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Pair production of Dirac particles in a $d+1$-dimensional noncommutative space–time

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Abstract This work addresses the computation of the probability of fermionic particle pair production in $(d+1)$-dimensional noncommutative Moyal space. Using Seiberg–Witten maps, which establish relations between noncommutative and commutative field variables, up to the first order in the noncommutative parameter θ, we derive the probability density of vacuum–vacuum pair production of Dirac particles. The cases of constant electromagnetic, alternating time-dependent, and space-dependent electric fields are considered and discussed.

1 Introduction

Noncommutative field theory (NCFT), arising from noncommutative (NC) geometry, has been the subject of intense studies, owing to its importance in the description of quantum gravity phenomena. More precisely, the concepts of the noncommutativity in fundamental physics have deep motivations which originated from the fundamental properties of the Snyder space–time [1]. Further, the results by Connes et al. [2–4] provided a clear definition of NC geometry, thus bringing about a new stimulus in this area. The NC geometry arises as a possible scenario for the short-distance behavior of physical theories (i.e. the Planck length scale $\lambda_p = \sqrt{\frac{G\hbar}{c^3}} \approx 1.6 \times 10^{-35}$ m); see [5–7] and references therein. This fundamental unit of length marks the scale of energies and distances at which the non-locality of interactions has to appear and the notion of continuous space–time becomes meaningless [5,6,8]. One of the important implications of noncommutativity is the Lorentz violation symmetry in higher than two-dimensional space–time [9–11], which, in part, modifies the dispersion relations [12–14]. It led to new developments in quantum electrodynamics (QED) and Yang–Mills (YM) theories in the NC variable function versions [15,16]. The same observation appears in the framework of string theory [17,18]. Also, the quantum Hall effect well illustrates the NC quantum mechanics of space–time [19,20] (and references therein).

In this work, we use a NC star product obtained by replacing the ordinary product of functions by the Moyal star product as follows:

$$f \star g = \mathbf{m}\left[\exp\left(\frac{i}{2}\theta^{\mu\nu}\,\partial_\mu \otimes \partial_\nu\right)(f\otimes g)\right], \tag{1}$$

where $f, g \in C^\infty(\mathbb{R}^D)$, $\mathbf{m}(f\otimes g) = f\cdot g$; $\theta^{\mu\nu}$ stands for a skew-symmetric tensor characterizing the NC behavior of the space–time, and it has the Planck length square dimension, i.e. $[\theta] \equiv [\lambda_p^2]$. The star product (1) satisfies the useful integral relation

$$\begin{aligned}\int d^Dx\,(f\star g)(x) &= \int d^Dx\,(g\star f)(x)\\ &= \int d^Dx\, f(x)\,g(x).\end{aligned} \tag{2}$$

It provides the following commutation relation between the coordinate functions:

$$[x^\mu, x^\nu]_\star = x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}, \quad x^\mu \in \mathbb{R}^D. \tag{3}$$

For convenience, we write the tensor $(\theta^{\mu\nu})$ in the following form:

$$(\theta^{\mu\nu}) = \begin{pmatrix} 0 & \theta & \cdots & 0 & 0\\ -\theta & 0 & \cdots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & \theta\\ 0 & 0 & \cdots & -\theta & 0\end{pmatrix}, \quad \theta \geq 0. \tag{4}$$

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Equation (4) means that the time does not commute with the NC spatial coordinates and the dimension D of the NC space–time is even. Recall that two main problems arise when one tries to implement electromagnetism in a NC geometry: the loss of causality due to the appearance of derivative couplings in the Lagrangian density and, more fundamentally, the violation of Lorentz invariance exhibited by plane wave solutions [12–14,21].

Like in ordinary quantum mechanics, the NC coordinates satisfy the coordinate–coordinate version of the Heisenberg uncertainty relation, namely $\Delta x^{\mu}\Delta x^{\nu} \geq \theta$, and then they make the space–time a quantum space. This idea leads to the concept of quantum gravity, since quantizing space–time leads to quantizing gravity. Apart from the overall results as regards QED and YM theory in NC space–time, it turns out to be important to understand how noncommutativity modifies the probability of pair production of fermionic particles. This is the task we shall deal with in this work.

Pair production refers to the creation of an elementary particle and its antiparticle, usually when a neutral boson interacts with a nucleus or another boson. Nevertheless a static electric field in empty space can create electron–positron pairs. This effect, called the Schwinger effect [22], is currently on the verge of being experimentally verified. Recently, the vacuum–vacuum transition amplitude and its probability density were computed in four-, three-, and two-dimensional space–time in constant and alternating electromagnetic (EM) fields ([23–31] and references therein). The related questions have been discussed and have gained considerable attention in the researchers community.

In this work, we provide the NC version of pair production of Dirac particles. Specially, we derive the exact expression for the probability density of particle production by an external field. This establishes a relation with important analytical results which were previously obtained in ordinary space–time, spread in the literature [22,23,25–27,31]. In particular, the case of $(d+1)$-dimensional space–time, which has been derived in [23,31], is extended to the noncommutative case.

The paper is organized as follows. In Sect. 2, we quickly review the Seiberg–Witten maps giving a relation between NC field variables and commutative ones [17,32,33]. Here we also expose the main result as regards gauge theory in NC space, which allows us to write the NC Lagrangian density of the Dirac particle (coupling to the EM field) with the commutative field variables. In Sect. 3 we compute the probability density of pair production of a Dirac particle in constant EM fields. In Sect. 4 we give the discussions and the comments of our result. This section also contains a similar analysis in the case of an alternating (EM) field. In Sect. 5, we conclude our work and make some remarks. Appendices A and B are for proofs of the key theorems in the main part of this paper.

2 NC gauge theory and Seiberg–Witten maps

Like in ordinary space–time, a gauge theory can be defined on a NC space–time [34] see also [35–39] and references therein. In the sequel, the NC variables are denoted with a "hat" notation. Let $\mathcal{A}_\theta$ be a Moyal algebra of functions and $\hat{X} \in \mathcal{A}_\theta$ be the covariant coordinate expressed in terms of the gauge potential $\hat{A} \in \mathcal{A}_\theta$ as

$$\hat{X} = \hat{x} + \hat{A}. \tag{5}$$

For an arbitrary function $\hat{\psi} \in \mathcal{A}_\theta$, the infinitesimal gauge transformation with parameter $\hat{\Lambda} \in \mathcal{A}_\theta$ is $\hat{\delta}\hat{\psi} = i\hat{\Lambda} \star \hat{\psi}$. The infinitesimal variation of the gauge potential can be written as

$$\hat{\delta}_{\hat{\Lambda}}\hat{A}^{\mu} = i[\hat{\Lambda}, \hat{A}^{\mu}]_{\star} - i[\hat{x}^{\mu}, \hat{\Lambda}]_{\star}. \tag{6}$$

Also the NC Faraday tensor is given by

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i[\hat{A}_\mu, \hat{A}_\nu]_{\star}. \tag{7}$$

Its infinitesimal variation is

$$\hat{\delta}\hat{F}_{\mu\nu} = i[\hat{\lambda}, \hat{F}_{\mu\nu}]_{\star}. \tag{8}$$

Besides, the functional action for a Dirac particle on NC space–time can be defined as follows:

$$S = \int_{\mathbb{R}^D} d^D x\, \mathcal{L}(\hat{\bar{\psi}}, \hat{\psi}), \tag{9}$$

$$\mathcal{L}(\hat{\bar{\psi}}, \hat{\psi}) = \hat{\bar{\psi}} \star i\gamma^{\mu}\hat{D}_\mu\hat{\psi} - m\hat{\bar{\psi}} \star \hat{\psi}. \tag{10}$$

