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Pulsating strings on $(\mathrm{AdS}_3 \times \mathrm{S}^3)_\varkappa$

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Abstract: We derive the energy of pulsating strings as a function of adiabatic invariant oscillation number, which oscillates in $S^2_\varkappa$. We find similar solutions for the strings oscillating in deformed AdS_3. Furthermore, we generalize the result of the oscillating strings in anti-de Sitter space in the presence of extra angular momentum in $(AdS_3 \times S^1)_\varkappa$.

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Contents

1 Introduction

The conjectured duality between the supersymmetric Yang-Mills theory in four dimensions and type IIB superstring in the compactified AdS space [1] has been the major research area for recent few years. Though solving exact free string spectrum on a generic given background is highly non-trivial problem, robustness of integrability in the both side of the conjecture played a key role in reducing the problem of solving the spectra in the large charge limit to the the problem of solving a set of algebraic Bethe equations. The fact that the lagrangian field equations of the $AdS_5 \times S^5$ theory can be recast in the zero curvature form [2] introduces the integrability on the anti-de Sitter side of the correspondence which ensures the existence of an infinite number of conserved quantities. The integrability arises as a quantum symmetry of operator mixing in CFT side [3, 4] and as a classical symmetry on the string world-sheet in AdS space [2]. Under the assumption that integrability continues to hold at the quantum level, the spectrum of the $AdS_5 \times S^5$ superstring is determined by means of the thermodynamic Bethe ansatz applied to a doubly Wick rotated version of its world sheet theory [5, 6]. Precisely, the integrability has improved the understanding of the equivalence between the Bethe equation for the spin chain and the corresponding classical realization of Bethe equation for the classical $AdS_5 \times S^5$ string sigma model [7, 8]. The corresponding Bethe equations are based on the knowledge of the S-matrix which describes the scattering of world-sheet excitations of the gauge-fixed string sigma model or the excitations of a certain spin chain in the dual gauge theory [7, 9–13].

To improve our understanding of the relationship between integrability and the amount of global symmetries preserved by the target space-time, one should explore possibilities of various deformations of the string target space time that preserve the integrability of the two-dimensional quantum field theory on the world sheet. Integrable deformations of $AdS_5 \times S^5$ can be achieved by a combination of T-duality and shift transformations [14, 15]. This geometric approach results in a new class of deformations which can be described in

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terms of original string theory and the deformations result into quasi-periodic but keeping the integrability intact. The other way is an algebraic approach based on q-deformations of the world sheet S-matrix [16–23]. Recently one real deformed parametered integrable q-deformed $AdS_5 \times S^5$ super coset model with fermionic degree of freedom was found in [24]. The deformed background breaks the symmetry of $AdS_5 \times S^5$ to $[\mathrm{U}(1)]^6$, which urges to the new insight of its dual field theory which has to be explored yet. In order to understand the various aspects of the background one can look in to [25–34]. The perturbative world sheet scattering matrix of bosonic particles of the model was computed in [25]. The maximal deformation limit of this model is T-dual to a flipped double Wick rotation of the target space and in the imaginary limit it becomes that of a pp-wave background with a curved transverse part [26]. Thermodynamic Bethe Ansatz description of exact finite size spectra concludes that this model maps on to itself under double Wick rotation [27]. The classical integrable structure of anisotropic Landau-Lifshitz sigma models has been derived by taking fast moving string limits in the bosonic sub sector of this model [28]. This background is formally related to $dS_5 \times H^5$ by a double T-duality with hidden supersymmetry [31]. The bosonic spinning strings on this background can be viewed as the solution to a deformed Neumann model [32]. In this deformed supercoset model corresponding type IIB supergravity solutions in the subset of $AdS_2 \times S^2$ and $AdS_3 \times S^3$ have been computed with non-trivial dilaton and RR scalar with a free parameter dependency on the solution [35]. Further by following the Yang-Baxter sigma model approach with classical r-matrices which satisfy the classical Yang-Baxter equation and carry two parameters and three-parameter generalization, type IIB supergravity solutuions have been found in [36]. However, the existence and properties of a gauge theory dual to string theory in the deformed background is still an open question. In this connection giant magnons and its finite size correction [29, 30, 33] have been computed for rotating string in string theory side. Here we wish to study pulsating string solution in the sub sectors of the deformed background as they are more stable than rotating ones [37]. After the inception of the pulsating string in [38], they have been studied both in AdS and non-AdS background [39–56].

