescription is

$$
E(\{\mathcal{E}, K, \sigma\}) \;=\; \frac{(-1)^{n+1}}{\sigma_{1,n}} \mathrm{Pf}\ \Psi_{1,n}(\{\mathcal{E}, K, \sigma\}).
$$

(5.6)

It was shown in refs. [1, 32] that, with this definition, eq. (5.1) computes the amplitudes for massless gauge bosons in $(d + M)$ dimensions.

Amplitudes for massive gauge bosons. We now use dimensional reduction to obtain the amplitude for massless and (up to three) massive gauge bosons in d dimensions. As in section 3, we set $K_a = (k_a | \kappa_a)$, where the massless bosons ($a = 3, \cdots, n - 1$) have $\kappa_a = 0$. We restrict all of the $(d + M)$-dimensional polarization vectors $\mathcal{E}_a$ to the d-dimensional subspace: $\mathcal{E}_a = (\epsilon_a | 0, 0, \cdots)$. All of the $(d + M)$-dimensional dot products in the matrices B and C reduce to d-dimensional dot products, $\mathcal{E}_a \cdot \mathcal{E}_b = \epsilon_a \cdot \epsilon_b$ and $\mathcal{E}_a \cdot K_b = \epsilon_a \cdot k_b$, whereas those in A become $K_a \cdot K_b = k_a \cdot k_b - \kappa_a \cdot \kappa_b$. Every matrix element A_{ab} that appears in $\Psi_{1,n}$ (from which the 1st and nth rows and columns of Ψ have been eliminated) has either $\kappa_a = 0$ or $\kappa_b = 0$, so $K_a \cdot K_b = k_a \cdot k_b$, which belongs to the set of kinematic invariants (2.9). Thus, eq. (5.6) becomes

$$
E(\{\mathcal{E}, K, \sigma\}) \;=\; \frac{(-1)^{n+1}}{\sigma_{1,n}} \mathrm{Pf}\ \Psi_{1,n}(\{\epsilon, k, \sigma\}).
$$

(5.7)

so the d-dimensional amplitude for n gauge bosons, up to three of which may be massive, is

$$
\mathcal{A} \;=\; \sum_{\{\sigma\} \in \mathrm{solutions}} \frac{C(\{\sigma\})}{\mathrm{det}'\, \Phi(\{k, \sigma\})} \frac{\mathrm{Pf}\ \Psi_{1,n}(\{\epsilon, k, \sigma\})}{\sigma_{1,n}}.
$$

(5.8)

where $\{\sigma\}$ are solutions of the scattering equations (3.1). We showed in section 3 that these solutions, when expressed in terms of the invariants (2.9), are independent of the masses, and in section 4 that $\mathrm{det}'\, \Phi$ is also independent of the masses. Hence, we conclude that the scattering amplitude (5.8) for n gauge bosons, up to three of which may be massive, is identical to that for n massless gauge bosons, provided it is expressed in terms of the invariants (2.9).

Let us illustrate this explicitly for the four-gauge-boson amplitude, with $m_3 = 0$. We use $\mathrm{SL}(2,\mathbb{C})$ invariance to set $\sigma_1 = 0$, $\sigma_2 = 1$, and $\sigma_4 \to \infty$, then evaluate the various components of eq. (5.8) on the single solution $\sigma_3 = -k_1 \cdot k_3/k_3 \cdot k_4$ of the $n = 4$ scattering equations. Taking σ_4 large but finite, we obtain

$$\mathrm{det}'\Phi(\{k,\sigma\}) \longrightarrow \frac{2(k_3 \cdot k_4)^3}{\sigma_4^4 \; k_1 \cdot k_3 \; k_2 \cdot k_3} \tag{5.9}$$

$$C(\{\sigma\}) \longrightarrow -\frac{(k_3 \cdot k_4)^2}{\sigma_4^2 \; k_2 \cdot k_3}\left[\frac{\mathbf{c}_{1234}}{k_3 \cdot k_4} + \frac{\mathbf{c}_{1324}}{k_1 \cdot k_3}\right] \tag{5.10}$$