In this expression $\hat{\psi}$ and $\hat{\bar{\psi}}$ are the Dirac spinor and its associated Hermitian conjugate, respectively. The γ are the Dirac matrices, which satisfy the Clifford algebra, $\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta^{\mu\nu}$, and they are given explicitly in terms of the Pauli matrices σ^i, $i = 1, 2, 3$, by

$$\gamma^0 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{11}$$

The covariant derivative $\hat{D}_\mu$ is expressed as

$$\hat{D}_\mu = \partial_\mu - i\hat{A}_\mu \star. \tag{12}$$

We choose $\hbar = c = 1$ and take the charge of particle equal to the unit value, i.e. $q_e = 1$. The Lagrangian $\mathcal{L}(\bar{\psi}, \psi)$ describes the propagation of the massive fermion (an electron in this case) and the interaction with photons via the covariant derivative $\hat{D}_\mu$. In this work, we treat in detail the case when the dimension of the space–time is equal to $D = 3 + 1$. The results for the cases where $D = 1 + 1$ and, more generally, $D = d + 1$, computed in a similar way, are given. Note that, despite the singularity exhibited by the matrix $(\theta^{\mu\nu})$ in the case of odd dimensions, the probability of pair production is well defined with the same analysis.

In what follows we give the Seiberg–Witten maps at the first order of perturbation in θ [17,32,33]. We write the NC field variables as a function of the commutative variables:

$$\hat{\psi} = \psi - \frac{1}{4}\theta^{\kappa\lambda} A_\kappa(\partial_\lambda + D_\lambda)\psi, \tag{13}$$

$$\hat{\bar{\psi}} = \bar{\psi} - \frac{1}{4}\theta^{\kappa\lambda} A_\kappa(\partial_\lambda + D_\lambda)\bar{\psi}, \tag{14}$$

$$\hat{A}_\mu = A_\mu - \frac{1}{4}\theta^{\kappa\lambda}\{A_\kappa, \partial_\lambda A_\mu + F_{\lambda\mu}\}. \tag{15}$$

By substituting (13), (14), and (15) in the action (9), we get, at the first order in θ,

$$\begin{aligned}
&\mathcal{L}(\bar{\psi}, \psi) \\
&= i\gamma^\mu \Big[\bar{\psi}(\partial_\mu - iA_\mu)\psi + \frac{i}{2}\theta^{\alpha\beta}\partial_\alpha\bar{\psi}\partial_\beta(\partial_\mu - iA_\mu)\psi \\
&\quad -\frac{1}{4}\theta^{\alpha\beta}\bar{\psi}\partial_\mu(A_\alpha(\partial_\beta + D_\beta)\psi) + \frac{1}{2}\theta^{\alpha\beta}\bar{\psi}\partial_\alpha A_\mu\partial_\beta\psi \\
&\quad +\frac{i}{4}\theta^{\alpha\beta}\bar{\psi}A_\mu A_\alpha(\partial_\beta + D_\beta)\psi + \frac{i}{4}\theta^{\kappa\lambda}\bar{\psi}\{A_\kappa, \partial_\lambda A_\mu \\
&\quad +F_{\lambda\mu}\}\psi - \frac{1}{4}\theta^{\kappa\lambda}A_\kappa(\partial_\lambda + D_\lambda)\bar{\psi}(\partial_\mu - iA_\mu)\psi\Big] \\
&\quad -m\Big[\bar{\psi}\psi + \frac{i}{2}\theta^{\mu\nu}\partial_\mu\bar{\psi}\partial_\nu\psi - \frac{1}{4}\theta^{\mu\nu}\bar{\psi}A_\mu(\partial_\nu + D_\nu)(\psi) \\
&\quad -\frac{1}{4}\theta^{\mu\nu}A_\mu(\partial_\nu + D_\nu)(\bar{\psi})\psi\Big] + O(\theta^2).
\end{aligned} \tag{16}$$

In the commutative limit i.e. $\theta \to 0$, we recover, as expected, the Lagrangian density $\mathcal{L}_C$ of a Dirac field in an ordinary space–time associated to the functional action $\mathcal{S}[\psi, \bar{\psi}, A]$:

$$\begin{aligned}
\mathcal{S}[\psi, \bar{\psi}, A] &= \int d^D x\, \mathcal{L}(\bar{\psi}, \psi) \\
&= \int d^D x\, \left(\mathcal{L}_C(\bar{\psi}, \psi) + \mathcal{B}(\theta, A, \bar{\psi}, \psi)\right),
\end{aligned} \tag{17}$$

where the quantity $\mathcal{B}(\theta, A, \bar{\psi}, \psi)$ depending on θ is given, after some algebra, by

$$\begin{aligned}
\mathcal{B}(\theta, A, \bar{\psi}, \psi) = i\gamma^\mu\theta^{\kappa\lambda}\bar{\psi}\Big[&-\frac{1}{2}(\partial_\mu A_\kappa)\partial_\lambda \\
&+\frac{1}{2}\partial_\kappa A_\mu\partial_\lambda + \frac{i}{2}A_\mu A_\kappa\partial_\lambda + \frac{i}{2}A_k\partial_\lambda A_\mu \\
&-\frac{i}{2}A_k\partial_\mu A_\lambda + \frac{1}{2}(\partial_\lambda A_\kappa)\partial_\mu - \frac{i}{2}(\partial_\lambda A_k)A_\mu\Big]\psi \\
&-\frac{m\theta^{\kappa\lambda}}{2}\bar{\psi}(\partial_\kappa A_\lambda)\,\psi.
\end{aligned} \tag{18}$$

Now by performing the path integral over the background fields ψ and $\bar{\psi}$, the vacuum–vacuum transition amplitude $\mathcal{Z}(A)$ is

$$\begin{aligned}
&\mathcal{Z}(A) \\
&= \mathcal{N}\int D\psi D\bar{\psi}\exp i\Big\{\int \mathrm{d}^4x\,(i\gamma^\mu\bar{\psi}(\partial_\mu - iA_\mu)\psi \\
&\quad -m\bar{\psi}\psi + \mathcal{B}(\theta, A, \bar{\psi}, \psi))\Big\},
\end{aligned} \tag{19}$$

in which the normalization constant $\mathcal{N}$ is chosen such that $\mathcal{Z}(0) = 1$. Note that $\mathcal{B}(\theta, 0, 1, 1) = 0$.

Let $\mathcal{M} := i\gamma^\mu D_\mu - m + \mathcal{B}(\theta, A, 1, 1) + i\epsilon$. Then we get a simpler form:

$$\mathcal{Z}(A) = \exp\left[-\operatorname{tr}\ln\frac{i\gamma^\mu\partial_\mu - m + i\epsilon}{\mathcal{M}}\right]. \tag{20}$$

Having the above quantity at hand, we compute the probability density amplitude $|\mathcal{Z}(A)|^2$ for various electromagnetic fields.