The rest of the paper is organized as follows. In section 2, we preview the truncated models of $\varkappa$-deformed $AdS_5 \times S^5$. In section 3, we study the semiclassical oscillating string solution in the deformed $R \times S^2$. In section 4, we analyze the solution in terms of energy as function of oscillation number for a class of pulsating strings in the deformed AdS_3. In section 5, we generalize the previous section with an extra angular momentum in the S^1. In section 6, we conclude with some remarks.

2 Consistent truncations of $(AdS_5 \times S^5)_\varkappa$

As $(AdS_5 \times S^5)_\varkappa$ is a classically integrable background, its consistent truncations must be classically integrable. Truncated lower dimensional integrable string models have been computed in [26]. We write the relevant backgrounds here.

$$ds^2_{AdS_3 \times S^3} = -h(\rho)dt^2 + f(\rho)d\rho^2 + \rho^2 d\phi^2 + \tilde{h}(r)d\varphi^2 + \tilde{f}(r)dr^2 + r^2 d\psi^2, \qquad (2.1)$$

where

$$h(\rho) = \frac{1+\rho^2}{1-\varkappa^2\rho^2}, \qquad f(\rho) = \frac{1}{(1+\rho^2)(1-\varkappa^2\rho^2)}$$
$$\tilde{h}(r) = \frac{1-r^2}{1+\varkappa^2 r^2}, \qquad \tilde{f}(r) = \frac{1}{(1-r^2)(1+\varkappa^2 r^2)}.$$

With $\phi = \psi = 0$, we get the lower dimensional consistent background as

$$ds^2_{AdS_2\times S^2} = -h(\rho)dt^2 + f(\rho)d\rho^2 + \tilde{h}(r)d\varphi^2 + \tilde{f}(r)dr^2. \tag{2.2}$$

Here co-ordinates have their usual range as in case of undeformed one and $\varkappa \in [0,\infty)$.

3 Pulsating string in $S^2_\varkappa$

Here we wish to study the string solution to a class of pulsating string which is oscillating in deformed S^2. In order to get the metric, we substitute the followings in the equation (2.2)

$$\rho = 0, \qquad \varphi = \phi, \qquad r = \cos\psi, \tag{3.1}$$

and get

$$ds^2 = -dt^2 + \frac{d\psi^2}{1+\varkappa^2\cos^2\psi} + \frac{\sin^2\psi d\phi^2}{1+\varkappa^2\cos^2\psi}. \tag{3.2}$$

The Polyakov action of the metric is given by

$$I = \frac{\sqrt{\hat{\lambda}}}{4\pi}\int d\tau\sigma \left[-(\dot{t}^2 - t'^2) + \frac{\dot{\psi}^2 - \psi'^2}{1+\varkappa^2\cos^2\psi} + \frac{\dot{\phi}^2 - \phi'^2}{1+\varkappa^2\cos^2\psi}\sin^2\psi\right], \tag{3.3}$$

where the 'dot' and 'prime' denote the derivatives with respect to τ and σ respectively and $\hat{\lambda} = \lambda(1+\varkappa^2)$, where λ is the 't Hooft coupling constant. We write the following anstaz for the pulsating string

$$t = \kappa\tau, \qquad \psi = \psi(\tau), \qquad \phi = m\sigma. \tag{3.4}$$

Equation of motion for ψ is given by

$$(1+\varkappa^2\cos^2\psi)\left(\frac{2\ddot{\psi}}{\sin 2\psi} + m^2\right) + \varkappa^2\dot{\psi}^2 + \varkappa^2 m^2\sin^2\psi = 0. \tag{3.5}$$