$$\mathrm{Pf}\;\Psi_{1,4}(\{\epsilon,k,\sigma\}) \longrightarrow \frac{k_3 \cdot k_4}{\sigma_4 \; k_1 \cdot k_3 \; k_2 \cdot k_3}\mathbf{K} \tag{5.11}$$

where $\mathbf{c}_{1234} = f^{\mathsf{a_1 a_2 b}} f^{\mathsf{b a_3 a_4}}$ and $\mathbf{c}_{1324} = f^{\mathsf{a_1 a_3 b}} f^{\mathsf{b a_2 a_4}}$, with f^{abc} the three-gauge-boson vertex, and $\mathbf{K}$ is the totally permutation-symmetric expression that appears in the scattering amplitude for four open strings (cf. eq. (7.4.42) of ref. [57])

$$\begin{aligned}
\mathbf{K} = -\Big[&(k_1 \cdot k_3 \; k_2 \cdot k_3) \; \epsilon_1 \cdot \epsilon_2 \; \epsilon_3 \cdot \epsilon_4 \\
&+(k_2 \cdot k_3 \; k_3 \cdot k_4) \; \epsilon_1 \cdot \epsilon_3 \; \epsilon_2 \cdot \epsilon_4 \\
&+(k_1 \cdot k_3 \; k_3 \cdot k_4) \; \epsilon_1 \cdot \epsilon_4 \; \epsilon_2 \cdot \epsilon_3 \\
&+(k_1 \cdot k_3 \; \epsilon_1 \cdot k_4 \; \epsilon_2 \cdot k_3 + k_2 \cdot k_3 \; \epsilon_1 \cdot k_3 \; \epsilon_2 \cdot k_4) \; \epsilon_3 \cdot \epsilon_4 \\
&+(k_2 \cdot k_3 \; \epsilon_1 \cdot k_2 \; \epsilon_3 \cdot k_4 + k_3 \cdot k_4 \; \epsilon_1 \cdot k_4 \; \epsilon_3 \cdot k_2) \; \epsilon_2 \cdot \epsilon_4 \\
&+(k_1 \cdot k_3 \; \epsilon_1 \cdot k_2 \; \epsilon_4 \cdot k_3 + k_3 \cdot k_4 \; \epsilon_1 \cdot k_3 \; \epsilon_4 \cdot k_2) \; \epsilon_2 \cdot \epsilon_3 \\
&+(k_1 \cdot k_3 \; \epsilon_2 \cdot k_1 \; \epsilon_3 \cdot k_4 + k_3 \cdot k_4 \; \epsilon_2 \cdot k_4 \; \epsilon_3 \cdot k_1) \; \epsilon_1 \cdot \epsilon_4 \\
&+(k_2 \cdot k_3 \; \epsilon_2 \cdot k_1 \; \epsilon_4 \cdot k_3 + k_3 \cdot k_4 \; \epsilon_2 \cdot k_3 \; \epsilon_4 \cdot k_1) \; \epsilon_1 \cdot \epsilon_3 \\
&+(k_1 \cdot k_3 \; \epsilon_3 \cdot k_2 \; \epsilon_4 \cdot k_1 + k_2 \cdot k_3 \; \epsilon_3 \cdot k_1 \; \epsilon_4 \cdot k_2) \; \epsilon_1 \cdot \epsilon_2 \Big].
\end{aligned} \tag{5.12}$$

Assembling the pieces, and taking the limit $\sigma_4 \to \infty$, we obtain

$$\mathcal{A}_{gggg} \;=\; \frac{1}{2k_2 \cdot k_3}\left[\frac{\mathbf{c}_{1234}}{k_3 \cdot k_4} + \frac{\mathbf{c}_{1324}}{k_1 \cdot k_3}\right]\mathbf{K}, \qquad \text{when } m_3 = 0\,. \tag{5.13}$$

This expression exhibits the expected factorization into color-dependent and polarization-dependent factors [5, 6].

