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*Research Article*

# **Identities of Symmetry for Euler Polynomials Arising from Quotients of Fermionic Integrals Invariant under $S_3$**

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We derive eight basic identities of symmetry in three variables related to Euler polynomials and alternating power sums. These and most of their corollaries are new, since there have been results only about identities of symmetry in two variables. These abundances of symmetries shed new light even on the existing identities so as to yield some further interesting ones. The derivations of identities are based on the  $p$ -adic integral expression of the generating function for the Euler polynomials and the quotient of integrals that can be expressed as the exponential generating function for the alternating power sums.

## **1. Introduction and Preliminaries**

Let  $p$  be a fixed odd prime. Throughout this paper,  $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}_p$  will, respectively, denote the ring of  $p$ -adic integers, the field of  $p$ -adic rational numbers, and the completion of the algebraic closure of  $\mathbb{Q}_p$ . For a continuous function  $f : \mathbb{Z}_p \rightarrow \mathbb{C}_p$ , the  $p$ -adic fermionic integral of  $f$  is defined by

$$\int_{\mathbb{Z}_p} f(z) d\mu_{-1}(z) = \lim_{N \rightarrow \infty} \sum_{j=0}^{p^N-1} f(j) (-1)^j. \quad (1.1)$$

Then it is easy to see that

$$\int_{\mathbb{Z}_p} f(z+1) d\mu_{-1}(z) + \int_{\mathbb{Z}_p} f(z) d\mu_{-1}(z) = 2f(0). \quad (1.2)$$

Let  $|\cdot|_p$  be the normalized absolute value of  $\mathbb{C}_p$ , such that  $|p|_p = 1/p$ , and let

$$E = \left\{ t \in \mathbb{C}_p \mid |t|_p < p^{-1/(p-1)} \right\}. \quad (1.3)$$

Then, for each fixed  $t \in E$ , the function  $f(z) = e^{zt}$  is analytic on  $\mathbb{Z}_p$ , and by applying (1.2) to this  $f$ , we get the  $p$ -adic integral expression of the generating function for Euler numbers  $E_n$ :

$$\int_{\mathbb{Z}_p} e^{zt} d\mu_{-1}(z) = \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \quad (t \in E). \quad (1.4)$$

So we have the following  $p$ -adic integral expression of the generating function for the Euler polynomials  $E_n(x)$ :

$$\int_{\mathbb{Z}_p} e^{(x+z)t} d\mu_{-1}(z) = \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \quad (t \in E, x \in \mathbb{Z}_p). \quad (1.5)$$

Let  $T_k(n)$ , denote the alternating  $k$ th power sum of the first  $(n+1)$  nonnegative integers, namely,

$$T_k(n) = \sum_{i=0}^n (-1)^i i^k = (-1)^0 0^k + (-1)^1 1^k + (-1)^2 2^k + \cdots + (-1)^n n^k. \quad (1.6)$$

In particular,

$$T_0(n) = \begin{cases} 1, & \text{if } n \equiv 0 \pmod{2}, \\ 0, & \text{if } n \equiv 1 \pmod{2}, \end{cases} \quad T_k(0) = \begin{cases} 1, & \text{for } k = 0, \\ 0, & \text{for } k > 0. \end{cases} \quad (1.7)$$

From (1.4) and (1.6), one easily derives the following identities: for any odd positive integer  $w$ ,

$$\frac{\int_{\mathbb{Z}_p} e^{xt} d\mu_{-1}(x)}{\int_{\mathbb{Z}_p} e^{wy^t} d\mu_{-1}(y)} = \sum_{i=0}^{w-1} (-1)^i e^{it} = \sum_{k=0}^{\infty} T_k(w-1) \frac{t^k}{k!} \quad (t \in E). \quad (1.8)$$

In what follows, we will always assume that the  $p$ -adic fermionic integrals of the various exponential functions on  $\mathbb{Z}_p$  are defined for  $t \in E$  (cf., (1.3)), and therefore it will not be mentioned.

Many authors have done much work on identities of symmetry involving Bernoulli polynomials or Euler polynomials or  $q$ -Bernoulli polynomials or  $q$ -Euler polynomials. We let the reader refer to the papers in [1–20]. In connection with Bernoulli polynomials and power sums, these results were generalized in [21] to obtain identities of symmetry involving three variables in contrast to the previous works involving just two variables.

In this paper, we will produce 8 basic identities of symmetry in three variables  $w_1, w_2, w_3$  related to Euler polynomials and alternating power sums (cf., (4.8), (4.9), (4.12), (4.16),

(4.20), (4.23), (4.25), and (4.26)). These and most of their corollaries seem to be new, since there have been results only about identities of symmetry in two variables in the literature. These abundances of symmetries shed new light even on the existing identities. For instance, it has been known that (1.9) and (1.10) are equal and (1.11) and (1.12) are so (cf., [3, Theorems 5, 7]). In fact, (1.9)–(1.12) are all equal, as they can be derived from one and the same  $p$ -adic integral. Perhaps, this was neglected to mention in [3]. Also, we have a bunch of new identities in (1.13)–(1.16). All of these were obtained as corollaries (cf., Corollary 4.9, 4.12, 4.15) to some of the basic identities by specializing the variable  $w_3$  as 1. Those would not be unearthed if more symmetries had not been available.