From the Virassoro constraint we get

$$\frac{m^2\sin^2\psi}{1+\varkappa^2\cos^2\psi} - \kappa^2 + \frac{\dot{\psi}^2}{1+\varkappa^2\cos^2\psi} = 0. \tag{3.6}$$

The energy for this string configuration is given by

$$E = \sqrt{\hat{\lambda}}\,\varepsilon = \sqrt{\hat{\lambda}}\,\kappa. \tag{3.7}$$

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The canonical momentum associated with ψ is

$$\Pi_\psi = \frac{\dot\psi}{1+\varkappa^2\cos^2\psi} \quad . \tag{3.8}$$

From equation (3.6) we get

$$\dot\psi^2 = \varepsilon^2(1+\varkappa^2\cos^2\psi) - m^2\sin^2\psi\ . \tag{3.9}$$

We can compute the oscillation number which should take integer values in quantum theory as

$$\begin{aligned} N = \sqrt{\hat\lambda}\,\mathcal{N} &= \frac{\sqrt{\hat\lambda}}{2\pi}\oint d\psi\ \Pi_\psi \\ &= \frac{\sqrt{\hat\lambda}}{2\pi}\oint d\psi\sqrt{\frac{\varepsilon^2}{1+\varkappa^2\cos^2\psi} - \frac{m^2\sin^2\psi}{(1+\varkappa^2\cos^2\psi)^2}}\ . \end{aligned} \tag{3.10}$$

Substituting $\sin^2\psi = z$ in the above equation (3.10) we get,

$$\mathcal{N} = \frac{1}{2\pi}\int_0^b \frac{dz}{1+\varkappa^2-\varkappa^2 z}\ \sqrt{\frac{\varepsilon^2(1+\varkappa^2-\varkappa^2 z)-m^2 z}{(1-z)z}}\ . \tag{3.11}$$

To find out this integration, we have taken derivative of N with respect to m, i.e

$$\frac{\partial\mathcal{N}}{\partial m} = -\frac{m}{\pi}\int_0^b \frac{z\ dz}{(1+\varkappa^2-\varkappa^2 z)\sqrt{z\varepsilon^2(1-z)(1+\varkappa^2-\varkappa^2 z)-m^2z^2(1-z)}}, \tag{3.12}$$

where $a > b > c$ are roots of the polynomial

$$f(z) = z\varepsilon^2(1-z)(1+\varkappa^2-\varkappa^2 z) - m^2z^2(1-z). \tag{3.13}$$

And $a = 1$, $b = \frac{\varepsilon^2(1+\varkappa^2)}{m^2+\varepsilon^2\varkappa^2}$, $c = 0$, $d = \frac{1+\varkappa^2}{\varkappa^2}$.

The above integral in (3.12) can be written as sum of two integrals i.e.

$$\frac{\partial\mathcal{N}}{\partial m} = I_1 + I_2\,,$$

Where

$$\begin{aligned} I_1 &= \frac{m}{\pi\varkappa^2}\int_0^b \frac{dz}{\sqrt{z\varepsilon^2(1-z)(1+\varkappa^2-\varkappa^2 z)-m^2z^2(1-z)}} \\ &= \frac{m}{\pi\varkappa^2\sqrt{m^2+\varepsilon^2\varkappa^2}}\int_0^b \frac{dz}{\sqrt{(z-a)(z-b)(z-c)}} \\ &= \frac{m}{\varkappa^2\pi\sqrt{m^2+\varepsilon^2\varkappa^2}}\ \mathbb{K}\,[b]\,, \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} I_2 &= \frac{m}{\pi\varkappa^2}\int_0^b \frac{-d\ dz}{(d-z)\sqrt{z\varepsilon^2(1-z)(1+\varkappa^2-\varkappa^2 z)-m^2z^2(1-z)}} \\ &= \frac{-m}{\pi\varkappa^2\sqrt{m^2+\varepsilon^2\varkappa^2}}\int_0^b \frac{d\ dz}{(d-z)\sqrt{(z-a)(z-b)(z-c)}} \\ &= \frac{-m}{\varkappa^2\pi\sqrt{m^2+\varepsilon^2\varkappa^2}}\ \Pi\left[\frac{b}{r}, b\right]. \end{aligned} \tag{3.15}$$

