



PUBLISHED FOR SISSA BY SPRINGER

RECEIVED: May 10, 2016

ACCEPTED: July 23, 2016

PUBLISHED: August 2, 2016

Towards bootstrapping QED_3

Shai M. Chester and Silviu S. Pufu

*Joseph Henry Laboratories, Princeton University,
Princeton, NJ 08544, U.S.A.*

E-mail: schester@princeton.edu, spufu@princeton.edu

ABSTRACT: We initiate the conformal bootstrap study of Quantum Electrodynamics in $2 + 1$ space-time dimensions (QED_3) with N flavors of charged fermions by focusing on the 4-point function of four monopole operators with the lowest unit of topological charge. We obtain upper bounds on the scaling dimension of the doubly-charged monopole operator, with and without assuming other gaps in the operator spectrum. Intriguingly, we find a (gap-dependent) kink in these bounds that comes reasonably close to the large N extrapolation of the scaling dimensions of the singly-charged and doubly-charged monopole operators down to $N = 4$ and $N = 6$.

KEYWORDS: Conformal and W Symmetry, Gauge Symmetry, Solitons Monopoles and Instantons

ARXIV EPRINT: [1601.03476](https://arxiv.org/abs/1601.03476)

JHEP08(2016)019

Contents

1	Introduction and summary	1
2	3d QED and monopole operators	3
2.1	Lowest dimension monopole operators M_q	4
2.2	Lowest dimension scalar $q = 0$ operators in OPE $M_{1/2} \times M_{-1/2}$	5
2.3	Conserved-current and stress-tensor two-point functions	6
3	Crossing equations	6
3.1	Known results for $N = 2, 4$	8
3.2	General algorithm	9
3.2.1	$N = 2$	10
3.2.2	$N = 4$	11
3.2.3	$N = 6$	12
3.3	Reflection positivity	13
3.3.1	SU(2)	14
3.3.2	SU(4)	15
3.3.3	SU(6)	16
3.4	Constraints from space-time parity	16
4	Numerical bounds	18
4.1	Strategy	18
4.2	Numerical bounds for $N = 2, 4, 6$	19
4.2.1	Bounds on Δ_{M_1}	20
4.2.2	Bounds on c_T, c_J^f, c_J^t	21
5	Discussion	21
A	Crossing functions for $N = 8, 10, 12, 14$	23

1 Introduction and summary

Quantum electrodynamics in $2 + 1$ spacetime dimensions (QED₃) could be regarded as a toy model for the real world quantum chromodynamics in $3 + 1$ dimensions because it is an asymptotically free theory that may also exhibit analogs of chiral symmetry breaking [1] and confinement [2, 3]. In a Lagrangian description, the field content of QED₃ consists of a U(1) gauge field possibly coupled to several flavors of charged fermions. When there is no charged matter, the theory confines [2, 3]. When the number N of (two-component complex) fermion flavors is large, it can be argued using $1/N$ perturbation theory that

the infrared physics is described by an interacting conformal field theory [4, 5]. When N is small but non-zero, the precise dynamics remains uncertain, however, because the theory is strongly coupled, and there are only very few non-perturbative tools available.¹ It is believed that in this regime the theory may exhibit analogs of both chiral symmetry breaking and confinement.

In this work, we aim to initiate a study of QED₃ at small N using the conformal bootstrap technique [11], with the goal of eventually shedding light on the behavior of the theory in this regime. The conformal bootstrap is a non-perturbative technique that has yielded quite impressive results in other non-supersymmetric examples, such as the 3d Ising model [12, 13], the critical $O(N)$ vector model [13–15], or, more recently, the Gross-Neveu models [16], so it is natural to ask whether it can also be used to learn about 3d gauge theories as well. In its numerical implementation in terms of semi-definite programming, the conformal bootstrap makes use of unitarity and associativity of the operator algebra as applied to 4-point functions of certain operators in a conformal field theory.

In this paper, we assume that the conformal fixed point of QED₃ seen in $1/N$ perturbation theory extends to all values of N , and study this CFT using the conformal bootstrap. Explicitly, we derive and study numerically the crossing relations of four monopole operators (to be defined more precisely shortly) for $N = 2, 4$, and 6 . What we find are rigorous bounds on the scaling dimensions of these monopole operators and of some of the operators appearing in their OPE. We find that these bounds come close to the large N results when extrapolated to small N . In addition, we find certain features in our bounds that are similar to those that appeared in the bounds of the lowest-dimension operators in 3d CFTs with global \mathbb{Z}_2 symmetry when looking at the single 4-point function of \mathbb{Z}_2 odd operators. In that case, examining the crossing equation of a system of mixed correlators yielded an allowed region in the form of an island centered around the 3d Ising CFT. It would be interesting to see if a study of mixed correlators of monopole operators also yields an island-shaped allowed region, though such an analysis is of a numerical complexity beyond what is currently feasible.

Before we delve into the details of our analysis, let us comment on our choice of studying the crossing equations of monopole operators as opposed to those of other operators in the theory. QED₃ with N unit charged fermions ψ^i has $SU(N) \times U(1)$ flavor symmetry. The fermions transform as a fundamental of $SU(N)$ and are uncharged under $U(1)$. The monopole operators have non-zero $U(1)$ charge and also transform in fairly complicated representations of $SU(N)$. In implementing the conformal bootstrap program, one option would have been to consider the 4-point function of the simplest non-monopole scalar operators, the bilinears $\bar{\psi}_i \psi^j$ transforming in the adjoint of $SU(N)$. The crossing equations for such a four-point function were worked out in [17], and it should be straightforward to study the constraints they imply numerically using computer programs such as SDPB [18]. The disadvantage of studying this four-point function by itself, however, is that besides QED₃, there are other theories such as scalar QED, QCD₃ or supersymmetric analogs that

¹Recently, the ϵ -expansion was used to argue that spontaneous chiral symmetry breaking should occur when $N \leq 4$ [6]. The F -theorem suggests that it occurs when $N \leq 8$ [7]. Lattice studies suggest that it occurs when $N = 2$ [8, 9] or $N = 0$ [10], but the situation at larger N is unclear.

all have $SU(N)$ flavor adjoint operators with similar properties, and thus from an abstract CFT point of view, it may be hard a priori to distinguish these theories from one another.

What is specific to QED_3 and is not shared by its QCD or supersymmetric analogs is indeed the spectrum of monopole operators, and this is why we focus on them. It can be shown [19, 20] that the monopole operator M_q that carries $U(1)$ charge $q \in \mathbb{Z}/2$ also transforms under $SU(N)$ as an irreducible representation given by the Young diagram

$$N/2 \left\{ \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \right\}. \tag{1.1}$$

$\underbrace{\hspace{10em}}_{2|q|}$

This feature makes QED_3 different from the other similar theories for which the lowest-dimension non-monopole scalars are also $SU(N)$ adjoints. Note that without any Chern-Simons interactions, N is required to be even in order to avoid a parity anomaly [19], so the Young diagram (1.1) is indeed well-defined.

Monopole operators are interesting to study not just so that we can distinguish QED_3 from other theories. More generally, they are quite important for the dynamics of gauge theories in $2 + 1$ dimensions. The simplest example is pure $U(1)$ gauge theory, where it was shown by Polyakov that their proliferation provides a mechanism for confinement [2]. If one adds a sufficiently large number N of charged matter fields (bosons or fermions), the infrared physics is believed to be governed by an interacting conformal field theory (CFT), where, in certain condensed matter realizations, monopole operators can act as order parameters for quantum phase transitions that evade the Ginzburg-Landau paradigm [21–34]. In these interacting CFTs, the only available method² for studying the properties of the monopole operators is the $1/N$ expansion, which so far has been used to compute their scaling dimensions to next-to-leading order in $1/N$ [19, 20, 36–40]. Going to higher orders in the $1/N$ expansion appears to be very challenging with current techniques. It is nevertheless desirable to learn about monopole operators away from the large N limit, which serves as further motivation for studying them using the conformal bootstrap.

The rest of this paper is organized as follows. In section 2, we review some known facts about 3d QED and monopole operators. Sections 3 and 4 represent the main part of this paper, in the former we compute the crossing equations for the monopole operators in 3d QED, including explicit crossing relations for the cases $N = 2, 4, 6$, and in the latter we present the results of our numerical bootstrap. In section 5 we conclude and discuss further directions. In the appendix we include the crossing relations for the cases $N = 8, 10, 12, 14$.

2 3d QED and monopole operators

The Lagrangian for 3d QED with N complex two-component fermions is

$$\mathcal{L} = -\bar{\psi}_i \gamma^\mu (\partial_\mu - iA_\mu) \psi^i - \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu}, \tag{2.1}$$

where ψ_i are the fermion fields, A_μ is a $U(1)$ gauge field with field strength $F_{\mu\nu}$, and e is the gauge coupling. In the following discussion we restrict to the case where N is even so that

²For fermions, preliminary $4 - \epsilon$ expansion results are discussed in [35].

$ q $	Δ_{M_q}
0	0
1/2	$0.265 N - 0.0383 + O(1/N)$
1	$0.673 N - 0.194 + O(1/N)$
3/2	$1.186 N - 0.422 + O(1/N)$
2	$1.786 N - 0.706 + O(1/N)$
5/2	$2.462 N - 1.04 + O(1/N)$

Table 1. Monopole operator dimension Δ_{M_q} for monopole charge q in U(1) gauge theory with N flavors.

we may preserve parity and time reversal symmetry [19]. At large N one can show that this theory flows to an interacting CFT in the infrared where the Maxwell term in (2.1) is irrelevant [41, 42]. At small N the theory is strongly coupled and difficult to study, although lattice gauge theory studies [8, 9, 43] and other arguments [6, 44] suggest that there is a critical value estimated around $N^{\text{crit}} = 2$ below which the theory no longer flows to an interacting CFT.

As mentioned in the introduction, in this paper we will work under the assumption that the IR dynamics is governed by a non-trivial interacting CFT whose properties are the same as those derived from the large N expansion extrapolated to finite N . At the CFT fixed point, one can define gauge-invariant order operators built from the fields in the Lagrangian, as well as disorder operators (monopole operators) defined through boundary conditions on these fields.

2.1 Lowest dimension monopole operators M_q

A monopole operator M_q with topological charge q at the conformal fixed point of 3d QED with N flavors must transform as a representation of the global symmetry group, which includes the conformal group $\text{SO}(3, 2)$, the flavor symmetry group $\text{SU}(N)$, and the U(1) “topological” symmetry generated by the topological current

$$J_\mu^{\text{top}} = \frac{1}{8\pi} \epsilon_{\mu\nu\rho} F^{\nu\rho}, \tag{2.2}$$

which is conserved due to the Bianchi identity obeyed by F . Under the conformal group, M_q has zero spin and scaling dimensions dependent on q and N . See table 1 for a list of the scaling dimensions Δ_{M_q} for $q \leq 5/2$, as computed for large N [20]. The operator M_q transforms under $\text{SU}(N)$ with Young diagram (1.1). Parity maps monopoles M_q to antimonopoles M_{-q} .

In this bootstrap study we consider the four-point function $\langle M_{1/2} M_{-1/2} M_{1/2} M_{-1/2} \rangle$, so we should also review what is known about the conformal primary operators that appear in the OPEs $M_{\pm 1/2} \times M_{\pm 1/2}$ and $M_{\pm 1/2} \times M_{\mp 1/2}$. These operators can have topological charge $q = \pm 1$ and $q = 0$, respectively. Since under $\text{SU}(N)$, $M_{\pm 1/2}$ transform as (1.1), the

operators in both the $M_{\pm 1/2} \times M_{\pm 1/2}$ and $M_{\pm 1/2} \times M_{\mp 1/2}$ OPEs must transform as

$$\left(1^{N/2}\right) \otimes \left(1^{N/2}\right) = \bigoplus_{n=0}^{N/2} \left(1^{N-2n}, 2^n\right), \quad (2.3)$$

where $(\lambda_1^{\nu_1}, \lambda_2^{\nu_2}, \dots)$ denotes a Young tableau with ν_i rows of length λ_i . There are thus $1 + N/2$ $SU(N)$ irreps in both the $q = \pm 1$ and $q = 0$ sectors. Because of Bose symmetry, only operators with certain spins can appear in each such irrep, as will be discussed in detail in section 3. In this bootstrap study, we will be interested primarily in bounding the scaling dimension of the lowest scalar $q = 1$ monopole operator M_1 , which according to (2.3) transforms under $SU(N)$ as $(2^{N/2})$.

2.2 Lowest dimension scalar $q = 0$ operators in OPE $M_{1/2} \times M_{-1/2}$

In our bootstrap study, it would be useful to make use of more information on the operators in the $M_{\pm 1/2} \times M_{\pm 1/2}$ and $M_{\pm 1/2} \times M_{\mp 1/2}$ OPEs, such as their scaling dimensions.

