Published for SISSA by Springer

Received: June 17, 2014

Revised: September 2, 2014

Accepted: September 10, 2014

Published: October 8, 2014

Particle-vortex and Maxwell duality in the $\mathrm{AdS}_4 \times \mathbb{CP}^3$/ABJM correspondence

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Abstract: We revisit the notion of particle-vortex duality in abelian theories of complex scalar fields coupled to gauge fields, formulating the duality as a transformation at the level of the path integral. This transformation is then made symmetric and cast as a self-duality that maps the original theory into itself with the role of particles and vortices interchanged. After defining the transformation for a pure Chern-Simons gauge theory, we show how to embed it into (a sector of) the $(2+1)$-dimensional ABJM model, and argue that this duality can be understood as being related to 4-dimensional Maxwell duality in the $\mathrm{AdS}_4 \times \mathbb{CP}^3$ bulk.

Keywords: Duality in Gauge Field Theories, Solitons Monopoles and Instantons, AdS-CFT Correspondence

ArXiv ePrint: 1404.5926

Article funded by SCOAP[3].

doi:10.1007/JHEP10(2014)051

Contents

1 Introduction

Non-perturbative dualities in quantum field theories have delivered many profound insights over the past three or so decades. Most famous among these are the lessons that we have learned about the very nature of spacetime via the duality between strongly coupled quantum field theories and theories of gravity as manifested in the AdS/CFT correspondence [1]. Within the realm of quantum field theories alone, non-perturbative dualities rely on the fact that the generating functions of observables include an integration over the degrees of freedom. Consequently, the choice of degrees of freedom with which we describe the system may result in multiple possibilities. In four dimensions, for example, the electromagnetic duality, manifest in the Maxwell equations, allows us to describe a system in terms of electric or magnetic fields and charges and exchanges fundamental particles for solitonic degrees of freedom. We therefore have a choice as to how we describe the system, and at the perturbative level, one or the other may be more appropriate depending on the problem at hand. This electric-magnetic duality (and its extension by Witten and Olive [2]) has had a powerful impact, not only on our understanding of the structure of gauge theories, but also on some of the deepest mathematical puzzles of our time [3].

A 3-dimensional analogue of the 4-dimensional duality above is one which exchanges fundamental particles with solitonic vortices but, defined only for abelian gauge theories, this particle-vortex duality, as well as its physical implications, is much less understood than its 4-dimensional counterparts. To set the scene for what follows, we will first give an heuristic description of the particle-vortex duality elaborating on a discussion in the textbook of Zee [4], before embarking on a more technical treatment in the following section. Like many concepts commonplace in high energy theory, particle-vortex duality has its roots in the landscape of condensed matter; in this case in the theory of anyonic superconductivity [5]. After some limited further development in condensed matter physics, it was in the context of string theory that more development occured, starting with Intriligator and Seiberg [6]. Following [4] then, we start with an abelian Higgs model

$$\mathcal{L} = -\frac{1}{2}|(\partial_\mu - iqA_\mu)\phi|^2 - V\left(\phi^\dagger\phi\right), \tag{1.1}$$

with some well-behaved potential, $V\left(\phi^\dagger\phi\right)$ for the complex scalar ϕ. We will ignore the potential term from now on, but presume that the theory exhibits vortex solutions (and consequently restrict our attention to three dimensions). Writing $\phi = |\phi|e^{i\theta}$ and restricting to the solution for which $|\phi| = v$ minimizes the potential gives

$$\mathcal{L} = -\frac{1}{2}v^2(\partial_\mu\theta - qA_\mu)^2. \tag{1.2}$$

We can introduce a non-dynamical field ξ_μ and write the Lagrangian in first order form as

$$\mathcal{L} = +\frac{1}{2v^2}\xi_\mu^2 - \xi^\mu(\partial_\mu\theta - qA_\mu)\,. \tag{1.3}$$

The phase θ, characterizing the vortex is, in fact, singular at the origin for a vortex solution, allowing us to split it into a smooth part, and a vortex part:

$$\theta = \theta_{\text{smooth}} + \theta_{\text{vortex}}\,, \tag{1.4}$$

where the vortex monodromy $\Delta\theta_{\text{vortex}} = 2\pi$. Integrating out θ_{smooth} gives $\partial_\mu\xi^\mu = 0$ and implies that we can write ξ^μ as the curl of a vector field

$$\xi^\mu = \epsilon^{\mu\nu\rho}\partial_\nu a_\rho\,. \tag{1.5}$$

Having integrated out θ_{smooth} and substituted in the new expression for ξ^μ, we get the following Lagrangian

$$\mathcal{L} = -\frac{1}{4v^2}f_{\mu\nu}^2 + \epsilon^{\mu\nu\rho}\partial_\nu a_\rho(\partial_\mu\theta_{\text{vortex}} - qA_\mu)\,, \tag{1.6}$$

where $f_{\mu\nu}$ is the field strength tensor for a_μ. A subsequent integration by parts and rewriting of the resulting term as

$$a_\rho\epsilon^{\rho\mu\nu}\partial_\mu\partial_\nu\theta_{\text{vortex}} = 2\pi a_\mu j^\mu_{\text{vortex}}\,, \tag{1.7}$$

finally gives

$$\mathcal{L} = -\frac{1}{4v^2} f_{\mu\nu}^2 + 2\pi a_\mu j^\mu_{\text{vortex}} - A_\mu J^\mu, \tag{1.8}$$

where $J^\mu = q\epsilon^{\mu\nu\rho}\partial_\nu a_\rho$. Note that equation (1.7) is a crucial step in this derivation. Since the derivatives are contracted with the epsilon tensor, by symmetry arguments this expression will naively vanish. The only time this will not be the case is when there are singularities in θ. Thus, θ_{vortex} is explicitly that part of θ which is not smooth and whose second derivative is related through this equation to the vortex current. *If there are no vortices then this expression will vanish and there will be no duality.*

The introduction of an auxiliary vector field ξ_μ leads to the coupling of the vortex current to the gauge field a_μ. We have gone from a description where the fundamental excitations are the particles associated to the field θ to the vortex description where the fundamental degrees of freedom are the vortices associated to θ_{vortex}. However, to complete this description, we must have a field whose fundamental excitations themselves are vortices. To that end, we introduce a new field Φ which couples to a_μ precisely for this purpose. On adding this field, we can define an action which gives a dual description, with particle and vortex degrees of freedom swapped. The Lagrangian

$$\mathcal{L} = -\frac{1}{4v^2} f_{\mu\nu}^2 - \frac{1}{2}|(\partial_\mu - i2\pi a_\mu)\Phi|^2 - V\left(|\Phi|^2\right) - A_\mu(q\epsilon^{\mu\nu\rho}\partial_\nu a_\rho)\,, \tag{1.9}$$

then describes an abelian Higgs model for the vortex field Φ coupled to a_μ as opposed to the original field where θ was coupled to A_μ. The action of the transformation

$$\partial_\mu\theta - qA_\mu = \xi_\mu = \epsilon_{\mu\nu\rho}\partial^\nu a^\rho\,, \tag{1.10}$$

exchanges the scalar degree of freedom θ with the gauge field degree of freedom a_μ in the presence of the background gauge field A_μ. However, the necessity of introducing the new field Φ does not feel very satisfactory. We will see that there is a more complete way to formalise the duality.[1] The above transformation is also not strictly true in the presence of Φ.

A supersymmetric generalization of these ideas was proposed in [8] (see also [9]), however a path integral transformation realizing the particle-vortex duality could only be reduced to an unproven identity. Witten [10] later defined an $Sl(2,\mathbb{Z})$ transformation on a conformal field theory by combining an S-transformation (which adds an $\epsilon B\partial A$ term to the Lagrangian) with a T-transformation (which adds a Chern Simons term, $\epsilon A\partial A$). For example, starting with a charged scalar Lagrangian of the form $\tilde{L}(\Phi, A)$, the TS-transformation maps

$$\tilde{L}(\Phi, A) \xrightarrow{TS} L(\Phi, A, B) = \tilde{L}(\Phi, A) + \epsilon^{ijk}B_i\partial_j A_k + \epsilon^{ijk}A_i\partial_j A_k\,. \tag{1.11}$$

The current-current two-point function of this three-dimensional CFT is constrained by conformal symmetry to be of the form:

$$\langle J_i(k)J_j(-k)\rangle = \left(\delta_{ij}k^2 - k_ik_j\right)\frac{t}{2\pi\sqrt{k^2}} + \epsilon_{ijk}k_k\frac{w}{2\pi}\,, \tag{1.12}$$

[1] A more precise definition of particle-vortex duality, and an undertanding of how it arises in a path integral formulation was given by Burgess and Dolan in [7]. For completeness, we review their formulation in appendix A.

JHEP10(2014)051

where t and w form a complex coupling $\tau = w + it$. The action of the TS-transformations of the $Sl(2,\mathbb{Z})$ group on the complex parameter τ is then $\tau \to (a\tau + b)/(c\tau + d)$. Because this is an action on a conformal field theory, we can ask what the action of the transformation is on the gravity dual of this theory via the AdS/CFT correspondence. In this case the transformation acts on a U(1) gauge field with a Maxwell action plus a topological theta term.

A while later, the constraints imposed on correlators in gauge theories by the existence of a particle-vortex duality were analysed in [11]. Note that when the theory is changed by the action of the duality, (i.e. the theory is not self-dual), the correlators are themselves transformed. The authors also analyzed the $\mathrm{AdS}_4 \times S^7$ gravity dual of the $\mathcal{N} = 8$ three-dimensional SU(N) SYM in the large N limit, and found that Maxwell duality in the bulk leads to the same type of constraints on correlators as would be obtained from a self-dual field theory. In abelian models a similar relation was obtained, and a correspondence with $\mathrm{AdS}_4 \times S^7$ was proposed as an implicit relation coming from large N non-abelian gauge theories.

Today, the ABJM model [12] is understood as the correct description of the field theory living on M2-branes and is dual (in the appropriate limit) to a type IIA supergravity on $\mathrm{AdS}_4 \times \mathbb{C}P^3$. This begs the question as to whether the results of [11] can be reinterpreted from this point of view.[2]

The aim of this article is two-fold: first we seek to provide a more precise definition of the particle-vortex duality at the level of a path integral transformation then, using this, we attempt to embed the duality transformation in the ABJM model.

The structure of the paper is as follows. In section 2 we revisit the formulation of the particle-vortex duality by retaining some features of the relation of [7] (reviewed in appendix A) and defining it as an action on the path integral of the theory. In particular we find that, by combining it with the Mukhi-Papageorgakis Higgs mechanism for three-dimensional Chern-Simons theories [15] (see also [16]), we can define it as a self-duality of abelian Chern-Simons theories. In section 3 we look explicitly at vortex solutions and the conditions under which they exist in such theories.

In section 4, we embed the particle-vortex duality in ABJM, showing that the abelian duality is part of the (large N) non-abelian theory. Finally, in section 5, we show that the particle-vortex duality is naturally obtained as the boundary relation corresponding to Maxwell duality in the bulk, using the AdS/CFT prescription. Thus, as in [17, 18], we see that by using an abelian reduction of ABJM to an interesting non-conformal theory, we learn something about the structure of ABJM.

2 Abelian particle-vortex duality in the path integral

In this section we will extend the path-integral formulation of [7] to give a better definition of the particle-vortex duality in abelian theories. To this end, let us consider a path integral for an abelian Higgs model consisting of a complex scalar field $\Phi = \Phi_0 e^{i\theta}$ coupled to a U(1)

[2] The ABJM theory is also known to admit a maximally supersymmetric mass deformation [13, 14], which not only allows us to go away from the conformal limit but also contains a rich spectrum of solitonic excitations.

gauge field a_μ. Any kinetic term for the gauge field will be no more than a spectator for the transformation, as in the Burgess-Dolan formulation described in appendix A and we will not include it in what follows. There will also be a potential term for Φ_0, $V(\Phi_0^2)$, but this will also be a spectator so we will also choose to omit it now. When we want to explicitly discuss vortex solutions however, the potential will be important and will be included. As long as we are never integrating over a_μ or Φ_0 we do not need to consider these last terms we have mentioned.