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Now

$$\frac{\partial \mathcal{N}}{\partial m} = \frac{m}{\varkappa^2 \pi \sqrt{m^2 + \varepsilon^2 \varkappa^2}} \left(\mathbb{K}\left[b\right] - \Pi\left[\frac{b}{r}, b\right] \right), \tag{3.16}$$

where $\mathbb{K}$, Π are complete elliptical integral of first and third kind respectively and Expanding the equation (3.16) for small value ε in the short string limit

$$\frac{\partial \mathcal{N}}{\partial m} = \frac{-\varepsilon^2}{2m^2} + \frac{3(-1+\varkappa^2)}{16m^4}\varepsilon^4 - \frac{5(3 - 2\varkappa 2 + 3\varkappa 4)}{128m^6}\varepsilon^6 \mathcal{O}[\varepsilon^8]. \tag{3.17}$$

Taking integration with respect to m we get

$$\mathcal{N} = \frac{\varepsilon^2}{2m} + \frac{1-\varkappa^2}{16m^3}\varepsilon^4 + \frac{3 - 2\varkappa^2 + 3\varkappa^4}{128m^5}\varepsilon^6 + \mathcal{O}[\varepsilon^8]. \tag{3.18}$$

Reversing the series we get

$$\varepsilon = \sqrt{2m\mathcal{N}} \left(1 - \frac{1-\varkappa^2}{8m}\mathcal{N} - \frac{5 + 6\varkappa^2 + 5\varkappa^4}{128m^2}\mathcal{N}^2 + \mathcal{O}[\mathcal{N}^3] \right). \tag{3.19}$$

In the above dispersion relation $\varepsilon < m$ gives an upper bound for $\mathcal{N}$, so one cannot take the large $\mathcal{N}$ limit. This gives the short string or small oscillation number expansion of the classical energy. If we put $\varkappa \to 0$ in the above equation (3.19), we get the exact expression for undeformed S^2 as found in [53].

4 Pulsating string in deformed AdS_3

In this section we study the semiclassical quantization of a class of strings which is oscillating in the radial ρ direction of AdS_3. We get the relevant metric for this from equation (2.1) (taking only AdS part) with the substitution of $\rho = \sinh\rho$

$$ds^2 = -\frac{\cosh^2\rho}{1 - \varkappa^2 \sinh^2\rho}dt^2 + \frac{d\rho^2}{1 - \varkappa^2 \sinh^2\rho} + \sinh^2\rho d\phi^2. \tag{4.1}$$

We chose the ansatz for this configuration as

$$t = t(\tau), \qquad \rho = \rho(\tau), \qquad \phi = m\sigma. \tag{4.2}$$

The polyakov action of the given metric is given by

$$I = \frac{\sqrt{\hat{\lambda}}}{4\pi} \int d\tau d\sigma \left[-\frac{\cosh^2\rho}{1 - \varkappa^2 \sinh^2\rho}\dot{t}^2 + \frac{\dot{\rho}^2}{1 - \varkappa^2 \sinh^2\rho} + m^2 \sinh^2\rho \right]. \tag{4.3}$$

Equation of motion for t and ρ are given by

$$\ddot{t}\cosh^2\rho + \dot{\rho}\dot{t}\sinh 2\rho \left[1 + \frac{\varkappa^2 \cosh^2\rho}{1 - \varkappa^2 \sinh^2\rho} \right] = 0 \tag{4.4}$$

$$2\ddot{\rho}(1 - \varkappa^2 \sinh^2\rho) + \sinh\rho[m^2(1 - \varkappa^2 \sinh^2\rho)^2 + \varkappa^2 \dot{\rho}^2 + \dot{t}^2(1 + \varkappa^2)] = 0. \tag{4.5}$$