For simplicity, let us focus on the Lorentz scalars with $q = 0$ appearing in the $M_{\pm 1/2} \times M_{\mp 1/2}$ OPE. For a given index $n > 0$, for which the $SU(N)$ irrep is $(1^{N-2n}, 2^n)$, let us denote the lowest dimension primary by \mathcal{O}_n , the next lowest by \mathcal{O}'_n , and so on. As mentioned above, all these operators can be built from gauge invariant combinations of ψ_i and A_μ because they have zero topological charge.

As will be explained in more detail in [45], the operator \mathcal{O}_n has the form

$$\mathcal{O}_n = \psi_{(\alpha_1}^1 \dots \psi_{\alpha_n)}^n \bar{\psi}_{n+1}^{(\alpha_1} \dots \bar{\psi}_{2n)}^{\alpha_n)}, \quad (2.4)$$

where $\alpha_m = 1, 2$ are Lorentz spinor indices. This operator is parity even (odd) depending on whether n is even (odd). Its scaling dimension is [45]

$$\begin{aligned} \Delta_1 &= 2 - \frac{64}{3\pi^2 N} + O(1/N^2), & \Delta_2 &= 4 - \frac{64}{\pi^2 N} + O(1/N^2), \\ \Delta_3 &= 6 - \frac{128}{\pi^2 N} + O(1/N^2), & \Delta_4 &= 8 - \frac{640}{3\pi^2 N} + O(1/N^2), \end{aligned} \quad (2.5)$$

etc.

Note that in this expansion N is taken to infinity before all other quantities. In particular, the results corresponding to the n channel may break down when N is comparable to n . The next two operators \mathcal{O}'_n and \mathcal{O}''_n have opposite parity from \mathcal{O}_n and can be constructed from $n + 1$ ψ 's and $n + 1$ $\bar{\psi}$'s. Their scaling dimensions can also be calculated in the $1/N$ expansion and take the form $\Delta'_n = 2(n + 1) + O(1/N)$ and $\Delta''_n = 2(n + 1) + O(1/N)$.

The previous results are only for $n > 0$. For $n = 0$, i.e. the $SU(N)$ singlet case, the lowest dimension parity odd operator is $\mathcal{O}_0 \propto \bar{\psi}_i \psi^i$, whose scaling dimension is given by [25]

$$\Delta_0 = 2 + \frac{128}{3\pi^2 N} + O(1/N^2). \quad (2.6)$$

For the lowest dimension parity even $SU(N)$ singlet, we must consider the mixing between $(\bar{\psi}_i \psi^i)(\bar{\psi}_j \psi^j)$ and $F_{\mu\nu}^2$, which gives [45]

$$\Delta'_0 = 4 + \frac{64(2 - \sqrt{7})}{3\pi^2} \frac{1}{N} + O(1/N^2), \quad \Delta''_0 = 4 + \frac{64(2 + \sqrt{7})}{3\pi^2} \frac{1}{N} + O(1/N^2). \quad (2.7)$$

(See also [46].)

2.3 Conserved-current and stress-tensor two-point functions

Another set of quantities in 3d QED that have been computed in large N are the “central charges” c_T , c_J^f , and c_J^t , which are defined as the coefficients of the two-point functions of the conserved stress tensor $T_{\mu\nu}$, $SU(N)$ flavor current $J_\mu^{f i j}$, and $U(1)$ topological current J_μ^t , respectively, where $J_\mu^{f i j}$ and $T_{\mu\nu}$ are canonically normalized and J_μ^t is normalized so that $\int d^2x J_0^t = 2q$.³ The two-point functions take the form:

$$\begin{aligned} \langle J_\mu^t(x) J_\nu^t(0) \rangle &= c_J^t \frac{I_{\mu\nu}(x)}{8\pi^2} \frac{1}{|x|^4}, \\ \langle J_\mu^{f i j}(x) J_\nu^{f k l}(0) \rangle &= c_J^f \frac{I_{\mu\nu}(x)}{8\pi^2} \frac{1}{|x|^4} \left(\delta_l^i \delta_j^k - \frac{1}{N} \delta_j^i \delta_l^k \right), \\ \langle T_{\mu\nu}(x) T_{\rho\sigma}(0) \rangle &= c_T \frac{3}{16\pi^2} \left(\frac{1}{2} (I_{\mu\rho}(x) I_{\nu\sigma}(x) + I_{\mu\sigma}(x) I_{\nu\rho}(x)) - \frac{1}{3} \eta_{\mu\nu} \eta_{\rho\sigma} \right) \frac{1}{|x|^6}, \end{aligned} \tag{2.8}$$

where $I_{\mu\nu}(x) = \eta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}$.⁴

These central charges have been computed to next to leading order in [47] as well as [48, 49]. In our normalization (2.8) we have

$$\begin{aligned} c_J^f &\approx 1 + \frac{0.1429}{N} + O(1/N^2), \\ c_J^t &\approx \frac{6.4846}{N} - \frac{0.9267}{N^2} + O(1/N^3), \\ c_T/N &\approx 1 + \frac{0.7193}{N} + O(1/N^2). \end{aligned} \tag{2.9}$$

3 Crossing equations

We now show how to set up the conformal bootstrap for the four point function of monopole operators in 3d QED. We will focus on the four-point function of two $q = 1/2$ monopole operators and two $q = -1/2$ antimonopole operators, which as mentioned previously transform in the $(1^{N/2})$ representation of $SU(N)$, i.e. they are completely antisymmetric tensors of $SU(N)$ with $N/2$ indices. Let $M_{1/2}^I$ denote the monopole operator, where $I = \{i_1, \dots, i_{N/2}\}$ and $i = 1, \dots, N$ are $SU(N)$ fundamental indices. It is convenient to recast $U(1)$ as $SO(2)$ by writing $M_{1/2}^I = M_{1/2}^{1I} + iM_{1/2}^{2I}$ and $M_{-1/2}^I = M_{1/2}^{1I} - iM_{1/2}^{2I}$ and working with $M_{1/2}^{aI}$, where $a = 1, 2$ is a fundamental $SO(2)$ index. We consider the four-point function:

$$\langle M_{1/2}^{aI}(x_1) M_{1/2}^{bJ}(x_2) M_{1/2}^{cK}(x_3) M_{1/2}^{dL}(x_4) \rangle, \tag{3.1}$$

which includes all orderings of 2 $M_{1/2}$'s and two $M_{-1/2}$'s at once.

The conformal primaries $\mathcal{O}_{(R,n)}^{\Delta,\ell}$ appearing in the $M_{1/2}^{aI} \times M_{1/2}^{bJ}$ OPE can be classified according to their transformation properties under $SO(2) \times SU(N)$, which are labeled by

³We have $J_\mu^t = 2J_\mu^{t\text{op}}$, where $J_\mu^{t\text{op}}$ was defined in (2.2).

⁴These definitions are such that $c_T/N = c_J^t/N = c_J^f = 1$ for a theory of N free complex two-component fermions in 3d. In such a theory, $J_\mu^{f i j}$ would be the generator of the $SU(N)$ rotations under which the fermions would transform as a fundamental, and J_μ^t would be the $U(1)$ current under which all fermions have charge +1.

the index (R, n) . Here, R labels the $SO(2)$ representation, and it can take the values: $R = S$ for $SO(2)$ singlets; $R = A$ for rank-two anti-symmetric tensors⁵ of $SO(2)$; and $R = T$ for rank-two traceless symmetric tensors. (In terms of the topological charge q , we have that $R = S, A$ correspond to $q = 0$ and $R = T$ corresponds to $q = \pm 1$.) For $SU(N)$, we see from (2.3) that we have representations $(1^{N-2n}, 2^n)$ where $n = 0, \dots, N/2$. We will show shortly that for each (R, n) only operators with either even ℓ or odd ℓ can appear in the $M_{1/2}^{aI} \times M_{1/2}^{bJ}$ OPE.

Performing the s -channel OPE in (3.1), we have

$$\langle M_{1/2}^{aI}(x_1) M_{1/2}^{bJ}(x_2) M_{1/2}^{cK}(x_3) M_{1/2}^{dL}(x_4) \rangle = \sum_{R \in \{S, A, T\}} \sum_{n=0}^{N/2} f_R^{abcd} \mathbf{t}_n^{IJKL} s_{R,n} \sum_{\mathcal{O}_{(R,n)}^{\Delta, \ell}} \lambda_{\mathcal{O}_{(R,n)}^{\Delta, \ell}}^2 g_{\Delta, \ell}(u, v) \tag{3.2}$$

where we combined the contribution from each conformal multiplet into a conformal block, and where f_R^{abcd} , \mathbf{t}_n^{IJKL} , $s_{R,n}$, $\lambda_{\mathcal{O}_{(R,n)}^{\Delta, \ell}}^2$, and $g_{\Delta, \ell}(u, v)$ are defined as follows. The f_R^{abcd} are $SO(2)$ 4-point tensor structures corresponding to exchanging operators in representation R of $SO(2)$. They are given by [50]

$$\begin{aligned} f_S^{abcd} &\equiv \delta^{ab} \delta^{cd}, \\ f_A^{abcd} &\equiv \delta^{ad} \delta^{bc} - \delta^{ac} \delta^{bd}, \\ f_T^{abcd} &\equiv \delta^{ad} \delta^{bc} + \delta^{ac} \delta^{bd} - \delta^{ab} \delta^{cd}. \end{aligned} \tag{3.3}$$

The \mathbf{t}_n^{IJKL} are 4-point tensor structures corresponding to exchanging operators in $(1^{N-2n}, 2^n)$ of $SU(N)$. The $s_{R,n}$ are very important signs ($s_{R,n} = +1$ or -1) that are determined by unitarity, as we will discuss in section 3.3. The $\lambda_{\mathcal{O}_{(R,n)}^{\Delta, \ell}}^2$ are the squares of the OPE coefficients that must be positive by unitarity. (We can normalize the OPE coefficient of the identity operator $\lambda_{\text{Id}}=1$.) Lastly, $g_{\Delta, \ell}(u, v)$ are conformal blocks corresponding to the exchange of the operator $\mathcal{O}_{(R,n)}^{\Delta, \ell}$, normalized, for concreteness, as in [13].

Swapping $(1, I, a) \leftrightarrow (3, K, c)$ in the four point function (3.1) yields crossing equations of the form

$$\sum_{\mathcal{O} \in M_{1/2}^{aI} \times M_{1/2}^{bJ}} \lambda_{\mathcal{O}}^2 \vec{d}_{\Delta, \ell}^{\mathcal{O}, R, n}(\Delta_{M_{1/2}}, u, v) = 0, \tag{3.4}$$

where \mathcal{O} runs over all conformal primaries in the $M_{1/2}^{aI} \times M_{1/2}^{bJ}$ OPE. The crossing function $\vec{d}_{\Delta, \ell}^{\mathcal{O}, R, n}$ is a $3(N/2+1)$ component vector. (The number of components is determined according to [50] by the number of $SO(2) \times SU(N)$ representations R, n that occur in the $M_{1/2}^{aI} \times M_{1/2}^{bJ}$ OPE, where representations with both odd and even spins contribute twice.) The components of the crossing function are explicit functions of the conformally-invariant cross-ratios $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$ and $v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$. The form of $\vec{d}_{\Delta, \ell}^{\mathcal{O}, R, n}$ depends only on the dimension of both the external monopole operator $\Delta_{M_{1/2}}$ and on the dimension $\Delta_{\mathcal{O}}$, Lorentz spin ℓ ,

⁵The singlet (S) and rank-two antisymmetric tensor (A) representations of $SO(2)$ are of course isomorphic, but it is convenient to keep track of whether $\mathcal{O}_{(R,n)}^{\Delta, \ell}$ appears in the symmetric (S) or anti-symmetric (A) product of two $SO(2)$ fundamentals. As will be explained, the operators in S and A have spins of opposite parity, with those in S having spins of the same parity as that of the operators in T .

and $SO(2) \times SU(N)$ representation (R, n) of the operator \mathcal{O} . In the rest of this section we provide an efficient algorithm to compute $\vec{d}_{\Delta,\ell}^{R,n}$ for any N , which we demonstrate explicitly for the cases $N = 2, 4, 6$. The cases $N = 8, 10, 12, 14$ are given in appendix A.