The partition function for the theory is

$$\begin{aligned} Z &= \int \mathcal{D}a_\mu \mathcal{D}\Phi_0 \mathcal{D}\theta \exp\left\{-\frac{i}{2}\int d^3x |(\partial_\mu - iea_\mu)\Phi|^2\right\} \\ &= \int \mathcal{D}a_\mu \mathcal{D}\Phi_0 \mathcal{D}\theta \exp\left\{-\frac{i}{2}\int d^3x \left[(\partial_\mu\Phi_0)^2 + (\partial_\mu\theta_{\text{smooth}} + \partial_\mu\theta_{\text{vortex}} + ea_\mu)^2\Phi_0^2\right]\right\}, \end{aligned} \tag{2.1}$$

where, as in the previous section, we have split the θ field into a smooth part, and a topologically non-trivial and non-smooth vortex part. We define $\lambda_\mu = \partial_\mu\theta$, after which we promote it to an independent variable in a first order formulation. $\lambda_\mu = \partial_\mu\theta$ follows from the constraint $\epsilon^{\mu\nu\rho}\partial_\nu\lambda_\rho = 0$, which can be imposed via a Lagrange multiplier b_μ, giving the path integral for the 'master' action

$$\begin{aligned} Z = \int \mathcal{D}a_\mu \mathcal{D}\Phi_0 \mathcal{D}b_\mu \mathcal{D}\lambda_\mu \exp\left\{-\frac{i}{2}\int d^3x \left[(\partial_\mu\Phi_0)^2 + (\lambda_{\mu,\text{smooth}} + \lambda_{\mu,\text{vortex}} + ea_\mu)^2\Phi_0^2\right.\right. \\ \left.\left. + \frac{1}{e}\epsilon^{\mu\nu\rho}b_\mu\partial_\nu\lambda_\rho\right]\right\}. \end{aligned} \tag{2.2}$$

Integrating over b_μ returns us to the original formulation for the partition function, establishing the self-consistency of our procedure. If however we integrate over λ_μ first, we obtain the equation of motion

$$(\lambda_{\mu,\text{smooth}} + \lambda_{\mu,\text{vortex}} + ea_\mu)\, e\Phi_0^2 = -\epsilon^{\mu\nu\rho}\partial_\nu b_\rho\,, \tag{2.3}$$

which, on substitution back into the action produces the path integral for the *dual* action,

$$\begin{aligned} Z = \int \mathcal{D}a_\mu \mathcal{D}\Phi_0 \mathcal{D}b_\mu \exp\left\{-i\int d^3x \left[\frac{1}{4e^2\Phi_0^2} f^b_{\mu\nu} f^{b\mu\nu} + \epsilon^{\mu\nu\rho}b_\mu\partial_\nu a_\rho - \frac{2\pi}{e} j^\mu_{\text{vortex}}(t) b_\mu\right.\right. \\ \left.\left. + \frac{1}{2}(\partial_\mu\Phi_0)^2\right]\right\}, \end{aligned} \tag{2.4}$$

where $j^\mu_{\text{vortex}}(t)$ is the vortex current in (A.9), i.e.,

$$j^\mu_{\text{vortex}}(t) = \frac{1}{2\pi}\epsilon^{\mu\nu\rho}\partial_\nu\partial_\rho\theta = \frac{1}{2\pi}\epsilon^{\mu\nu\rho}\partial_\nu\partial_\rho\omega = \sum_a N_a\, \dot{y}^\mu_a\, \delta[x - y_a(t)]\,, \tag{2.5}$$

and is associated with the existence of vortex boundary conditions for θ in the original action with vortices positioned at $\vec{y}_a(t)$ in the two dimensional space (see equation (A.5) for a definition of ω). In the dual action, it appears as an explicit source term. Here, the sum is over all vortex positions labeled by the index a. Also note that, as in (A.19),

$j_\mu = e\Phi_0^2 \partial_\mu \theta$ is a scalar current, and we have then the duality relation between the vortex current and the scalar current:

$$j^\mu_{\rm vortex}(t) = \frac{1}{2\pi e \Phi_0^2} \epsilon^{\mu\nu\rho} \partial_\nu j_\rho \,. \tag{2.6}$$

Notice that here Φ_0 has the interpretation of a coupling constant for the field b_μ dual to θ, which itself becomes a dynamical Maxwell gauge field. In this sense this duality maps particles to vortices, justifying the name *particle-vortex duality*.

2.1 The Mukhi-Papageorgakis Higgs mechanism

There is a striking similarity between the particle-vortex duality described here and a version of the Higgs mechanism for three-dimensional Chern-Simons theories discovered by Mukhi and Papageorgakis in [15] in the context of ABJM theories, but valid more generally (see also [16] for more details about its implementation). The statement analogous to the usual Higgs mechanism statement that a massless gauge field eats a scalar and becomes massive, is now that a Chern-Simons gauge field (with no dynamical degrees of freedom) eats a scalar and becomes dynamical, i.e. of Maxwell (or Yang-Mills) form with one dynamical degree of freedom.

The mechanism itself goes as follows. We start with an action for a complex scalar, Ψ, coupled to a Chern-Simons gauge field, a_μ,

$$S = -\int d^3x \left[\frac{k}{2\pi} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu \tilde{a}_\rho + \frac{1}{2} |\left(\partial_\mu - iea_\mu\right)\Psi|^2 + V\left(|\Psi|^2\right)\right] , \tag{2.7}$$

with a vacuum solution $\Psi = b$. We can then expand the scalar degrees of freedom around the ground state

$$\Psi = (b + \delta\psi) e^{-i\delta\theta}; \quad \delta\theta = \theta_{\rm smooth} + \theta_{\rm vortex} \,, \tag{2.8}$$

and plug it back in the action to find

$$S = -\int d^3x \left[\frac{k}{2\pi} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu \tilde{a}_\rho + \frac{1}{2}(\partial_\mu \delta\psi)^2 + \frac{1}{2}(\partial_\mu \theta_{\rm smooth} + \partial_\mu \theta_{\rm vortex} + ea_\mu)^2 b^2 + \dots \right] . \tag{2.9}$$

Here, the omitted terms come from the $\delta\psi$ self-interaction in $V\left(|\Psi|^2\right)$ and the $\delta\theta$-$\delta\psi$ interaction. Note that, for the purposes of making a comparison, we have allowed for the possibility that $\delta\theta$ contains a vortex piece $\theta_{\rm vortex}$. The mechanism by which the Chern-Simons vector eats the scalar and becomes a dynamical Maxwell vector happens through exactly the same redefinition as in the the usual Higgs mechanism. Here we write

$$ea_\mu + \partial_\mu \theta_{\rm smooth} + \partial_\mu \theta_{\rm vortex} = ea'_\mu \,, \tag{2.10}$$

trivially integrate out θ and add a boundary term to the action to obtain

$$S = -\int d^3x \left[\frac{k}{2\pi} \epsilon^{\mu\nu\rho} a'_\mu \partial_\nu \tilde{a}_\rho + \frac{1}{2}\left(\partial_\mu \delta\psi\right)^2 + \frac{1}{2}\left(ea'_\mu\right)^2 b^2 - \frac{k}{e} j^\mu_{\rm vortex} \tilde{a}_\mu + \dots \right] . \tag{2.11}$$

Solving for a'_μ gives

$$a^\mu + \frac{1}{e}\partial^\mu \delta\theta = a'^\mu = -\frac{k}{2\pi b^2} \epsilon^{\mu\nu\rho} \partial_\nu \tilde{a}_\rho \,, \tag{2.12}$$

which is similar to (2.3) for the particle-vortex duality. Defining $\tilde{f}_{\mu\nu} = \partial_\mu \tilde{a}_\nu - \partial_\nu \tilde{a}_\mu$, we find

$$S = \int d^3x \left[-\frac{k^2}{16\pi^2 b^2} \left(\tilde{f}_{\mu\nu} \right)^2 - \frac{1}{2} \left(\partial_\mu \delta\psi \right)^2 + \frac{k}{e} j^\mu_{\text{vortex}} \tilde{a}_\mu + \dots \right] , \tag{2.13}$$

where again some nonlinear terms in the fluctuations — specifically, the self-interactions of $\delta\psi$ coming from $V\left(|\Psi|^2\right)$ and terms that would appear when replacing b in the Maxwell coupling with $|\Psi| = b + \delta\psi$ — are omitted with impunity, since they could be reintroduced by simply writing $|\Psi|$ instead of b and retaining $V\left(|\Psi|^2\right)$. We close this section with two points of note. Firstly, the addition of a term $-\epsilon^{\mu\nu\rho}\tilde{a}_\mu \partial_\nu b_\rho$ to either (2.7) or (2.13) can be made without changing anything since the transformations don't act on either $\tilde{a}$ or b. Second, assuming vortex boundary conditions in the initial action gives a vortex current coupling in the final action. Again, this is as in the case of particle-vortex duality, although here we can assume regular boundary conditions and thus avoid the vortex current j^μ_{vortex}.

2.2 A symmetric duality

As described in the previous section, particle-vortex duality is not a self-duality, in that it maps the original action to a manifestly different action. In particular it dualizes the scalar angle θ to the gauge field b_μ. For our purposes of embedding the duality in the ABJM model, it will be useful to 'symmetrize' this duality. As we demonstrate now, this may be acheived by adding a gauge field and a real scalar, and dualizing them to a complex scalar. This means that the original and final action will look the same. As before, we may also add vortex currents. We will also omit a possible kinetic term for a_μ and explicitly write the self-interactions of the scalars Φ and χ. Our launching point, again, will be the path integral

$$\begin{aligned} Z = \int \mathcal{D}a_\mu \mathcal{D}\Phi_0 \mathcal{D}\chi_0 \mathcal{D}\theta \mathcal{D}\tilde{b}_\mu \exp\left\{ -i \int d^3x \left[\frac{1}{2} |(\partial_\mu - iea_\mu)\Phi_0 e^{-i\theta}|^2 + \frac{1}{2}(\partial_\mu \chi_0)^2 \right.\right. \\ \left.\left. + \frac{1}{4e^2\chi_0^2} f^{(\tilde{b})}_{\mu\nu} f^{(\tilde{b})\mu\nu} + \epsilon^{\mu\nu\rho} a_\mu \partial_\nu \tilde{b}_\rho - \frac{2\pi}{e} \tilde{b}_\mu \tilde{j}^\mu_{\text{vortex}}(t) + V\left(\Phi_0^2\right) + V\left(\chi_0^2\right) \right] \right\} , \end{aligned} \tag{2.14}$$

where $\tilde{j}^\mu_{\text{vortex}}(t)$ is a source term that, in the dual version, will be associated to vortex boundary conditions for the dual scalar. $\tilde{b}_\mu$ is our new gauge field and χ_0, the new scalar. It is the addition of these two that will lead to a self-dual action. We again write a first order formulation for $\lambda_\mu = \partial_\mu \theta$ and then impose this relation as the constraint $\epsilon^{\mu\nu\rho}\partial_\nu \lambda_\rho = 0$ through a Lagrange multiplier b_μ. Conversely, we can define $\tilde{\lambda}_\mu$ via a tilde version of (2.3), namely

$$\left(\tilde{\lambda}_{\mu,\text{smooth}} + \tilde{\lambda}_{\mu,\text{vortex}} + ea_\mu \right) e\chi_0^2 = -\epsilon^{\mu\nu\rho}\partial_\nu \tilde{b}_\rho \,, \tag{2.15}$$

and then introduce $\tilde{\lambda}_\mu$ in the action such that we have the above equation as its equation of motion. Either way, we obtain the path integral for the master action

$$\begin{aligned} Z = \int \mathcal{D}a_\mu \mathcal{D}\Phi_0 \mathcal{D}\chi_0 \mathcal{D}\lambda_\mu \mathcal{D}b_\mu \mathcal{D}\tilde{\lambda}_\mu \mathcal{D}\tilde{b}_\mu & \qquad (2.16) \\ \exp\left\{ -i \int d^3x \left[\frac{1}{2}(\partial_\mu \Phi_0)^2 + \frac{1}{2}(\partial_\mu \chi_0)^2 + \frac{1}{e}\epsilon^{\mu\nu\rho} \left(b_\mu \partial_\nu \lambda_\rho + \tilde{b}_\mu \partial_\nu \tilde{\lambda}_\rho \right) \right.\right. & \\ \left.\left. + \frac{1}{2}(\lambda_\mu + \lambda_{\mu,\text{vortex}} + ea_\mu)^2 \Phi_0^2 + \frac{1}{2}\left(\tilde{\lambda}_\mu + \tilde{\lambda}_{\mu,\text{vortex}} + e\tilde{a}_\mu \right)^2 \chi_0^2 + V\left(\Phi_0^2\right) + V\left(\chi_0^2\right) \right] \right\} . & \end{aligned}$$