Let  $w_1, w_2$  be any odd positive integers. Then we have

$$\sum_{k=0}^n \binom{n}{k} E_k(w_1 y_1) T_{n-k}(w_2 - 1) w_1^{n-k} w_2^k \quad (1.9)$$

$$= \sum_{k=0}^n \binom{n}{k} E_k(w_2 y_1) T_{n-k}(w_1 - 1) w_2^{n-k} w_1^k \quad (1.10)$$

$$= w_1^n \sum_{i=0}^{w_1-1} (-1)^i E_n \left( w_2 y_1 + \frac{w_2}{w_1} i \right) \quad (1.11)$$

$$= w_2^n \sum_{i=0}^{w_2-1} (-1)^i E_n \left( w_1 y_1 + \frac{w_1}{w_2} i \right) \quad (1.12)$$

$$= \sum_{k+l+m=n} \binom{n}{k, l, m} E_k(y_1) T_l(w_1 - 1) T_m(w_2 - 1) w_1^{k+m} w_2^{k+l} \quad (1.13)$$

$$= w_1^n \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{w_1-1} (-1)^i E_k \left( y_1 + \frac{i}{w_1} \right) T_{n-k}(w_2 - 1) w_2^k \quad (1.14)$$

$$= w_2^n \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{w_2-1} (-1)^i E_k \left( y_1 + \frac{i}{w_2} \right) T_{n-k}(w_1 - 1) w_1^k \quad (1.15)$$

$$= (w_1 w_2)^n \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} (-1)^{i+j} E_n \left( y_1 + \frac{i}{w_1} + \frac{j}{w_2} \right). \quad (1.16)$$

The derivations of identities will be based on the  $p$ -adic integral expression of the generating function for the Euler polynomials in (1.5) and the quotient of integrals in (1.8) that can be expressed as the exponential generating function for the alternating power sums. We indebted this idea to the paper in [3].

## 2. Several Types of Quotients of Fermionic Integrals

Here we will introduce several types of quotients of  $p$ -adic fermionic integrals on  $\mathbb{Z}_p$  or  $\mathbb{Z}_p^3$  from which some interesting identities follow owing to the built-in symmetries in  $w_1, w_2, w_3$ . In the following,  $w_1, w_2, w_3$  are all positive integers and all of the explicit expressions of integrals in (2.2), (2.4), (2.6), and (2.8) are obtained from the identity in (1.4).

(a) Type  $\Lambda_{23}^i$  (for  $i = 0, 1, 2, 3$ ). One has

$$I(\Lambda_{23}^i) = \frac{\int_{\mathbb{Z}_p^3} e^{(w_2 w_3 x_1 + w_1 w_3 x_2 + w_1 w_2 x_3 + w_1 w_2 w_3 (\sum_{j=1}^{3-i} y_j))t} d\mu_{-1}(x_1) d\mu_{-1}(x_2) d\mu_{-1}(x_3)}{\left(\int_{\mathbb{Z}_p} e^{w_1 w_2 w_3 x_4 t} d\mu_{-1}(x_4)\right)^i} \quad (2.1)$$

$$= \frac{2^{3-i} e^{w_1 w_2 w_3 (\sum_{j=1}^{3-i} y_j)t} (e^{w_1 w_2 w_3 t} + 1)^i}{(e^{w_2 w_3 t} + 1)(e^{w_1 w_3 t} + 1)(e^{w_1 w_2 t} + 1)} \quad (2.2)$$

(b) Type  $\Lambda_{13}^i$  (for  $i = 0, 1, 2, 3$ ). One has

$$I(\Lambda_{13}^i) = \frac{\int_{\mathbb{Z}_p^3} e^{(w_1 x_1 + w_2 x_2 + w_3 x_3 + w_1 w_2 w_3 (\sum_{j=1}^{3-i} y_j))t} d\mu_{-1}(x_1) d\mu_{-1}(x_2) d\mu_{-1}(x_3)}{\left(\int_{\mathbb{Z}_p} e^{w_1 w_2 w_3 x_4 t} d\mu_{-1}(x_4)\right)^i} \quad (2.3)$$

$$= \frac{2^{3-i} e^{w_1 w_2 w_3 (\sum_{j=1}^{3-i} y_j)t} (e^{w_1 w_2 w_3 t} + 1)^i}{(e^{w_1 t} + 1)(e^{w_2 t} + 1)(e^{w_3 t} + 1)} \quad (2.4)$$

(c-0) Type  $\Lambda_{12}^0$ . One has

$$I(\Lambda_{12}^0) = \int_{\mathbb{Z}_p^3} e^{(w_1 x_1 + w_2 x_2 + w_3 x_3 + w_2 w_3 y + w_1 w_3 y + w_1 w_2 y)t} d\mu_{-1}(x_1) d\mu_{-1}(x_2) d\mu_{-1}(x_3) \quad (2.5)$$

$$= \frac{8e^{(w_2 w_3 + w_1 w_3 + w_1 w_2)yt}}{(e^{w_1 t} + 1)(e^{w_2 t} + 1)(e^{w_3 t} + 1)} \quad (2.6)$$

(c-1) Type  $\Lambda_{12}^1$ . One has

$$I(\Lambda_{12}^1) = \frac{\int_{\mathbb{Z}_p^3} e^{(w_1 x_1 + w_2 x_2 + w_3 x_3)t} d\mu_{-1}(x_1) d\mu_{-1}(x_2) d\mu_{-1}(x_3)}{\int_{\mathbb{Z}_p^3} e^{(w_2 w_3 z_1 + w_1 w_3 z_2 + w_1 w_2 z_3)t} d\mu_{-1}(z_1) d\mu_{-1}(z_2) d\mu_{-1}(z_3)} \quad (2.7)$$

$$= \frac{(e^{w_2 w_3 t} + 1)(e^{w_1 w_3 t} + 1)(e^{w_1 w_2 t} + 1)}{(e^{w_1 t} + 1)(e^{w_2 t} + 1)(e^{w_3 t} + 1)}. \quad (2.8)$$

All of the above  $p$ -adic integrals of various types are invariant under all permutations of  $w_1, w_2, w_3$  as one can see either from  $p$ -adic integral representations in (2.1), (2.3), (2.5), and (2.7) or from their explicit evaluations in (2.2), (2.4), (2.6), and (2.8).

### 3. Identities for Euler Polynomials

In the following  $w_1, w_2, w_3$  are all odd positive integers except for (a-0) and (c-0), where they are any positive integers.