Consider two amplitudes involving massive gauge bosons: $W\gamma \to W\gamma$ and $W\gamma \to WZ^0$. In each case we have $\mathbf{c}_{1234} = \mathbf{c}_{1324}$, so that eq. (5.13) reduces to

$$\mathcal{A}_{W\gamma\gamma W} \;\propto\; \frac{e^2}{k_1 \cdot k_3 \; k_3 \cdot k_4}\mathbf{K}, \qquad \mathcal{A}_{WZ^0\gamma W} \;\propto\; \frac{e^2 \cot\theta_W}{k_1 \cdot k_3 \; k_3 \cdot k_4}\mathbf{K} \tag{5.14}$$

both of which can be verified by a Feynman diagram calculation.[3] A Feynman diagram calculation also shows that eq. (5.13) is *not* valid for $WZ^0 \to WZ^0$, which is not surprising since this amplitude violates the condition $m_3 = 0$.

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[3]Holstein [22] calculated the Compton scattering amplitude from a massive spin-one particle with an arbitrary value g for its magnetic moment. Our result (5.14) agrees with his when $g = 2$, the standard model magnetic moment for the W boson.

Amplitudes for two adjoint scalars and $n-2$ gauge bosons. Next we turn to the amplitude for two adjoint scalars (with masses m_1 and m_n) and $n-2$ gauge bosons, one of which (m_2) may be massive. This amplitude may be obtained by dimensional reduction of eq. (5.1) with $\mathcal{E}_1 = \mathcal{E}_n = (0|1, 0, \cdots)$ and $\mathcal{E}_a = (\epsilon_a|0, 0, \cdots)$ for $a = 2, \cdots, n-1$, as was done in ref. [25]. In that case, where all the particles were massless, Ψ became block diagonal, but the situation here is slightly more complicated because $K_a = (k_a|\kappa_a)$ with $\kappa_a \neq 0$ for $a = 1$, 2, and n.

We can evaluate the Pfaffian of a $2m \times 2m$ matrix Z by using the relation

$$\text{Pf } Z = \sum_{p=1}^{2m}(-1)^{p+1}z_{p,2m}\text{Pf } Z_{p,2m} \tag{5.15}$$

where z_{pq} is a matrix element of Z, and we define Z_{pq} as the matrix obtained from Z by omitting the pth and qth rows and columns. Observe that only four entries in the $(2n)$th column of Ψ are non-zero: $\psi_{1,2n}$, $\psi_{2,2n}$, $\psi_{n,2n}$, and $\psi_{n+1,2n}$. Two of these, $\psi_{1,2n}$ and $\psi_{n,2n}$, have already been omitted from $\Psi_{1,n}$, leaving only two terms in the sum (5.15) for $Z = \Psi_{1,n}(\{\mathcal{E}, K, \sigma\})$. Next observe that only four entries in the $(n+1)$th column of Ψ are non-zero: $\psi_{1,n+1}$, $\psi_{2,n+1}$, $\psi_{n,n+1}$, and $\psi_{2n,n+1}$. All four of these have been omitted from $\Psi_{1,2,n,2n}$, and the Pfaffian of a matrix with an entire column of zeros vanishes. Thus, only one term in the sum (5.15) survives, namely (using $\mathcal{E}_1 \cdot \mathcal{E}_n = -1$)

$$\text{Pf } \Psi_{1,n}(\{\mathcal{E}, K, \sigma\}) = (-1)^n\psi_{n+1,2n}\text{Pf } \Psi_{1,n,n+1,2n} = \frac{(-1)^{n+1}}{\sigma_{1,n}}\text{Pf } \Psi_{1,n,n+1,2n}(\{\epsilon, k, \sigma\}) \tag{5.16}$$

and therefore eq. (5.6) becomes

$$E(\{\mathcal{E}, K, \sigma\}) = \frac{1}{\sigma_{1,n}^2}\text{Pf } \Psi_{1,n,n+1,2n}(\{\epsilon, k, \sigma\}) \tag{5.17}$$

where $\Psi_{1,n,n+1,2n}$ denotes the matrix obtained from Ψ by removing the 1st, nth, $(n+1)$st and $(2n)$th rows and columns.