JHEP10(2014)051

Now repeating the same procedure for the fields with tilde replaced with untilde (or, equivalently, integrating over λ_μ and $\tilde{b}_\mu$, to write $\tilde{\lambda}_\mu = \partial_\mu \tilde{\theta}$), we obtain the path integral for the dual action

$$Z = \int \mathcal{D}a_\mu \mathcal{D}\Phi_0 \mathcal{D}\chi_0 \mathcal{D}\tilde{\theta}\mathcal{D}b_\mu \exp\left\{-i\int d^3x \left[\frac{1}{2}|(\partial_\mu - iea_\mu)\chi_0 e^{-i\tilde{\theta}}|^2 + \frac{1}{2}(\partial_\mu\Phi_0)^2 \right.\right.$$
$$\left.\left. + \frac{1}{4e^2\Phi_0^2} f^{(b)}_{\mu\nu} f^{(b)\mu\nu} + \epsilon^{\mu\nu\rho} a_\mu \partial_\nu b_\rho - \frac{2\pi}{e} b_\mu j^\mu_{\text{vortex}}(t) + V\left(\Phi_0^2\right) + V\left(\chi_0^2\right)\right]\right\}. \tag{2.17}$$

Assuming that a_μ has no kinetic term, we can now actually integrate it out in both the original and dual actions. Indeed, the terms containing a_μ in the Lagrangian (equation (2.14)) are

$$\mathcal{L}^{(a)} = -\frac{1}{2}e^2 a_\mu^2 |\Phi_0|^2 - a^\mu (j_\mu + J_\mu)\,, \tag{2.18}$$

with $j_\mu = -\frac{ie}{2}(\Phi\partial_\mu\Phi^* - \Phi^*\partial_\mu\Phi) = e\partial_\mu\theta$, the scalar current and topological (vortex-like) current $J^\mu = \epsilon^{\mu\nu\rho}\partial_\nu\tilde{b}_\rho$. Solving for a_μ we obtain

$$a_\mu = -\frac{1}{e^2\Phi_0^2}(j_\mu + J_\mu)\,, \tag{2.19}$$

and substituting back into $\mathcal{L}^{(a)}$, produces an extra contribution

$$\mathcal{L}_{\text{extra}} = +\frac{1}{2e^2\Phi_0^2}(j_\mu + J_\mu)^2 = -\frac{1}{4e^2\Phi_0^2}\left(f^{\tilde{b}}_{\mu\nu} - \epsilon_{\mu\nu\rho} j^\rho\right)^2. \tag{2.20}$$

Having thus eliminated a_μ from the picture, we are now in a position to realize the duality as a map from

$$Z = \int \mathcal{D}\Phi_0 \mathcal{D}\chi_0 \mathcal{D}\theta \mathcal{D}\tilde{b}_\mu \exp\left\{-i\int d^3x \left[\frac{1}{2}|\partial_\mu\left(\Phi_0 e^{-i\theta}\right)|^2 + \frac{1}{2}(\partial_\mu\chi_0)^2 + \frac{1}{4e^2\chi_0^2} f^{(\tilde{b})}_{\mu\nu} f^{(\tilde{b})\mu\nu}\right.\right.$$
$$\left.\left. + \frac{1}{4e^2\Phi_0^2}\left(f^{(\tilde{b})}_{\mu\nu} - \epsilon_{\mu\nu\rho} j^\rho\right)^2 - \frac{2\pi}{e}\tilde{b}_\mu \tilde{j}^\mu_{\text{vortex}}(t) + V\left(\Phi_0^2\right) + V\left(\chi_0^2\right)\right]\right\}, \tag{2.21}$$

into

$$Z = \int \mathcal{D}\Phi_0 \mathcal{D}\chi_0 \mathcal{D}\tilde{\theta} \mathcal{D}b_\mu \exp\left\{-i\int d^3x \left[\frac{1}{2}|\partial_\mu\left(\chi_0 e^{-i\tilde{\theta}}\right)|^2 + \frac{1}{2}(\partial_\mu\Phi_0)^2 + \frac{1}{4e^2\Phi_0^2} f^{(b)}_{\mu\nu} f^{(b)\mu\nu}\right.\right.$$
$$\left.\left. + \frac{1}{4e^2\chi_0^2}\left(f^{(b)}_{\mu\nu} - \epsilon_{\mu\nu\rho}\tilde{j}^\rho\right)^2 - \frac{2\pi}{e} b_\mu j^\mu_{\text{vortex}}(t) + V\left(\Phi_0^2\right) + V\left(\chi_0^2\right)\right]\right\}, \tag{2.22}$$

that furnishes a formulation of the particle-vortex duality with an explicitly self-dual action.

Of course, since our aim is to embed the particle-vortex duality into the ABJM model and, in this case we have only scalars and a Chern-Simons gauge field at our disposal we will need to combine the symmetric form of the duality above with the Mukhi-Papageorgakis Higgs mechanism of the previous section. Moreover, in order for the duality to be nontrivial, we need to retain the vortex boundary conditions only in the original scalar, not the one that gets Higgsed. Starting from the path integral

$$Z = \int \mathcal{D}a_\mu \mathcal{D}\Phi_0 \mathcal{D}\theta \mathcal{D}\tilde{b}_\mu \mathcal{D}\chi \mathcal{D}\chi^* \mathcal{D}\mathcal{A}_\mu \exp\left\{-i\int d^3x \left[\frac{1}{2}|(\partial_\mu - iea_\mu)\Phi_0 e^{-i\theta}|^2\right.\right.$$
$$\left.\left. + \frac{1}{2}|(\partial_\mu - ie\mathcal{A}_\mu)\chi_0 e^{-i\phi}|^2 + \epsilon^{\mu\nu\rho}\left(\frac{1}{e}\mathcal{A}_\mu\partial_\nu\tilde{b}_\rho + a_\mu\partial_\nu\tilde{b}_\rho\right) + V\left(\phi_0^2\right) + V\left(\chi_0^2\right)\right]\right\}, \tag{2.23}$$

we first implement the Mukhi-Papageorgakis Higgs mechanism by shifting $\mathcal{A}_\mu \to \mathcal{A}'_\mu$ as in equation (2.10), absorbing ϕ and performing the (now trivial) path integral over ϕ. Subsequently, we integrate over $\mathcal{A}'_\mu$ using the equation of motion

$$e\mathcal{A}_\mu + \partial_\mu\phi + \partial_\mu\phi_{\text{vortex}} \equiv e\mathcal{A}'_\mu = \frac{1}{e^2\chi_0^2}\epsilon_\mu{}^{\nu\rho}\partial_\nu\tilde{b}_\rho\,, \tag{2.24}$$

and get exactly the path integral in (2.14) which, as we saw previously, is dual to that in (2.17). We now undo the Mukhi-Papageorgakis Higgs mechanism, by writing a first order formalism for $f^{(b)}_{\mu\nu}$ in terms of a field $\tilde{\mathcal{A}}'_\mu$, then introducing a trivial path integration over a variable $\tilde{\phi}$ and shifting $\mathcal{A}'_\mu$ by

$$e\tilde{\mathcal{A}}_\mu + \partial_\mu\tilde{\phi} + \partial_\mu\tilde{\phi}_{\text{vortex}} \equiv e\tilde{\mathcal{A}}'_\mu = \frac{1}{e^2\Phi_0^2}\epsilon^{\mu\nu\rho}\partial_\nu b_\rho\,, \tag{2.25}$$

so that we finally arrive at the path integral

$$\begin{aligned}Z = \int \mathcal{D}a_\mu\mathcal{D}\chi_0\mathcal{D}\tilde{\theta}\mathcal{D}b_\mu\mathcal{D}\Phi\mathcal{D}\Phi^*\mathcal{D}\tilde{\mathcal{A}}_\mu \exp\Bigg\{-i\int d^3x\,\Bigg[\frac{1}{2}|(\partial_\mu - iea_\mu)\chi|^2 + \frac{1}{2}|\left(\partial_\mu - ie\tilde{\mathcal{A}}_\mu\right)\Phi|^2 \\ +\epsilon^{\mu\nu\rho}\left(\frac{1}{e}\tilde{\mathcal{A}}_\mu\partial_\nu b_\rho + a_\mu\partial_\nu b_\rho\right) + V\left(\phi_0^2\right) + V\left(\chi_0^2\right)\Bigg]\Bigg\}\,,\end{aligned} \tag{2.26}$$

where now $\chi = \chi_0 e^{-i\tilde{\theta}}$ and $\Phi = \Phi_0 e^{-i\tilde{\phi}}$. Naively, it would seem that (2.24) undoes the duality transformation but it does not, since the interpretation is different. In the Higgs mechanism, we solve for $\mathcal{A}_\mu$ and ϕ, while retaining $\tilde{b}_\mu$ in the theory. In the particle-vortex duality, we exchange $\tilde{b}_\mu$ for $\tilde{\theta}$ and similarly for quantities with tilde and untilde exchanged.

3 Vortex solutions

To summarize the story so far; we have formulated a manifest duality in the path integral formalism and argued that such a duality should exchange particles with vortices. Obviously, in order to do so, we need to have vortex solutions in the theory. Until now we have simply *presumed* the existence of such vortices in the field theories under investigation. Clearly this will not be the case for all field theories of the form we have been discussing. Here, therefore, we devote some time to discuss constraints on the form of the potential which will lead to such solutions. Thus, we consider the action in the path integral (2.23). In order to do this one first writes down the full equations of motion, and *only afterwards* will sets $\chi = \tilde{b}_\mu = \mathcal{A}_\mu = 0$ (which is itself a solution of these equations). The remaining equations of motion then become

$$\begin{aligned}\epsilon^{\mu\nu\rho}\partial_\nu a_\rho &= 0\,, \\ \Phi(D_\mu\Phi)^\dagger - \Phi^\dagger D_\mu\Phi &= 0\,,\end{aligned} \tag{3.1}$$

and the equation of motion for Φ, which depends on the potential is

$$D_\mu D^\mu\Phi = \frac{dV}{d|\Phi|^2}\,. \tag{3.2}$$

Note that the first of equations (3.1) implies that a_μ must be pure gauge while the second equation means that

$$D_\mu \theta = 0 \Rightarrow \partial_\alpha \theta = a_\alpha \,, \tag{3.3}$$

where α is the polar angle in the complex plane, and θ is the argument of Φ, i.e. $\Phi = |\Phi| e^{i\theta}$. In particular this relation is valid at infinity. This gives the usual charge quantization condition $\oint d\alpha \, a_\alpha = \oint d\theta$ which, in turn, implies that $\theta = N\alpha$. From the 0-component of equation (3.3) we get for static solutions that $a_0 = 0$.