(a-0) First, let us consider Type  $\Lambda_{23}^i$ , for each  $i = 0, 1, 2, 3$ . The following results can be easily obtained from (1.5) and (1.8):

$$\begin{aligned}
 I(\Lambda_{23}^0) &= \int_{\mathbb{Z}_p} e^{w_2 w_3 (x_1 + w_1 y_1) t} d\mu_{-1}(x_1) \int_{\mathbb{Z}_p} e^{w_1 w_3 (x_2 + w_2 y_2) t} d\mu_{-1}(x_2) \int_{\mathbb{Z}_p} e^{w_1 w_2 (x_3 + w_3 y_3) t} d\mu_{-1}(x_3) \\
 &= \left( \sum_{k=0}^{\infty} \frac{E_k(w_1 y_1)}{k!} (w_2 w_3 t)^k \right) \left( \sum_{l=0}^{\infty} \frac{E_l(w_2 y_2)}{l!} (w_1 w_3 t)^l \right) \left( \sum_{m=0}^{\infty} \frac{E_m(w_3 y_3)}{m!} (w_1 w_2 t)^m \right) \\
 &= \sum_{n=0}^{\infty} \left( \sum_{k+l+m=n} \binom{n}{k, l, m} E_k(w_1 y_1) E_l(w_2 y_2) E_m(w_3 y_3) w_1^{l+m} w_2^{k+m} w_3^{k+l} \right) \frac{t^n}{n!},
 \end{aligned} \tag{3.1}$$

where the inner sum is over all nonnegative integers  $k, l, m$ , with  $k + l + m = n$ , and

$$\binom{n}{k, l, m} = \frac{n!}{k!l!m!}. \tag{3.2}$$

(a-1) Here we write  $I(\Lambda_{23}^1)$  in two different ways:

(1) One has

$$\begin{aligned}
 I(\Lambda_{23}^1) &= \int_{\mathbb{Z}_p} e^{w_2 w_3 (x_1 + w_1 y_1) t} d\mu_{-1}(x_1) \\
 &\quad \times \int_{\mathbb{Z}_p} e^{w_1 w_3 (x_2 + w_2 y_2) t} d\mu_{-1}(x_2) \times \frac{\int_{\mathbb{Z}_p} e^{w_1 w_2 x_3 t} d\mu_{-1}(x_3)}{\int_{\mathbb{Z}_p} e^{w_1 w_2 w_3 x_4 t} d\mu_{-1}(x_4)} \\
 &= \left( \sum_{k=0}^{\infty} E_k(w_1 y_1) \frac{(w_2 w_3 t)^k}{k!} \right) \left( \sum_{l=0}^{\infty} E_l(w_2 y_2) \frac{(w_1 w_3 t)^l}{l!} \right) \left( \sum_{m=0}^{\infty} T_m(w_3 - 1) \frac{(w_1 w_2 t)^m}{m!} \right) \\
 &= \sum_{n=0}^{\infty} \left( \sum_{k+l+m=n} \binom{n}{k, l, m} E_k(w_1 y_1) E_l(w_2 y_2) T_m(w_3 - 1) w_1^{l+m} w_2^{k+m} w_3^{k+l} \right) \frac{t^n}{n!}.
 \end{aligned} \tag{3.3}$$

(2) Invoking (1.8), (3.3) can also be written as

$$\begin{aligned}
 I(\Lambda_{23}^1) &= \sum_{i=0}^{w_3-1} (-1)^i \int_{\mathbb{Z}_p} e^{w_2 w_3 (x_1 + w_1 y_1) t} d\mu_{-1}(x_1) \int_{\mathbb{Z}_p} e^{w_1 w_3 (x_2 + w_2 y_2 + (w_2/w_3)i)t} d\mu_{-1}(x_2) \\
 &= \sum_{i=0}^{w_3-1} (-1)^i \left( \sum_{k=0}^{\infty} E_k(w_1 y_1) \frac{(w_2 w_3 t)^k}{k!} \right) \left( \sum_{l=0}^{\infty} E_l \left( w_2 y_2 + \frac{w_2}{w_3} i \right) \frac{(w_1 w_3 t)^l}{l!} \right) \\
 &= \sum_{n=0}^{\infty} \left( w_3^n \sum_{k=0}^n \binom{n}{k} E_k(w_1 y_1) \sum_{i=0}^{w_3-1} (-1)^i E_{n-k} \left( w_2 y_2 + \frac{w_2}{w_3} i \right) w_1^{n-k} w_2^k \right) \frac{t^n}{n!}.
 \end{aligned} \tag{3.5}$$

(a-2) Here we write  $I(\Lambda_{23}^2)$  in three different ways:

(1) One has

$$I(\Lambda_{23}^2) = \int_{\mathbb{Z}_p} e^{w_2 w_3 (x_1 + w_1 y_1) t} d\mu_{-1}(x_1) \\ \times \frac{\int_{\mathbb{Z}_p} e^{w_1 w_3 x_2 t} d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p} e^{w_1 w_2 w_3 x_4 t} d\mu_{-1}(x_4)} \times \frac{\int_{\mathbb{Z}_p} e^{w_1 w_2 x_3 t} d\mu_{-1}(x_3)}{\int_{\mathbb{Z}_p} e^{w_1 w_2 w_3 x_4 t} d\mu_{-1}(x_4)} \quad (3.6)$$

$$= \left( \sum_{k=0}^{\infty} E_k(w_1 y_1) \frac{(w_2 w_3 t)^k}{k!} \right) \left( \sum_{l=0}^{\infty} T_l(w_2 - 1) \frac{(w_1 w_3 t)^l}{l!} \right) \left( \sum_{m=0}^{\infty} T_m(w_3 - 1) \frac{(w_1 w_2 t)^m}{m!} \right) \\ = \sum_{n=0}^{\infty} \left( \sum_{k+l+m=n} \binom{n}{k, l, m} E_k(w_1 y_1) T_l(w_2 - 1) T_m(w_3 - 1) w_1^{l+m} w_2^{k+m} w_3^{k+l} \right) \frac{t^n}{n!}. \quad (3.7)$$