The Lorentz scalars $M_{1/2}^{aI}$ transforms in the fundamental of $SO(2)$ and in the representation $(1^{N/2})$ of $SU(N)$. The crossing equations of an operator such as $M_{1/2}^{aI}$ that transforms under a product group can be expressed, roughly, as a tensor product of the crossing equations under each group factor. In this case, we rewrite (3.4) more explicitly as

$$0 = \sum_{\mathcal{O} \in n, \ell^+} \lambda_{\mathcal{O}}^2 \vec{d}_{\Delta,\ell}^{S,n} + \sum_{\mathcal{O} \in n, \ell^-} \lambda_{\mathcal{O}}^2 \vec{d}_{\Delta,\ell}^{A,n} + \sum_{\mathcal{O} \in n, \ell^+} \lambda_{\mathcal{O}}^2 \vec{d}_{\Delta,\ell}^{T,n}, \quad (3.5)$$

where $\vec{d}_{\Delta,\ell}^{R,n}$ are given by the $O(2)$ fundamental crossing functions [50]

$$\vec{d}_{\Delta,\ell}^{S,n} = s_{S,n} \begin{pmatrix} 0 \\ \vec{d}_{\Delta,\ell}^{-,n} \\ \vec{d}_{\Delta,\ell}^{+,n} \end{pmatrix}, \quad \vec{d}_{\Delta,\ell}^{A,n} = s_{A,n} \begin{pmatrix} -\vec{d}_{\Delta,\ell}^{-,n} \\ \vec{d}_{\Delta,\ell}^{-,n} \\ -\vec{d}_{\Delta,\ell}^{+,n} \end{pmatrix}, \quad \vec{d}_{\Delta,\ell}^{T,n} = s_{T,n} \begin{pmatrix} \vec{d}_{\Delta,\ell}^{-,n} \\ 0 \\ -2\vec{d}_{\Delta,\ell}^{+,n} \end{pmatrix}, \quad (3.6)$$

with $\vec{d}_{\Delta,\ell}^{\pm,n}$ being the crossing functions under $SU(N)$ that we will describe next. In (3.5), the notation ℓ^+ (ℓ^-) means that we sum over the same (opposite) set of spins as the component $SU(N)$ crossing functions.

3.1 Known results for $N = 2, 4$

In the cases $N = 2, 4$, the crossing functions $\vec{d}_{\Delta,\ell}^{\pm,n}$ appearing in (3.6) are already known. When $N = 2$, the representation $(1^{N/2}) = (1)$ of the external operator is the fundamental representation of $SU(2)$. The corresponding crossing functions are a reduced version of the general fundamental $SU(N)$ crossing functions written in [50], and they are given by⁶

$$N = 2 : \quad \vec{d}_{\Delta,\ell}^{\mp,0} = \begin{pmatrix} F_{\Delta,\ell}^{\mp} \\ F_{\Delta,\ell}^{\pm} \end{pmatrix}, \quad \vec{d}_{\Delta,\ell}^{\mp,1} = \begin{pmatrix} -F_{\Delta,\ell}^{\mp} \\ 3F_{\Delta,\ell}^{\pm} \end{pmatrix}. \quad (3.7)$$

Here, the operators in the $n = 0$ singlet ($n = 1$ adjoint) representations can have odd (even) spins, and the functions $F_{\Delta,\ell}^{\pm}$ are defined in terms of the conformal blocks $g_{\Delta,\ell}(u, v)$, the conformal cross ratios $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$ and $v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$, and the scaling dimension Δ_{ext} of the external operator:

$$F_{\Delta,\ell}^{\pm}(u, v) = v^{\Delta_{\text{ext}}} g_{\Delta,\ell}(u, v) \pm u^{\Delta_{\text{ext}}} g_{\Delta,\ell}(v, u). \quad (3.8)$$

Recall that the external operator dimension in our case is $\Delta_{\text{ext}} = \Delta_{M_{1/2}}$.

For $N = 4$, the six dimensional (1^2) representation of $SU(4)$ is isomorphic to the six dimensional fundamental representation of $SO(6)$, so the crossing functions are given by

⁶We multiplied $\vec{d}_{\Delta,\ell}^{\mp,1}$ by an overall minus sign in order to agree with the conventions we use in section 3.2.1. For now, we can think of this minus sign as a redefinition of the $s_{R,1}$ coefficients in (3.6). These coefficients will be determined in section 3.3.

the $O(6)$ fundamental crossing functions [50]:

$$N = 4: \quad \bar{d}_{\Delta,\ell}^{\mp,0} = \begin{pmatrix} 0 \\ F_{\Delta,\ell}^{\mp} \\ F_{\Delta,\ell}^{\pm} \end{pmatrix}, \quad \bar{d}_{\Delta,\ell}^{\mp,1} = \begin{pmatrix} -F_{\Delta,\ell}^{\mp} \\ F_{\Delta,\ell}^{\mp} \\ -F_{\Delta,\ell}^{\pm} \end{pmatrix}, \quad \bar{d}_{\Delta,\ell}^{\mp,2} = \begin{pmatrix} F_{\Delta,\ell}^{\mp} \\ \frac{2}{3}F_{\Delta,\ell}^{\mp} \\ -\frac{4}{3}F_{\Delta,\ell}^{\pm} \end{pmatrix}. \quad (3.9)$$

Here, the operators in the singlet $n = 0$, antisymmetric $n = 1$, and traceless symmetric $n = 2$ representations of $O(6)$ can have even, odd, and even spins, respectively.

For $N \geq 6$ there are no results in the literature for the crossing equations, but they can be efficiently derived using the algorithm described below. As a check on our algorithm, we recover the known results given above for $N = 2, 4$.

3.2 General algorithm

We begin by considering the four point function of operators \mathcal{O}_I where $I = \{i_1, \dots, i_{N/2}\}$ and $i = 1, \dots, N$ are $SU(N)$ fundamental indices:

$$x_{12}^{2\Delta_{\text{ext}}} x_{34}^{2\Delta_{\text{ext}}} \langle \mathcal{O}^I(x_1) \mathcal{O}^J(x_2) \mathcal{O}^K(x_3) \mathcal{O}^L(x_4) \rangle = \sum_{n=0}^{N/2} \mathbf{t}_n^{IJKL} \sum_{\mathcal{O}_n} \lambda_{\mathcal{O}_n}^2 g_{\Delta,\ell}(u, v), \quad (3.10)$$

where \mathbf{t}_n^{IJKL} is the four-point tensor structure that corresponds to the exchange of a conformal multiplet whose primary transforms as $(1^{N-2n}, 2^n)$ for $n = 0, \dots, N/2$, and we will suppress the sets of $SU(N)$ indices $IJKL$ for now on. Using explicit expressions for \mathbf{t}_n , it will be straightforward to implement the crossings $(1, I) \leftrightarrow (3, K)$ and $(1, I) \leftrightarrow (2, J)$. The former crossing will give us the crossing functions, while the latter will give us the allowed spins in each representation.

All the indices on the l.h.s. of (3.10) are fundamentals of $SU(N)$, which implies that \mathbf{t}_n can be written as

$$\mathbf{t}_n = \mathbf{b}_{m'} \mathbf{U}_{m'n}, \quad \mathbf{b}_{m'} = \epsilon^{p_1 \dots p_N} \epsilon^{p_{N+1} \dots p_{2N}}, \quad (3.11)$$

where $p \in \{i_1, \dots, i_{N/2}, j_1, \dots, j_{N/2}, k_1, \dots, k_{N/2}, l_1, \dots, l_{N/2}\}$ and $\mathbf{b}_{m'}$ form a basis for all tensor structures of this form.

Our first step is to exchange $(I) \leftrightarrow (K)$ or $(I) \leftrightarrow (J)$ for each $\mathbf{b}_{m'}$ and express the result as a linear combination of $\mathbf{b}_{m'}$'s:

$$(\mathbf{b}_{m'})_{(I) \leftrightarrow (K)} = \mathbf{b}_{n'} \mathbf{X}_{n'm'}, \quad (\mathbf{b}_{m'})_{(I) \leftrightarrow (J)} = \mathbf{b}_{n'} \mathbf{Y}_{n'm'}. \quad (3.12)$$

Our second step is to compute the matrix $\mathbf{U}_{m'n}$ that transforms between the bases \mathbf{t}_n and $\mathbf{b}_{m'}$. For this purpose we will use the $SU(N)$ rank-2 Casimir, which we define in our case as

$$C_2 = \left(\left(T^{(1)} \oplus \dots \oplus T^{(N/2)} \right)^2 \right), \quad (3.13)$$

where $(T^{(q)})_{i_q}^{j_q}$ are fundamental $SU(N)$ generators for each index i_q , so that C_2 acts on $SU(N)$ tensors with $N/2$ fundamental indices i_q . C_2 acts on the (suppressed) first N fundamental $SU(N)$ indices of $\mathbf{b}_{m'}$ as

$$C_2 \mathbf{b}_{m'} = \mathbf{b}_{n'} \mathbf{D}_{n'm'}. \quad (3.14)$$

The eigenvectors $(\mathbf{t}_n)_{m'}$ of $\mathbf{D}_{n'm'}$ are eigenvectors of C_2

$$(C_2 \mathbf{t}_n)_{n'} = \mathbf{D}_{n'm'}(\mathbf{t}_n)_{m'} = (c_2)_n (\mathbf{t}_n)_{n'}. \quad (3.15)$$

The eigenvalues $(c_2)_n$ of an $SU(N)$ tensor in representation $(1^{N-2n}, 2^n)$ for $n = 0, \dots, N/2$ can be calculated by standard group theory formulae and are given by

$$(c_2)_n = n(2N + 1 - n), \quad (3.16)$$

so that indexing \mathbf{t}_n by order of increasing $(c_2)_n$ is consistent with the original definition of \mathbf{t}_n in (3.10). Note that each \mathbf{t}_n as defined above can be multiplied by any real constant and still obeys (3.15). Here, we just make a choice of some \mathbf{t}_n that obey (3.15).

The transformation matrix $\mathbf{U}_{m'n}$ in (3.11) between the bases \mathbf{t}_n and $\mathbf{b}_{m'}$ is then given by

$$\mathbf{U}_{m'n} = ((\mathbf{t}_n)_{m'}) \quad (3.17)$$

where we compute $((\mathbf{t}_n)_{m'})$ in (3.15).

Putting everything together, the crossing function $\vec{d}_{\Delta,\ell}^{\pm,n}$ for the exchange $(1, I) \leftrightarrow (3, K)$ acting on the four point function (3.10) is an $(N/2 + 1) \times (N/2 + 1)$ matrix given by

$$\vec{d}_{\Delta,\ell}^{\pm,n} = \mathbf{b}_{m'} [\mathbf{X}_{m'n'} \mathbf{U}_{n'n} u^{\Delta_{\text{ext}}} g_{\Delta,\ell}(v, u) - \mathbf{U}_{m'n} v^{\Delta_{\text{ext}}} g_{\Delta,\ell}(u, v)], \quad (3.18)$$

which we can rewrite in terms of $F_{\Delta,\ell}^{\pm}(u, v)$ using the definition (3.8). When expressing $\vec{d}_{\Delta,\ell}^{\pm,n}$ as a column vector, it is convenient to do so in a basis different from $\mathbf{b}_{m'}$ that is chosen such that some components involve only $F_{\Delta,\ell}^+(u, v)$ and some only $F_{\Delta,\ell}^-(u, v)$.

The analogous equation for the exchange $(1, I) \leftrightarrow (2, J)$, with $\mathbf{X} \leftrightarrow \mathbf{Y}$, will yield equations of form $F_{\Delta,\ell}^{\pm}(u, v) \lambda_{\mathcal{O}_n}^2 = 0$ for each representation n , which for F^- , F^+ imposes even, odd spins for that representation.

To demonstrate this algorithm, we will now perform it explicitly for the cases $N = 2, 4, 6$. The crossing functions for $N = 8, 10, 12, 14$ are given in appendix A.

3.2.1 $N = 2$

We choose the $\mathbf{b}_{m'}$ basis:

$$\mathbf{b}_0 = \epsilon^{ij} \epsilon^{kl}, \quad \mathbf{b}_1 = \epsilon^{ik} \epsilon^{jl} \quad (3.19)$$

The exchanges $(I) \leftrightarrow (K)$ or $(I) \leftrightarrow (J)$ yield the transformation matrices:

$$\mathbf{X} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \quad \mathbf{Y} = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}. \quad (3.20)$$

Acting with the Casimir C_2 on (3.19) gives the matrix

$$\mathbf{D} = \begin{pmatrix} 0 & -1 \\ 0 & 2 \end{pmatrix} \quad (3.21)$$

whose eigenvectors form the matrix

$$\mathbf{U} = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}. \quad (3.22)$$

Quite nicely, this matrix gives the basis of 4-point structures $\mathbf{t}_0^{ijkl} = \epsilon^{ij}\epsilon^{kl}$, which is anti-symmetric under $i \leftrightarrow j$ and $k \leftrightarrow l$, as appropriate for the exchange of an SU(2) singlet, and $\mathbf{t}_1^{ijkl} = -\epsilon^{ij}\epsilon^{kl} + 2\epsilon^{ik}\epsilon^{jl} = \epsilon^{ik}\epsilon^{jl} + \epsilon^{il}\epsilon^{jk}$, which is symmetric under $i \leftrightarrow j$ and $k \leftrightarrow l$, as appropriate for the exchange of an SU(2) triplet.