3 Transition amplitude in the case of a constant external EM field

In this section, we consider the EM field, defined in x direction as $\mathbf{B} = B\mathbf{e_x}$ and $\mathbf{E} = E\mathbf{e_x}$, $E > 0$ and $B \geq 0$. The position and momentum operators $X_\mu = (X_0, X_1, X_2, X_3) =: (X_0, X, Y, Z)$ and $P_\mu = i\partial_\mu = (P_0, P_1, P_2, P_3)$ satisfy the commutation relation:

$$[X_\mu, P_\mu] = i\eta_{\mu\nu}. \tag{21}$$

The covariant vector V_μ is expressed with the contravariant V^μ as $V_\mu = \eta_{\mu\nu}V^\nu$, where $(\eta) = \mathrm{diag}(1, -1, -1, -1)$. The covariant Faraday tensor $F_{\mu\nu} =: \partial_\mu A_\nu - \partial_\nu A_\mu$ can be expressed as

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}, \tag{22}$$

with $A_\mu = (-EX, 0, 0, BY)$. Then $\mathcal{B}(\theta, A, 1, 1)$ is obtained as

$$\begin{aligned}
\mathcal{B}(\theta, A, 1, 1) = \frac{m\theta}{2}(B - E) + \frac{i\theta}{2}\gamma^\mu[&i(E + B)A_\mu \\
&-(E + B)\partial_\mu - iA_\mu(EX\partial_1 + BY\partial_2) \\
&-(\partial_1 A_\mu)\partial_0 \\
&+(\partial_2 A_\mu)\partial_3 + \partial_\mu(EX)\partial_1 + \partial_\mu(BY)\partial_2].
\end{aligned} \tag{23}$$

Using the charge conjugation matrix $C = i\gamma^2\gamma^0$, using the identity $C\gamma_\mu C^{-1} = -\gamma_\mu^t$, and taking into account the fact that the trace of an operator is invariant under a matrix transposition lead to

$$\mathcal{Z}^t(A) = \exp\left[-\operatorname{tr}\ln\frac{iC\gamma^\mu C^{-1}\partial_\mu + m - i\epsilon}{\mathcal{M}^t}\right], \tag{24}$$

where $\mathcal{M}^t = iC\gamma^\mu C^{-1}D_\mu + m - \mathcal{B}^t(\theta, A, 1, 1) - i\epsilon$. The probability density is defined by the module of $Z(A)$ as

$$|\mathcal{Z}(A)|^2 := \exp\left[-\operatorname{tr}\ln\frac{P^2 - m^2 + i\epsilon}{\mathcal{M}\mathcal{M}^t}\right], \tag{25}$$

with

$$\mathcal{M}\mathcal{M}^t = [\gamma^\mu(P_\mu + A_\mu)]^2 - m^2 - m^2\theta(B-E) \\ +\mathcal{B}\gamma^\mu(P_\mu + A_\mu) - \gamma^\mu(P_\mu + A_\mu)\mathcal{B}^t + i\epsilon. \tag{26}$$

The conjugate of $\mathcal{B}(\theta, A, 1, 1)$, denoted by $\mathcal{B}^t(\theta, A, 1, 1)$, can then be written

$$\mathcal{B}^t(\theta, A, 1, 1) = \frac{m\theta}{2}(B-E) + \frac{i\theta}{2}C\gamma^\mu C^{-1} \\ \times[i(E+B)A_\mu - iA_\mu(EX\partial_1 + BY\partial_2) \\ -(E+B)\partial_\mu - (\partial_1 A_\mu)\partial_0 + (\partial_2 A_\mu)\partial_3 \\ +\partial_\mu(EX)\partial_1 + \partial_\mu(BY)\partial_2]. \tag{27}$$

At this point it would be worth using the identity

$$\ln\frac{a+i\epsilon}{b+i\epsilon} = \int_0^\infty \frac{\mathrm{d}s}{s}[\mathrm{e}^{is(b+i\epsilon)} - \mathrm{e}^{is(a+i\epsilon)}] \tag{28}$$

to get

$$\ln\frac{P^2 - m^2 + i\epsilon}{\mathcal{M}\mathcal{M}^t} = \int_0^\infty \frac{\mathrm{d}s}{s}\mathrm{e}^{-is(m^2 - i\epsilon)} \\ \times[\mathrm{e}^{is[(P+A)^2 + \frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu} - m^2\theta(B-E) + \mathcal{X}(\theta)]} - \mathrm{e}^{isP^2}] \tag{29}$$

where the operator $\mathcal{X}(\theta)$ should be Hermitian. We use the following commutation relations:

$$[X^n, P_1] = -niX^{n-1}, \quad [P_1^n, X] = niP^{n-1}, \tag{30}$$

also valid when one replaces X by Y and P_1 by P_2. For an arbitrary operator A, we can define the associated Hermitian operator denoted by A_H as

$$A_H = \frac{(A + A^\dagger)}{2}. \tag{31}$$

From now, the H symbol indexing any operator A, e.g. A_H, refers to the Hermitian operator associated with A. We then have the following.

Proposition 1 *The Hermitian operator associated with $\mathcal{X}(\theta)$, denoted $\mathcal{X}_H(\theta)$, is given by*

$$\mathcal{X}_H(\theta) = \frac{\theta}{2}\Big[iEB\gamma^3\gamma^2 + iE^2\gamma^0\gamma^1 + iB^2\gamma^3\gamma^2 \\ +\frac{1}{2}i(\gamma^0\gamma^1 + \gamma^3\gamma^2)EB + \gamma^0\gamma^1 EBYP_2 \\ +2E^2\gamma^0\gamma^1 XP_1 + \gamma^0\gamma^3(E^2B - EB^2)XY \\ -\gamma^1\gamma^3 EBYP_1 - (4E^3 + 3BE^2)X^2 \\ +(2B^3 + B^2E)Y^2 + (4E^2 + 5EB)XP_0 \\ +(2B^2 + 3EB)YP_3 - 2BP_0^2 + 2BP_1^2 \\ +2EP_2^2 + 2EP_3^2\Big]. \tag{32}$$

Further,

$$\mathcal{X}_H(\theta) = \mathcal{X}_H^\dagger(\theta). \tag{33}$$

Proof Taking into account the fact that the trace is invariant under matrix transposition, and using the relation (31), the operator $\mathcal{X}_H(\theta)$ takes the given form. □

Now we focus on the computation of the following quantity:

$$\mathcal{O} = \langle\mathbf{x}|\mathrm{e}^{is[(P+A)^2 + \frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu} - m^2\theta(B-E) + \mathcal{X}_H(\theta)]}|\mathbf{x}\rangle \\ = \mathrm{e}^{\frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu} - m^2\theta(B-E)}\langle\mathbf{x}|\mathrm{e}^{is[(P+A)^2 + \mathcal{X}_H(\theta)]}|\mathbf{x}\rangle, \tag{34}$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. We use the relation

$$[\gamma(P+A)]^2 = (P+A)^2 + \frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu}, \tag{35}$$

and choose the 4-vectors $|\mathbf{x}\rangle = |x_\mu\rangle$ such that $X_\mu|\mathbf{x}\rangle = x_\mu|\mathbf{x}\rangle$. In the momentum representation, we get a similar relation for $P_\mu|\mathbf{k}\rangle = k_\mu|\mathbf{k}\rangle$, and we obtain

$$\langle\mathbf{k}|\mathbf{x}\rangle = \frac{1}{(2\pi)^2}\mathrm{e}^{i\langle\mathbf{x}, \mathbf{k}\rangle}, \quad \langle\mathbf{x}, \mathbf{k}\rangle =: \sum_{i=1}^4 x_i k_i. \tag{36}$$