Note however that this result would imply as usual that $|\Phi(r=0)| = 0$ for consistency of the vortex ansatz. This in turn means that the second equation in (3.1) is already satisfied at $r = 0$, hence we don't need $a_\alpha = N$ at $r = 0$. That would be good, since substituting $a_\alpha = N$, requires that

$$\epsilon^{\mu\nu\rho} \partial_\nu \partial_\rho \theta \propto j^\mu_{\text{vortex}} \propto \delta(r) \,, \tag{3.4}$$

so the first of equations (3.1) would be satisfied everywhere except at $r = 0$ which would in turn imply a discontinuous form for a_μ at $r = 0$, necessitating some kind of regularization at this point. In fact, as we will soon demonstrate, in order to have a solution we need $|\Phi| \neq 0$ at $r = 0$. Consequently, the solution as it stands will be valid everywhere except at $r = 0$. It remains now to satisfy the $|\Phi|$ equation of motion in order to determine the vortex profile. We know already from (3.2) that any vortex solution must satisfy

$$\frac{|\Phi|''}{|\Phi|} = \frac{dV}{d|\Phi|^2} \,, \tag{3.5}$$

where, from general considerations about vortices, the one-vortex solution should behave like $|\Phi| \sim Ar$ as $r \to 0$. If in addition, we consider the most general renormalizable potential in three dimensions, namely the sextic, $V = C_1 |\Phi|^6 + \lambda |\Phi|^4 + m^2 |\Phi|^2$ for which $\frac{dV}{d|\Phi|^2} = m^2 + 2\lambda |\Phi|^2 + 3C_1 |\Phi|^4$, several cases of interest for the asymptotic behaviour of these solutions present themselves. They are (in no particular order):

- $m \neq 0$ and $\lambda \neq 0$. In this case, $V = C_1 |\Phi|^6 + \lambda |\Phi|^4 + m^2 |\Phi|^2$. Near the origin, we take as an ansatz for the field

$$|\Phi| \sim Ar + Cr^p + \dots \,. \tag{3.6}$$

This reduces the equation of motion in this region to

$$\frac{p(p-1)Cr^{p-2}}{Ar} = m^2 \,, \tag{3.7}$$

which fixes p to be 3 and $C = Am^2/6$. Therefore the small$-r$ form of the field is

$$|\Phi| \sim Ar \left(1 + \frac{m^2}{6} r^2 + \dots \right) \,. \tag{3.8}$$

Clearly we could go to any order analytically if needed. Taking the other asymptotic limit, if we chose that as $r \to \infty$, $|\Phi| \sim \tilde{A}/r^n$, there is an inconsistency for non-zero n as $|\Phi|''/|\Phi| \sim 1/r^2 \to 0$, whereas $dV/d|\Phi|^2 = m^2 + \dots$. To avoid this, we choose instead

$$|\Phi| \sim \tilde{A} + \frac{\tilde{B}}{r^n} + \dots \,. \tag{3.9}$$

With this ansatz, the equation of motion reduces to

$$\frac{\tilde{B}}{\tilde{A}}\frac{n(n+1)}{r^{n+2}}=\left(m^2+2\lambda\tilde{A}^2+3C_1\tilde{A}^4\right)+\left(\frac{4\lambda\tilde{A}\tilde{B}+12C_1\tilde{A}^3\tilde{B}}{r^n}\right)+\frac{4\lambda\tilde{B}^2}{r^{2n}}+\frac{18C_1\tilde{A}^2\tilde{B}^2}{r^{2n}}\,, \tag{3.10}$$

and we see that we need $n = 2$ to satisfy the radial behaviour, along with the constraint that the two parentheses must vanish separately. From the first of these we find,

$$m^2 + 2\lambda\tilde{A}^2 + 3C_1\tilde{A}^4 = 0\,, \tag{3.11}$$

which says that $|\Phi| = \tilde{A}$ is the nontrivial vacuum of the theory, satisfying $dV/d|\Phi|^2 = 0$. The vanishing of the second parenthesis requires $\lambda + 3C_1\tilde{A}^2 = 0$. Taken together, these two constraints give that,

$$\tilde{A}^2 = -\frac{m^2}{\lambda}\,, \qquad C_1 = \frac{\lambda^2}{3m^2}\,. \tag{3.12}$$

Note that the latter is a constraint on the potential, allowing for only a certain class of sixth order potentials with non-zero quadratic and quartic terms to lead to vortex solutions. This tells us that λ and thus C_1 need to be nonzero — i.e. the potential must be truly sextic. We then solve to the next order in r in the equation of motion, i.e. $1/r^4$, giving

$$\tilde{B} = \frac{3}{\tilde{A}\left(2\lambda + 9C_1\tilde{A}^2\right)}, \tag{3.13}$$

so that

$$|\Phi| \sim \tilde{A} + \frac{3}{\tilde{A}r^2\left(2\lambda + 9C_1\tilde{A}^2\right)} + \dots\,. \tag{3.14}$$

Clearly as $m \to 0$ this solution vanishes.

- $m = 0$ and $\lambda \neq 0$. In this case $V = C_1|\Phi|^6 + \lambda|\Phi|^4$. As before we take the asymptotics close to the vortex origin to be

$$|\Phi| \sim Ar + Cr^p + \dots\,. \tag{3.15}$$

The equation of motion is now

$$\frac{p(p-1)Cr^{p-2}}{Ar} \simeq 2\lambda|\Phi|^2 \simeq 2\lambda A^2 r^2\,, \tag{3.16}$$

which gives $p = 5$ and $C = \lambda\frac{A^3}{10}$ so that

$$|\Phi| \sim Ar\left(1 + \frac{\lambda A^2}{10}r^4 + \dots\right)\,. \tag{3.17}$$

Far away from the vortex we take $|\Phi| \sim \frac{\tilde{A}}{r^n}$ which reduces the equation of motion to

$$\frac{n(n+1)}{r^2} \simeq 2\lambda|\Phi|^2 = 2\lambda\frac{\tilde{A}^2}{r^{2n}}\,. \tag{3.18}$$

This fixes $n = 1$ and $\tilde{A} = 1/\sqrt{\lambda}$, meaning that

$$|\Phi| \sim \frac{1}{\sqrt{\lambda} r} + \dots \,. \tag{3.19}$$

Note that $\tilde{A} + \tilde{B}/r^n$ leads to a contradiction in the equations of motion and thus the leading term must be $\sim \frac{1}{r}$. In contrast to the first case above, there is no constraint on the potential.

- $m = 0$ and $\lambda = 0$. In this case $V = C_1|\Phi|^6$, a purely sextic potential). At $r = 0$, as above, we find

$$\frac{p(p-1)Cr^{p-2}}{Ar} = 3C_1|\Phi|^4 \sim 3C_1 A^4 r^4 \,, \tag{3.20}$$

which gives $p = 7$ and $C = C_1 A^5/14$, so

$$|\Phi| \sim Ar\left(1 + \frac{C_1 A^4}{14} r^6 + \dots\right) . \tag{3.21}$$

At infinity, with $|\Phi| \sim \tilde{A}/r^n$, the equation of motion is

$$\frac{n(n+1)}{r^2} = 3C_1 \frac{\tilde{A}^4}{r^{4n}} \,, \tag{3.22}$$

which gives $n = 1/2$ and $\tilde{A}^4 = 1/(4C_1)$, so that there

$$|\Phi| \sim \frac{1}{(4C_1)^{1/4}\sqrt{r}} \,. \tag{3.23}$$

Evidently, in the case of a massive potential in order to find a non-trivial solution the constraint ($C_1 = \lambda^2/(3m^2)$ and $m^2/\lambda < 0$) must be satisfied, whereas for the two massless scenarios there are always solutions. A simple check that will be carried out in the next section finds that the constraint is *not* satisfied in the case of the massive ABJM model. This will mean that an embedding of the duality into massive ABJM will not be possible and within massless ABJM, only the purely sextic potential will be relevant.

3.1 Pure sextic potential

It turns out that in the pure sextic case, $V = C_1|\Phi|^6$, we can solve everything explicitly using some simple considerations. The equation of motion is

$$|\Phi|'' = 3C_1|\Phi|^5 \,, \tag{3.24}$$

and we write it in terms of $v = |\Phi|'$ as

$$v\frac{dv}{d|\Phi|} = 3C_1|\Phi|^5 \,, \tag{3.25}$$

solved by

$$v^2 = C_1|\Phi|^6 + K_1 \Rightarrow |\Phi|' = \pm\sqrt{C_1|\Phi|^6 + K_1} \,. \tag{3.26}$$

The general solution is then

$$r + K_2/\sqrt{C_1} = \pm \int \frac{d|\Phi|}{\sqrt{C_1|\Phi|^6 + K_1}} . \tag{3.27}$$

Note however that if $|\Phi| \sim Ar$ at $r \sim 0$ and $|\Phi| \sim A/r^n$ at $r \sim \infty$ (for positive n), there must exist at least one place where $d|\Phi|/dr = 0$ in the middle, or where $v = 0$, which implies $C_1|\Phi|^6 + K_1 = 0$, i.e. $|\Phi|_{\rm mid} = (-K_1/C_1)^{\frac{1}{6}}$, which in turn means that[3] $K_1/C_1 < 0$. For the branch connected with $r = 0$ however, it is clear from equation (3.26) that having $K_1 < 0, C_1 > 0$ would mean that $|\Phi|'$ was imaginary and therefore that $C_1 < 0$ and $K_1 > 0$. However, this is inconsistent as we would have a runaway potential with no stable vacuum.

We must therefore choose $C_1 < 0$ and $K_1 > 0$. This choice is, if anything, worse since it implies that the potential is negative definite. In fact even if the vacuum were stable in this case there would be a problem because the solution

$$|\Phi|' = +\sqrt{K_1 - |C_1||\Phi|^6} , \tag{3.28}$$

until we reach $|\Phi|_{\rm mid}$, and thereafter

$$|\Phi|' = -\sqrt{K_1 - |C_1||\Phi|^6} . \tag{3.29}$$

This means that we would reach $|\Phi| = 0$ with nonzero derivative, $|\Phi|' = -\sqrt{K_1}$. Since $|\Phi| \geq 0$, this results in a singularity at this point, as $|\Phi|'$ would jump discontinously.

In other words, there is *no normal smooth solution for the vortex.* This will however not be a problem as the smoothness constraint is not required. We saw that in any case the solution is not valid at $r = 0$ itself, so we can ignore the constraint that $|\Phi| = 0$ there. With a little more thought it is clear that, with $C_1 > 0$ as it should be, the only solution that makes sense (which goes to zero at infinity) is one with $K_1 = 0$, since if $K_1 < 0$, $|\Phi|'$ must become imaginary before reaching $r = \infty$, and if $K_1 > 0$, $|\Phi|'$ must remain finite as $|\Phi| = 0$, which means it is again reached before $r = \infty$. Then the solution is

$$\sqrt{C_1} r + K_2 = -\int \frac{d|\Phi|}{|\Phi|^3} = \frac{1}{2|\Phi|^2} , \tag{3.30}$$

(we can easily see that the + in front of the integral also doesn't make sense), so that

$$|\Phi| = \frac{1}{\sqrt{\sqrt{C_1} 2r + 2K_2}} , \tag{3.31}$$

which has

$$|\Phi|'(0) = -\frac{\sqrt{C_1}}{\sqrt{2K_2}} , \tag{3.32}$$

which is finite, but as we said, we must excise and regularize an infinitesimal region around $r = 0$. To conclude this section, there is a strong constraint on the form of the sextic potential in the massive case which leads to a vortex solution, whereas for a purely sextic potential, there will be non-smooth solutions which, with excision of the irregular core, will correspond to vortices.