The Virasoro constraint gives us

$$m^2 \sinh^2 \rho - \frac{\cosh^2 \rho}{1 - \varkappa^2 \sinh^2 \rho} \dot{t}^2 + \frac{1}{1 - \varkappa^2 \sinh^2 \rho} \dot{\rho}^2 = 0. \tag{4.6}$$

The energy of the oscillating string is given by

$$E = \sqrt{\hat{\lambda}} \varepsilon = \frac{\cosh^2 \rho}{1 - \varkappa^2 \sinh^2 \rho} \dot{t}. \tag{4.7}$$

The canonical momentum associated with ρ is

$$\Pi_\rho = \frac{\dot{\rho}}{1 - \varkappa^2 \sinh^2 \rho}. \tag{4.8}$$

Using the equations (4.6) and (4.7), we can get

$$\dot{\rho}^2 - \frac{\varepsilon^2 (1 - \varkappa^2 \sinh^2 \rho)^2}{\cosh^2 \rho} + m^2 \sinh^2 \rho (1 - \varkappa^2 \sinh^2 \rho) = 0. \tag{4.9}$$

With the help of equation (4.8), we can write

$$\Pi_\rho^2 + V(\rho) = 0, \qquad V(\rho) = -\frac{\varepsilon^2}{\cosh^2 \rho} + \frac{m^2 \sinh^2 \rho}{1 - \varkappa^2 \sinh^2 \rho}. \tag{4.10}$$

This may be interpreted as an equation for a particle moving in a potential which is growing to infinity at $\rho \to \infty$. The coordinate $\rho(\tau)$ thus oscillates between 0 and a maximal ρ value ($\rho_{\max}$). Since the string is oscillating along ρ direction, we can define the oscillation number as

$$\begin{aligned} N = \sqrt{\hat{\lambda}}\, \mathcal{N} &= \frac{\sqrt{\hat{\lambda}}}{2\pi} \oint d\rho\, \Pi_\rho \\ &= \frac{\sqrt{\hat{\lambda}}}{\pi} \int_0^{\rho_{\max}} \sqrt{\frac{\varepsilon^2}{\cosh^2 \rho} - \frac{m^2 \sinh^2 \rho}{1 - \varkappa^2 \sinh^2 \rho}}\,. \end{aligned} \tag{4.11}$$

Taking $\sinh^2 \rho = z$

$$\mathcal{N} = \frac{1}{2\pi} \int_0^{R_2} \frac{dz}{1+z} \sqrt{\frac{\varepsilon^2 (1 - \varkappa^2 z) - m^2 z (1+z)}{z(1 - \varkappa^2 z)}}. \tag{4.12}$$

To make the integration simple we make the derivative with respect to m

$$\frac{\partial \mathcal{N}}{\partial m} = -\frac{m}{2\pi} \int_0^{R_2} dz \frac{\sqrt{z}}{\sqrt{\varepsilon^2 (1 - \varkappa^2 z)^2 - m^2 z (1+z)(1 - \varkappa^2 z)}}, \tag{4.13}$$

where $R_1 > R_2 > R_3$ are roots of the polynomial

$$f(z) = \varepsilon^2 (1 - \varkappa^2 z)^2 - m^2 z (1+z)(1 - \varkappa^2 z). \tag{4.14}$$

And

$$R_1 = \frac{1}{\varkappa^2}, \qquad R_2 = \frac{-m^2 - \varepsilon^2\varkappa^2 + \sqrt{4m^2\varepsilon^2 + (m^2 + \varepsilon^2\varkappa^2)^2}}{2m^2},$$
$$R_3 = \frac{-m^2 - \varepsilon^2\varkappa^2 - \sqrt{4m^2\varepsilon^2 + (m^2 + \varepsilon^2\varkappa^2)^2}}{2m^2}.$$