Using eq. (5.17) in eq. (5.1), we obtain the d-dimensional amplitude for two massive adjoint scalars and $n-2$ gauge bosons, one of which may be massive:

$$\mathcal{A} = (-1)^{n-1}\sum_{\{\sigma\}\in\text{solutions}}\frac{C(\{\sigma\})}{\det' \Phi(\{k, \sigma\})}\frac{\text{Pf } \Psi_{1,n,n+1,2n}(\{\epsilon, k, \sigma\})}{\sigma_{1,n}^2}. \tag{5.18}$$

For the same reasons as before, this amplitude is independent of the masses m_1, m_2, and m_n when written in terms of the invariants (2.9).

Amplitudes for two fundamental scalars and $n-2$ gauge bosons. The scalars in the amplitude (5.18) transform in the adjoint representation. The amplitude for scalars in the fundamental representation is essentially the same except that we replace $\mathbf{c}_{1\gamma n}$ in eq. (5.2) with [50]

$$\mathbf{t}_{1\gamma n} = (T^{\mathsf{a}_{\gamma(2)}}T^{\mathsf{a}_{\gamma(3)}}\cdots T^{\mathsf{a}_{\gamma(n-1)}})^{\mathsf{i}_1}{}_{\mathsf{i}_n} \tag{5.19}$$

where T^{a} denote generators in the fundamental representation. In this way, we recover the results of ref. [50] for amplitudes with two massive scalars in the fundamental representation and $n - 2$ massless gauge bosons.[4]

Again, we illustrate our result for the four-point amplitude with two fundamental scalars and two gauge bosons. On the single solution of the $n = 4$ scattering equations, we obtain

$$\text{Pf } \Psi_{1,4,5,8}(\{\epsilon, k, \sigma\}) \ \longrightarrow \ \frac{k_3 \cdot k_4}{k_1 \cdot k_3 \ k_2 \cdot k_3} \mathbf{L} \tag{5.20}$$

where

$$\mathbf{L} = (k_1 \cdot k_3 \ k_3 \cdot k_4) \ \epsilon_2 \cdot \epsilon_3 + k_1 \cdot k_3 \ \epsilon_2 \cdot k_1 \ \epsilon_3 \cdot k_4 + k_3 \cdot k_4 \ \epsilon_2 \cdot k_4 \ \epsilon_3 \cdot k_1 \,. \tag{5.21}$$

Putting this together with eqs. (5.9) and (5.10), we obtain the amplitude

$$\mathcal{A}_{\psi g g \bar{\psi}} \ = \ \frac{1}{2 k_2 \cdot k_3} \left[\frac{\mathbf{t}_{1234}}{k_3 \cdot k_4} + \frac{\mathbf{t}_{1324}}{k_1 \cdot k_3} \right] \mathbf{L}, \qquad \text{when } m_3 = 0 \,. \tag{5.22}$$

Again we see the expected factorization into a product of a color factor and a polarization-dependent factor [5, 6], valid provided one of the gauge bosons is massless.