(2) Invoking (1.8), (3.6) can also be written as

$$I(\Lambda_{23}^2) = \sum_{i=0}^{w_2-1} (-1)^i \int_{\mathbb{Z}_p} e^{w_2 w_3 (x_1 + w_1 y_1 + (w_1/w_2)i) t} d\mu_{-1}(x_1) \times \frac{\int_{\mathbb{Z}_p} e^{w_1 w_2 x_3 t} d\mu_{-1}(x_3)}{\int_{\mathbb{Z}_p} e^{w_1 w_2 w_3 x_4 t} d\mu_{-1}(x_4)} \quad (3.8)$$

$$= \sum_{i=0}^{w_2-1} (-1)^i \left( \sum_{k=0}^{\infty} E_k \left( w_1 y_1 + \frac{w_1}{w_2} i \right) \frac{(w_2 w_3 t)^k}{k!} \right) \left( \sum_{l=0}^{\infty} T_l(w_3 - 1) \frac{(w_1 w_2 t)^l}{l!} \right) \\ = \sum_{n=0}^{\infty} \left( w_2^n \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{w_2-1} (-1)^i E_k \left( w_1 y_1 + \frac{w_1}{w_2} i \right) T_{n-k}(w_3 - 1) w_1^{n-k} w_3^k \right) \frac{t^n}{n!}. \quad (3.9)$$

(3) Invoking (1.8) once again, (3.8) can be written as

$$I(\Lambda_{23}^2) = \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} (-1)^{i+j} \int_{\mathbb{Z}_p} e^{w_2 w_3 (x_1 + w_1 y_1 + (w_1/w_2)i + (w_1/w_3)j) t} d\mu_{-1}(x_1) \\ = \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} (-1)^{i+j} \sum_{n=0}^{\infty} E_n \left( w_1 y_1 + \frac{w_1}{w_2} i + \frac{w_1}{w_3} j \right) \frac{(w_2 w_3 t)^n}{n!} \quad (3.10) \\ = \sum_{n=0}^{\infty} \left( (w_2 w_3)^n \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} (-1)^{i+j} E_n \left( w_1 y_1 + \frac{w_1}{w_2} i + \frac{w_1}{w_3} j \right) \right) \frac{t^n}{n!}.$$

(a-3) One has

$$\begin{aligned}
 I(\Lambda_{23}^3) &= \frac{\int_{\mathbb{Z}_p} e^{w_2 w_3 x_1 t} d\mu_{-1}(x_1)}{\int_{\mathbb{Z}_p} e^{w_1 w_2 w_3 x_4 t} d\mu_{-1}(x_4)} \times \frac{\int_{\mathbb{Z}_p} e^{w_1 w_3 x_2 t} d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p} e^{w_1 w_2 w_3 x_4 t} d\mu_{-1}(x_4)} \times \frac{\int_{\mathbb{Z}_p} e^{w_1 w_2 x_3 t} d\mu_{-1}(x_3)}{\int_{\mathbb{Z}_p} e^{w_1 w_2 w_3 x_4 t} d\mu_{-1}(x_4)} \\
 &= \left( \sum_{k=0}^{\infty} T_k(w_1 - 1) \frac{(w_2 w_3 t)^k}{k!} \right) \left( \sum_{l=0}^{\infty} T_l(w_2 - 1) \frac{(w_1 w_3 t)^l}{l!} \right) \left( \sum_{m=0}^{\infty} T_m(w_3 - 1) \frac{(w_1 w_2 t)^m}{m!} \right) \\
 &= \sum_{n=0}^{\infty} \sum_{k+l+m=n} \binom{n}{k, l, m} T_k(w_1 - 1) T_l(w_2 - 1) T_m(w_3 - 1) w_1^{l+m} w_2^{k+m} w_3^{k+l} \frac{t^n}{n!}.
 \end{aligned} \tag{3.11}$$

(b) For Type  $\Lambda_{13}^i$  ( $i = 0, 1, 2, 3$ ), we may consider the analogous things to the ones in (a-0), (a-1), (a-2), and (a-3). However, these do not lead us to new identities. Indeed, if we substitute  $w_2 w_3, w_1 w_3, w_1 w_2$ , respectively, for  $w_1, w_2, w_3$  in (2.1), this amounts to replacing  $t$  by  $w_1 w_2 w_3 t$  in (2.3). So, upon replacing  $w_1, w_2, w_3$ , respectively, by  $w_2 w_3, w_1 w_3, w_1 w_2$ , and then dividing by  $(w_1 w_2 w_3)^n$ , in each of the expressions of Theorem 4.1 through Corollary 4.15, we will get the corresponding symmetric identities for Type  $\Lambda_{13}^i$  ( $i = 0, 1, 2, 3$ ).

(c-0) One has

$$\begin{aligned}
 I(\Lambda_{12}^0) &= \int_{\mathbb{Z}_p} e^{w_1(x_1+w_2 y)t} d\mu_{-1}(x_1) \int_{\mathbb{Z}_p} e^{w_2(x_2+w_3 y)t} d\mu_{-1}(x_2) \int_{\mathbb{Z}_p} e^{w_3(x_3+w_1 y)t} d\mu_{-1}(x_3) \\
 &= \left( \sum_{n=0}^{\infty} \frac{E_n(w_2 y)}{n!} (w_1 t)^n \right) \left( \sum_{l=0}^{\infty} \frac{E_l(w_3 y)}{l!} (w_2 t)^l \right) \left( \sum_{m=0}^{\infty} \frac{E_m(w_1 y)}{m!} (w_3 t)^m \right) \\
 &= \sum_{n=0}^{\infty} \left( \sum_{k+l+m=n} \binom{n}{k, l, m} E_k(w_2 y) E_l(w_3 y) E_m(w_1 y) w_1^k w_2^l w_3^m \right) \frac{t^n}{n!}.
 \end{aligned} \tag{3.12}$$