The above integral can be written in the standard elliptical integrals as

$$\frac{\partial \mathcal{N}}{\partial m} = \frac{R_3}{\pi\sqrt{R_1(R_2 - R_3)}\varkappa}\left[\Pi\left(\frac{R_2}{R_2 - R_3}, \frac{R_2(R_1 - R_3)}{R_1(R_2 - R_3)}\right) - \mathbb{K}\left(\frac{R_2(R_1 - R_3)}{R_1(R_2 - R_3)}\right)\right]. \tag{4.15}$$

Now expanding the above equation for a small oscillation number with small ε we will get

$$\frac{\partial \mathcal{N}}{\partial m} = \frac{-\varepsilon^2}{4m^2} + \frac{3(5 + 3\varkappa^2)}{32m^4}\varepsilon^4 - \frac{5(63 + 70\varkappa^2 + 15\varkappa^4)}{256m^5}\varepsilon^6 + \mathcal{O}\left[\varepsilon^8\right]. \tag{4.16}$$

Integrating with respect to m we get

$$\mathcal{N} = \frac{\varepsilon^2}{4m} - \frac{(5 + 3\varkappa^2)}{32m^3}\varepsilon^4 + \frac{(63 + 70\varkappa^2 + 15\varkappa^4)}{256m^5}\varepsilon^6 + \mathcal{O}\left[\varepsilon^8\right]. \tag{4.17}$$

Reversing the series

$$\varepsilon = 2\sqrt{m\mathcal{N}} + \frac{5 + 3\varkappa^2}{2\sqrt{m}}\mathcal{N}^{3/2} + \frac{(-77 - 70 + 3\varkappa^4)}{16m^{3/2}}\mathcal{N}^{5/2} + \mathcal{O}\left[\mathcal{N}^{7/2}\right]. \tag{4.18}$$

This is the classical energy expression in the small energy limit for short string configuration in deformed AdS_3. After substituting $\varkappa = 0$, we can get the the flat-space dependence which is expected in the small-energy limit where the string oscillates near the center of AdS_3 which can be found in [47]. The result found in [53] differs by a factor 2 as they have defined the oscillation number accordingly.

5 Pulsating string in $(AdS_3 \times S^1)_\varkappa$

In this section we generalize the previous section where we study a class of oscillating string solution which is oscillating in the radial ρ direction of AdS_3 with an extra angular momentum along S^1. In order to get the consistent truncated metric, we substitute the following in the equation (2.1)

$$\rho = \sinh\rho, \qquad r = \psi = 0. \tag{5.1}$$

Now the relevant background is given by

$$ds^2 = -\frac{\cosh^2\rho}{1 - \varkappa^2\sinh^2\rho}dt^2 + \frac{d\rho^2}{1 - \varkappa^2\sinh^2\rho} + \sinh^2\rho d\phi^2 + d\varphi^2. \tag{5.2}$$

Choosing the ansatz as

$$t = t(\tau), \qquad \rho = \rho(\tau), \qquad \phi = m\sigma, \qquad \varphi = \varphi(\tau), \tag{5.3}$$

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we write down the Polyakov action of the above metric

$$I = \frac{\sqrt{\hat{\lambda}}}{4\pi} \int d\tau d\sigma \left[-\frac{\cosh^2\rho}{1-\varkappa^2\sinh^2\rho}\dot{t}^2 + \frac{\dot{\rho}^2}{1-\varkappa^2\sinh^2\rho} + m^2\sinh^2\rho + \dot{\varphi}^2 \right]. \tag{5.4}$$

Equation of motion for t is

$$\ddot{t}\cosh^2\rho + 2\dot{\rho}\dot{t}\sinh\rho\cosh\rho\left[1 + \frac{\varkappa^2\cosh^2\rho}{1-\varkappa^2\sinh^2\rho}\right] = 0. \tag{5.5}$$

Equation of motion for ρ is

$$-2\ddot{\rho}(1-\varkappa^2\sinh^2\rho) + \sinh\rho[m^2(1-\varkappa^2\sinh^2\rho)^2 + \varkappa^2\dot{\rho}^2 + \dot{t}^2(1+\varkappa^2)] = 0. \tag{5.6}$$