To achieve our goal, we use the Baker–Campbell–Hausdorff formula given by

$$\mathrm{e}^{t(U+V)} = \mathrm{e}^{tU}\ \mathrm{e}^{tV}\mathrm{e}^{t^2C_2}\mathrm{e}^{t^3C_3}\mathrm{e}^{t^4C_4}\ldots \\ = \mathrm{e}^{tU}\ \mathrm{e}^{tV}\prod_{n=2}^\infty \mathrm{e}^{t^nC_n} \tag{37}$$

where the constants C_n are given by the Zassenhaus formula [35,40],

$$C_{n+1} = \frac{1}{n+1}\sum_{j=0}^{n-1}\frac{(-1)^n}{j!(n-j)!}\mathrm{ad}_V^j\ \mathrm{ad}_U^{n-j}V \tag{38}$$

with

$$ad_U V = [U, V],\ ad_U^j V = [U, ad_U^{j-1}V],\ ad_U^0 V = V. \tag{39}$$

Explicitly, we get

$$\mathrm{e}^{t(U+V)} = \mathrm{e}^{tU}\ \mathrm{e}^{tV}\ \mathrm{e}^{-\frac{t^2}{2}[U,V]}\ \mathrm{e}^{\frac{t^3}{6}(2[V,[U,V]]+[U,[U,V]])} \\ \times\ \mathrm{e}^{\frac{-t^4}{24}([[[U,V],U],U]+3[[[XU,V],U],V]+3[[[U,V],V],V])}\ldots, \tag{40}$$

where the exponents of higher order in t are likewise nested. Then we take into account the first approximation of θ in the expansion of all quantities and we arrive at the expression

$$\mathrm{e}^{is[(P+A)^2 + \mathcal{X}(\theta)]} = \mathrm{e}^{is(P+A)^2}\mathrm{e}^{is\mathcal{X}(\theta)}\mathrm{e}^{T(\theta)} \\ = \mathrm{e}^{is(P+A)^2}(1 + is\mathcal{X}_H(\theta) + T_H(\theta) + O(\theta^2)) \tag{41}$$

where, for $t = is$, $U = (P + A)^2$, and $V = \mathcal{X}_H(\theta)$, we have

$$T_H(\theta) = -\frac{t^2}{2}[U, V]_H + \frac{t^3}{6}([U, [U, V]_H]_H) - \frac{t^4}{24}([[[U, V]_H, U]_H, U]_H) + \cdots . \tag{42}$$

The expectation value of the operator $e^{is[(P+A)^2+\mathcal{X}_H(\theta)]}$ is then evaluated as

$$\langle \mathbf{x}|e^{is[(P+A)^2+\mathcal{X}_H(\theta)]}|\mathbf{x}\rangle = \int d\mathbf{y}\, \langle \mathbf{x}|e^{is(P+A)^2}|\mathbf{y}\rangle\langle \mathbf{y}|\mathcal{J}(\theta)|\mathbf{x}\rangle, \tag{43}$$

where $\mathcal{J}(\theta) = (1+is\mathcal{X}_H(\theta)+T_H(\theta))$. Now after expanding U as

$$U = P_0^2 - P_1^2 - P_2^2 - P_3^2 - 2EP_0X - 2BP_3Y + E^2X^2 - B^2Y^2, \tag{44}$$

we can easily observe that $U = U^\dagger$. Fortunately, we get the following statement.

Proposition 2 *Let $U = (P + A)^2$, and $V = \mathcal{X}_H(\theta)$. The commutation relations between U and V vanish, i.e.*

$$[U, V]_H = 0,\ [[[U, V]_H, U]_H, \ldots]_H, U]_H = 0, \tag{45}$$

and therefore $T_H(\theta) = 0$.

Proof The proof of this proposition is simply obtained by using (31) and (32). □

Finally the quantity $\mathcal{O}$ is reduced to

$$\mathcal{O} = \mathcal{O}_c + \mathcal{O}_{nc}(\theta), \quad \mathcal{O}_{nc}(0) = 0 \tag{46}$$

where

$$\mathcal{O}_c = e^{\frac{is}{2}\sigma^{\mu\nu}F_{\mu\nu}}\langle \mathbf{x}|e^{is(P+A)^2}|\mathbf{x}\rangle \tag{47}$$

and

$$\mathcal{O}_{nc}(\theta) = e^{\frac{is}{2}\sigma^{\mu\nu}F_{\mu\nu}}\int d\mathbf{y}\, \langle \mathbf{x}|e^{is(P+A)^2}|\mathbf{y}\rangle \times\langle \mathbf{y}|is[m^2\theta(E-B) + \mathcal{X}_H(\theta)]|\mathbf{x}\rangle. \tag{48}$$

We then obtain the following result.

Theorem 1 *Let $\theta =: \wp \cdot \theta_0$ where $\wp$ is a dimensionless quantity which is bounded by two numbers a_1, and a_2 and such that $\theta_0 << 1$. The mass dimension of θ_0 is obviously $\theta_0 \equiv [M^{-2}]$. Let $M \subset \mathbb{R}^2$ be the compact subset of $\mathbb{R}^2$ in which the following integral is convergent:*

$$\int_{M\subset\mathbb{R}^2}\frac{dt}{t_0}dz = b \equiv [M^{-2}]. \tag{49}$$

$t_0 \neq 0$ is arbitrary initial time. The trace of the expectation value $\mathcal{O}$ is given by

$$\mathrm{tr}\,\mathcal{O} = (1 - \wp - \sigma(\theta_0, E, B))\,\mathrm{tr}\,\mathcal{O}_c, \tag{50}$$

where

$$\sigma(\theta_0, E, B) = \frac{16\pi^3 b\,\wp\exp[is\theta_0\mathcal{G}_0]}{\theta_0^3 s^3 EB}\sqrt{\frac{B}{E}f(E, B)}, \tag{51}$$

$$\mathrm{tr}\,\mathcal{O}_c = -\frac{1}{4\pi^2 i}EB\cos h(Es)\cot(Bs), \tag{52}$$

$f(E, B)$ being a positive function given by

$$f(E, B) = [4B^6 + 76EB^5 + 258E^2B^4 + 494E^3B^3 + 224E^4B^2 + 12E^5B]^{-1}. \tag{53}$$

Proof The proof of this theorem is given in Appendix A. □

Note that the quantity $\sigma(\theta_0, E, B)$ leads to the divergence in the limit where $B = 0$ and in the limit where $\theta_0 = 0$. This expression does not contribute to the physical solution and then the trace of $\mathcal{O}$ is reduced to $\mathrm{tr}\,\mathcal{O} = (1 - \wp)\,\mathrm{tr}\,\mathcal{O}_c$.

Theorem 2 *The vacuum–vacuum transition probability is $|Z(A)|^2 = \exp\left[-\int dx\, \omega_{3+1}(x)\right]$ where*

$$\omega_{3+1}(x) = \frac{1}{4\pi^2 i}\int_0^\infty ds\,\frac{e^{ism^2}}{s}\left[(1-\wp)EB\cot h(Es) \times\cot(Bs) - \frac{1}{s^2}\right] \tag{54}$$

whose real part, denoted by $\Re_e\omega(x) = \frac{\omega+\omega*}{2}$, is given by

$$\Re_e\omega_{3+1}(x) = -\frac{1}{8\pi^2 i}\int_{-\infty}^\infty ds\,\frac{e^{ism^2}}{s} \times\left[(1-\wp)EB\cot h(Es)\cot(Bs) - \frac{1}{s^2}\right] = -\frac{m^4\wp}{16\pi} + \frac{EB}{4\pi^2}(1-\wp)\sum_{k=1}^\infty\frac{1}{k}\cot h\left(k\pi\frac{B}{E}\right) \times\exp\left(-\frac{k\pi m^2}{E}\right). \tag{55}$$

Proof The proof of this statement is given in Appendix B. □

4 Discussion of the results

In this section, we discuss the reported results in Theorems 1 and 2 and make some more comments in the framework of $d + 1$-dimensional space–time.