(c-1) One has

$$\begin{aligned}
 &\frac{\int_{\mathbb{Z}_p} e^{w_1 x_1 t} d\mu_{-1}(x_1)}{\int_{\mathbb{Z}_p} e^{w_1 w_2 z_3 t} d\mu_{-1}(z_3)} \times \frac{\int_{\mathbb{Z}_p} e^{w_2 x_2 t} d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p} e^{w_2 w_3 z_1 t} d\mu_{-1}(z_1)} \times \frac{\int_{\mathbb{Z}_p} e^{w_3 x_3 t} d\mu_{-1}(x_3)}{\int_{\mathbb{Z}_p} e^{w_3 w_1 z_2 t} d\mu_{-1}(z_2)} \\
 &= \left( \sum_{k=0}^{\infty} T_k(w_2 - 1) \frac{(w_1 t)^k}{k!} \right) \left( \sum_{l=0}^{\infty} T_l(w_3 - 1) \frac{(w_2 t)^l}{l!} \right) \left( \sum_{m=0}^{\infty} T_m(w_1 - 1) \frac{(w_3 t)^m}{m!} \right) \\
 &= \sum_{n=0}^{\infty} \left( \sum_{k+l+m=n} \binom{n}{k, l, m} T_k(w_2 - 1) T_l(w_3 - 1) T_m(w_1 - 1) w_1^k w_2^l w_3^m \right) \frac{t^n}{n!}.
 \end{aligned} \tag{3.13}$$

#### 4. Main Theorems

As we noted earlier in the last paragraph of Section 2, the various types of quotients of  $p$ -adic fermionic integrals are invariant under any permutation of  $w_1, w_2, w_3$ . So the corresponding expressions in Section 3 are also invariant under any permutation of  $w_1, w_2, w_3$ . Thus our results about identities of symmetry will be immediate consequences of this observation.

However, not all permutations of an expression in Section 3 yield distinct ones. In fact, as these expressions are obtained by permuting  $w_1, w_2, w_3$  in a single one labelled by them, they can be viewed as a group in a natural manner and hence it is isomorphic to a quotient of  $S_3$ . In particular, the numbers of possible distinct expressions are 1, 2, 3, or 6. (a-0), (a-1(1)), (a-1(2)), and (a-2(2)) give the full six identities of symmetry, (a-2(1)) and (a-2(3)) yield three identities of symmetry, and (c-0) and (c-1) give two identities of symmetry, while the expression in (a-3) yields no identities of symmetry.

Here we will just consider the cases of Theorems 4.8 and 4.17 leaving the others as easy exercises for the reader. As for the case of Theorem 4.8, in addition to (4.15)–(4.17), we get the following three ones:

$$\sum_{k+l+m=n} \binom{n}{k, l, m} E_k(w_1 y_1) T_l(w_3 - 1) T_m(w_2 - 1) w_1^{l+m} w_3^{k+m} w_2^{k+l}, \quad (4.1)$$

$$\sum_{k+l+m=n} \binom{n}{k, l, m} E_k(w_2 y_1) T_l(w_1 - 1) T_m(w_3 - 1) w_2^{l+m} w_1^{k+m} w_3^{k+l}, \quad (4.2)$$

$$\sum_{k+l+m=n} \binom{n}{k, l, m} E_k(w_3 y_1) T_l(w_2 - 1) T_m(w_1 - 1) w_3^{l+m} w_2^{k+m} w_1^{k+l}. \quad (4.3)$$

But, by interchanging  $l$  and  $m$ , we see that (4.1), (4.2), and (4.3) are, respectively, equal to (4.15), (4.16), and (4.17).

As to Theorem 17, in addition to (4.26) and (4.27), we have

$$\sum_{k+l+m=n} \binom{n}{k, l, m} T_k(w_2 - 1) T_l(w_3 - 1) T_m(w_1 - 1) w_1^k w_2^l w_3^m, \quad (4.4)$$

$$\sum_{k+l+m=n} \binom{n}{k, l, m} T_k(w_3 - 1) T_l(w_1 - 1) T_m(w_2 - 1) w_2^k w_3^l w_1^m, \quad (4.5)$$

$$\sum_{k+l+m=n} \binom{n}{k, l, m} T_k(w_3 - 1) T_l(w_2 - 1) T_m(w_1 - 1) w_1^k w_3^l w_2^m, \quad (4.6)$$

$$\sum_{k+l+m=n} \binom{n}{k, l, m} T_k(w_2 - 1) T_l(w_1 - 1) T_m(w_3 - 1) w_3^k w_2^l w_1^m. \quad (4.7)$$

However, (4.4) and (4.5) are equal to (4.26), as we can see by applying the permutations  $k \rightarrow l, l \rightarrow m$ , and  $m \rightarrow k$  for (4.4) and  $k \rightarrow m, l \rightarrow k$ , and  $m \rightarrow l$  for (4.5). Similarly,



we see that (4.6) and (4.7) are equal to (4.27), by applying permutations  $k \rightarrow l, l \rightarrow m$ , and  $m \rightarrow k$  for (4.6) and  $k \rightarrow m, l \rightarrow k$ , and  $m \rightarrow l$  for (4.7).

**Theorem 4.1.** *Let  $w_1, w_2, w_3$  be any positive integers. Then the following expression is invariant under any permutation of  $w_1, w_2, w_3$ , so that it gives us six symmetries:*

$$\begin{aligned}
 & \sum_{k+l+m=n} \binom{n}{k, l, m} E_k(w_1 y_1) E_l(w_2 y_2) E_m(w_3 y_3) w_1^{l+m} w_2^{k+m} w_3^{k+l} \\
 &= \sum_{k+l+m=n} \binom{n}{k, l, m} E_k(w_1 y_1) E_l(w_3 y_2) E_m(w_2 y_3) w_1^{l+m} w_3^{k+m} w_2^{k+l} \\
 &= \sum_{k+l+m=n} \binom{n}{k, l, m} E_k(w_2 y_1) E_l(w_1 y_2) E_m(w_3 y_3) w_2^{l+m} w_1^{k+m} w_3^{k+l} \\
 &= \sum_{k+l+m=n} \binom{n}{k, l, m} E_k(w_2 y_1) E_l(w_3 y_2) E_m(w_1 y_3) w_2^{l+m} w_3^{k+m} w_1^{k+l} \\
 &= \sum_{k+l+m=n} \binom{n}{k, l, m} E_k(w_3 y_1) E_l(w_1 y_2) E_m(w_2 y_3) w_3^{l+m} w_1^{k+m} w_2^{k+l} \\
 &= \sum_{k+l+m=n} \binom{n}{k, l, m} E_k(w_3 y_1) E_l(w_2 y_2) E_m(w_1 y_3) w_3^{l+m} w_2^{k+m} w_1^{k+l}.
 \end{aligned} \tag{4.8}$$