Consider an amplitude involving a photon, W, and a pair of scalar quarks ψ_d and $\psi_{\bar{u}}$. In this case, each color factor $\mathbf{t}_{1\cdots 4}$ is proportional to the $\psi_d W \psi_{\bar{u}}$ coupling times the charge of the quark interacting with the photon, giving

$$\mathcal{A}_{\psi_d W \gamma \psi_{\bar{u}}} \ \propto \ \frac{1}{k_2 \cdot k_3} \left[\frac{Q_u}{k_3 \cdot k_4} + \frac{Q_d}{k_1 \cdot k_3} \right] \mathbf{L} \tag{5.23}$$

agreeing with a Feynman diagram evaluation [10]. Note that this amplitude vanishes when

$$\frac{k_1 \cdot k_3}{k_3 \cdot k_4} \ = \ -\frac{Q_d}{Q_u} = \frac{1}{2} \tag{5.24}$$

independent of the polarizations. This implies that the amplitude for $\psi_d \psi_{\bar{u}} \to W^- \gamma$, vanishes when $\cos \theta = -1/3$ (for massless quarks), where θ is the angle between ψ_d and W^- in the center-of-mass frame. Such "radiation zeros" were first found in calculations of $d \bar{u} \to W^- \gamma$ processes [7–9], leading to the discovery of four-point factorization [5, 6], which was the forerunner of the BCJ relations [4]. Radiation zeros were also found to be present in higher-point amplitudes under certain conditions [11, 12].

6 Gravity amplitudes

In this section, we demonstrate that various gravitational amplitudes involving up to three massive particles can be expressed as a sum over the $(n - 3)!$ solutions of the scattering equations, and that the expressions for these amplitudes, when written in terms of the kinematic invariants (2.9), are independent of the masses of the particles.

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[4]The matrix $\Psi_{1,n,n+1,2n}$ is denoted as Ψ in ref. [50] (note the clarification in v2).

We begin in $d + M$ dimensions with the CHY representation for the tree-level gravitational scattering amplitude [1, 2]

$$\mathcal{A}_{\text{grav}} = (-1)^{n-1} \sum_{\{\sigma\} \in \text{solutions}} \frac{E(\{\mathcal{E}, K, \sigma\}) \, E(\{\tilde{\mathcal{E}}, K, \sigma\})}{\det' \Phi(\{K, \sigma\})} \qquad (6.1)$$

for particles with momenta K_a and whose polarization state is described by a tensor $\mathcal{E}_a^{\mu\nu} = \mathcal{E}_a^\mu \tilde{\mathcal{E}}_a^\nu$, which includes the graviton, B-field, and dilaton. The factors $\det' \Phi(\{K, \sigma\})$ and $E(\{\mathcal{E}, K, \sigma\})$ were defined in eqs. (4.2) and (5.6), and the sum is over the solutions of the scattering equations (3.7).

In ref. [25], dimensional reduction from $d + M$ to d dimensions was used to obtain mixed amplitudes in Einstein-Maxwell theory with gauge group $U(1)^M$. We will use the same approach to obtain various mixed amplitudes involving massive particles.

Amplitudes for two gauge bosons and $n - 2$ gravitons. To obtain amplitudes for two (possibly massive) gauge bosons and $n - 2$ gravitons, we set $K_a = (k_a | \kappa_a)$, with $\kappa_a = 0$ for the gravitons ($a = 2, \cdots, n-1$). We choose the $(d+M)$-dimensional polarization vectors to be $\mathcal{E}_1 = \mathcal{E}_n = (0|1, 0, \cdots)$ and $\mathcal{E}_a = (\epsilon_a | 0, 0, \cdots)$ for $a = 2, \cdots, n-1$, so that $E(\{\mathcal{E}, K, \sigma\})$ is given by eq. (5.17), while we set $\tilde{\mathcal{E}}_a = (\tilde{\epsilon}_a | 0, 0, \cdots)$ for all a so that $E(\{\tilde{\mathcal{E}}, K, \sigma\})$ is given by eq. (5.7). This yields the amplitude in d dimensions

$$\mathcal{A} = \sum_{\{\sigma\} \in \text{solutions}} \frac{\text{Pf } \Psi_{1,n}(\{\epsilon, k, \sigma\}) \, \text{Pf } \Psi_{1,n,n+1,2n}(\{\tilde{\epsilon}, k, \sigma\})}{\det' \Phi(\{k, \sigma\}) \, \sigma_{1,n}^3} . \qquad (6.2)$$

Although dimensional reduction gives the amplitude for two photons and $n - 2$ gravitons, it remains valid for any gauge theory since no non-abelian vertices are involved. For the same reasons described previously, this amplitude has no dependence (when expressed in terms of the invariants (2.9)) on the mass of the gauge bosons.