(1) Let

$$U_k = \frac{1}{k}\cot h\left(k\pi\frac{B}{E}\right)\exp\left(-\frac{k\pi m^2}{E}\right), \quad k \in \mathbb{N}\backslash\{0\}. \tag{56}$$

We get

$$\lim_{k\to\infty}\left|\frac{U_{k+1}}{U_k}\right| = \exp\left(-\frac{\pi m^2}{E}\right) < 1 \tag{57}$$

and conclude that the corresponding series, i.e. $\sum_{k=1}^{\infty} U_k$ is obviously convergent. Recall that there exist two positive constants a_1 and a_2 such that for $a_1 \leq \wp \leq a_2$, $\theta_0 << 1$. Then there exists a bound on θ in which the solution (55) is well defined. We have

$$-\frac{m^4 a_2}{16\pi} + \frac{EB}{4\pi^2}(1-a_2)\sum_{k=1}^{\infty} U_k \leq \Re_e\omega_{3+1}(x)$$
$$\leq -\frac{m^4 a_1}{16\pi} + \frac{EB}{4\pi^2}(1-a_1)\sum_{k=1}^{\infty} U_k. \quad (58)$$

(2) Consider $\Re_e\omega_{c,3+1}(x)$ as the probability density provided with Eq. (55), in the limit where $\theta \to 0$ i.e.

$$\lim_{\theta\to 0} \Re_e\omega_{3+1}(x) = \Re_e\omega_{c,3+1}(x). \quad (59)$$

This expression corresponds to the commutative limit derived by Q-G. Lin (see [25]) and is given by

$$\Re_e\omega_{c,3+1}(x) = \frac{EB}{4\pi^2}\sum_{k=1}^{\infty} U_k. \quad (60)$$

We get

$$\Re_e\omega_{3+1}(x) < \Re_e\omega_{c,3+1}(x), \quad (61)$$

and we conclude that the noncommutativity increases the amplitude $|Z(A)|^2$. This shows the importance of noncommutativity at the high energy regime in which the creation of particles is manifest. The same conclusion can be drawn as in Ref. [28], in which pair production by a constant external field on NC space is also considered. Note that $\Re_e\omega_{3+1}(x) = -\frac{m^4\wp}{16\pi}$ if $E = 0$. For $B = 0$, we use the Taylor expansion (108) given in Appendix B, and we get

$$\Re_e\omega_{3+1}^{(B=0)}(x) = \frac{E^2}{4\pi^3}(1-\wp)\sum_{k=1}^{\infty}\frac{1}{k^2}\exp\left[-\frac{k\pi m^2}{E}\right]$$
$$-\frac{m^4\wp}{16\pi}. \quad (62)$$

The commutative limit which corresponds to the case where $\wp = 0$ is restored.

(3) In the case of 1+1 dimensions we consider the electric field $\mathbf{E} = E\mathbf{e}_x$, with $E > 0$ and $\mathbf{B} = \mathbf{0}$. The nonvanishing component of the tensor $F_{\mu\nu}$ is given by $F_{01} = E$. The quantity $B(\theta, A, 1, 1)$ given in Eq. (18) takes the form

$$B(\theta, A, 1, 1) = -\frac{m\theta E}{2} + \frac{i\theta\gamma^\mu}{2}[iEA_\mu - E\partial_\mu$$
$$-iA_\mu EX\partial_1 - \partial_1 A_\mu\partial_0 + \partial_\mu(EX)\partial_1]. \quad (63)$$

We remark that the result (63) is also obtained by taking in (23) the magnetic field B to be zero and by deleting the coordinate components Y and Z. Now, we refer to (29); the Hermitian operator $\mathcal{X}(\theta)$ is obviously

$$\mathcal{X}_H(\theta) = \frac{\theta}{2}[iE^2\gamma^0\gamma^1 + 2E^2\gamma^0\gamma^1 XP_1 - 4E^3X^2$$
$$+4E^2XP_0]. \quad (64)$$

We use the following results which are applicable in $1+1$ dimensions:

$$\langle\mathbf{x}|e^{is(P+A)^2}|\mathbf{x}\rangle = \frac{E}{4\pi\sin h(Es)} \quad (65)$$

and

$$\langle\mathbf{x}|e^{isP^2}|\mathbf{x}\rangle = \frac{1}{4\pi s}. \quad (66)$$

By performing the same computation in Appendix A and B, we obtain

$$\Re_e\omega_{1+1}^{(B=0)}(x) = \frac{E}{2\pi}(1-\wp)\sum_{k=1}^{\infty}\frac{1}{k}\exp\left(-\frac{k\pi m^2}{E}\right)$$
$$-\frac{m^2\wp}{8}. \quad (67)$$

In the limit where $\theta \to 0$ Eq. (67) restores the commutative limit given in [25].

(4) Let us discuss the case of $(2+1)$-dimensional space–time. The matrix $(\theta)^{\mu\nu}$ becomes singular and takes the form

$$\theta^{\mu\nu} = \begin{pmatrix} 0 & \theta & 0 \\ -\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (68)$$

Then the noncommutativity of space–time is described by the commutation relation

$$[x^0, x^1]_\star = i\theta,\ [x^0, x^2]_\star = 0,\ [x^1, x^2]_\star = 0. \quad (69)$$

Despite this singularity of the matrix $(\theta^{\mu\nu})$, the SW application (13), (14), and (15) are well satisfied. Therefore using the fact that

$$\langle\mathbf{x}|e^{is(P+A)^2}|\mathbf{x}\rangle = \frac{(1-i)E}{2(2\pi)^{\frac{3}{2}}s^{\frac{1}{2}}\sin h(Es)} \quad (70)$$

and

$$\langle\mathbf{x}|e^{isP^2}|\mathbf{x}\rangle = \frac{1-i}{4(2\pi)^{\frac{3}{2}}s^{\frac{3}{2}}}, \quad (71)$$

we can derive the probability density of particle creation from the vacuum by external constant EM fields. We obtain

$$\Re_e\omega_{2+1}^{(B=0)}(x) = \frac{E^{\frac{3}{2}}}{4\pi^2}(1-\wp)\sum_{k=1}^{\infty}\frac{1}{k^{\frac{3}{2}}}\exp\left(-\frac{k\pi m^2}{E}\right)$$
$$-\frac{m^3\wp}{8(2\pi)^{\frac{1}{2}}} \quad (72)$$

and obviously the limit $\theta = 0$ restores the result of Ref. [25].

(5) Furthermore, the previous investigation [31], devoted to such an EM field as $\mathbf{E} = E\cos(t)\mathbf{e_x}$ and $\mathbf{B} = B\mathbf{e_x}$, has also been considered here in the framework of the NCFT.