The Virassoro constraint is given by

$$m^2\sinh^2\rho - \frac{\cosh^2\rho}{1-\varkappa^2\sinh^2\rho}\dot{t}^2 + \frac{1}{1-\varkappa^2\sinh^2\rho}\dot{\rho}^2 + \dot{\varphi}^2 = 0. \tag{5.7}$$

Conserved quantities are

$$\begin{aligned} E &= \sqrt{\hat{\lambda}}\varepsilon = \sqrt{\hat{\lambda}}\frac{\cosh^2\rho}{1-\varkappa^2\sinh^2\rho}\dot{t} \\ J &= \sqrt{\hat{\lambda}}j = \sqrt{\hat{\lambda}}\dot{\varphi}. \end{aligned} \tag{5.8}$$

The canonical momentum associated with ρ is

$$\Pi_\rho = \frac{\dot{\rho}}{1-\varkappa^2\sinh^2\rho}. \tag{5.9}$$

From the equation (5.7), with the help of the equation (5.8) we get

$$\dot{\rho}^2 = \frac{\varepsilon^2(1-\varkappa^2\sinh^2\rho)^2}{\cosh^2\rho} - (m^2\sinh^2\rho + j^2)(1-\varkappa^2\sinh^2\rho). \tag{5.10}$$

With the help of the equation (5.9), the above equation can be written as

$$\Pi_\rho^2 + V(\rho) = 0, \qquad V(\rho) = -\frac{\varepsilon^2}{\cosh^2\rho} + \frac{m^2\sinh^2\rho}{1-\varkappa^2\sinh^2\rho} + \frac{j^2}{1-\varkappa^2\sinh^2\rho}. \tag{5.11}$$

This is similar to the previous section (eq. (4.10)) with an extra additive term. This is similar to an equation for a particle moving in such a potential so that the coordinate $\rho(\tau)$ oscillates between 0 and a maximal ρ value ($\rho_{\max}$). Now we can write the oscillation number as

$$\begin{aligned} N = \sqrt{\hat{\lambda}}\,\mathcal{N} &= \frac{\sqrt{\hat{\lambda}}}{2\pi}\oint d\rho\,\Pi_\rho \\ &= \frac{\sqrt{\hat{\lambda}}}{2\pi}\oint d\rho\sqrt{\frac{\varepsilon^2}{\cosh^2\rho} - \frac{m^2\sinh^2\rho}{1-\varkappa^2\sinh^2\rho} - \frac{j^2}{1-\varkappa^2\sinh^2\rho}}. \end{aligned} \tag{5.12}$$

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Taking $\sinh^2\rho = z$, then differentiating with respect to m

$$\frac{\partial\mathcal{N}}{\partial m} = -\frac{m}{2\pi}\int_0^{R_2} dz\frac{\sqrt{z}}{\sqrt{\varepsilon^2(1-\varkappa^2 z)^2-(m^2 z+j^2)(1-\varkappa^2 z)(1+z)}}, \tag{5.13}$$

where $R_1 > R_2 > R_3$ are roots of the polynomial

$$f(z) = \varepsilon^2(1-\varkappa^2 z)^2-(m^2 z+j^2)(1-\varkappa^2 z)(1+z). \tag{5.14}$$

And

$$R_1 = \frac{1}{\varkappa^2}, \qquad R_2 = \frac{-j^2-m^2-\varepsilon^2\varkappa^2+\sqrt{4m^2(\varepsilon^2-j^2)+(j^2+m^2+\varepsilon^2\varkappa^2)^2}}{2m^2},$$
$$R_3 = \frac{-j^2-m^2-\varepsilon^2\varkappa^2-\sqrt{4m^2(\varepsilon^2-j^2)+(j^2+m^2+\varepsilon^2\varkappa^2)^2}}{2m^2}.$$