We illustrate eq. (6.2) for the case of the four-point amplitude $gh \to gh$, describing the scattering of gravitons (h) from gauge bosons (g). Assembling eqs. (5.9), (5.11), and (5.20), we obtain

$$\mathcal{A}_{ghhg} = -\frac{1}{2k_1 \cdot k_3 \, k_2 \cdot k_3 \, k_3 \cdot k_4} \mathbf{L} \, \tilde{\mathbf{K}} \qquad (6.3)$$

where $\mathbf{L}$ was defined in eq. (5.21) and $\tilde{\mathbf{K}}$ indicates that ϵ is replaced by $\tilde{\epsilon}$ in eq. (5.12). Equation (6.3) agrees with the result of a Feynman diagram calculation [20] in which the factorization into $\mathbf{L}$ and $\tilde{\mathbf{K}}$ was noted. As emphasized in ref. [22], factorization relates graviton scattering from W's to Compton scattering from W's, but only when the W magnetic moment is given by its standard model value $g = 2$.

Amplitudes for two scalars and $n - 2$ gravitons. To obtain amplitudes for two massive scalars and $n - 2$ gravitons, we choose K_a and $\mathcal{E}_a$ to be the same as above, but now set $\tilde{\mathcal{E}}_1 = \tilde{\mathcal{E}}_n = (0|1, 0, \cdots)$ and $\tilde{\mathcal{E}}_a = (\tilde{\epsilon}_a | 0, 0, \cdots)$ for $a = 2, \cdots, n-1$. Hence both both $E(\{\mathcal{E}, K, \sigma\})$ and $E(\{\tilde{\mathcal{E}}, K, \sigma\})$ are given by eq. (5.17), and the d-dimensional amplitude becomes

$$\mathcal{A} = (-1)^{n-1} \sum_{\{\sigma\} \in \text{solutions}} \frac{\text{Pf } \Psi_{1,n,n+1,2n}(\{\epsilon, k, \sigma\}) \, \text{Pf } \Psi_{1,n,n+1,2n}(\{\tilde{\epsilon}, k, \sigma\})}{\det' \Phi(\{k, \sigma\}) \, \sigma_{1,n}^4} \qquad (6.4)$$

thus recovering the expression obtained in ref. [50]. As we have seen before, the amplitude is independent of the scalar mass, when expressed in terms of the invariants (2.9).

We illustrate eq. (6.4) for the case of the scattering $\psi h \to \psi h$ of gravitons from massive scalars [58]. Assembling eqs. (5.9) and (5.20), we obtain

$$\mathcal{A}_{\psi h h \bar{\psi}} = - \frac{1}{2 k_1 \cdot k_3 \; k_2 \cdot k_3 \; k_3 \cdot k_4} \mathbf{L} \, \tilde{\mathbf{L}} \tag{6.5}$$

where $\tilde{\mathbf{L}}$ indicates that ϵ is replaced by $\tilde{\epsilon}$ in eq. (5.21). The factorization of this amplitude was first noted by the authors of ref. [20].