Indeed, following step by step the approach displayed earlier in this work, and using the following relation:

$$\langle \mathbf{x}|e^{is(P+A)^2}|\mathbf{x}\rangle = \frac{-iEB\cos(t)}{16\pi^2 \sin h(Es)\sin(Bs)}, \tag{73}$$

after some algebra we get

$$\widetilde{\omega}_{3+1}(t) = \frac{1}{4\pi^2 i}\int_0^\infty \mathrm{d}s\, \frac{e^{ism^2}}{s}\left[(1-\wp) \times \frac{EB\cos(t)\cos h[E\cos(t)s]\cot(Bs)}{\sin h(Es)} - \frac{1}{s^2}\right]. \tag{74}$$

We found that the probability density of pair production of Dirac particles in NC space–time with an alternating EM field is given by

$$\Re_e\widetilde{\omega}_{3+1}(t) = -\frac{m^4\wp}{16\pi} + \frac{EB}{4\pi^2}(1-\wp) \times \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\cos(n\pi\cos(t))\cot h\left(n\pi\frac{B}{E}\right) \times \exp\left(-\frac{n\pi m^2}{E}\right), \tag{75}$$

from which, in the limit where the NC parameter $\theta = 0$, we recover the formula of Hounkonnou et al. [31] i.e.

$$\widetilde{\omega}(t) = \frac{EB}{4\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\cos(n\pi\cos(t))\cot h\left(n\pi\frac{B}{E}\right) \times \exp\left(-\frac{n\pi m^2}{E}\right). \tag{76}$$

A more compact form of Eq. (75) in the case of arbitrary $D = d + 1$ dimensions can be given in the same way. We get for $B = 0$ the following results:

$$\widetilde{\omega}_{d+1}(t) = (1-\wp)\frac{E^{\frac{d+1}{2}}\cos(t)}{(2\pi)^d}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{\frac{d+1}{2}}} \times \cos\left(\frac{n\pi}{\cos(t)}\right)\exp\left(-\frac{n\pi m^2}{E}\right) - \frac{m^{d+1}\wp}{4(2)^{\frac{d+1}{2}}(\pi)^{\frac{d-1}{2}}}. \tag{77}$$

Also, by replacing the vector field A_μ by $\mathcal{A}_\mu = A_\mu + f_\mu$, where $A_\mu = (-EX, 0, 0, 0)$ and $f_\mu = (-E\sin(x), 0, 0, 0)$ correspond to the plane wave function, we get

$$\widetilde{\omega}_{d+1}(x) = (1-\wp)\frac{(2E)^{\frac{d+1}{2}}(1+\cos(x))}{(2\pi)^d}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{\frac{d+1}{2}}} \times \cos\left(n\pi\frac{1+cos(x)}{2}\right)\exp\left(-\frac{n\pi m^2}{2E}\right) - \frac{m^{d+1}\wp}{4(2)^{\frac{d+1}{2}}(\pi)^{\frac{d-1}{2}}}. \tag{78}$$

All these results use the computations performed in Appendices A and B. In the limit where $\theta = 0$, Eqs. (77) and (78) lead to the results of [31]—see also [23], in which the generalization in arbitrary dimensions is given in the case of quasiconstant external EM fields.

5 Concluding remarks

In this paper, we have considered NC theory of the fermionic field interacting with its corresponding boson. We have used the Seiberg–Witten expansion describing the relation between the NC and commutative variables, to compute the probability density of pair production of NC fermions. We have showed that, in the limit where the NC parameter $\theta = 0$, we recover the result of Qiong–Gui Lin [25]. Our study has highlighted that the noncommutativity of space–time increases the density ω of the probability of pair creation of the fermion particle. Our results can easily be extended to take into account the case where $D = 1 + 1$.

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Appendix A: Proof of Theorem 1

We give the Proof of Theorem 1. We have introduced the two quantities: θ_0 and $\wp$, and we decompose the NC parameter θ as

$$\theta = \wp\,\theta_0, \quad \text{such that } \theta_0 << 1. \tag{79}$$

The first step is to re-express $\mathcal{O}_{\mathrm{nc}}(\theta)$ as follows:

$$\begin{aligned}\mathcal{O}_{\mathrm{nc}}(\theta) &= e^{\frac{is}{2}\sigma^{\mu\nu}F_{\mu\nu}}\int \mathrm{d}\mathbf{y}\,\langle\mathbf{x}|e^{is(P+A)^2}|\mathbf{y}\rangle\langle\mathbf{y}| \\ &\quad -is[m^2\theta(B-E) - \mathcal{X}_H(\theta)]|\mathbf{x}\rangle. \\ &= \wp\, e^{\frac{is}{2}\sigma^{\mu\nu}F_{\mu\nu}}\int \mathrm{d}\mathbf{k}\,\mathrm{d}\mathbf{y}\,\langle\mathbf{x}|e^{is(P+A)^2}|\mathbf{y}\rangle\langle\mathbf{y}| \\ &\quad \times is\theta_0\mathcal{G}(E,B,\theta)|\mathbf{k}\rangle\langle\mathbf{k}|\mathbf{x}\rangle\end{aligned}$$

$$= \wp\, e^{\frac{is}{2}\sigma^{\mu\nu}F_{\mu\nu}} \int \mathrm{d}\mathbf{k}\,\mathrm{d}\mathbf{x}\,\langle \mathbf{x}|e^{is(P+A)^2}|\mathbf{x}\rangle \times (is\theta_0\mathcal{G}(E,B,\theta)), \tag{80}$$

where

$$\begin{aligned}\mathcal{G}(E,B,\theta) = \Big\{ & m^2(E-B) + \frac{1}{2}\Big[iEB\gamma^3\gamma^2 \\ & + iE^2\gamma^0\gamma^1 + iB^2\gamma^3\gamma^2 \\ & + \frac{1}{2}i(\gamma^0\gamma^1+\gamma^3\gamma^2)EB + \gamma^0\gamma^1 EByk_2 \\ & + 2E^2\gamma^0\gamma^1 xk_1 + \gamma^0\gamma^3(E^2B - EB^2)xy \\ & - \gamma^1\gamma^3 EByk_1 - (4E^3 + 3BE^2)x^2 \\ & + (2B^3 + B^2E)y^2 + (4E^2+5EB)xk_0 \\ & + (2B^2 + 3EB)yk_3 - 2Bk_0^2 + 2Bk_1^2 \\ & + 2Ek_2^2 + 2Ek_3^2\Big]\Big\}. \end{aligned} \tag{81}$$

Equation (81) is subdivided into three contributions, namely

$$\begin{aligned}\mathcal{G}_0 = m^2(E-B) + \frac{1}{2}\Big[& iEB\gamma^3\gamma^2 + iE^2\gamma^0\gamma^1 \\ & + iB^2\gamma^3\gamma^2 + \frac{1}{2}i(\gamma^0\gamma^1+\gamma^3\gamma^2)EB\Big],\end{aligned} \tag{82}$$

$$\begin{aligned}\mathcal{G}_1 = \frac{1}{2}[& \gamma^0\gamma^1 EByk_2 + 2E^2\gamma^0\gamma^1 xk_1 - \gamma^1\gamma^3 EByk_1 \\ & + (4E^2+5EB)xk_0 + (2B^2+3EB)yk_3 - 2Bk_0^2 \\ & + 2Bk_1^2 + 2Ek_2^2 + 2Ek_3^2],\end{aligned} \tag{83}$$

and

$$\begin{aligned}\mathcal{G}_2 = \frac{1}{2}[& \gamma^0\gamma^3(E^2B - EB^2)xy - (4E^3+3BE^2)x^2 \\ & + (2B^3 + B^2E)y^2].\end{aligned} \tag{84}$$