**Theorem 4.2.** *Let  $w_1, w_2, w_3$  be any odd positive integers. Then the following expression is invariant under any permutation of  $w_1, w_2, w_3$ , so that it gives us six symmetries:*

$$\begin{aligned}
 & \sum_{k+l+m=n} \binom{n}{k, l, m} E_k(w_1 y_1) E_l(w_2 y_2) T_m(w_3 - 1) w_1^{l+m} w_2^{k+m} w_3^{k+l} \\
 &= \sum_{k+l+m=n} \binom{n}{k, l, m} E_k(w_1 y_1) E_l(w_3 y_2) T_m(w_2 - 1) w_1^{l+m} w_3^{k+m} w_2^{k+l} \\
 &= \sum_{k+l+m=n} \binom{n}{k, l, m} E_k(w_2 y_1) E_l(w_1 y_2) T_m(w_3 - 1) w_2^{l+m} w_1^{k+m} w_3^{k+l} \\
 &= \sum_{k+l+m=n} \binom{n}{k, l, m} E_k(w_2 y_1) E_l(w_3 y_2) T_m(w_1 - 1) w_2^{l+m} w_3^{k+m} w_1^{k+l} \\
 &= \sum_{k+l+m=n} \binom{n}{k, l, m} E_k(w_3 y_1) E_l(w_2 y_2) T_m(w_1 - 1) w_3^{l+m} w_2^{k+m} w_1^{k+l} \\
 &= \sum_{k+l+m=n} \binom{n}{k, l, m} E_k(w_3 y_1) E_l(w_1 y_2) T_m(w_2 - 1) w_3^{l+m} w_1^{k+m} w_2^{k+l}.
 \end{aligned} \tag{4.9}$$

Putting  $w_3 = 1$  in (4.9), we get the following corollary.

**Corollary 4.3.** *Let  $w_1, w_2$  be any odd positive integers. Then one has*

$$\begin{aligned}
 & \sum_{k=0}^n \binom{n}{k} E_k(w_1 y_1) E_{n-k}(w_2 y_2) w_1^{n-k} w_2^k \\
 &= \sum_{k=0}^n \binom{n}{k} E_k(w_2 y_1) E_{n-k}(w_1 y_2) w_2^{n-k} w_1^k \\
 &= \sum_{k+l+m=n} \binom{n}{k, l, m} E_k(y_1) E_l(w_2 y_2) T_m(w_1 - 1) w_2^{k+m} w_1^{k+l} \\
 &= \sum_{k+l+m=n} \binom{n}{k, l, m} E_k(w_2 y_1) E_l(y_2) T_m(w_1 - 1) w_2^{l+m} w_1^{k+l} \\
 &= \sum_{k+l+m=n} \binom{n}{k, l, m} E_k(y_1) E_l(w_1 y_2) T_m(w_2 - 1) w_1^{k+m} w_2^{k+l} \\
 &= \sum_{k+l+m=n} \binom{n}{k, l, m} E_k(w_1 y_1) E_l(y_2) T_m(w_2 - 1) w_1^{l+m} w_2^{k+l}.
 \end{aligned} \tag{4.10}$$

Letting further  $w_2 = 1$  in (4.10), we have the following corollary.

**Corollary 4.4.** *Let  $w_1$  be any odd positive integer. Then one has*

$$\begin{aligned}
 & \sum_{k=0}^n \binom{n}{k} E_k(w_1 y_1) E_{n-k}(y_2) w_1^{n-k} \\
 &= \sum_{k=0}^n \binom{n}{k} E_k(y_1) E_{n-k}(w_1 y_2) w_1^k \\
 &= \sum_{k+l+m=n} \binom{n}{k, l, m} E_k(y_1) E_l(y_2) T_m(w_1 - 1) w_1^{k+l}.
 \end{aligned} \tag{4.11}$$

**Theorem 4.5.** Let  $w_1, w_2, w_3$  be any odd positive integers. Then the following expression is invariant under any permutation of  $w_1, w_2, w_3$ , so that it gives us six symmetries:

$$\begin{aligned}
 & w_1^n \sum_{k=0}^n \binom{n}{k} E_k(w_3 y_1) \sum_{i=0}^{w_1-1} (-1)^i E_{n-k} \left( w_2 y_2 + \frac{w_2}{w_1} i \right) w_3^{n-k} w_2^k \\
 &= w_1^n \sum_{k=0}^n \binom{n}{k} E_k(w_2 y_1) \sum_{i=0}^{w_1-1} (-1)^i E_{n-k} \left( w_3 y_2 + \frac{w_3}{w_1} i \right) w_2^{n-k} w_3^k \\
 &= w_2^n \sum_{k=0}^n \binom{n}{k} E_k(w_3 y_1) \sum_{i=0}^{w_2-1} (-1)^i E_{n-k} \left( w_1 y_2 + \frac{w_1}{w_2} i \right) w_3^{n-k} w_1^k \\
 &= w_2^n \sum_{k=0}^n \binom{n}{k} E_k(w_1 y_1) \sum_{i=0}^{w_2-1} (-1)^i E_{n-k} \left( w_3 y_2 + \frac{w_3}{w_2} i \right) w_1^{n-k} w_3^k \\
 &= w_3^n \sum_{k=0}^n \binom{n}{k} E_k(w_2 y_1) \sum_{i=0}^{w_3-1} (-1)^i E_{n-k} \left( w_1 y_2 + \frac{w_1}{w_3} i \right) w_2^{n-k} w_1^k \\
 &= w_3^n \sum_{k=0}^n \binom{n}{k} E_k(w_1 y_1) \sum_{i=0}^{w_3-1} (-1)^i E_{n-k} \left( w_2 y_2 + \frac{w_2}{w_3} i \right) w_1^{n-k} w_2^k.
 \end{aligned} \tag{4.12}$$

Letting  $w_3 = 1$  in (4.12), we obtain alternative expressions for the identities in (4.10).