[3]Note that we can (and in general should) glue different branches of the solution at the point where $v = 0$.

The non-smooth solution will get smoothed out near $r = 0$ in a physical situation, in such a way that the simple variable separation ansatz $\Phi = |\Phi|(r)e^{i\theta}$ is not valid anymore in the vicinity of $r = 0$. This is very likely, since the energy of the solution, $3C_1/2 \int rdr \left[\sqrt{C_1}2r + 2K_2\right]^{-3}$, is finite (convergent both at $r = 0$ and $r = \infty$), so it should be related to a physical state.

4 Embedding particle-vortex duality in ABJM

In order to formulate the particle-vortex duality within ABJM we must be able to find an abelian reduction of the ABJM model which can both be mapped to the path integral in equation (2.23) as well as shown to fulfill the constraint which leads to vortex solutions. We will show below that while the mass-deformed ABJM theory has the appropriate mapping to the self-dual action with non-zero mass, the vortex constraints on the potential are not fulfilled and thus we can only get a self-dual theory with vortices in the massless case. See appendix A for a brief overview of the ABJM formalism.

4.1 Constructing a self-dual abelian reduction of ABJM

For the two bifundamental scalars, Φ and χ, of ABJM we split the N-dimensional matrix space into two (block-diagonal) $N/2$ dimensional subspaces. In doing so we will be able to use each of the sub-spaces to construct a self-duality under the particle-vortex transformation. In the first subspace, we write the ansatz

$$\begin{aligned}
A_\mu &= a_\mu^{(1)} \mathbf{1}_{N/2\times N/2}\,, \\
\hat{A}_\mu &= \hat{a}_\mu^{(1)} \mathbf{1}_{N/2\times N/2}\,, \\
Q^1 &= \phi G^1_{N/2\times N/2}\,, \\
Q^2 &= \phi G^2_{N/2\times N/2}\,, \\
R^\alpha &= 0\,,
\end{aligned} \tag{4.1}$$

where A_μ and $\hat{A}_\mu$ live in the two gauge groups making up the $\mathrm{U}(N)\times\mathrm{U}(N)$ gauge symmetry of ABJM and Q^α and R^α with $\alpha = 1, 2$ are the two (first sub-space) bifundamental scalars. The combination of Q^α and R^α, often labeled N^α can be shown with this choice of R^α to vanish while the other combination will be non-zero. The covariant derivative on the scalar Q^α is given by

$$D_\mu Q^\alpha = G^\alpha \left(\partial_\mu \phi + i\left(a_\mu^{(1)} - \hat{a}_\mu^{(1)}\right)\phi\right)\,, \tag{4.2}$$

where the $N/2 \times N/2$ subscript is now left off the G^α for brevity. This leads to the kinetic terms

$$\mathrm{Tr}\left[|D_\mu Q^\alpha|^2\right] = 2\frac{N}{2}\left(\frac{N}{2} - 1\right)|\partial_\mu + i\left(a_\mu^{(1)} - \hat{a}_\mu^{(1)}\right)\phi|^2\,. \tag{4.3}$$

The second contribution to the mass deformed potential is

$$M^\alpha = \mu Q^\alpha + \frac{2\pi}{k}\left(Q^\alpha Q^\dagger_\beta Q^\beta - Q^\beta Q^\dagger_\beta Q^\alpha\right) = G^\alpha\left(\mu\phi + \frac{2\pi}{k}\phi^3\right)\,, \tag{4.4}$$

and thus the full potential

$$V = \mathrm{Tr}\left[|M^\alpha|^2\right] = \frac{N}{2}\left(\frac{N}{2}-1\right)|\phi|^2\left|\mu+\frac{2\pi}{k}\phi^2\right|^2 = \frac{N}{2}\left(\frac{N}{2}-1\right)|\phi|^2\left(\mu+\frac{2\pi}{k}|\phi|^2\right)^2. \quad (4.5)$$

With this ansatz, the Chern-Simons terms reduce to

$$\frac{k}{4\pi}\frac{N}{2}\epsilon^{\mu\nu\rho}\left(a_\mu^{(1)}\partial_\nu a_\rho^{(1)} - \hat{a}_\mu^{(1)}\partial_\nu \hat{a}_\rho^{(1)}\right) = \frac{k}{4\pi}\frac{N}{2}\epsilon^{\mu\nu\rho}\left(a_\mu^{(1)}+\hat{a}_\mu^{(1)}\right)\partial_\nu\left(a_\rho^{(1)}-\hat{a}_\rho^{(1)}\right), \quad (4.6)$$

so we obtain the first half of the required action, with the additional identification $\Phi \to \phi$, $a_\mu \to a_\mu^{(1)} - \hat{a}_\mu^{(1)}$ and $\tilde{b}_\mu \to a_\mu^{(1)} + \hat{a}_\mu^{(1)}$. The other half, for χ and $\mathcal{A}_\mu$, is obtained from the second $N/2$ subspace, now with the constraint $\tilde{b}_\mu = a_\mu^{(1)} + \hat{a}_\mu^{(1)} = a_\mu^{(2)} + \hat{a}_\mu^{(2)}$.

4.2 Vortex constraints on the ABJM potential

We are now in a position to construct a duality in this constrained sector of ABJM by mapping the action to a known self-dual action. However, to prove that this is a particle-vortex duality, we first need to show that there is enough freedom in the sextic potential to provide vortex solutions. In the massive case there is a constraint on the potential that $C_1 = \lambda^2/3m^2$ in order to have a vortex, which means that we must have

$$V = \frac{|\phi|^2}{3m^2}\left(\lambda^2|\phi|^4 + 3m^2\lambda|\phi|^2 + 3m^4\right) = \frac{|\phi|^2}{3m^2}\left[\left(\lambda|\phi|^2+\frac{3m^2}{2}\right)^2 + \frac{3m^4}{4}\right], \quad (4.7)$$

in order to have solitons. Clearly, this is not the case for the mass-deformed ABJM model. Therefore at $\mu \neq 0$, the mechanism doesn't work. However at $\mu = 0$ (i.e. the purely sextic potential), as shown in the previous section, vortex solutions do actually exist. This, along with the field identifications in section (4.1), suffices to demonstrate that at $\mu = 0$ we can construct a reduction of ABJM which exhibits a particle-vortex self-duality.

4.3 Toward a non-abelian extension

To close this section we speculate on a possible extension of the particle-vortex duality to non-abelian vortices starting with the observation that with the embedding of the particle vortex duality, we can write it *on the reduction ansatz* in the invariant form

$$\frac{1}{2}\mathrm{Tr}\left[Q_\alpha^\dagger D^\mu Q^\alpha - Q^\alpha(D^\mu Q^\alpha)^\dagger\right] = \frac{1}{e}\epsilon^{\mu\nu\rho}\partial_\nu \mathrm{Tr}\left(A_\rho + \hat{A}_\rho\right) = \frac{1}{2}\mathrm{Tr}\left[\tilde{Q}_\alpha^\dagger \tilde{D}^\mu \tilde{Q}^\alpha - \tilde{Q}^\alpha\left(\tilde{D}^\mu\tilde{Q}^\alpha\right)^\dagger\right], \quad (4.8)$$

where the trace is taken only on half the matrix space. With the caveat that we have not been able to prove that this holds in general (i.e. not on the reduction ansatz), it is tempting to think that one can write a nonabelian generalization of the type

$$\frac{1}{2}\left[Q_\alpha^\dagger D^\mu Q^\alpha - Q^\alpha(D^\mu Q^\alpha)^\dagger\right] = \frac{1}{e}\epsilon^{\mu\nu\rho}\partial_\nu\left(A_\rho + \hat{A}_\rho\right) = \frac{1}{2}\left[\tilde{Q}_\alpha^\dagger \tilde{D}^\mu \tilde{Q}^\alpha - \tilde{Q}^\alpha\left(\tilde{D}^\mu\tilde{Q}^\alpha\right)^\dagger\right], \quad (4.9)$$

for half the matrix space, and a similar one for the other half. Showing that this is indeed that case in general would go a long way toward generalizing the (self-dual) particle-vortex duality and we leave it as an open problem that we will return to in the future.

5 Particle vortex duality from Maxwell duality in the bulk, via AdS/CFT

Having established a framework to understand particle-vortex duality in (at least a reduction of) the ABJM model, we now relate the duality with Maxwell duality in the bulk, via the AdS/CFT correspondence. The partition function in a three-dimensional conformal field theory for a gauge field with a source is generically (in Euclidean signature)

$$Z_{\rm CFT}[a_i] = \int \mathcal{D}\phi e^{-S[\phi]+\int d^3x J^i a_i}\,, \tag{5.1}$$

($i = 1,2,3$), where ϕ represents all of the fields in the gauge theory, J^i the U(1) current that couples to the source a_i which is itself the boundary value for the bulk gauge field A_μ.

The corresponding supergravity partition function in the bulk (in Euclidean signature) is given by the bulk Maxwell action in an AdS geometry

$$Z_{\rm sugra}[a_i] = e^{-\int d^4x\sqrt{-g}\left[+\frac{1}{4g^2}F^2_{\mu\nu}\right]}\,, \tag{5.2}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the bulk gauge field field strength and Φ corresponds to all dynamical fields in the bulk. We work in the radial gauge $A_z = 0$, so $A_i \to a_i$ on the boundary. We define the four-dimensional Maxwell duality,

$$\tilde{F}^{\mu\nu} = \frac{1}{2\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\,, \tag{5.3}$$

in terms of which we can rewrite the partition function as

$$Z_{\rm sugra}[a_i] = Z_{\rm sugra}[\tilde{a}_i] = e^{-\int d^4x\sqrt{-g}\left[+\frac{1}{4g^2}\tilde{F}^2_{\mu\nu}\right]}\,. \tag{5.4}$$

The question is how to relate this to the particle-vortex duality we have already found, in a theory with a known gravity dual. The field theory dual to the self-dual Maxwell theory in the bulk can itself be rewritten, defining a particle-vortex type duality for currents similar to (2.6)

$$J^i = \frac{1}{2}\epsilon^{ijk}\partial_j\tilde{J}_k\,, \tag{5.5}$$

as

$$Z_{\rm CFT}[a_i] = \int \mathcal{D}\phi e^{-S[\phi]+\int d^3x\frac{1}{2}\epsilon^{ijk}(\partial_j\tilde{J}_k)a_i} = \int \mathcal{D}\phi e^{-S[\phi]+\int d^3x\tilde{J}^i\left(\frac{1}{2}\epsilon^{ijk}\partial_j a_k\right)}\,, \tag{5.6}$$

so that, if

$$\tilde{a}^i = \frac{1}{2}\epsilon^{ijk}\partial_j a_k\,, \tag{5.7}$$

it would be written in exactly the form to match $Z_{\rm sugra}[\tilde{a}_i]$, thus relating the particle-vortex duality (5.5) in the CFT with the Maxwell duality (5.3) in the bulk.