We are especially interested in the first order Taylor expansion of the form

$$is\theta_0\mathcal{G}(E,B,\theta) \equiv \exp(is\theta_0\mathcal{G}(E,B,\theta)) - 1. \tag{85}$$

Going back to Eq. (34), this equation can be expressed as

$$\begin{aligned}\mathcal{O} = (1-\wp)\mathcal{O}_c + \wp\, e^{\frac{is}{2}\sigma^{\mu\nu}F_{\mu\nu}} \\ \times \int \mathrm{d}\mathbf{k}\,\mathrm{d}\mathbf{x}\,\langle \mathbf{x}|e^{is(P+A)^2}|\mathbf{x}\rangle \exp(is\theta_0\mathcal{G}(E,B,\theta)).\end{aligned} \tag{86}$$

First we consider the contribution $\exp(is\theta_0\mathcal{G}_1)$ of $\exp(is\theta_0\mathcal{G}(E,B,\theta))$ in (86) and the integral relation

$$\int_{-\infty}^{\infty} e^{isx^2}\mathrm{d}x = \begin{cases} e^{\frac{\pi}{4}}\sqrt{\frac{\pi}{s}} & \text{for } s>0, \\ e^{-\frac{\pi}{4}}\sqrt{\frac{\pi}{s}} & \text{for } s<0. \end{cases} \tag{87}$$

We get, respectively,

$$\begin{aligned}\mathcal{K}_1 &= \int \mathrm{d}k_0 \exp\left[\frac{is\theta_0}{2}(-2Bk_0^2 + (4E^2+5EB)xk_0)\right] \\ &= e^{-\frac{\pi}{4}}\sqrt{\frac{\pi}{sB\theta_0}}\exp\left[\frac{is\theta_0}{16B}(4E^2+5EB)^2x^2\right],\end{aligned} \tag{88}$$

$$\begin{aligned}\mathcal{K}_2 &= \int \mathrm{d}k_1 \exp\Big[\frac{is\theta_0}{2}(2Bk_1^2 + 2\gamma^0\gamma^1E^2xk_1 \\ &\quad -\gamma^1\gamma^3EByk_1)\Big] \\ &= e^{\frac{\pi}{4}}\sqrt{\frac{\pi}{sB\theta_0}}\exp\Big[-\frac{is\theta_0}{16B}(2\gamma^0\gamma^1E^2x \\ &\quad -\gamma^1\gamma^3EBy)^2\Big],\end{aligned} \tag{89}$$

$$\begin{aligned}\mathcal{K}_3 &= \int \mathrm{d}k_1 \exp\left[\frac{is\theta_0}{2}(2Ek_2^2 + \gamma^0\gamma^1EByk_2)\right] \\ &= e^{\frac{\pi}{4}}\sqrt{\frac{\pi}{sE\theta_0}}\exp\left[\frac{is\theta_0}{16E}E^2B^2y^2\right],\end{aligned} \tag{90}$$

$$\begin{aligned}\mathcal{K}_4 &= \int \mathrm{d}k_3 \exp\Big[\frac{is\theta_0}{2}(2Ek_3^2 \\ &\quad +(2B^2+3EB)yk_3)\Big] \\ &= e^{\frac{\pi}{4}}\sqrt{\frac{\pi}{sE\theta_0}}\exp\left[\frac{is\theta_0}{16E}(2B^2+3EB)^2y^2\right].\end{aligned} \tag{91}$$

Using the properties of the gamma matrices and the results of [25,31] we get the following:

$$\mathrm{tr}\, e^{\frac{is}{2}\sigma^{\mu\nu}F_{\mu\nu}} = 4\cos h(Es)\cos(Bs) \tag{92}$$

$$\langle \mathbf{x}|e^{is(P+A)^2}|\mathbf{x}\rangle = -\frac{iEB}{16\pi^2\sin h(Es)\sin(Bs)}, \tag{93}$$

$$\langle x|e^{isP^2}|x\rangle = -\frac{i}{16\pi^2s^2}. \tag{94}$$

We can evaluate the trace of relevant quantities in Eq. (86). On the other hand, before obtaining (50), we compute the trace of $\int \mathrm{d}\mathbf{k}\,\exp[is\theta_0\langle\mathbf{k}|\mathcal{G}(E,B,\theta)|\mathbf{k}\rangle]$, i.e.

$$\begin{aligned}&\int \mathrm{d}\mathbf{x}\int \mathrm{d}\mathbf{k}\,\exp[is\theta_0\langle\mathbf{k}|\mathcal{G}(E,B,\theta)|\mathbf{k}\rangle] \\ &= \exp[is\theta_0\mathcal{G}_0]\int \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\prod_{j=1}^{4}\mathcal{K}_j\exp[is\theta_0\mathcal{G}_2].\end{aligned} \tag{95}$$

This is obtained by using the Gaussian integral. Moreover, taking into account Eqs. (88), (89), (90), and (91) we get

$$\prod_{j=1}^{4}\mathcal{K}_j \exp[is\theta_0\mathcal{G}_2]=\frac{i\pi^2}{s^2\theta_0^2 EB}\exp\left\{\frac{is\theta_0}{16B}(4E^2+5EB)^2x^2 -\frac{is\theta_0}{16B}(2\gamma^0\gamma^1E^2x-\gamma^1\gamma^3EBy)^2+\frac{is\theta_0}{16E}E^2B^2y^2 +\frac{is\theta_0}{16E}(2B^2+3EB)^2y^2+\frac{is\theta_0}{2}[\gamma^0\gamma^3(E^2B-EB^2)xy -(4E^3+3BE^2)x^2+(2B^3+B^2E)y^2]\right\}. \quad (96)$$

In this relation, the quantity

$$\exp\left\{\frac{is\theta_0}{16B}[(4E^2+5EB)^2x^2-4E^4x^2 +8B\gamma^0\gamma^3(E^2B-EB^2)xy-8B(4E^3+3BE^2)x^2]\right\} \quad (97)$$

contributes to the integration respect to x. We get after integration

$$e^{\frac{i\pi}{4}}\sqrt{\frac{16B\pi}{s\theta_0(12E^4+8E^3B+E^2B^2)}} \times\exp\left[is\theta_0\frac{B(E^2B-EB^2)^2y^2}{(12E^4+8E^3B+E^2B^2)}\right]. \quad (98)$$

Considering Eq. (98), the expression

$$\exp\left\{is\theta_0\frac{B(E^2B-EB^2)^2y^2}{(12E^4+8E^3B+E^2B^2)}+\frac{is\theta_0}{16B}E^2B^2y^2 +\frac{is\theta_0}{16E}E^2B^2y^2+\frac{is\theta_0}{16E}(2B^2+3EB)^2y^2 +\frac{is\theta_0}{2}(2B^3+B^2E)y^2\right\} =\exp\left\{is\theta_0\frac{E(4B^6+76EB^5+258E^2B^4+494E^3B^3+224E^4B^2+12E^5B)y^2}{16(12E^4+8E^3B+E^2B^2)}\right\} \quad (99)$$

contributes to the integration with respect to y. We get after integration:

$$e^{\frac{i\pi}{4}}\sqrt{\frac{16\pi(12E^4+8E^3B+E^2B^2)}{s\theta_0E(4B^6+76EB^5+258E^2B^4+494E^3B^3+224E^4B^2+12E^5B)}}. \quad (100)$$