Amplitudes for three gauge bosons and $n-3$ gravitons. Since the amplitude for three (possibly) massive gauge bosons and $n-3$ gravitons cannot be obtained directly from the dimensional reduction of eq. (6.1) to an Einstein-Maxwell theory (since the triple photon vertex vanishes), we start instead with the mixed amplitude for three massless gauge bosons and $n-3$ gravitons in $d+M$ dimensions [24]

$$\mathcal{A} = (-1)^{n-1} \sum_{\{\sigma\} \in \text{solutions}} \frac{C(\sigma_1, \sigma_2, \sigma_n) \, E(\mathcal{E}_3, \cdots, \mathcal{E}_{n-1}, \{K, \sigma\}) \, E(\{\tilde{\mathcal{E}}, K, \sigma\})}{\det' \Phi(\{K, \sigma\})} \tag{6.6}$$

where

$$C(\sigma_1, \sigma_2, \sigma_n) = \frac{\text{Tr}(T^{\mathsf{a}_1} T^{\mathsf{a}_2} T^{\mathsf{a}_n})}{\sigma_{12} \sigma_{2n} \sigma_{n1}}, \quad E(\mathcal{E}_3, \cdots, \mathcal{E}_{n-1}, \{K, \sigma\}) = \text{Pf } \Psi_{1,2,n,n+1,n+2,2n}(\{\mathcal{E}, K, \sigma\}) \tag{6.7}$$

and $\Psi_{1,2,n,n+1,n+2,2n}$ denotes Ψ with the six rows and columns indicated removed. Equation (6.6) can be obtained from a pure graviton amplitude using a "squeezing" process [25].

To obtain the amplitude for three *massive* gauge bosons and $n-3$ gravitons in d dimensions, we dimensionally reduce eq. (6.6) by taking $\mathcal{E}_a = (\epsilon_a|0, 0, \cdots)$, $\tilde{\mathcal{E}}_a = (\tilde{\epsilon}_a|0, 0, \cdots)$, and $K_a = (k_a|\kappa_a)$ for all a, with $\kappa_a = 0$ for $a = 3, \cdots, n-1$. Since $K_a \cdot K_b = k_a \cdot k_b$, $K_a \cdot \mathcal{E}_b = k_a \cdot \epsilon_b$, and $\mathcal{E}_a \cdot \mathcal{E}_b = \epsilon_a \cdot \epsilon_b$ for all entries of the matrices $\Psi_{1,2,n,n+1,n+2,2n}$ and $\Psi_{1,n}$, eq. (6.6) reduces to

$$\mathcal{A} = \sum_{\{\sigma\} \in \text{solutions}} \frac{C(\sigma_1, \sigma_2, \sigma_n) \, \text{Pf } \Psi_{1,2,n,n+1,n+2,2n}(\{\epsilon, k, \sigma\}) \, \text{Pf } \Psi_{1,n}(\{\tilde{\epsilon}, k, \sigma\})}{\det' \Phi(\{k, \sigma\}) \, \sigma_{1,n}} \tag{6.8}$$

which is independent of the gauge boson masses (when expressed in terms of invariants (2.9)).

To illustrate eq. (6.8) for the four-point function, we evaluate its components on the single solution of the $n=4$ scattering equations to obtain

$$C(\sigma_1, \sigma_2, \sigma_4) \longrightarrow \frac{\text{Tr}(T^{\mathsf{a}_1} T^{\mathsf{a}_2} T^{\mathsf{a}_4})}{\sigma_4^2}, \tag{6.9}$$

$$\text{Pf } \Psi_{1,2,4,5,6,8}(\{\epsilon, k, \sigma\}) = \sum_{c \neq 3} \frac{\epsilon_3 \cdot k_c}{\sigma_{3,c}} \longrightarrow \frac{k_3 \cdot k_4 \, (k_1 \cdot k_3 \, \epsilon_3 \cdot k_2 - k_2 \cdot k_3 \, \epsilon_3 \cdot k_1)}{k_1 \cdot k_3 \; k_2 \cdot k_3}$$

together with eqs. (5.9) and (5.11). Assembling the pieces and taking $\sigma_4 \to \infty$, we obtain

$$\mathcal{A}_{gghg} = \frac{(k_1 \cdot k_3 \, \epsilon_3 \cdot k_2 - k_2 \cdot k_3 \, \epsilon_3 \cdot k_1)}{2 k_1 \cdot k_3 \; k_2 \cdot k_3 \; k_3 \cdot k_4} \text{Tr}(T^{\mathsf{a}_1} T^{\mathsf{a}_2} T^{\mathsf{a}_4}) \, \tilde{\mathbf{K}} \tag{6.10}$$

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where all three gauge bosons can be massive. This result agrees with Feynman diagram computations of graviton photoproduction $W\gamma \to Wh$ from massive vector bosons [20, 23]. Equation (6.10) implies that $WZ^0 \to Wh$ should also factorize.