Constructing the $(1, I) \leftrightarrow (3, K)$ SU(2) crossing function as in (3.18) yields the expected result given in (3.7) for the coefficients of $\vec{d}_{\Delta, \ell}^{-, n}$ in the basis $\mathbf{b}'_0 = -\mathbf{b}_0 + \frac{1}{2}\mathbf{b}_1$ and $\mathbf{b}'_1 = -\frac{1}{2}\mathbf{b}_1$ — after rewriting (3.18) in terms of \mathbf{b}'_0 and \mathbf{b}'_1 , one can identify the coefficients of \mathbf{b}'_0 with the first row of (3.7) and the coefficients of \mathbf{b}'_1 with the second row of (3.7). It can be checked that the $(1, I) \leftrightarrow (2, J)$ SU(2) crossing equations are consistent with the expected spin parities required by Bose symmetry, namely odd and even for \mathbf{t}_0 (singlet) and \mathbf{t}_1 (adjoint), respectively.⁷

3.2.2 $N = 4$

We choose the $\mathbf{b}_{m'}$ basis:

$$\mathbf{b}_0 = \epsilon^{i_1 i_2 j_1 j_2} \epsilon^{k_1 k_2 l_1 l_2}, \quad \mathbf{b}_1 = \epsilon^{i_1 i_2 k_1 k_2} \epsilon^{j_1 j_2 l_1 l_2}, \quad \mathbf{b}_2 = \epsilon^{i_1 i_2 l_1 l_2} \epsilon^{j_1 j_2 k_1 k_2}. \quad (3.23)$$

The exchanges $(I) \leftrightarrow (K)$ or $(I) \leftrightarrow (J)$ yield the transformation matrices:

$$\mathbf{X} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{Y} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (3.24)$$

Acting with the Casimir C_2 on (3.19) gives the matrix

$$\mathbf{D} = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 5 & 1 \\ 0 & 1 & 5 \end{pmatrix} \quad (3.25)$$

whose eigenvectors form the matrix

$$\mathbf{U} = \begin{pmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix}. \quad (3.26)$$

⁷Bose symmetry requires that only even (odd) spin operators appear in the symmetric (anti-symmetric) product of the representations of the external operators. It is not hard to see that the representations with $N - n$ even (odd) appear in the symmetric product of $(1^{N/2})$ with itself, so they should contain operators with even (odd) spins if no other flavor symmetries are present. If other flavor symmetries are present (such as SO(2) in our case), then the spin parity of the operators for each n is the same as above in the symmetric product of the representations of the other flavor symmetries, and opposite to above in the anti-symmetric product.

Constructing the $(1, I) \leftrightarrow (3, K)$ $SU(4)$ crossing function as in (3.18) yields the expected result for $\vec{d}_{\Delta, \ell}^{\mp, n}$ given in (3.9), when expressing the components of $\vec{d}_{\Delta, \ell}^{\mp, n}$ in the basis $\mathbf{b}'_0 = -\mathbf{b}_1$, $\mathbf{b}'_1 = -(\mathbf{b}_0 + \mathbf{b}_2)/2$, $\mathbf{b}'_2 = -(\mathbf{b}_0 - \mathbf{b}_2)/2$. The $(1, I) \leftrightarrow (2, J)$ $SU(4)$ crossing equations are consistent with the expected spin parities required by Bose symmetry: even, odd, even for \mathbf{t}_0 (singlet), \mathbf{t}_1 (adjoint/antisymmetric), \mathbf{t}_2 (symmetric), respectively — see footnote 7.

3.2.3 $N = 6$

We choose the $\mathbf{b}_{m'}$ basis:

$$\begin{aligned} \mathbf{b}_0 &= \epsilon^{i_1 i_2 i_3 j_1 j_2 j_3} \epsilon^{k_1 k_2 k_3 l_1 l_2 l_3}, & \mathbf{b}_1 &= \epsilon^{i_1 i_2 i_3 k_1 k_2 k_3} \epsilon^{j_1 j_2 j_3 l_1 l_2 l_3}, \\ \mathbf{b}_2 &= \epsilon^{i_1 i_2 i_3 l_1 l_2 l_3} \epsilon^{j_1 j_2 j_3 k_1 k_2 k_3}, & \mathbf{b}_3 &= \epsilon^{i_1 i_2 i_3 j_1 j_2 k_1} \epsilon^{j_3 k_2 k_3 l_1 l_2 l_3}. \end{aligned} \quad (3.27)$$

The exchanges $(I) \leftrightarrow (K)$ or $(I) \leftrightarrow (J)$ yield the transformation matrices:

$$\mathbf{X} = \begin{pmatrix} 0 & 0 & -1 & -2 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{Y} = \begin{pmatrix} -1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (3.28)$$

Acting with the Casimir C_2 on (3.27) gives the matrix

$$\mathbf{D} = \begin{pmatrix} 0 & 1 & 2 & -9 \\ 0 & 11 & 1 & 0 \\ 0 & 1 & 11 & 0 \\ 0 & -1 & -1 & 6 \end{pmatrix} \quad (3.29)$$

whose eigenvalues form the matrix

$$\mathbf{U} = \begin{pmatrix} 1 & -3 & 1 & -3 \\ 0 & 0 & -10 & -6 \\ 0 & 0 & 10 & -6 \\ 0 & 2 & 0 & 2 \end{pmatrix}. \quad (3.30)$$

Constructing the $(1, I) \leftrightarrow (3, K)$ $SU(6)$ crossing function as in (3.18) yields:

$$\begin{aligned} N = 6 : \quad \vec{d}_{\Delta, \ell}^{\mp, 0} &= \begin{pmatrix} 0 \\ F_{\Delta, \ell}^{\mp} \\ F_{\Delta, \ell}^{\pm} \\ 0 \end{pmatrix}, & \vec{d}_{\Delta, \ell}^{\mp, 1} &= \begin{pmatrix} F_{\Delta, \ell}^{\mp} \\ 0 \\ 0 \\ F_{\Delta, \ell}^{\pm} \end{pmatrix}, \\ \vec{d}_{\Delta, \ell}^{\mp, 2} &= \begin{pmatrix} 0 \\ -9F_{\Delta, \ell}^{\mp} \\ 21F_{\Delta, \ell}^{\pm} \\ -10F_{\Delta, \ell}^{\pm} \end{pmatrix}, & \vec{d}_{\Delta, \ell}^{\mp, 3} &= \begin{pmatrix} F_{\Delta, \ell}^{\mp} \\ 6F_{\Delta, \ell}^{\mp} \\ 0 \\ -5F_{\Delta, \ell}^{\pm} \end{pmatrix}. \end{aligned} \quad (3.31)$$

Here, the components of the column vectors are the components of $\vec{d}_{\Delta,\ell}^{\mp,n}$ in the basis $\mathbf{b}'_0 = (7\mathbf{b}_0 + 2\mathbf{b}_1 + \mathbf{b}_2 - 4\mathbf{b}_3)/2$, $\mathbf{b}'_1 = -(\mathbf{b}_0 - \mathbf{b}_2)/2$, $\mathbf{b}'_2 = -(\mathbf{b}_0 + \mathbf{b}_2)/2$, $\mathbf{b}'_3 = -(\mathbf{b}_0 + 2\mathbf{b}_1 + \mathbf{b}_2)/2$. The $(1, I) \leftrightarrow (2, J)$ $SU(6)$ crossing equations are consistent with the expected spin parities required by Bose symmetry: odd, even, odd, even for \mathbf{t}_0 (singlet), \mathbf{t}_1 (adjoint), \mathbf{t}_2 , and \mathbf{t}_3 , respectively — see footnote 7.

3.3 Reflection positivity

Reflection positivity is the Euclidean version of the unitarity constraints on a Lorentzian CFT. These constraints fix the sign of $\lambda_{\mathcal{O}}^2$, by demanding that when we consider the four-point function of scalar operators $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_2^\dagger \mathcal{O}_1^\dagger \rangle$, the coefficients multiplying the conformal blocks in the s -channel OPE should be positive [11]. $SU(N)$ has complex generators, so to enforce this condition in our case, we must define what we mean by the complex conjugate of an operator \mathcal{O}^{aI} transforming under $SO(2) \times SU(N)$. In fact, we will consider \mathcal{O}^{aI} to be real under this notion of complex conjugation.

The subtlety in defining the reality properties of our operators comes from the fact that the $SU(N)$ irrep $(1^{N/2})$ under which these operators transform is real when $N/2$ is even and pseudo-real when $N/2$ is odd. We thus have two different reality conditions depending on whether $N/2$ is even or odd:

$$\begin{aligned} N/2 \text{ Odd} : \quad (\mathcal{O}^{aI})^\dagger &= \frac{1}{(N/2)!} \epsilon_{IJ} \epsilon^{ab} \mathcal{O}^{bJ}, \\ N/2 \text{ Even} : \quad (\mathcal{O}^{aI})^\dagger &= \frac{1}{(N/2)!} \epsilon_{IJ} \delta^{ab} \mathcal{O}^{bJ}, \end{aligned} \tag{3.32}$$

where $\epsilon^{IJ} \equiv \epsilon^{i_1 \dots i_{N/2} j_1 \dots j_{N/2}}$. The overall coefficient as well as the dependence on whether $N/2$ is even or odd in (3.32) can be determined (up to a sign) from the requirement that $(\mathcal{O}^{aI})^{\dagger\dagger} = \mathcal{O}^{aI}$. These reality conditions together with the reflection positivity requirement $\langle \mathcal{O}(x) \mathcal{O}^\dagger(-x) \rangle > 0$ imply that we can normalize our operators \mathcal{O}^{aI} to have the following 2-point functions:

$$\begin{aligned} N/2 \text{ Odd} : \quad \langle \mathcal{O}^{aI}(x_1) \mathcal{O}^{bJ}(x_2) \rangle &= \epsilon^{IJ} \epsilon^{ab} \frac{1}{|x_{12}|^{2\Delta_{\mathcal{O}}}}, \\ N/2 \text{ Even} : \quad \langle \mathcal{O}^{aI}(x_1) \mathcal{O}^{bJ}(x_2) \rangle &= \epsilon^{IJ} \delta^{ab} \frac{1}{|x_{12}|^{2\Delta_{\mathcal{O}}}}. \end{aligned} \tag{3.33}$$

There are several ways of determining the signs $s_{R,n}$ appearing in (3.6). We choose to do so by looking at an example, namely the one where \mathcal{O}^{aI} represent free fields obeying (3.33) with $\Delta_{\mathcal{O}} = 1/2$. In this free theory, the four-point function can be obtained from Wick contractions using (3.33):

$$\begin{aligned} \langle \mathcal{O}^{aI}(x_1) \mathcal{O}^{bJ}(x_2) \mathcal{O}^{cK}(x_3) \mathcal{O}^{dL}(x_4) \rangle_{\text{free}} &= \\ N/2 \text{ Odd} : \quad &\frac{1}{x_{12} x_{34}} \left[\mathbf{b}_0 \epsilon^{ab} \epsilon^{cd} + \sqrt{u} \mathbf{b}_1 \epsilon^{ac} \epsilon^{bd} + \frac{\sqrt{u}}{\sqrt{v}} \mathbf{b}_2 \epsilon^{ad} \epsilon^{bc} \right], \\ N/2 \text{ Even} : \quad &\frac{1}{x_{12} x_{34}} \left[\mathbf{b}_0 \delta^{ab} \delta^{cd} + \sqrt{u} \mathbf{b}_1 \delta^{ac} \delta^{bd} + \frac{\sqrt{u}}{\sqrt{v}} \mathbf{b}_2 \delta^{ad} \delta^{bc} \right], \end{aligned} \tag{3.34}$$

where $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2$ are defined as

$$\begin{aligned}\mathbf{b}_0 &= \epsilon^{i_1 \dots i_{N/2} j_1 \dots j_{N/2}} \epsilon^{k_1 \dots k_{N/2} l_1 \dots l_{N/2}}, \\ \mathbf{b}_1 &= \epsilon^{i_1 \dots i_{N/2} k_1 \dots k_{N/2}} \epsilon^{j_1 \dots j_{N/2} l_1 \dots l_{N/2}}, \\ \mathbf{b}_2 &= \epsilon^{i_1 \dots i_{N/2} l_1 \dots l_{N/2}} \epsilon^{j_1 \dots j_{N/2} k_1 \dots k_{N/2}}.\end{aligned}\tag{3.35}$$

We should express this four-point function in terms of the SO(2) four-point structures (3.3) using

$$\begin{aligned}\epsilon^{ab} \epsilon^{cd} &= -f_A^{abcd}, & \delta^{ab} \delta^{cd} &= f_S^{abcd}, \\ \epsilon^{ac} \epsilon^{bd} &= \frac{f_S^{abcd} - f_A^{abcd} - f_T^{abcd}}{2}, & \delta^{ac} \delta^{bd} &= \frac{f_S^{abcd} - f_A^{abcd} + f_T^{abcd}}{2}, \\ \epsilon^{ad} \epsilon^{bc} &= \frac{f_S^{abcd} + f_A^{abcd} - f_T^{abcd}}{2}, & \delta^{ad} \delta^{bc} &= \frac{f_S^{abcd} + f_A^{abcd} + f_T^{abcd}}{2}.\end{aligned}\tag{3.36}$$