5.1 Maxwell duality in AdS$_4$

Having identified a link between a generic $d+1$-dimensional Maxwell duality and a d-dimensional particle-vortex-like duality we turn specifically to the Maxwell duality in AdS$_4$. In Poincaré coordinates,

$$ds^2 = \frac{-dt^2 + dx^2 + dy^2 + dz^2}{z^2}\,, \tag{5.8}$$

the Maxwell duality (5.3) becomes

$$\tilde{F}_{01} = -F_{23}; \quad \tilde{F}_{23} = -F_{01}, \dots \tag{5.9}$$

i.e. exchanging electric and magnetic components, including in the radial direction. In the radial gauge $A_3 = \tilde{A}_3 = 0$ now, we have $F_{23} = -\partial_3 A_2$ and $\tilde{F}_{23} = -\partial_3 \tilde{A}_2$, so

$$\tilde{F}_{01}(z=0) = \partial_z A_2(z=0); \quad F_{01}(z=0) = \partial_z \tilde{A}_2(z=0)\,, \tag{5.10}$$

where $z = 0$ is the boundary of AdS. Expanding near the boundary

$$\begin{aligned} A_i &= a_i + z\bar{a}_i + \frac{z^2}{2} a_i^{(2)} + \frac{z^3}{3!} a_i^{(3)} + \dots \\ \tilde{A}_i &= \tilde{a}_i + z\bar{\tilde{a}}_i + \frac{z^2}{2} \tilde{a}_i^{(2)} + \frac{z^3}{3!} \tilde{a}_i^{(3)} + \dots\,, \end{aligned} \tag{5.11}$$

the above Maxwell duality relations give

$$\tilde{f}_{ij} = \frac{1}{2}\epsilon_{ijk}\bar{a}_k; \quad f_{ij} = \frac{1}{2}\epsilon_{ijk}\bar{\tilde{a}}_k\,, \tag{5.12}$$

where f_{ij} corresponds to the field strength coming only from the leading term in the expansion on the boundary (i.e. $f_{ij} = \partial_i a_j - \partial_j a_i$) as well as

$$\begin{aligned} \bar{\tilde{f}}_{ij} &= -\frac{1}{2}\epsilon_{ijk}\partial_l^2 a_k = \frac{1}{2}\epsilon_{ijk} a_k^{(2)}\,, \\ \bar{f}_{ij} &= -\frac{1}{2}\epsilon_{ijk}\partial_l^2 \tilde{a}_k = \frac{1}{2}\epsilon_{ijk} \tilde{a}_k^{(2)}\,, \end{aligned} \tag{5.13}$$

etc. Again $\bar{f}_{ij} = \partial_i \bar{a}_j - \partial_j \bar{a}_i$. This result is obtained from two applications of the duality transformations. For the first equality, we first write the duality for $\bar{a}_i$ in terms of f_{ij} and then take a derivative. For the second, we look at the order z term in the duality for $\tilde{F}_{ij}$ vs. A_k. Equating the two results gives $\partial_l^2 a_k = -a_k^{(2)}$. We will see shortly that this appears from the Maxwell equations.

Normally, in $d \neq 4$, one should be able to give only the a_i as boundary condition, but not $\bar{a}_i$, the subleading term in the z expansion. In $d = 4$ however, as a result of the Maxwell duality, we can specify both a_i and $\bar{a}_i$, or equivalently, both the source a_i and the source for the Maxwell dual, $\tilde{a}_i$. In Poincaré coordinates the Maxwell equations

$$\partial_\rho\left[\sqrt{g} g^{\rho\mu} g^{\sigma\nu} \partial_\mu\right] A_\nu - \partial_\rho\left[\sqrt{g} g^{\rho\mu} g^{\sigma\nu} \partial_\nu\right] A_\mu = 0\,, \tag{5.14}$$

and since $g_{\mu\nu} = 1/z^2 \delta_{\mu\nu}$, this reduces to

$$\partial_\rho \delta^{\rho\mu} \partial_\mu A_\nu \delta^{\sigma\nu} - \partial_\rho \delta^{\mu\rho} \delta^{\sigma\nu} \partial_\nu A_\mu = 0\,. \tag{5.15}$$

JHEP10(2014)051

Notice that all explicit factors of z have disappeared from the equation! This happens only in four dimensions. More generally there will be an extra contribution of $(4-d)/z\partial_z A_\sigma$, which means that in particular, the leading term in this equation is of order $1/z$, namely, for $\sigma = i$, it is $(d-4)\bar{a}_i/z$, implying that $\bar{a}_i = 0$. In the gauge $A_z = 0$, we obtain $\partial_i\partial_z A_i = 0$ for $\sigma = z$. This constrains $\partial_i\bar{a}_i = 0$, $\partial_i a_i^{(n)} = 0$ for $n \geq 2$, leaving only $\partial_i a_i$ possibly nonzero. However, since it is perfectly consistent to set it to zero, we will do so. This is equivalent to the usual radiation gauge with time replaced by z, $a_z = 0$ and $\partial_i a_i = 0$. For $\sigma = i$ we obtain

$$\partial_j^2 A_i + \partial_z^2 A_i - \partial_i(\partial_j A_j) = 0\,, \tag{5.16}$$

which when expanded in z (and taking into account the conditions above for $\sigma = z$), results in the system of equations

$$\begin{aligned}
\partial_j^2 a_i + a_i^{(2)} - \partial_i(\partial_j a_j) &= 0\,,\\
\partial_j^2 \bar{a}_i + a_i^{(3)} &= 0\,,\\
\partial_j^2 a_i^{(n)} + a_i^{(n+2)} &= 0\,.
\end{aligned} \tag{5.17}$$

Note that the first relation also implies $\partial_i a_i^{(2)} = 0$, as it should. Thus, in the radiation gauge for a_i we have $a_i^{(n+2)} = -\partial_j^2 a_i^{(n)}$. Specifying a_i and $\bar{a}_i$ (or equivalently, $\tilde{a}_i$) then completely fixes the solution to the Maxwell equation in AdS$_4$.

Returning to the gauge theory side of the correspondence, we need to specify a_i and $\tilde{a}_i$ as sources for the path integral (5.1), or exchange a_i with $\tilde{\tilde{a}}_i$ and $\tilde{a}_i$ with $\bar{a}_i$. As claimed earlier, this exchange of a_i with $\tilde{\tilde{a}}_i$ corresponds to a particle-vortex duality exchanging dual currents as in (5.5). These currents however, need to be currents of global symmetries that can couple to the gravity dual gauge fields. We need to have two currents, one for particles and one for vortices, that can be replaced by their corresponding particle-vortex dual currents. According to our embedding of particle-vortex duality in ABJM (4.1), the scalar ϕ appears in half of the U(N) space and χ in the other half. With this ansatz $j^\mu = \hat{j}^\mu$ from (B.4) but splits into two currents (for each of the two $N/2$ subspaces) of $\tilde{J}$ type in (5.5), $\tilde{J}_i^{(1)}$ and $\tilde{J}_i^{(2)}$ that couple to $\bar{a}_k$ and $\tilde{\tilde{a}}_k$ respectively.

6 Conclusions

This article details our exploration of holographic particle-vortex duality. In particular we have foccused on its realization in the ABJM model and a possible relation to Maxwell duality in AdS$_4$ via the AdS/CFT correspondence. By combining a path integral version of particle-vortex duality with the Mukhi-Papageorgakis Higgs mechanism we have formulated a symmetric version of the transformation that acts as a self-duality. We then proceeded to show how to embed it as an abelian duality in the $(2+1)$-dimensional, $\mathcal{N} = 6$ super Chern-Simons-matter theory that is the ABJM model and speculated on a possible non-abelian extension. Going to the gravity side of the correspondence, Maxwell duality in AdS$_4$ is found to reduce on the boundary to a particle-vortex duality acting on two independent gauge field sources $\bar{a}$ and $\tilde{\tilde{a}}$ and their associated currents $\tilde{J}^{(1)}$ and $\tilde{J}^{(2)}$.

JHEP10(2014)051

Our primary motivation for this work was two-fold; first we simply wanted to understand if particle-vortex duality is realized in the (mass-deformed) ABJM model with its rich solitonic spectrum and second, we wanted to see if the phenomenological work of [11] could be embedded in the concrete setting of the $\mathrm{AdS}_4 \times \mathbb{CP}^3$/ABJM correspondence. This work paves the way for both these directions but there remains much to be done. Among the possible extensions of this work are

- the development of our speculations on a non-abelian version of the particle-vortex duality. To the best of our knowledge the duality has thus far been formulated only of vortices of the conventional Nielsen-Olesen type exhibited by the abelian Higgs model and variants thereof. Vortices, however, come in many different forms and flavors[4] such as non-abelian as well as semi-local kinds. It would be of great interest to understand if and how the duality applies to these.

- An understanding of the manifestation of the full particle-vortex duality on the gravity side of the correspondence. In particular, having established, in this article, that the duality can actually be embedded into (at least some reduction of) the ABJM model, an important development would be to establish precisely how it acts on states of the type IIA superstring on $\mathrm{AdS}_4 \times \mathbb{CP}^3$.

- The extraction of the phenomenological results for quantum critical transport uncovered in [11].

- A more complete understanding of how the particle-vortex duality of this article relates to level-rank duality and its generalizations discovered by Kutasov and collaborators in recent years.

It is quite clear that particle-vortex duality should of great interest to both the holographic condensed matter as well as more formal string theoretic communities and we hope that this article will stimulate further work in this area.

Acknowledgments

We would like to thank Robert de Mello Koch, Dimitrios Giataganis, Ken Intriligator and Sean Hartnoll for useful conversations at various stages of this work. The work of HN is supported in part by CNPq grant 301219/2010-9 and FAPESP grant 2013/14152-7. JM acknowledges support from the National Research Foundation (NRF) of South Africa under its IPRR and CPRR programs. NR is supported by a DAAD-AIMS Scholarship; he wishes to thank the Arnold Sommerfeld Centre for Theoretical Physics and the Max Planck Institute for Physics for hospitality, and the DAAD for a Research Grant for Doctoral Candidates and Young Academics which made his visit to Munich possible.

[4]Pardon the pun.

A Particle-vortex duality à la Burgess and Dolan

In this appendix we review the duality of [7], ignoring some terms that are not essential to our argument.

A.1 First derivation

The starting point is (1.2), the action of an abelian Higgs system of constant modulus, with an external gauge field A_μ. One also introduces a statistical (Chern-Simons) gauge field[5] a_μ:

$$\begin{aligned}\mathcal{L}(\phi, a, A) = &-\frac{\kappa}{2}\left[\partial_\mu\phi - q_\phi\left(a_\mu + A_\mu\right)\phi\right]^2 - \frac{\pi}{2\theta}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho \\ &+ \mathcal{L}_p(\xi, a+A)\ ,\end{aligned} \tag{A.1}$$

where

$$\mathcal{L}_p(\xi, a+A) = \sum_k\left[\frac{m}{2}\dot{\xi}^\mu_k\dot{\xi}_{k,\mu} + q_k\dot{\xi}^\mu_k(a+A)_\mu\right]\delta[x-\xi_k(t)] \tag{A.2}$$

is the particle Lagrangian. Here, $\theta = 2\pi n$ for bosons and $\theta = (2n+1)\pi$ for fermions, and ϕ is the phase angle of $\Phi = |\Phi|e^{-i\phi}$.

As is usual for dualities in the path integral formulation, we lift this action to a *master action* through the coupling of ϕ to a new gauge field $\mathcal{A}_\mu$ constrained by a Lagrange-multiplier field b_μ to be pure gauge:

$$\begin{aligned}\mathcal{L} = &-\frac{\kappa}{2}[\partial_\mu\phi - q_\phi(a_\mu + A_\mu + \mathcal{A}_\mu)\phi]^2 - \frac{\pi}{2\theta}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho \\ &+ \mathcal{L}_p(\xi, a+A) + \epsilon_{\mu\nu\rho}b_\mu\partial_\nu\mathcal{A}_\rho + \dots\ .\end{aligned} \tag{A.3}$$

Indeed, integrating over b_μ, we find $\partial_{[\nu}\mathcal{A}_{\rho]} = 0$, and then integrating over $\mathcal{A}_\mu$ is equivalent to putting it to zero.[6] On the other hand, integrating first over ϕ instead, and then over $\mathcal{A}$, will lead to a dual action in terms of the Lagrange multiplier b_μ.