Consider $f(E,B)$ as a positive defined function given by the following:

$$f(E,B)=[4B^6+76EB^5+258E^2B^4+494E^3B^3 +224E^4B^2+12E^5B]^{-1}. \quad (101)$$

Hence, it is straightforward to obtain

$$\int dx\,dy\prod_{j=1}^{4}\mathcal{K}_j\exp[is\theta_0\mathcal{G}_2]=-\frac{16\pi^3}{s^3EB\theta_0^3}\sqrt{\frac{B}{E}f(E,B)}. \quad (102)$$

We choose the subset $M\in\mathbb{R}^2$ such that $\int\frac{dt}{t_0}dz=b$ (for example M could be the unit disk of $\mathbb{R}^2$). We find

$$\int dt\,dz\,dx\,dy\int d\mathbf{k}\,[is\theta_0\langle\mathbf{k}|\mathcal{G}(E,B,\theta)|\mathbf{k}\rangle] =-\frac{16\pi^3b}{s^3EB\theta_0^3}\sqrt{\frac{B}{E}f(E,B)}\exp[is\theta_0\mathcal{G}_0], \quad (103)$$

with $\int\frac{dt}{t_0}dz=b\equiv[L^2]$. Finally the following expression is obviously satisfied:

$$\operatorname{tr}\mathcal{O}=(1-\wp-\sigma(\theta_0,E,B))\operatorname{tr}\mathcal{O}_c, \quad (104)$$

where

$$\operatorname{tr}\mathcal{O}_c=\frac{1}{4\pi^2i}EB\cos h(Es)\cot(Bs). \quad (105)$$

This ends the Proof of Theorem 1.

Appendix B: Proof of Theorem 2

This section is devoted to the proof of Theorem 2. To evaluate the integral (54) before getting (55), we need to collect information as regards the physical properties in the limit where the magnetic field B tends to zero. This is clearly given in [25]. However, we think that it may be instructive to collect here all the arguments and rewrite the complete proof for our purpose. All the integrals will be performed in the complex half-plane. We will select only the positive half-plane. Consider first the integral $\int_{-\infty}^{\infty}ds\,\frac{e^{ism^2}}{s^3}$. Using the residue theorem, we simply get

$$\int_{-\infty}^{\infty}ds\,\frac{e^{ism^2}}{s^3}=-i\pi\frac{m^4}{2}. \quad (106)$$

Let us consider $\int_{-\infty}^{\infty}ds\,\frac{e^{ism^2}}{s}\cot h(Es)\cot(Bs)$. Let $h(z)=\frac{e^{izm^2}}{z}\cot h(Ez)\cot(Bz),\ z\in\mathbb{C}$. The integrand has singularities at point $z=0$ (poles of order 3), at $z=\frac{in\pi}{E}$ and $z=\frac{n\pi}{B}$

(simple poles). Let $\mathrm{Res}(z_0)$ be the residue of $h(z)$ at the point $z_0 \in \mathbb{C}$. We write the Taylor expansion of $\cot(z)$ and $\cot h(z)$ at point z_0 as

$$\cot(z) = \frac{1}{z-z_0} + \sum_{k=1}^{\infty} (-1)^k 2^{2k} \frac{B_{2k}}{(2k)!}(z-z_0)^{2k-1} = \frac{1}{z-z_0} - \frac{z-z_0}{3} - \frac{(z-z_0)^3}{45} + \cdots \tag{107}$$

and

$$\cot h(z) = \frac{1}{z-z_0} + \sum_{k=1}^{\infty} 2^{2k} \frac{B_{2k}}{(2k)!}(z-z_0)^{2k-1} = \frac{1}{z-z_0} + \frac{z-z_0}{3} - \frac{(z-z_0)^3}{45} + \cdots, \tag{108}$$

where B_n stands for the Bernoulli numbers with the initial values ($B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, $B_{2n-1} = 0, n = 2, 3, \ldots$). After taking into account the Taylor expansion of $\cot h(Ez)\cot(Bz)$ in Eqs. (107) and (108), and the fact that $\cot(iz) = -i \tan h(z)$, we get simply

$$\mathrm{Res}(0) = -\frac{m^4}{2BE}, \tag{109}$$

$$\mathrm{Res}\left(\frac{n\pi}{B}\right) = \frac{1}{n\pi} \exp\left(im^2 \frac{n\pi}{B}\right) \cot h\left(\frac{n\pi E}{B}\right), \tag{110}$$

and

$$\mathrm{Res}\left(\frac{in\pi}{E}\right) = -\frac{1}{\pi n} \exp\left(-\frac{n\pi m^2}{E}\right) \cot h\left(\frac{n\pi B}{E}\right). \tag{111}$$

Then

$$\int_{-\infty}^{\infty} \mathrm{d}s \frac{\mathrm{e}^{ism^2}}{s} \cot h(Es)\cot(Bs) = 2i\pi \sum_{n=1}^{\infty} \left[\frac{1}{n\pi} \exp\left(im^2 \frac{n\pi}{B}\right) \cot h\left(\frac{n\pi E}{B}\right) - \frac{1}{n\pi} \exp\left(-\frac{n\pi m^2}{E}\right) \cot h\left(\frac{n\pi B}{E}\right)\right] - i\pi \frac{m^4}{2BE}. \tag{112}$$

By multiplying the above result by EB it is clear that the limit $B \to 0$ is not well defined. This is why Eq. (110) cannot be taken into account in the physical situation. Therefore (112) reduces to

$$\int_{-\infty}^{\infty} \mathrm{d}s \frac{\mathrm{e}^{ism^2}}{s} \cot h(Es)\cot(Bs) = -i\pi \frac{m^4}{2BE} - 2i\pi \sum_{n=1}^{\infty} \left[\frac{1}{n\pi} \mathrm{e}^{-\frac{n\pi m^2}{E}} \cot h\left(\frac{n\pi B}{E}\right)\right]. \tag{113}$$

Now let $k(z) = \frac{\mathrm{e}^{izm^2}}{z^4} \cot h(Ez)\cot(Bz)$, $z \in \mathbb{C}$. The integrand has singularities at point $z = 0$ (poles of order 6), at $z = \frac{in\pi}{E}$ and $z = \frac{n\pi}{B}$ (simple poles). Using the same argument as (109), (110), and (111) we get, respectively,

$$\mathrm{Res}(0) = \frac{im^{10}}{120BE}, \tag{114}$$

$$\mathrm{Res}\left(\frac{n\pi}{B}\right) = \frac{B^3}{(n\pi)^4} \exp\left(im^2 \frac{n\pi}{B}\right) \cot h\left(\frac{n\pi E}{B}\right) \tag{115}$$

and

$$\mathrm{Res}\left(\frac{in\pi}{E}\right) = \frac{iE^3}{(\pi n)^4} \exp\left(-\frac{n\pi m^2}{E}\right) \cot h\left(\frac{n\pi B}{E}\right). \tag{116}$$

Now we come to the interpretation of Eqs. (114), (115), and (116).

- Equations (114) and (116) lead to a complex probability density and then cannot be taken into account.
- As we have seen in Eq. (110), Eq. (115) leads to a singularity at the limit $B \to 0$. This pointless expression also will not contribute to $\Re_e(\omega)$.

The same analysis can be provided, using the holomorphic function

$$\frac{\cot h(Ez)\cot(Bz)}{z^4} \exp[iz(m^2 + \theta_0 \mathcal{G}_0)].$$

Taking into account the above two comments, we conclude that the function $\sigma(\theta_0, E, B)$ does not contribute to the probability density, and therefore the positive integer n disappears in Eq. (55). Finally, by taking into account only Eq. (113), Theorem 2 is proved.

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