Plugging (3.36) into (3.34), we obtain

$$\begin{aligned}\langle \mathcal{O}^{aI}(x_1) \mathcal{O}^{bJ}(x_2) \mathcal{O}^{cK}(x_3) \mathcal{O}^{dL}(x_4) \rangle_{\text{free}} = \\ N/2 \quad \text{Odd : } & \frac{1}{x_{12} x_{34}} \left[\frac{\frac{\mathbf{b}_2}{\sqrt{v}} + \mathbf{b}_1}{2} \sqrt{u} (f_S - f_T) + \left(\frac{\frac{\mathbf{b}_2}{\sqrt{v}} - \mathbf{b}_1}{2} \sqrt{u} - \mathbf{b}_0 \right) f_A \right], \\ N/2 \quad \text{Even : } & \frac{1}{x_{12} x_{34}} \left[\left(\frac{\frac{\mathbf{b}_2}{\sqrt{v}} + \mathbf{b}_1}{2} \sqrt{u} + \mathbf{b}_0 \right) f_S + \left(\frac{\frac{\mathbf{b}_2}{\sqrt{v}} - \mathbf{b}_1}{2} \right) \sqrt{u} f_A + \left(\frac{\frac{\mathbf{b}_2}{\sqrt{v}} + \mathbf{b}_1}{2} \right) \sqrt{u} f_T \right].\end{aligned}\tag{3.37}$$

Finally, we change from the $\mathbf{b}_{m'}$ to the \mathbf{t}_m basis using the inverse of the transformation matrix \mathbf{U} (3.17), and expand (3.37) in terms of the lowest conformal blocks, using the relations

$$\begin{aligned}1/\sqrt{v} &= 1 + 4\eta r + 8\eta^2 r^2 + 4(4\eta^3 - \eta)r^3 + O(r^4), & \sqrt{u} &= 4r - 8\eta r^2 + 4(4\eta^2 - 1)r^3 + O(r^4), \\ g_{0,0} &= 1 + O(r^2), & g_{1,0} &= r + O(r^3), & g_{2,1} &= r^2 \eta + O(r^4), & g_{3,2} &= \frac{r^3}{2}(3\eta^2 - 1) + O(r^5).\end{aligned}\tag{3.38}$$

where r, η are functions of u, v defined in [51]. We can now read off the signs multiplying the conformal blocks of each tensor structure from this example. These signs must be the same in all theories where the reality conditions (3.32) are satisfied. We now carry out this program explicitly for the cases $N = 2, 4, 6$.

3.3.1 SU(2)

Computing the inverse of \mathbf{U} for SU(2) (3.22) we get

$$\begin{aligned}\mathbf{b}_0 &= \mathbf{t}_0, \\ \mathbf{b}_1 &= \frac{\mathbf{t}_0 + \mathbf{t}_1}{2}, \\ \mathbf{b}_2 &= \mathbf{b}_1 - \mathbf{b}_0 = \frac{-\mathbf{t}_0 + \mathbf{t}_1}{2},\end{aligned}\tag{3.39}$$

where the third equation follows as an identity. So

$$\frac{\mathbf{b}_2}{\sqrt{v}} \pm \mathbf{b}_1 = -\frac{\mathbf{t}_0}{2} \left[\frac{1}{\sqrt{v}} \mp 1 \right] + \frac{\mathbf{t}_1}{2} \left[\frac{1}{\sqrt{v}} \pm 1 \right].\tag{3.40}$$

SU(2)	SO(2)	spin	$s_{R,n}$
0	S	odd	-1
1	S	even	1
0	A	even	-1
1	A	odd	1
0	T	odd	1
1	T	even	-1

Table 2. Properties of conformal blocks and signs $s_{R,n}$ from (3.6) for the case $N = 2$.

Using the relations (3.38), we express the four point function (3.37) for the $N/2$ odd case in terms of conformal blocks:

$$\begin{aligned}
 4\text{-pt} = \frac{1}{x_{12}x_{34}} & \left[(2g_{1,0}\mathbf{t}_1 - 4g_{2,1}\mathbf{t}_0 + \dots) (f_S - f_T) \right. \\
 & \left. + (-\mathbf{t}_0g_{0,0} - 2g_{1,0}\mathbf{t}_0 + 4g_{2,1}\mathbf{t}_1 - 4g_{3,2}\mathbf{t}_0 + \dots) f_A \right]. \tag{3.41}
 \end{aligned}$$

Table 2 follows. As a consistency check, the spin parities in this table match our computation in section 3.2.1.

3.3.2 SU(4)

Computing the inverse of \mathbf{U} for SU(4) (3.26) we get

$$\begin{aligned}
 \mathbf{b}_0 &= \mathbf{t}_0, \\
 \mathbf{b}_1 &= \frac{\mathbf{t}_0}{6} - \frac{\mathbf{t}_1 - \mathbf{t}_2}{2}, \\
 \mathbf{b}_2 &= \frac{\mathbf{t}_0}{6} + \frac{\mathbf{t}_1 + \mathbf{t}_2}{2}. \tag{3.42}
 \end{aligned}$$

So

$$\frac{\mathbf{b}_2}{\sqrt{v}} \pm \mathbf{b}_1 = \frac{\mathbf{t}_1}{2} \left[\frac{1}{\sqrt{v}} \mp 1 \right] + \frac{3\mathbf{t}_2 + \mathbf{t}_0}{6} \left[\frac{1}{\sqrt{v}} \pm 1 \right]. \tag{3.43}$$

Using the relations (3.38), we express the four point function (3.37) for the $N/2$ odd case in terms of conformal blocks:

$$\begin{aligned}
 4\text{-pt} = \frac{1}{x_{12}x_{34}} & \left[\left(\mathbf{t}_0g_{0,0} + 2g_{1,0} \frac{\mathbf{t}_0 + 3\mathbf{t}_2}{3} + 4\mathbf{t}_1g_{2,1} + 4g_{3,2} \frac{\mathbf{t}_0 + 3\mathbf{t}_2}{3} + \dots \right) f_S \right. \\
 & \left. + \left(2\mathbf{t}_1g_{1,0} + 4g_{2,1} \frac{3\mathbf{t}_2 + \mathbf{t}_0}{3} + \dots \right) f_A + \left(2g_{1,0} \frac{\mathbf{t}_0 + 3\mathbf{t}_2}{3} + 4\mathbf{t}_1g_{2,1} + \dots \right) f_T \right]. \tag{3.44}
 \end{aligned}$$

Table 3 follows. As a consistency check, the spin parities in this table match our computation in section 3.2.2.

SU(4)	SO(2)	spin	$s_{R,n}$
0	S	even	1
1	S	odd	1
2	S	even	1
0	A	odd	1
1	A	even	1
2	A	odd	1
0	T	even	1
1	T	odd	1
2	T	even	1

Table 3. Properties of conformal blocks and signs $s_{R,n}$ from (3.6) for the case $N = 4$.

3.3.3 SU(6)

Computing the inverse of \mathbf{U} for SU(6) (3.30) we get

$$\begin{aligned}
 \mathbf{b}_0 &= \mathbf{t}_0, \\
 \mathbf{b}_1 &= -\frac{\mathbf{t}_2 - \mathbf{t}_0}{20} + \frac{\mathbf{t}_1 - \mathbf{t}_3}{12}, \\
 \mathbf{b}_2 &= \frac{\mathbf{t}_2 - \mathbf{t}_0}{20} + \frac{\mathbf{t}_1 - \mathbf{t}_3}{12}.
 \end{aligned} \tag{3.45}$$

So

$$\frac{\mathbf{b}_2}{\sqrt{v}} \pm \mathbf{b}_1 = \frac{\mathbf{t}_2 - \mathbf{t}_0}{20} \left[\frac{1}{\sqrt{v}} \mp 1 \right] + \frac{\mathbf{t}_1 - \mathbf{t}_3}{12} \left[\frac{1}{\sqrt{v}} \pm 1 \right]. \tag{3.46}$$

Using the relations (3.38), we express the four point function (3.37) for the $N/2$ odd case in terms of conformal blocks:

$$\begin{aligned}
 4\text{-pt} &= \frac{1}{x_{12}x_{34}} \left[\left(g_{1,0} \frac{\mathbf{t}_1 - \mathbf{t}_3}{3} + 2g_{2,1} \frac{\mathbf{t}_2 - \mathbf{t}_0}{5} + \dots \right) (f_S - f_T) \right. \\
 &\quad \left. + \left(-\mathbf{t}_0 g_{0,0} + g_{1,0} \frac{\mathbf{t}_2 - \mathbf{t}_0}{5} + 2g_{2,1} \frac{\mathbf{t}_1 - \mathbf{t}_3}{3} + 2g_{3,2} \frac{\mathbf{t}_2 - \mathbf{t}_0}{5} + \dots \right) f_A \right].
 \end{aligned} \tag{3.47}$$

Table 4 follows. As a consistency check, the spin parities in this table match our computation in section 3.2.3.

3.4 Constraints from space-time parity

As described in [19], space-time parity maps a monopole operator M_q to an anti-monopole operator with opposite charge M_{-q} . In terms of SO(2) indices, parity acts by sending $1 \rightarrow 1$ and $2 \rightarrow -2$, thus the S sector is parity even, the A sector is parity odd, and the T sector can transform as both even or odd for different operators.

To find the parity of the uncharged spin 0 operators in each SU(N) sector, we must determine whether they are in the A or S sector. Operators appearing in the $M_{1/2}^{aI} \times M_{1/2}^{bJ}$ OPE

SU(6)	SO(2)	spin	$s_{R,n}$
0	S	odd	-1
1	S	even	1
2	S	odd	1
3	S	even	-1
0	A	even	-1
1	A	odd	1
2	A	even	1
3	A	odd	-1
0	T	odd	1
1	T	even	-1
2	T	odd	-1
3	T	even	1

Table 4. Properties of conformal blocks and signs $s_{R,n}$ from (3.6) for the case $N = 6$.

$n =$	$N/2 - 0, 2, \dots$	$N/2 - 1, 3, \dots$
$N/2$ Even	\mathcal{O}_n	\mathcal{O}'_n
$N/2$ Odd	\mathcal{O}'_n	\mathcal{O}_n

Table 5. Composite fermion operator in representation $(1^{N-2n}, 2^n)$ with required parity for $N/2$ even or odd.

have even/odd spins depending on whether they appear in the symmetric/antisymmetric product of the combined $SO(2) \times SU(N)$ representation. Thus operators in representations S, T and $n = N/2, N/2 - 2, \dots$ or A and $n = N/2 - 1, N/2 - 3, \dots$ all have even spin, while the rest have odd spin. The parity of spin 0 uncharged operators must therefore be even for $n = N/2, N/2 - 2, \dots$, and odd for $n = N/2 - 1, N/2 - 3, \dots$.

As described in section 2, the two lowest dimension spin 0 operators \mathcal{O}_n and \mathcal{O}'_n in $SU(N)$ representations $(1^{N-2n}, 2^n)$ are composed of $2n$ and $2n + 2$ fermions, respectively. The parity of a $2n$ fermion operator is even/odd for n even/odd, so the lowest dimension spin 0 operator $\mathcal{O}_{(n)}$ in $SU(N)$ representations $(1^{N-2n}, 2^n)$ with the required parity depends on whether $N/2$ is even or odd. In table 5 we show which operator \mathcal{O}_n or \mathcal{O}'_n is the lowest dimension operator with the required parity for each $SU(N)$ sectors for $N/2$ even or odd. The scaling dimensions of these operators presented in section 2 will be used to motivate the gaps we impose in the subsequent section 4.2.

4 Numerical bounds

4.1 Strategy

After deriving the precise form of the crossing equations (3.4), in order to find bounds on the scaling dimensions of operators appearing in the $M_{1/2}^{aI} \times M_{1/2}^{bJ}$ OPE, one can consider linear functionals α satisfying the following conditions:

$$1. \quad \alpha(\vec{d}_{\text{Id}}) = 1, \quad (4.1)$$

$$2. \quad \alpha\left(\vec{d}_{\Delta,\ell}^{R,n}\right) \geq 0, \quad \text{for all } \Delta \geq \Delta_{R,n,\ell}^* \quad (4.2)$$

where $\Delta_{R,n,\ell}^*$ are the assumed lower bounds for spin- ℓ conformal primaries (other than the identity) that appear in the $M_{1/2}^{aI} \times M_{1/2}^{bJ}$ OPE and transform in the $\text{SO}(2) \times \text{SU}(N)$ representation (R, n) . The existence of any such α would contradict (3.4), and thereby would allow us to find an upper bound on the lowest-dimension $\Delta_{R,n,\ell}^*$ of the spin- ℓ conformal primary in representation R, n . In particular, if we set $\Delta_{T,N/2,0}^* = \Delta_{M_1}$ and all other $\Delta_{R,n,\ell}^*$ equal to either their unitarity value or some gap value, then we can then find a disallowed region in the $(\Delta_{M_{1/2}}, \Delta_{M_1})$ plane for our chosen gap assumptions.