To do that, care must be taken about the periodicity of ϕ in the presence of vortices for the original complex scalar field Φ. We have

$$\phi(\theta + 2\pi) = \phi(\theta) + 2\pi\sum_a N_a\ , \tag{A.4}$$

where N_a is the *vorticity* or *winding number* of vortex a. We then write $\phi = \omega + \varphi$, where φ satisfies periodic boundary conditions, $\varphi(\theta + 2\pi) = \varphi(\theta)$, and $\omega(x)$ is an explicit vortex solution,

$$\omega(x) = \sum_a N_a \arctan\left(\frac{x^1 - y^1_a}{x^2 - y^2_a}\right) \equiv \sum_a N_a\theta_a\ , \tag{A.5}$$

where

$$\frac{x^1 - y^1_a}{x^2 - y^2_a} = \tan\theta_a \tag{A.6}$$

defines the angle of rotation around a particular vortex. In the notation of [4] described in the introduction, ω corresponds to θ_{vortex}, and φ to θ_{smooth}.

[5]This field, which arises from the combinatorics of the charged particles, has no dynamical degrees of freedom.

[6]Performing the integration over b_μ produces a functional delta function which enforces the constraint $\epsilon^{\mu\nu\rho}\partial_\nu\mathcal{A}_\rho = 0$; this, together with the gauge fixing condition, implies that integrating over $\mathcal{A}_\mu$ is equivalent to setting $\mathcal{A}_\mu = 0$.

We then define $v_\mu = \partial_\mu \omega$, obtaining

$$v_\mu = \sum_a N_a \frac{1}{1+\tan^2\theta_a} \partial_\mu \tan\theta_a = \sum_a N_a \partial_\mu \theta_a \ , \tag{A.7}$$

which means that

$$\epsilon^{\mu\nu\rho} b_\mu \partial_\nu v_\rho = b_\mu \sum_a N_a \ \epsilon^{\mu\nu\rho} \partial_\nu \partial_\rho \theta_a = 2\pi b_\mu \sum_a N_a \ \dot{y}_a^\mu \ \delta[x - y_a(t)] = 2\pi b_\mu j^\mu(t) \ , \tag{A.8}$$

where

$$j^\mu(t) = j^\mu_{\text{vortex}}(t) = \sum_a N_a \ \dot{y}_a^\mu \ \delta[x - y_a(t)] \tag{A.9}$$

is the vortex current. Note that $\epsilon^{ij}\partial_i\partial_j\theta_a = 2\pi\delta^2(x)$, so we can indeed verify the above formula for static $y_a^i(t) = y_a^i$, when $\dot{y}_a^0 = 1$ and the rest are 0, giving

$$\epsilon^{\mu\nu\rho}\partial_\nu\partial_\rho\theta_a = \delta^{\mu 0}\epsilon^{ij}\partial_i\partial_j\theta_a = 2\pi\delta^{\mu 0}\delta^2(x - y_a) \ . \tag{A.10}$$

Note now that (A.3) has a gauge invariance

$$\delta A_\mu = \partial_\mu \lambda; \quad \delta\phi = q_\phi \lambda \ , \tag{A.11}$$

which we can gauge-fix by putting $\varphi = 0$ (i.e. $\phi = \omega$), thus making the path integration over ϕ trivial. We are thus left with only the path integral over $\mathcal{A}_\mu$ to do, and since $\partial_\mu\phi = \partial_\mu\omega = v_\mu$, the path integral we need to determine is

$$\int \mathcal{D}\mathcal{A}_\mu \exp\left\{ i \int \left[-\frac{\kappa}{2}(v_\mu - q_\phi(a_\mu + A_\mu + \mathcal{A}_\mu))^2 + \epsilon^{\mu\nu\rho}\partial_\mu b_\nu \mathcal{A}_\rho \right] \right\} \ , \tag{A.12}$$

and of course, we still have the particle action and the statistical gauge field part of the action outside the path integral. Then, defining

$$J^\rho \equiv \epsilon^{\mu\nu\rho}\partial_\mu b_\nu \ , \tag{A.13}$$

we get the path integral

$$\int \mathcal{D}\mathcal{A}_\mu \exp\left[i \int \left(-\frac{\kappa}{2} q_\phi^2 \left(\mathcal{A}_\mu + a_\mu + A_\mu - \frac{v_\mu}{q_\phi} - \frac{J_\mu}{k q_\phi^2} \right)^2 - J_\mu \left(a_\mu + A_\mu - \frac{v_\mu}{q_\phi} \right) + \frac{J_\mu^2}{2\kappa q_\phi^2} \right) \right]$$
$$= \mathcal{N} \exp\left[i \int \left(-J_\mu \left(a_\mu + A_\mu - \frac{v_\mu}{q_\phi} \right) + \frac{J_\mu^2}{2\kappa q_\phi^2} \right) \right] .$$

Given

$$\int \frac{J_\mu v^\mu}{q_\phi} = \int \frac{1}{q_\phi} \epsilon^{\mu\nu\rho}\partial_\mu b_\nu v_\rho = \int \frac{1}{q_\phi} \epsilon^{\mu\nu\rho} b_\mu \partial_\nu v_\rho = \frac{2\pi}{q_\phi} b_\mu j^\mu(t) \tag{A.14}$$

and

$$J_\mu^2 = 2\delta^{\rho\sigma}_{\mu\nu} \ \partial^\mu \ b^\nu \partial_\rho b_\sigma = \frac{1}{2} f_{\mu\nu}^{(b)2} \ , \tag{A.15}$$

where $f_{\mu\nu}^{(b)} = \partial_\mu b_\nu - \partial_\nu b_\mu$, we have as the dual action

$$\mathcal{L}_{\text{dual}}(a, b, A) = -\frac{1}{4\kappa q_\phi^2} f_{\mu\nu}^{(b)2} - \epsilon^{\mu\nu\rho} b_\mu \partial_\nu (a_\rho + A_\rho) + \sum_a \frac{2\pi}{q_\phi} N_a \ \dot{y}_\mu^a b^\mu \ \delta(x - y_a(t))$$
$$- \frac{\pi}{2\theta} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho + \mathcal{L}_p(\xi, a, A) \ . \tag{A.16}$$

Note that, besides the dualization from the field ϕ to the field b_μ, we have also obtained an explicit action for moving vortices, with positions $y_\mu^a(t)$, namely $2\pi b_\mu j^\mu_{\rm vortex}(t)$. Therefore we explicitly see that the dualization of ϕ to b_μ also exchanges particles with vortices, deserving the name of particle-vortex duality.

A.2 Second derivation

We now review a second derivation from [7], which is closer to what we use in the bulk of the paper. We start with an abelian Higgs action where the complex scalar field is coupled to a Chern-Simons gauge field a and an external gauge field A, with an arbitrary scalar potential depending only on $|\Phi|$,

$$S = -\frac{1}{2}\int \left[[(i\partial_\mu - e\tilde{a}_\mu)\Phi]^\dagger [(i\partial^\mu - e\tilde{a}^\mu)\Phi] + \frac{\pi e^2}{\theta}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho \right] + S_{\rm int}\left[|\Phi|^2\right] , \tag{A.17}$$

where $\tilde{a} \equiv a + A$. We rewrite it as

$$S = -\frac{1}{2}\int \left[(\partial_\mu\Phi)^\dagger(\partial^\mu\Phi) + e^2|\Phi|^2\tilde{a}_\mu\tilde{a}^\mu - 2\tilde{a}_\mu j^\mu + \frac{\pi e^2}{\theta}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho \right] + S_{\rm int}\left[|\Phi|^2\right] , \tag{A.18}$$

where the scalar current is

$$j_\mu = \frac{ie}{2}\left[\Phi^\dagger\partial_\mu\Phi - \left(\partial_\mu\Phi^\dagger\right)\Phi\right] = e|\Phi|^2\partial_\mu\theta . \tag{A.19}$$

Then we split Φ as before into a vortex part v and a smooth part,

$$\Phi(\vec{r}) = \Phi_0(\vec{r})e^{-i\theta(\vec{r})}v(\vec{r}) , \tag{A.20}$$

where

$$v(\vec{r}) = \exp\left[\frac{2\pi i}{q_\phi}\sum_a N_a \arctan\left(\frac{x^1 - y_a^1}{x^2 - y_a^2}\right)\right] . \tag{A.21}$$

Then we have

$$\begin{aligned} S^a[\Phi_0, \theta, a, A] &= -\frac{1}{2}\int \left[(\partial_\mu\Phi_0)^2 + (\partial_\mu\theta - iv^*\partial_\mu v - e\tilde{a}_\mu)^2\Phi_0^2 \right] + S_{CS}[a] + S_{\rm int}\left[|\Phi|^2\right] \\ &= -\frac{1}{2}\int \left[(\partial_\mu\Phi_0)^2 + e^2\Phi_0^2\tilde{a}_\mu\tilde{a}^\mu + \frac{1}{e^2\Phi_0^2}j_\mu j^\mu - 2\tilde{a}_\mu j^\mu \right] - \frac{\pi e^2}{2\theta}\int \epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho \\ &\quad + S_{\rm int}[\Phi_0^2] , \end{aligned} \tag{A.22}$$

where the current is

$$j_\mu = e\Phi_0^2(\partial_\mu\theta + iv^*\partial_\mu v) . \tag{A.23}$$

Now we define $\lambda_\mu = \partial_\mu\theta$.[7] We substitute the integration over θ with integration over λ_μ, subject to the constraint $\epsilon^{\mu\nu\rho}\partial_\nu\lambda_\rho = 0$ imposed with a Lagrange multiplier $\tilde{b}_\mu$, i.e.

$$\begin{aligned} &\int \mathcal{D}\theta \exp\left[-\frac{i}{2}\int (\partial_\mu\theta + iv^*\partial_\mu v - e\tilde{a}_\mu)^2\Phi_0^2\right] \\ &\quad = \int \mathcal{D}\lambda_\mu\mathcal{D}\tilde{b}_\mu \exp\left[-\frac{i}{2}\int \left(\lambda_\mu + iv^*\partial_\mu v - e\tilde{a}_\mu\right)^2\Phi_0^2 + \epsilon^{\mu\nu\rho}\tilde{b}_\mu\partial_\nu\lambda_\rho\right] . \end{aligned} \tag{A.24}$$

[7] This is the $\partial_\mu\varphi$ from the last subsection, whereas $-iv^*\partial_\mu v$ is the $\partial_\mu\omega$ from there.

Then doing the integral over λ_μ first, we obtain

$$S^b\left[\Phi_0, A, a, \tilde{b}\right] = \int\left[-\frac{1}{4e^2\Phi_0^2}\tilde{f}^{(b)}_{\mu\nu}\tilde{f}^{(b)\mu\nu} + \tilde{j}^\mu\tilde{b}_\mu - \epsilon^{\mu\nu\rho}\tilde{a}_\rho\partial_\nu\tilde{b}_\rho - \frac{\pi e^2}{2\theta}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho\right] \\ -\frac{1}{2}\int\partial_\mu\Phi_0\partial^\mu\Phi_0 + S'_{\text{int}}\left[\Phi_0^2\right], \tag{A.25}$$

where as usual, $\tilde{f}^{(b)}_{\mu\nu} = \partial_\mu\tilde{b}_\nu - \partial_\nu\tilde{b}_\mu$, and

$$\tilde{j}^\mu = \frac{i}{e}\epsilon^{\mu\nu\rho}\partial_\nu v^*\partial_\rho v \tag{A.26}$$

is the *vortex* current.

The duality between the actions S^a and S^b is exactly the same from last subsection, with the kinetic term for Φ_0 and the interaction term being spectators, but here it was derived from an abelian Higgs action by path integration.

Note that the classical solution for λ_μ is

$$\partial_\mu\theta \equiv \lambda_\mu = -iv^*\partial_\mu v + e\tilde{a}_\mu + \frac{i}{e\Phi_0^2}\epsilon^{\mu\nu\rho}\partial_\nu\tilde{b}_\rho\,, \tag{A.27}$$

which matches with the duality transform of Zee (from the introduction) with $v =$ constant, $\tilde{a} = 0$.