**Corollary 4.6.** Let  $w_1, w_2$  be any odd positive integers. Then one has

$$\begin{aligned}
 & \sum_{k=0}^n \binom{n}{k} E_k(w_1 y_1) E_{n-k}(w_2 y_2) w_1^{n-k} w_2^k \\
 &= \sum_{k=0}^n \binom{n}{k} E_k(w_2 y_1) E_{n-k}(w_1 y_2) w_2^{n-k} w_1^k \\
 &= w_1^n \sum_{k=0}^n \binom{n}{k} E_k(y_1) \sum_{i=0}^{w_1-1} (-1)^i E_{n-k} \left( w_2 y_2 + \frac{w_2}{w_1} i \right) w_2^k \\
 &= w_1^n \sum_{k=0}^n \binom{n}{k} E_k(w_2 y_1) \sum_{i=0}^{w_1-1} (-1)^i E_{n-k} \left( y_2 + \frac{i}{w_1} \right) w_2^{n-k} \\
 &= w_2^n \sum_{k=0}^n \binom{n}{k} E_k(y_1) \sum_{i=0}^{w_2-1} (-1)^i E_{n-k} \left( w_1 y_2 + \frac{w_1}{w_2} i \right) w_1^k \\
 &= w_2^n \sum_{k=0}^n \binom{n}{k} E_k(w_1 y_1) \sum_{i=0}^{w_2-1} (-1)^i E_{n-k} \left( y_2 + \frac{i}{w_2} \right) w_1^{n-k}.
 \end{aligned} \tag{4.13}$$

Putting further  $w_2 = 1$  in (4.13), we have the alternative expressions for the identities for (4.11).

**Corollary 4.7.** *Let  $w_1$  be any odd positive integer. Then one has*

$$\begin{aligned} & \sum_{k=0}^n \binom{n}{k} E_k(y_1) E_{n-k}(w_1 y_2) w_1^k \\ &= \sum_{k=0}^n \binom{n}{k} E_k(y_2) E_{n-k}(w_1 y_1) w_1^k \\ &= w_1^n \sum_{k=0}^n \binom{n}{k} E_k(y_1) \sum_{i=0}^{w_1-1} (-1)^i E_{n-k}\left(y_2 + \frac{i}{w_1}\right). \end{aligned} \quad (4.14)$$

**Theorem 4.8.** *Let  $w_1, w_2, w_3$  be any odd positive integers. Then one has the following three symmetries in  $w_1, w_2, w_3$ :*

$$\sum_{k+l+m=n} \binom{n}{k, l, m} E_k(w_1 y_1) T_1(w_2 - 1) T_m(w_3 - 1) w_1^{l+m} w_2^{k+m} w_3^{k+l} \quad (4.15)$$

$$= \sum_{k+l+m=n} \binom{n}{k, l, m} E_k(w_2 y_1) T_1(w_3 - 1) T_m(w_1 - 1) w_2^{l+m} w_3^{k+m} w_1^{k+l} \quad (4.16)$$

$$= \sum_{k+l+m=n} \binom{n}{k, l, m} E_k(w_3 y_1) T_1(w_1 - 1) T_m(w_2 - 1) w_3^{l+m} w_1^{k+m} w_2^{k+l}. \quad (4.17)$$

Putting  $w_3 = 1$  in (4.15)–(4.17), we get the following corollary.

**Corollary 4.9.** *Let  $w_1, w_2$  be any odd positive integers. Then one has*

$$\begin{aligned} & \sum_{k=0}^n \binom{n}{k} E_k(w_1 y_1) T_{n-k}(w_2 - 1) w_1^{n-k} w_2^k \\ &= \sum_{k=0}^n \binom{n}{k} E_k(w_2 y_1) T_{n-k}(w_1 - 1) w_2^{n-k} w_1^k \\ &= \sum_{k+l+m=n} \binom{n}{k, l, m} E_k(y_1) T_1(w_1 - 1) T_m(w_2 - 1) w_1^{k+m} w_2^{k+l}. \end{aligned} \quad (4.18)$$

Letting further  $w_2 = 1$  in (4.18), we get the following corollary. This is also obtained in [20, Corollary 2] and mentioned in [3].

**Corollary 4.10.** *Let  $w_1$  be any odd positive integer. Then one has*

$$E_n(w_1 y_1) = \sum_{k=0}^n \binom{n}{k} E_k(y_1) T_{n-k}(w_1 - 1) w_1^k. \quad (4.19)$$

**Theorem 4.11.** Let  $w_1, w_2, w_3$  be any odd positive integers. Then the following expression is invariant under any permutation of  $w_1, w_2, w_3$ , so that it gives us six symmetries:

$$\begin{aligned}
& w_1^n \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{w_1-1} (-1)^i E_k \left( w_2 y_1 + \frac{w_2}{w_1} i \right) T_{n-k}(w_3 - 1) w_2^{n-k} w_3^k \\
&= w_1^n \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{w_1-1} (-1)^i E_k \left( w_3 y_1 + \frac{w_3}{w_1} i \right) T_{n-k}(w_2 - 1) w_3^{n-k} w_2^k \\
&= w_2^n \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{w_2-1} (-1)^i E_k \left( w_1 y_1 + \frac{w_1}{w_2} i \right) T_{n-k}(w_3 - 1) w_1^{n-k} w_3^k \\
&= w_2^n \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{w_2-1} (-1)^i E_k \left( w_3 y_1 + \frac{w_3}{w_2} i \right) T_{n-k}(w_1 - 1) w_3^{n-k} w_1^k \\
&= w_3^n \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{w_3-1} (-1)^i E_k \left( w_1 y_1 + \frac{w_1}{w_3} i \right) T_{n-k}(w_2 - 1) w_1^{n-k} w_2^k \\
&= w_3^n \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{w_3-1} (-1)^i E_k \left( w_2 y_1 + \frac{w_2}{w_3} i \right) T_{n-k}(w_1 - 1) w_2^{n-k} w_1^k.
\end{aligned} \tag{4.20}$$

Putting  $w_3 = 1$  in (4.20), we obtain the following corollary. In Section 1, the identities in (4.18), (4.21), and (4.24) are combined to give those in (1.9)–(1.16).