The above procedure allows us to put gaps for operators that do not have both the same representation and spin as the operator we are bounding. If we would like to put a gap above the operator $\mathcal{O}_{(R',n'),\ell'}$ that we are bounding, then we must add the following condition:

$$3. \quad \alpha\left(\vec{d}_{\Delta',\ell'}^{R',n'}(\Delta_{M_{1/2}})\right) \geq 0, \quad (4.3)$$

as well as make sure in condition (4.2) that $\Delta_{R',n',\ell'}^* > \Delta'_{R',n',\ell'}$.

To find lower bounds on the central charges of conserved currents, we relate these charges to OPE coefficients of conformal primaries appearing in the $M_{1/2}^{aI} \times M_{1/2}^{bJ}$ OPE, for which we can find upper bounds using the bootstrap. On general grounds, the relation must take the form

$$c_J^t \propto \frac{\lambda_{R,0,0,0}^2}{\lambda_{R,0,2,1}^2}, \quad c_J^f \propto \frac{\lambda_{R,0,0,0}^2}{\lambda_{R,1,2,1}^2}, \quad c_T \propto \frac{\Delta_{M_{1/2}}^2 \lambda_{R,0,0,0}^2}{\lambda_{R,0,3,2}^2}, \quad (4.4)$$

where the OPE coefficient $\lambda_{R,n,\Delta,\ell}$ has R either S or A depending on which $\text{SO}(2)$ representation gives the prescribed spin for the given $\text{SU}(N)$ representation, and $n = 0, 1$ are the singlet, adjoint representations of $\text{SU}(N)$. The OPE coefficient $\lambda_{R,0,0,0}^2$ of the identity operator can be chosen to be equal to 1 as a normalization condition for the external operator. The coefficients of proportionality in (4.4) can be found from the free theory presented in section 3.3. A theory of free scalars transforming in representation R of $\text{SU}(N)$ and fundamental representation of $\text{SO}(2)$, with the reality condition (3.32) has⁸

$$c_T^{\text{free}} = c_J^{t,\text{free}} = \dim R, \quad c_J^{f,\text{free}} = \frac{2C_2(R) \dim R}{N^2 - 1}, \quad (4.5)$$

where $\dim R$ is the dimension of R and $C_2(R)$ is the value of the quadratic Casimir of the representation. For us, $R = (1^{N/2})$, which has $C_2(R) = N(N+1)/8$ and

⁸The definitions of the central charges are those of footnote 4 or eq. (2.8).

$\dim R = \binom{N}{N/2}$. Comparing these values with the explicit four-point function decompositions in (3.41), (3.44), and (3.47), we find

$$c_J^t = \frac{8\lambda_{R,0,0,0}^2}{\lambda_{R,0,2,1}^2}, \quad c_J^f = \frac{A_N \lambda_{R,0,0,0}^2}{\lambda_{R,1,2,1}^2}, \quad c_T = \frac{32\Delta_{M_{1/2}}^2 \lambda_{R,0,0,0}^2}{\lambda_{R,0,3,2}^2}, \quad (4.6)$$

with $A_2 = 4$, $A_4 = 8$, and $A_6 = 2$.

Using (4.6), the lower bounds on the central charges can be recast as upper bounds on certain OPE coefficients. Upper bounds on the OPE coefficient of an operator \mathcal{O}^* can be determined by considering linear functionals $\vec{\alpha}$ satisfying the following conditions:

$$1. \quad \alpha(\vec{d}_{\mathcal{O}^*}) = 1, \quad (4.7)$$

$$2. \quad \alpha\left(\vec{d}_{\Delta,\ell}^{R,n}\right) \geq 0, \quad \text{for all } \Delta \geq \Delta_{R,n,\ell}^* \quad (4.8)$$

where $\Delta_{R,n,\ell}^*$ are the assumed lower bounds for spin- ℓ conformal primaries (other than the identity) that appear in the $M_{1/2}^{aI} \times M_{1/2}^{bJ}$ OPE and transform in the $\text{SO}(2) \times \text{SU}(N)$ representation R . If such a functional $\vec{\alpha}$ exists, then this α applied to (3.4) along with the positivity of all $\lambda_{\mathcal{O}}^2$ except, possibly, for that of $\lambda_{\mathcal{O}^*}^2$ implies that

$$\lambda_{\mathcal{O}^*}^2 \leq -\lambda_{\text{Id}}^2 \alpha(\vec{d}_{\text{Id}}) \quad (4.9)$$

provided that the scaling dimensions of each $\mathcal{O} \neq \mathcal{O}^*$ satisfies $\Delta \geq \Delta_{R,n,\ell}^*$. We can choose the spectrum to only satisfy unitarity bounds, or impose gaps on various sectors. To obtain the most stringent upper bound on $\lambda_{\mathcal{O}^*}^2$, and therefore lower bound on its associated central charges, one should then minimize the r.h.s. of (4.9) under the constraints (4.8).

The numerical implementation of the above problems requires two truncations: one in the number of derivatives used to construct α and one in the range of spins ℓ that we consider, whose contributions to the conformal blocks are exponentially suppressed for large ℓ . We denote the maximum derivative order by Λ (as in [52]) and the maximum spin by ℓ_{\max} . The truncated constraint problem can be rephrased as a semidefinite programming problem using the method developed in [11]. This problem can then be solved efficiently using `sdpb` [18]. In this study, we set $\Lambda = 19$ and $\ell_{\max} = 25$. We checked that increasing Λ and ℓ_{\max} did not change the values of $\Delta_{M_{1/2}}$ or Δ_{M_1} by more than .01 for $N = 2, 4$, and .02 for $N = 6$. In terms of computing time, `sdpb` took approximately 4 cpu hour for $N = 2$, 12 cpu hours for $N = 3$, and 18 cpu hours for $N = 6$.

4.2 Numerical bounds for $N = 2, 4, 6$

We now present bounds on scaling dimensions and central charges using the numerical conformal bootstrap. The number of crossing equations, and therefore the numerical complexity, increases as $3(N/2 + 1)$, so we will only focus on the cases $N = 2, 4, 6$. We use the crossing functions and spin parities computed in the previous section. We will also impose gaps on operators in the uncharged $\text{U}(1)$ sector, motivated by the operator scaling dimensions in section 2. The parity constraints discussed in section 3.4 require that for $N = 2, 6$ the lowest dimension operators in $\text{SU}(N)$ representation $(1^{N-n}, 2^{2n})$ are the

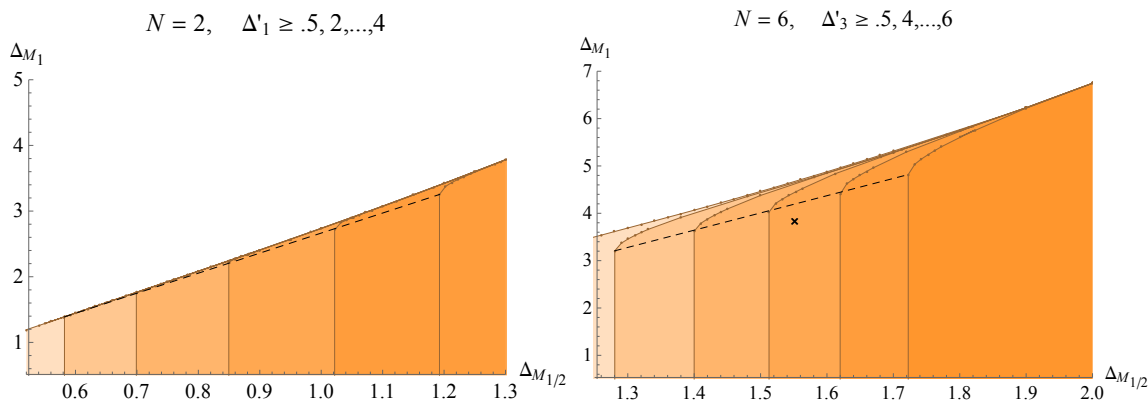


Figure 1. Bounds on basic $q = 1$ monopole operator scaling dimension Δ_{M_1} in terms of basic $q = 1/2$ monopole operator scaling dimension $\Delta_{M_{1/2}}$ in $d = 3$ for $N = 2, 6$ (left, right) with gaps $\Delta'_1 \geq .5, 2, 2.5, 3, 3.5, 4$ for $N = 2$ and $\Delta'_3 \geq .5, 2, 2.5, 3, 3.5, 4$ for $N = 6$ in the uncharged sector in the same $SU(N)$ representation ($2^{N/2}$) as M_1 . These bounds were computed with $\ell_{\max} = 25$ and $\Lambda = 19$. The black cross denotes the large N expansion values of $(\Delta_{M_{1/2}}, \Delta_{M_1})$.

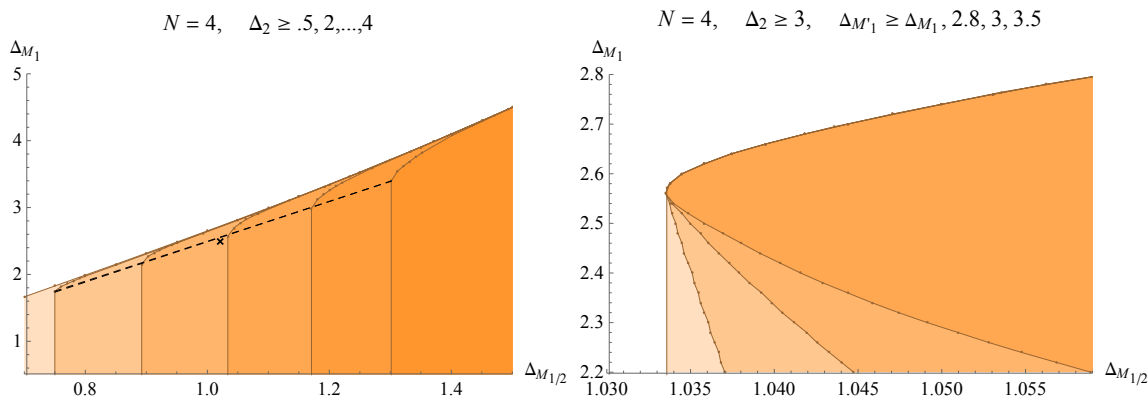


Figure 2. Bounds on basic $q = 1$ monopole operator scaling dimension Δ_{M_1} in terms of basic $q = 1/2$ monopole operator scaling dimension $\Delta_{M_{1/2}}$ in $d = 3$ for $N = 4$ with gaps $\Delta_2 \geq .5, 2, 2.5, 3, 3.5, 4$ in the uncharged sector in the same $SU(N)$ representation ($2^{N/2}$) as M_1 . The righthand plot focuses on the $\Delta_2 \geq 3$ case, and shows that placing an additional gap $\Delta_{M'_1}$ above Δ_{M_1} creates a peninsula around the kinks seen in the lefthand plots. These bounds were computed with $\ell_{\max} = 25$ and $\Lambda = 19$. The black cross denotes the large N expansion values of $(\Delta_{M_{1/2}}, \Delta_{M_1})$.

$(2n + 2)$ -fermion operators of dimension Δ'_n , while for $N = 4$ they are the $2n$ -fermion operators of dimension Δ_n . In the singlet $n = 0$ sector, $N = 2, 6$ has the 2-fermion operator of dimension Δ_0 , while $N = 4$ has the 4-fermion operator of dimension Δ'_0 . As the $1/N$ expansion for these values still seems rather large for $N = 2, 3, 4$, the precise numerical values obtained from the large N expansion will serve more as rough guides than exact inputs.

4.2.1 Bounds on Δ_{M_1}

In figures 1 and 2 we show bounds on the basic $q = 1$ monopole operator scaling dimension Δ_{M_1} in terms of basic $q = 1/2$ monopole operator scaling dimension $\Delta_{M_{1/2}}$. As seen from

figure 1 and the left plot of figure 2, when a sufficiently large gap is imposed in the uncharged U(1) sector in the same SU(N) representation ($2^{N/2}$) as M_1 , then a lower bound on $\Delta_{M_{1/2}}$ with an associated Δ_{M_1} value appears. This feature (kink) seems to depend linearly on this gap — see the dotted lines in figures 1 and 2. Moreover, the slope of this line of kinks has the same value, ≈ 3 , for all of the values of N that we considered. It is a reassuring check on our crossing equations, which differ drastically in form, that all these plots show the same qualitative features.