B Review of ABJM and its massive deformation

The ABJM model [12] is obtained as the low-energy limit of the theory of N coincident M2-branes in a $\mathbb{C}^4/\mathbb{Z}_k$ background. It is a supersymmetric $\mathcal{N} = 6$ U(N) × U(N) (or SU(N) × SU(N)) Chern-Simons (CS) gauge theory, with bifundamental scalars Y^I and fermions ψ_I, $I = 1,\ldots,4$ in the fundamental of the SU$(4)_R$ symmetry group, and the two CS gauge fields, A_μ and $\hat{A}_\mu$, have equal and opposite levels k and $-k$. Its action is

$$\begin{aligned} S = \int d^3x\, \Bigg[&\frac{k}{4\pi}\epsilon^{\mu\nu\lambda}\,\mathrm{Tr}\left(A_\mu\partial_\nu A_\lambda + \frac{2i}{3}A_\mu A_\nu A_\lambda - \hat{A}_\mu\partial_\nu\hat{A}_\lambda - \frac{2i}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\lambda\right) \\ &-\mathrm{Tr}\left(D_\mu Y_I^\dagger D^\mu Y^I + i\psi^{I\dagger}\gamma^\mu D_\mu\psi_I\right) + \frac{4\pi^2}{3k^2}\,\mathrm{Tr}\Big(Y^I Y_I^\dagger Y^J Y_J^\dagger Y^K Y_K^\dagger \\ &+Y_I^\dagger Y^I Y_J^\dagger Y^J Y_K^\dagger Y^K + 4Y^I Y_J^\dagger Y^K Y_I^\dagger Y^J Y_K^\dagger - 6Y^I Y_J^\dagger Y^J Y_I^\dagger Y^K Y_K^\dagger Y^K\Big) \\ &+\frac{2\pi i}{k}\,\mathrm{Tr}\Big(Y_I^\dagger Y^I\psi^{J\dagger}\psi_J - \psi^{J\dagger}Y^I Y_I^\dagger\psi_J - 2Y_I^\dagger Y^J\psi^{I\dagger}\psi_J + 2\psi^{J\dagger}Y^I Y_J^\dagger\psi_J \\ &+\epsilon^{IJKL}Y_I^\dagger\psi_J Y_K^\dagger\psi_L - \epsilon_{IJKL}Y^I\psi^{J\dagger}Y^K\psi^{L\dagger}\Big)\Bigg]. \end{aligned} \tag{B.1}$$

Here the covariant derivative acts like

$$D_\mu Y^I = \partial_\mu Y^I + i\left(A_\mu Y^I - Y^I\hat{A}_\mu\right).$$

The action (B.1) has SU(4)×U(1) R-symmetry associated with the $\mathcal{N}=6$ supersymmetries.

B.1 Massive ABJM

There exists a unique supersymmetry-preserving massive deformation of the model [19], parametrised by μ, that breaks the R-symmetry down to $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)_A\times\mathrm{U}(1)_B\times\mathbb{Z}_2$ as a consequence of splitting the scalars as

$$Y^I = (Q^\alpha, R^\alpha), \qquad \alpha = 1,2 \ .$$

The $\mathbb{Z}_2$ action interchanges Q^α and R^α, the SU(2) factors act each only on one of the doublets Q^α and R^α, and the $\mathrm{U}(1)_A$ symmetry rotates Q^α with a charge $+1$ and R^α with a charge -1. The mass deformation gives mass to the fermions and changes the potential of the theory. The bosonic part of the action in the mass deformed case is

$$\begin{aligned}\mathcal{L}_{\text{bosonic}} = \frac{k}{4\pi}\epsilon^{\mu\nu\lambda}\,\mathrm{Tr}\left[A_\mu\partial_\nu A_\lambda - \hat{A}_\mu\partial_\nu\hat{A}_\lambda + \frac{2i}{3}\left(A_\mu A_\nu A_\lambda - \hat{A}_\mu\hat{A}_\nu\hat{A}_\lambda\right)\right] \\ - \mathrm{Tr}\,|D_\mu Q^\alpha|^2 - \mathrm{Tr}\,|D_\mu R^\alpha|^2 - V \ . \end{aligned} \tag{B.2}$$

The sextic scalar potential in (B.2) is

$$V = \mathrm{Tr}\left(|M^\alpha|^2 + |N^\alpha|^2\right),$$

where

$$\begin{aligned} M^\alpha &= \mu Q^\alpha + \frac{2\pi}{k}\left(2Q^{[\alpha}Q^\dagger_\beta Q^{\beta]} + R^\beta R^\dagger_\beta Q^\alpha - Q^\alpha R^\dagger_\beta R^\beta + 2Q^\beta R^\dagger_\beta R^\alpha - 2R^\alpha R^\dagger_\beta Q^\beta\right), \\ N^\alpha &= -\mu R^\alpha + \frac{2\pi}{k}\left(2R^{[\alpha}R^\dagger_\beta R^{\beta]} + Q^\beta Q^\dagger_\beta R^\alpha - R^\alpha Q^\dagger_\beta Q^\beta + 2R^\beta Q^\dagger_\beta Q^\alpha - 2Q^\alpha Q^\dagger_\beta R^\beta\right). \end{aligned}$$

The equations of motion of the bosonic fields are

$$\begin{aligned} D_\mu D^\mu Q^\alpha &= \frac{\partial V}{\partial Q^\dagger_\alpha}\,, & D_\mu D^\mu R^\alpha &= \frac{\partial V}{\partial R^\dagger_\alpha}\,, \\ F_{\mu\nu} &= \frac{2\pi}{k}\epsilon_{\mu\nu\lambda}J^\lambda\,, & \hat{F}_{\mu\nu} &= \frac{2\pi}{k}\epsilon_{\mu\nu\lambda}\hat{J}^\lambda\,, \end{aligned} \tag{B.3}$$

where $F_{\mu\nu} = \partial_{[\mu}A_{\nu]} + i[A_\mu, A_\nu]$, and the two gauge currents J^μ and $\hat{J}^\mu$, expressed as

$$\begin{aligned} J^\mu &= i\left[Q^\alpha(D^\mu Q^\alpha)^\dagger - (D^\mu Q^\alpha)Q^\dagger_\alpha + R^\alpha(D^\mu R^\alpha)^\dagger - (D^\mu R^\alpha)R^\dagger_\alpha\right], \\ \hat{J}^\mu &= -i\left[Q^\dagger_\alpha(D^\mu Q^\alpha) - (D^\mu Q^\alpha)^\dagger Q^\alpha + R^\dagger_\alpha(D^\mu R^\alpha) - (D^\mu R^\alpha)^\dagger R^\alpha\right], \end{aligned}$$

are covariantly conserved, i.e. $\nabla_\mu J^\mu = \nabla_\mu \hat{J}^\mu = 0$. The trace parts of those gauge currents yields two abelian currents j^μ and $\hat{j}^\mu$ corresponding to the global $\mathrm{U}(1)_A$ and $\mathrm{U}(1)_B$ invariances

$$j^\mu = \mathrm{Tr}\,J^\mu\,, \qquad \hat{j}^\mu = \mathrm{Tr}\,\hat{J}^\mu\,, \tag{B.4}$$

which are ordinarily conserved, i.e. $\partial_\mu j^\mu = \partial_\mu \hat{j}^\mu = 0$. The gauge choice $A_0 = \hat{A}_0 = 0$ implies that the energy density is given by

$$H = \mathrm{Tr}\left[(\partial_0 Q^\alpha)^\dagger(\partial_0 Q^\alpha) + (D_i Q^\alpha)^\dagger(D_i Q^\alpha) + (\partial_0 R^\alpha)^\dagger(\partial_0 R^\alpha) + (D_i R^\alpha)^\dagger(D_i R^\alpha) + V\right] \ .$$

JHEP10(2014)051

Since this is a Chern-Simons theory, varying with respect to A_0 and $\hat{A}_0$ gives the Gauss law constraints

$$F_{12} = \frac{2\pi i}{k} J^0 = \frac{2\pi i}{k} \left[Q^\alpha \left(\partial^0 Q^\alpha\right)^\dagger - \left(\partial^0 Q^\alpha\right) Q^\dagger_\alpha + R^\alpha \left(\partial^0 R^\alpha\right)^\dagger - \left(\partial^0 R^\alpha\right) R^\dagger_\alpha \right],$$
$$\hat{F}_{12} = \frac{2\pi i}{k} \hat{J}^0 = -\frac{2\pi i}{k} \left[Q^\dagger_\alpha \left(\partial^0 Q^\alpha\right) - \left(\partial^0 Q^\alpha\right)^\dagger Q^\alpha + R^\dagger_\alpha \left(\partial^0 R^\alpha\right) - \left(\partial^0 R^\alpha\right)^\dagger R^\alpha \right]. \tag{B.5}$$

Note as an aside that the gauge choice does not uniquely specify the Hamiltonian. Choosing A_0 and $\hat{A}_0$ different from zero introduces an extra term in the Hamiltonian, $\epsilon^{\mu\nu\lambda}\,\mathrm{Tr}\left[A_\mu A_\nu A_\lambda - \hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda\right]$. In the abelianisation ansatz of [18], this vanishes anyway since it is proportional to $\epsilon^{\mu\nu\lambda} a^{(i)}_\mu a^{(j)}_\nu a^{(k)}_\lambda$ and there are only two $a^{(i)}_\mu$'s. So in the abelian case, the Hamiltonian is the same even away from the gauge $A_0 = \hat{A}_0 = 0$.

The mass deformed theory has fuzzy sphere ground states given by[8]

$$R^\alpha = cG^\alpha; \quad Q^\alpha = 0 \quad \text{and} \quad Q^\dagger_\alpha = cG^\alpha; \quad R^\alpha = 0 \tag{B.6}$$

where $c \equiv \sqrt{\frac{\mu k}{2\pi}}$ and the matrices G^α, $\alpha = 1, 2$, satisfy the equations [13, 14]

$$G^\alpha = G^\alpha G^\dagger_\beta G^\beta - G^\beta G^\dagger_\beta G^\alpha. \tag{B.7}$$

In [20, 21], it was shown that this solution corresponds to a fuzzy 2-sphere, not a 3-sphere as originally thought.

An explicit solution of these equations, which is the unique irreducible one up to a $\mathrm{U}(N) \times \mathrm{U}(N)$ gauge transformation, is given by

$$\begin{aligned} \left(G^1\right)_{m,n} &= \sqrt{m-1}\;\delta_{m,n}\,, \\ \left(G^2\right)_{m,n} &= \sqrt{(N-m)}\;\delta_{m+1,n}\,, \\ \left(G^\dagger_1\right)_{m,n} &= \sqrt{m-1}\;\delta_{m,n}\,, \\ \left(G^\dagger_2\right)_{m,n} &= \sqrt{(N-n)}\;\delta_{n+1,m}\,. \end{aligned} \tag{B.8}$$

In particular, $G^1 = G^\dagger_1$. In the case of pure ABJM, instead of a fuzzy sphere ground state, there is a fuzzy funnel *BPS solution* with c replaced by

$$c(s) = \sqrt{\frac{k}{4\pi s}}\,. \tag{B.9}$$

Here s is one of the two spatial coordinates of the ABJM model. The matrices G^α are bifundamental under $\mathrm{U}(N) \times \mathrm{U}(N)$, implying that $G^1 G^\dagger_1$ and $G^2 G^\dagger_2$ are in the adjoint of the first $\mathrm{U}(N)$, and that $G^\dagger_1 G^1$ and $G^\dagger_2 G^2$ are in the adjoint of the second $\mathrm{U}(N)$.

[8]General vacuum configurations could also be direct sums of these irreducible solutions.

JHEP10(2014)051

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JHEP10(2014)051

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