**Corollary 4.12.** Let  $w_1, w_2$  be any odd positive integers. Then one has

$$\begin{aligned}
& w_1^n \sum_{i=0}^{w_1-1} (-1)^i E_n \left( w_2 y_1 + \frac{w_2}{w_1} i \right) \\
&= w_2^n \sum_{i=0}^{w_2-1} (-1)^i E_n \left( w_1 y_1 + \frac{w_1}{w_2} i \right) \\
&= \sum_{k=0}^n \binom{n}{k} E_k(w_2 y_1) T_{n-k}(w_1 - 1) w_2^{n-k} w_1^k \\
&= \sum_{k=0}^n \binom{n}{k} E_k(w_1 y_1) T_{n-k}(w_2 - 1) w_1^{n-k} w_2^k \\
&= w_1^n \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{w_1-1} (-1)^i E_k \left( y_1 + \frac{i}{w_1} \right) T_{n-k}(w_2 - 1) w_2^k \\
&= w_2^n \sum_{k=0}^n \binom{n}{k} \sum_{i=0}^{w_2-1} (-1)^i E_k \left( y_1 + \frac{i}{w_2} \right) T_{n-k}(w_1 - 1) w_1^k.
\end{aligned} \tag{4.21}$$

Letting further  $w_2 = 1$  in (4.21), we get the following corollary. This is the multiplication formula for Euler polynomials together with the relatively new identity mentioned in (4.19).

**Corollary 4.13.** *Let  $w_1$  be any odd positive integer. Then one has*

$$\begin{aligned} E_n(w_1 y_1) &= w_1^n \sum_{i=0}^{w_1-1} (-1)^i E_n\left(y_1 + \frac{i}{w_1}\right) \\ &= \sum_{k=0}^n \binom{n}{k} E_k(y_1) T_{n-k}(w_1 - 1) w_1^k. \end{aligned} \quad (4.22)$$

**Theorem 4.14.** *Let  $w_1, w_2, w_3$  be any odd positive integers. Then one has the following three symmetries in  $w_1, w_2, w_3$ :*

$$\begin{aligned} & (w_1 w_2)^n \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} (-1)^{i+j} E_n\left(w_3 y_1 + \frac{w_3}{w_1} i + \frac{w_3}{w_2} j\right) \\ &= (w_2 w_3)^n \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} (-1)^{i+j} E_n\left(w_1 y_1 + \frac{w_1}{w_2} i + \frac{w_1}{w_3} j\right) \\ &= (w_3 w_1)^n \sum_{i=0}^{w_3-1} \sum_{j=0}^{w_1-1} (-1)^{i+j} E_n\left(w_2 y_1 + \frac{w_2}{w_3} i + \frac{w_2}{w_1} j\right). \end{aligned} \quad (4.23)$$

Letting  $w_3 = 1$  in (4.23), we have the following corollary.

**Corollary 4.15.** *Let  $w_1, w_2$  be any odd positive integers. Then one has*

$$\begin{aligned} & w_1^n \sum_{j=0}^{w_1-1} (-1)^j E_n\left(w_2 y_1 + \frac{w_2}{w_1} j\right) \\ &= w_2^n \sum_{i=0}^{w_2-1} (-1)^i E_n\left(w_1 y_1 + \frac{w_1}{w_2} i\right) \\ &= (w_1 w_2)^n \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} (-1)^{i+j} E_n\left(y_1 + \frac{i}{w_1} + \frac{j}{w_2}\right). \end{aligned} \quad (4.24)$$

**Theorem 4.16.** Let  $w_1, w_2, w_3$  be any positive integers. Then one has the following two symmetries in  $w_1, w_2, w_3$ :

$$\begin{aligned} & \sum_{k+l+m=n} \binom{n}{k, l, m} E_k(w_1 y) E_l(w_2 y) E_m(w_3 y) w_3^k w_1^l w_2^m \\ &= \sum_{k+l+m=n} \binom{n}{k, l, m} E_k(w_1 y) E_l(w_3 y) E_m(w_2 y) w_2^k w_1^l w_3^m. \end{aligned} \quad (4.25)$$

**Theorem 4.17.** Let  $w_1, w_2, w_3$  be any odd positive integers. Then one has the following two symmetries in  $w_1, w_2, w_3$ :

$$\sum_{k+l+m=n} \binom{n}{k, l, m} T_k(w_1 - 1) T_l(w_2 - 1) T_m(w_3 - 1) w_3^k w_1^l w_2^m \quad (4.26)$$

$$= \sum_{k+l+m=n} \binom{n}{k, l, m} T_k(w_1 - 1) T_l(w_3 - 1) T_m(w_2 - 1) w_2^k w_1^l w_3^m. \quad (4.27)$$

Putting  $w_3 = 1$  in (4.26) and (4.27), we get the following corollary.

**Corollary 4.18.** Let  $w_1, w_2$  be any odd positive integers. Then one has

$$\sum_{k=0}^n \binom{n}{k} T_k(w_2 - 1) T_{n-k}(w_1 - 1) w_1^k = \sum_{k=0}^n \binom{n}{k} T_k(w_1 - 1) T_{n-k}(w_2 - 1) w_2^k. \quad (4.28)$$

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