For the cases $N = 4, 6$ we mark the large N prediction listed in table 1 for $(\Delta_{M_{1/2}}, \Delta_{M_1})$ with a cross in the corresponding plots.⁹ For $N = 4$, the large N extrapolation seems to lie almost exactly on the dotted line connecting the kinks, which implies that a certain value of the gap Δ_2 will give a feature at exactly the predicted value in the $(\Delta_{M_{1/2}}, \Delta_{M_1})$ plane. We note that imposing reasonable gaps¹⁰ in the other uncharged sectors for $N = 4$ does not noticeably change the plots. For $N = 6$, the large N value lies somewhat below the dotted line connecting the kinks. We found that for $N = 6$, unlike $N = 4$, imposing gaps in the other uncharged sectors does change the location of the kinks and brings the line joining the kinks down closer to the large N extrapolation value.

In figure 2, the righthand plot focuses on the gap $\Delta_2 = 3$ case, which from the lefthand plot seems to match the large N values of $(\Delta_{M_{1/2}}, \Delta_{M_1})$ best. The righthand plot puts an additional gap $\Delta_{M'_1}$ above Δ_{M_1} . We find that any value of $\Delta_{M'_1} > \Delta_{M_1}$ creates a peninsular allowed region around the kink seen in the lefthand plot. In previous bootstrap studies [12, 14], it was found that such a peninsula leads to islands once mixed correctors are used — see, for instance, figure 3 in [12]. It would be interesting to see whether a similar phenomenon occurs here.

4.2.2 Bounds on c_T, c_J^f, c_J^t

In figures 3, 4, and 5, we show bounds on the stress tensor central charge c_T , topological U(1) current charge c_J^t , and SU(N) flavor current charge c_J^f , respectively, plotted versus the basic monopole scaling dimensions $\Delta_{M_{1/2}}$. As with the bounds on M_1 , we show these bounds for various gaps in the uncharged sector in the same SU(N) representation ($2^{N/2}$) as M_1 . For $N = 4, 6$ we show the large N values for c_T, c_J^f , and c_J^t (2.9). The numerical bounds for c_J^f and c_T/N are not very restrictive, as they lie below the free theory value of 1. On the other hand, the bound for c_J^t is close to being saturated by the large N expansion values for $N = 4$.

5 Discussion

In this work, we studied constraints coming from crossing symmetry and unitarity in 3d CFTs with SU(N) \times U(1) flavor symmetry that contain operators transforming as rank- $N/2$ anti-symmetric tensors of SU(N) that have unit U(1) charge. An example of such a CFT is 3d QED, in which the most basic monopole operators transform under SU(N) \times U(1) as

⁹For $N = 2$ this value lies below unitarity, which could mean that there is no corresponding CFT.

¹⁰For instance, $\Delta'_0 \approx 3.5$ in the $n = 0$ sector and $\Delta_1 \approx 1.5$ in the $n = 1$ sector, as suggested by the large N expressions (2.5) and (2.7), respectively.

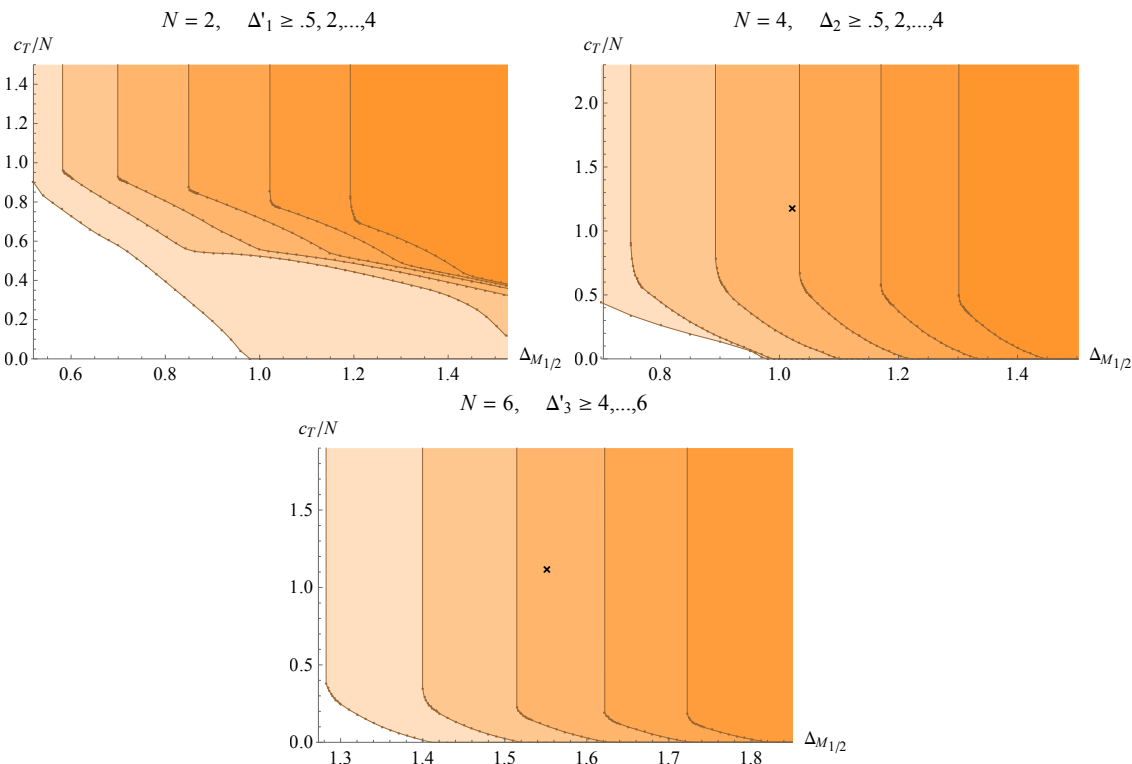


Figure 3. Bounds on stress tensor central charge c_T in terms of basic $q = 1/2$ monopole operator scaling dimension $\Delta_{M_{1/2}}$ in $d = 3$ for $N = 2, 4, 6$ with gaps $\Delta'_2 \geq .5, 2, 2.5, 3, 3.5, 4$ for $N = 2$, $\Delta_4 \geq .5, 2, 2.5, 3, 3.5, 4$ for $N = 4$, and $\Delta'_3 \geq 2, 2.5, 3, 3.5, 4$ for $N = 6$ in the uncharged sector in the same $SU(N)$ representation ($2^{N/2}$) as M_1 . These bounds were computed with $\ell_{\max} = 25$ and $\Lambda = 19$. The black crosses denote the large N expansion values of c_T .

above. Interpreted in the context of 3d QED, we obtained bounds on the scaling dimension of the doubly-charged monopole operators in terms of the scaling dimension of the singly-charged one (figures 1 and 2), and also on the coefficients c_T , c_J^t , and c_J^f appearing in the two-point function of the canonically normalized stress tensor, U(1) flavor current, and $SU(N)$ flavor current (figures 3, 4, and 5).

We hope that our work represents the first steps toward a more systematic study of QED₃ using the conformal bootstrap. We observed that when we impose certain gaps in the operator spectrum, we obtain a kink in our scaling dimension bounds (figure 2) that is at the edge of an allowed region whose shape is similar to that seen in the study of theories with \mathbb{Z}_2 global symmetry. In a further mixed correlator study, such a region turned into an island centered around the 3d Ising CFT, so it would be interesting to see if a mixed correlator study in the present setup would also lead to an island-shaped allowed region. In this study we also assumed that a CFT exists for all N , which is still an unsettled question. Perhaps by looking at mixed correlators one could exclude the existence of such a CFT for low N . We hope to report on such a mixed correlator study in an upcoming work.

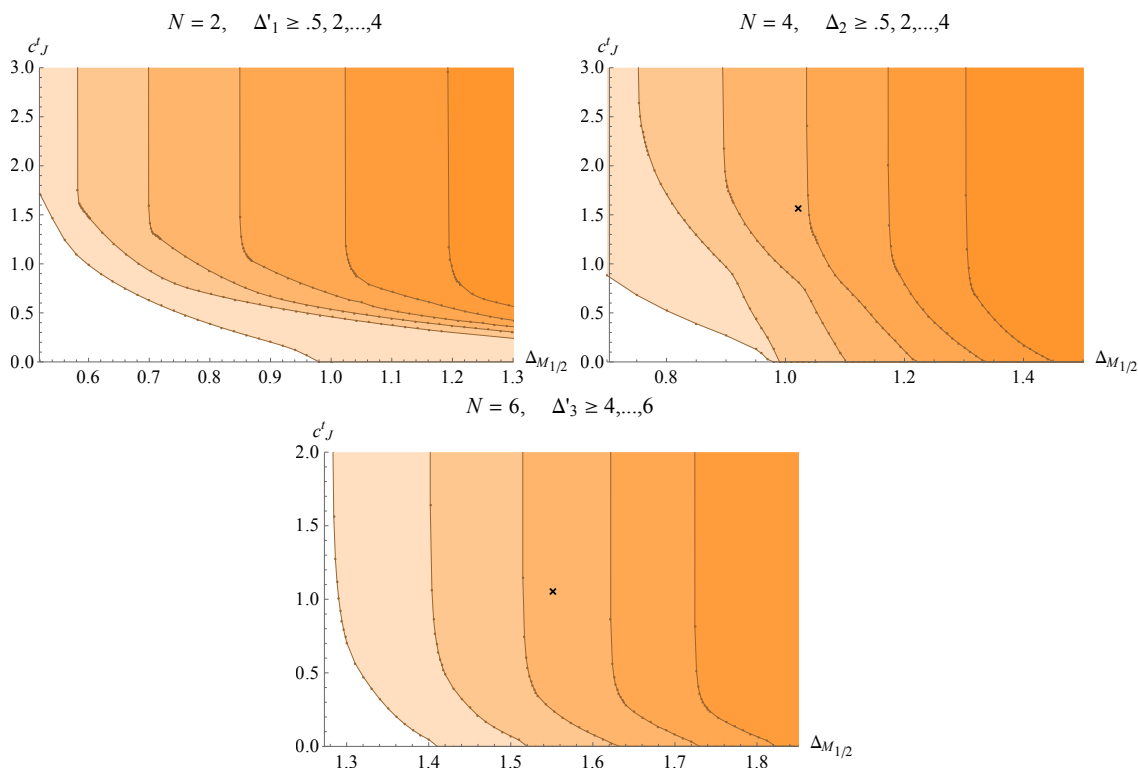


Figure 4. Bounds on topological U(1) current charge c_J^t in terms of basic $q = 1/2$ monopole operator scaling dimension $\Delta_{M_{1/2}}$ in $d = 3$ for $N = 2, 4, 6$ with gaps $\Delta'_2 \geq .5, 2, 2.5, 3, 3.5, 4$ for $N = 2$, $\Delta_4 \geq .5, 2, 2.5, 3, 3.5, 4$ for $N = 4$, and $\Delta'_6 \geq 2, 2.5, 3, 3.5, 4$ for $N = 6$ in the uncharged sector in the same $SU(N)$ representation ($2^{N/2}$) as M_1 . These bounds were computed with $\ell_{\max} = 25$ and $\Lambda = 19$. The black crosses denote the large N expansion values of c_J^t .

Acknowledgments

We thank Luca Iliesiu and David Simmons-Duffin for useful discussions. This work was supported in part by the US NSF under Grant No. PHY-1418069.

A Crossing functions for $N = 8, 10, 12, 14$

The crossing functions given in (3.5) for the four-point function of Lorentz scalar operators in the $SU(N)$ irrep ($1^{N/2}$) and the fundamental $SO(2)$ irrep require the input of the $SU(N)$ crossing functions, which differ with N . Below we list these functions for $N = 8, 10, 12, 14$, along with the signs $s_{R,n}$ defined in (3.6). When $N/2$ is even $s_{S,n} = s_{T,n} = s_{A,n} = s_n$, when $N/2$ is odd $s_{S,n} = -s_{T,n} = s_{A,n} = s_n$. The allowed spins are even in the following cases, and odd otherwise: $SO(2)$ irrep is S, T and $SU(N)$ irrep n is even, $SO(2)$ irrep is A