Amplitudes for two scalars, one gauge boson, and $n - 3$ gravitons. Finally, to obtain the amplitude for two massless scalars, a (possibly) massive gauge boson, and $n - 3$ gravitons in d dimensions, we choose K_a and $\mathcal{E}_a$ to be the same as in the previous example, but now set $\tilde{\mathcal{E}}_1 = \tilde{\mathcal{E}}_n = (0|1, 0, \cdots)$ and $\tilde{\mathcal{E}}_a = (\tilde{\epsilon}_a|0, 0, \cdots)$ for $a = 2, \cdots, n - 1$. Then eq. (6.6) reduces to

$$\mathcal{A} = (-1)^{n-1} \sum_{\{\sigma\} \in \text{solutions}} \frac{C(\sigma_1, \sigma_2, \sigma_n) \text{Pf } \Psi_{1,2,n,n+1,n+2,2n}(\{\epsilon, k, \sigma\}) \text{ Pf } \Psi_{1,n,n+1,2n}(\{\tilde{\epsilon}, k, \sigma\})}{\text{det}' \Phi(\{k, \sigma\}) \sigma_{1,n}^2}$$

(6.11)

once again independent of the masses (when expressed in terms of invariants (2.9)). For the four-point amplitude, this yields

$$\mathcal{A}_{\psi g h \bar{\psi}} = \frac{(k_1 \cdot k_3 \ \epsilon_3 \cdot k_2 - k_2 \cdot k_3 \ \epsilon_3 \cdot k_1)}{2k_1 \cdot k_3 \ k_2 \cdot k_3 \ k_3 \cdot k_4} \text{Tr}(T^{\mathsf{a}_1} T^{\mathsf{a}_2} T^{\mathsf{a}_4}) \ \tilde{\mathbf{L}}$$

(6.12)

which agrees with Feynman diagram computations of graviton photoproduction $\psi\gamma \to \psi h$ from scalars [20, 21]. Equation (6.12) implies that $\psi Z^0 \to \psi h$ and $\psi_d W^+ \to \psi_u h$ should also factorize.

7 Conclusions

In this paper we have derived CHY representations for a wide class of tree-level gauge and gravitational amplitudes containing up to three massive particles. These mixed amplitudes involve gravitons, massive and massless gauge bosons, and adjoint or fundamental massive scalar particles. All of these amplitudes can be obtained through the dimensional reduction of massless gauge and gravitational amplitudes in a higher dimension. Further, we demonstrated that amplitudes containing up to three massive particles, when written in terms of the kinematic invariants (2.9), are in fact independent of the masses, and so have the same form as purely massless amplitudes.

It is not so straightforward to obtain CHY representations for gauge and gravitational amplitudes with four or more massive particles. As explained in section 3, such amplitudes do not generically result from the dimensional reduction of massless amplitudes because of the difficulty of satisfying momentum conservation in the internal dimensions. Moreover, the propagator matrix in this case generally has rank greater than $(n - 3)!$ and so cannot be represented as a sum over $(n - 3)!$ solutions.

On the other hand, there is good reason to suspect the existence of CHY representations involving fermions, since the factorization of four-point gauge amplitudes also holds for amplitudes containing spin-one-half particles [5, 6] and gravitational scattering from spin-one-half particles [20–23] has also been shown to factorize. For progress in this area, see refs. [33, 40, 53].

Acknowledgments

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