The above integral in (5.13) can be written as

$$\frac{\partial\mathcal{N}}{\partial m} = \frac{R_3}{\pi\sqrt{R_1(R_2-R_3)}\varkappa}\left[\Pi\left(\frac{R_2}{R_2-R_3},\frac{R_2(R_1-R_3)}{R_1(R_2-R_3)}\right)-\mathbb{K}\left(\frac{R_2(R_1-R_3)}{R_1(R_2-R_3)}\right)\right], \tag{5.15}$$

$\mathbb{K}$ and Π are complete elliptical integral of first and third kind respectively. Expanding the above equation with small ε and small j

$$\begin{aligned}\frac{\partial\mathcal{N}}{\partial m} =& \left[\frac{j^2}{4m^2}+\frac{3(1-\varkappa^2)}{32m^2}j^4+\mathcal{O}[j^6]\right]\\
&+\left[-\frac{1}{4m^2}-\frac{3(3+\varkappa^2)}{16m^4}j^2-\frac{15(15+6\varkappa^2-\varkappa^4)}{256m^6}j^4+\mathcal{O}[j^6]\right]\varepsilon^2\\
&+\left[\frac{3(5+3\varkappa^2)}{32m^4}+\frac{15(35+30\varkappa^2+3\varkappa^4)}{256m^6}j^2+\frac{105(105+105\varkappa^2+15\varkappa^4-\varkappa^6)}{2048m^8}j^4+\mathcal{O}[j^6]\right]\varepsilon^4\\
&+\mathcal{O}[\varepsilon^6].\end{aligned} \tag{5.16}$$

Integrating with respect to m and reversing the series we get

$$\varepsilon = 2\sqrt{m\mathcal{M}}\;K_1(j)\left[1+K_2(j)\frac{5\mathcal{M}}{4m}+\mathcal{O}[\mathcal{M}^2]\right], \tag{5.17}$$

where

$$\begin{aligned}\mathcal{M} &= \mathcal{N}+\frac{j^2}{4m}+\frac{j^4(1-\varkappa^2)}{32m^3}+\mathcal{O}[j^6],\\
K_1(j) &= 1+\frac{3+\varkappa^2}{4m^2}j^2+\frac{3(15+6\varkappa^2-\varkappa^4)}{64m^4}j^4+\mathcal{O}[j^6],\\
K_2(j) &= \frac{5+3\varkappa^2}{5}+\frac{3(35+30\varkappa^2+3\varkappa^4)}{40m^2}j^2+\frac{3(105+105\varkappa^2+15\varkappa^4-\varkappa^6)}{64m^4}j^4+\mathcal{O}[j^6].\end{aligned} \tag{5.18}$$

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This is the classical energy expression for the small energy and angular momentum in the $\varkappa$-deformed $AdS_3 \times S^1$. After putting $\varkappa = 0$, we can get the energy for the short string which oscillates near the center of AdS_3 with an angular momentum in S^1 in undeformed $AdS_3 \times S^1$ as computed in [55]. With both $\varkappa$ and angular momentum as zero we can get back the energy expression for the strings oscillating in one plane for small energy limit as in the [47].

6 Conclusion

We have studied various pulsating string in the so called $\varkappa$ deformed $AdS_3 \times S^3$ background. We find the energy of the short string in the small energy limit for the pulsating strings in the $\varkappa$-deformed $S^3_\varkappa$ subspace of the full $(AdS_5 \times S^5)_\varkappa$ background. $\varkappa = 0$ limit agrees with the computation of the undeformed case and the $\varkappa$ infact enters in a vary natural way in the expression. It is perhaps along the expected lines as a theory with non-zero $\varkappa$ also provides an exact integrable sigma model background (with a redefined string tension) and hence the string configurations in the undeformed background must also have correspondence with the ones in the deformed case as well. We have further found out the short string energy as a function of $\mathcal{N}, m, \varkappa$. We have also analyzed case for the string with an extra angular momentum along the deformed S^1. We wish to look for the field theory duals in these theories in future.

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