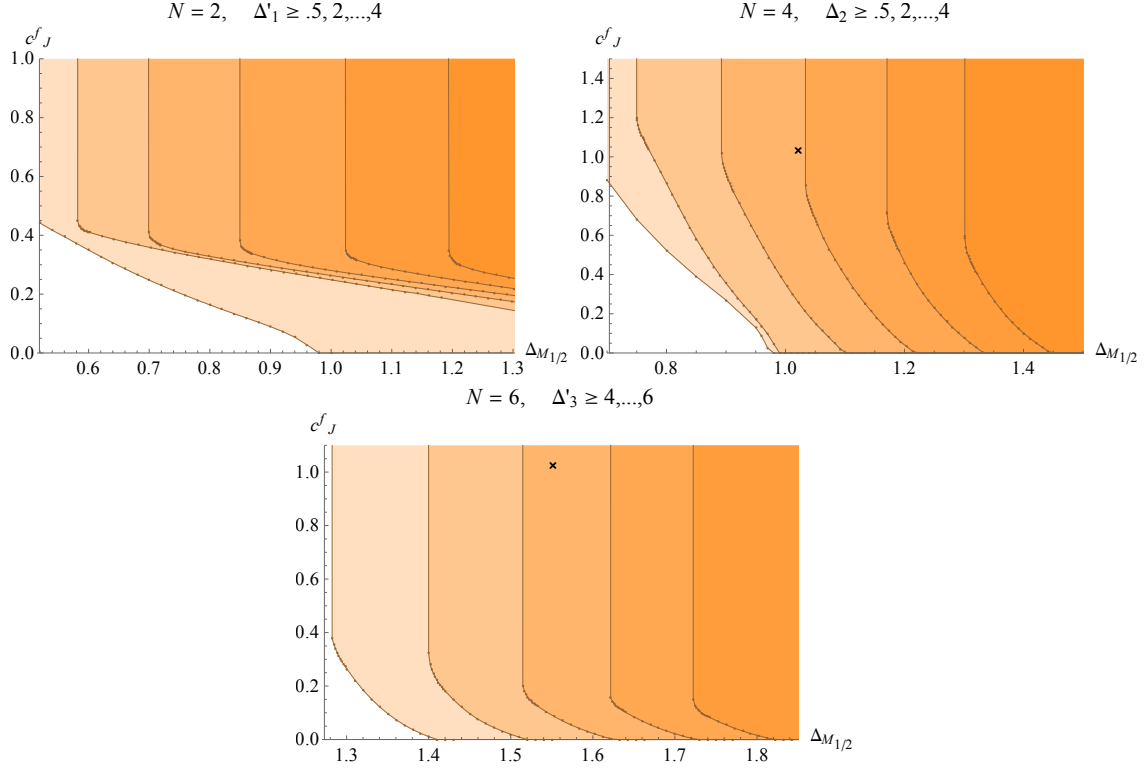


Figure 5. Bounds on $SU(N)$ flavor current charge c_J^f in terms of basic $q = 1/2$ monopole operator scaling dimension $\Delta_{M_{1/2}}$ in $d = 3$ for $N = 2, 4, 6$ with gaps $\Delta'_2 \geq .5, 2, 2.5, 3, 3.5, 4$ for $N = 2$, $\Delta_4 \geq .5, 2, 2.5, 3, 3.5, 4$ for $N = 4$, and $\Delta'_6 \geq 2, 2.5, 3, 3.5, 4$ for $N = 6$ in the uncharged sector in the same $SU(N)$ representation ($2^{N/2}$) as M_1 . These bounds were computed with $\ell_{\max} = 25$ and $\Lambda = 19$. The black crosses denote the large N expansion values of c_J^f .

and n odd.

$$\begin{aligned}
 N = 8 : \quad & \vec{d}_{\Delta,\ell}^{\mp,0} = \begin{pmatrix} 7F_{\Delta,\ell}^{\mp} \\ 0 \\ 0 \\ F_{\Delta,\ell}^{\pm} \\ 0 \end{pmatrix}, \quad \vec{d}_{\Delta,\ell}^{\mp,1} = \begin{pmatrix} 0 \\ F_{\Delta,\ell}^{\mp} \\ 0 \\ 0 \\ F_{\Delta,\ell}^{\pm} \end{pmatrix}, \quad \vec{d}_{\Delta,\ell}^{\mp,2} = \begin{pmatrix} 0 \\ 0 \\ 7F_{\Delta,\ell}^{\mp} \\ -18F_{\Delta,\ell}^{\pm} \\ 21F_{\Delta,\ell}^{\pm} \end{pmatrix}, \\
 & \vec{d}_{\Delta,\ell}^{\mp,3} = \begin{pmatrix} 120F_{\Delta,\ell}^{\mp} \\ -4F_{\Delta,\ell}^{\mp} \\ -5F_{\Delta,\ell}^{\mp} \\ -10F_{\Delta,\ell}^{\pm} \\ F_{\Delta,\ell}^{\pm} \end{pmatrix}, \quad \vec{d}_{\Delta,\ell}^{\mp,4} = \begin{pmatrix} -54F_{\Delta,\ell}^{\mp} \\ -15F_{\Delta,\ell}^{\mp} \\ -10F_{\Delta,\ell}^{\mp} \\ 18F_{\Delta,\ell}^{\pm} \\ 15F_{\Delta,\ell}^{\pm} \end{pmatrix}, \quad s_n = (1 \ -1 \ 1 \ 1 \ -1).
 \end{aligned} \tag{A.1}$$

$$\begin{aligned}
 N = 10 : \quad & \vec{d}_{\Delta,\ell}^{\mp,0} = \begin{pmatrix} 9F_{\Delta,\ell}^{\mp} \\ 0 \\ 0 \\ 9F_{\Delta,\ell}^{\pm} \\ 0 \\ 0 \end{pmatrix}, \quad \vec{d}_{\Delta,\ell}^{\mp,1} = \begin{pmatrix} 0 \\ F_{\Delta,\ell}^{\mp} \\ 0 \\ 0 \\ F_{\Delta,\ell}^{\pm} \\ 0 \end{pmatrix}, \quad \vec{d}_{\Delta,\ell}^{\mp,2} = \begin{pmatrix} 0 \\ 0 \\ 9F_{\Delta,\ell}^{\mp} \\ 0 \\ 0 \\ 9F_{\Delta,\ell}^{\pm} \end{pmatrix}, \\
 & \vec{d}_{\Delta,\ell}^{\mp,3} = \begin{pmatrix} 175F_{\Delta,\ell}^{\mp} \\ -5F_{\Delta,\ell}^{\mp} \\ -7F_{\Delta,\ell}^{\mp} \\ -385F_{\Delta,\ell}^{\pm} \\ 23F_{\Delta,\ell}^{\pm} \\ -35F_{\Delta,\ell}^{\pm} \end{pmatrix}, \quad \vec{d}_{\Delta,\ell}^{\mp,4} = \begin{pmatrix} -625F_{\Delta,\ell}^{\mp} \\ 0 \\ 7F_{\Delta,\ell}^{\mp} \\ -275F_{\Delta,\ell}^{\pm} \\ 70F_{\Delta,\ell}^{\pm} \\ -133F_{\Delta,\ell}^{\pm} \end{pmatrix}, \quad \vec{d}_{\Delta,\ell}^{\mp,5} = \begin{pmatrix} -145F_{\Delta,\ell}^{\mp} \\ -4F_{\Delta,\ell}^{\mp} \\ -5F_{\Delta,\ell}^{\mp} \\ 385F_{\Delta,\ell}^{\pm} \\ -14F_{\Delta,\ell}^{\pm} \\ 35F_{\Delta,\ell}^{\pm} \end{pmatrix}, \\
 & s_n = (-1 \ 1 \ -1 \ -1 \ 1 \ 1) .
 \end{aligned} \tag{A.2}$$

$$\begin{aligned}
 N = 12 : \quad & \vec{d}_{\Delta,\ell}^{\mp,0} = \begin{pmatrix} 11F_{\Delta,\ell}^{\mp} \\ 0 \\ 0 \\ 0 \\ 11F_{\Delta,\ell}^{\pm} \\ 0 \\ 0 \end{pmatrix}, \quad \vec{d}_{\Delta,\ell}^{\mp,1} = \begin{pmatrix} 0 \\ 3F_{\Delta,\ell}^{\mp} \\ 0 \\ 0 \\ 0 \\ F_{\Delta,\ell}^{\pm} \\ 0 \end{pmatrix}, \quad \vec{d}_{\Delta,\ell}^{\mp,2} = \begin{pmatrix} 0 \\ 0 \\ 11F_{\Delta,\ell}^{\mp} \\ 0 \\ 0 \\ 0 \\ 11F_{\Delta,\ell}^{\pm} \end{pmatrix}, \\
 & \vec{d}_{\Delta,\ell}^{\mp,3} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 3F_{\Delta,\ell}^{\mp} \\ -180F_{\Delta,\ell}^{\pm} \\ 20F_{\Delta,\ell}^{\pm} \\ -9F_{\Delta,\ell}^{\pm} \end{pmatrix}, \quad \vec{d}_{\Delta,\ell}^{\mp,4} = \begin{pmatrix} 455F_{\Delta,\ell}^{\mp} \\ -70F_{\Delta,\ell}^{\mp} \\ -2F_{\Delta,\ell}^{\mp} \\ -7F_{\Delta,\ell}^{\mp} \\ -175F_{\Delta,\ell}^{\pm} \\ 0 \\ 5F_{\Delta,\ell}^{\pm} \end{pmatrix}, \quad \vec{d}_{\Delta,\ell}^{\mp,5} = \begin{pmatrix} 0 \\ -165F_{\Delta,\ell}^{\mp} \\ 0 \\ -21F_{\Delta,\ell}^{\mp} \\ 630F_{\Delta,\ell}^{\pm} \\ 15F_{\Delta,\ell}^{\pm} \\ -7F_{\Delta,\ell}^{\pm} \end{pmatrix}, \\
 & \vec{d}_{\Delta,\ell}^{\mp,6} = \begin{pmatrix} 2340F_{\Delta,\ell}^{\mp} \\ -140F_{\Delta,\ell}^{\mp} \\ 7F_{\Delta,\ell}^{\mp} \\ -35F_{\Delta,\ell}^{\mp} \\ -1440F_{\Delta,\ell}^{\pm} \\ 0 \\ -28F_{\Delta,\ell}^{\pm} \end{pmatrix}, \quad s_n = (1 \ -1 \ 1 \ 1 \ 1 \ -1 \ 1) .
 \end{aligned} \tag{A.3}$$

$$\begin{aligned}
 N = 14 : \quad & \vec{d}_{\Delta,\ell}^{\mp,0} = \begin{pmatrix} 26F_{\Delta,\ell}^{\mp} \\ 0 \\ 0 \\ 0 \\ 26F_{\Delta,\ell}^{\pm} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{d}_{\Delta,\ell}^{\mp,1} = \begin{pmatrix} 0 \\ 22F_{\Delta,\ell}^{\mp} \\ 0 \\ 0 \\ 0 \\ 22F_{\Delta,\ell}^{\pm} \\ 0 \\ 0 \end{pmatrix}, \quad \vec{d}_{\Delta,\ell}^{\mp,2} = \begin{pmatrix} 0 \\ 0 \\ 26F_{\Delta,\ell}^{\mp} \\ 0 \\ 0 \\ 0 \\ 26F_{\Delta,\ell}^{\pm} \\ 0 \end{pmatrix}, \\
 & \vec{d}_{\Delta,\ell}^{\mp,3} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 22F_{\Delta,\ell}^{\mp} \\ 0 \\ 0 \\ 0 \\ 22F_{\Delta,\ell}^{\pm} \end{pmatrix}, \quad \vec{d}_{\Delta,\ell}^{\mp,4} = \begin{pmatrix} 11025F_{\Delta,\ell}^{\mp} \\ 3465F_{\Delta,\ell}^{\mp} \\ 55F_{\Delta,\ell}^{\mp} \\ -99F_{\Delta,\ell}^{\mp} \\ -23625F_{\Delta,\ell}^{\pm} \\ -11385F_{\Delta,\ell}^{\pm} \\ -935F_{\Delta,\ell}^{\pm} \\ -297F_{\Delta,\ell}^{\pm} \end{pmatrix}, \quad \vec{d}_{\Delta,\ell}^{\mp,5} = \begin{pmatrix} 1029F_{\Delta,\ell}^{\mp} \\ 49F_{\Delta,\ell}^{\mp} \\ -7F_{\Delta,\ell}^{\mp} \\ 3F_{\Delta,\ell}^{\mp} \\ 1323F_{\Delta,\ell}^{\pm} \\ 1351F_{\Delta,\ell}^{\pm} \\ 147F_{\Delta,\ell}^{\pm} \\ 45F_{\Delta,\ell}^{\pm} \end{pmatrix}, \quad (\text{A.4}) \\
 & \vec{d}_{\Delta,\ell}^{\mp,6} = \begin{pmatrix} 8575F_{\Delta,\ell}^{\mp} \\ -735F_{\Delta,\ell}^{\mp} \\ -15F_{\Delta,\ell}^{\mp} \\ 21\bar{\Delta}_{\ell} \\ -28175F_{\Delta,\ell}^{\pm} \\ -10605F_{\Delta,\ell}^{\pm} \\ -1065F_{\Delta,\ell}^{\pm} \\ -357F_{\Delta,\ell}^{\pm} \end{pmatrix}, \quad \vec{d}_{\Delta,\ell}^{\mp,7} = \begin{pmatrix} -5775F_{\Delta,\ell}^{\mp} \\ 255F_{\Delta,\ell}^{\mp} \\ -35F_{\Delta,\ell}^{\mp} \\ 21\bar{\Delta}_{\ell} \\ -4725F_{\Delta,\ell}^{\pm} \\ -6255F_{\Delta,\ell}^{\pm} \\ -525F_{\Delta,\ell}^{\pm} \\ -189F_{\Delta,\ell}^{\pm} \end{pmatrix}, \\
 s_n = & \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \end{pmatrix}.
 \end{aligned}$$

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