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A note on Hardy's inequality

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Abstract

In this paper, we prove that the inequality $\sum_{n=1}^{\infty}(\frac{1}{n}\sum_{k=1}^{n}a_k)^p \leq (\frac{p}{p-1})^p\sum_{n=1}^{\infty}(1-\frac{d(p)}{(n-1/2)^{1-1/p}})a_n^p$ holds for $p\leq -1$ and $d(p)=(1+(2^{-1/p}-1)p)/[8(1+(2^{-1/p}-1)p)+2]$ if $a_n>0$ $(n=1,2,\ldots)$, and $\sum_{n=1}^{\infty}a_n^p<+\infty$.

MSC: 26D15

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1 Introduction

Let $p>1$ and $a_n>0$ $(n=1,2,\ldots)$ with $\sum_{n=1}^{\infty}a_n{}^p<+\infty$, then Hardy's well-known inequality [1] is given by

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^p < \left(\frac{p}{p-1}\right)^p\sum_{n=1}^{\infty}a_n^p. \tag{1.1}$$

Recently, the refinement, improvement, generalization, extension, and application for Hardy's inequality have attracted the attention of many researchers [2–10].

Yang and Zhu [11] presented an improvement of Hardy's inequality (1.1) for $p=2$ as follows:

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^2 < 4\sum_{n=1}^{\infty}\left(1-\frac{1}{3\sqrt{n}+5}\right)a_n^2.$$

For $7/6\leq p\leq 2$, Huang [12] proved that

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^p < \left(\frac{p}{p-1}\right)^p\sum_{n=1}^{\infty}\left(1-\frac{15}{196(n^{1-1/p}+3{,}436)}\right)a_n^p.$$

In [13], Wen and Zhang proved that the inequality

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^p < \left(\frac{p}{p-1}\right)^p\sum_{n=1}^{\infty}\left(1-\frac{C_p}{2n^{1-1/p}}\right)a_n^p \tag{1.2}$$

holds for $p>1$ if $a_n>0$ $(n=1,2,\ldots)$, with $\sum_{n=1}^{\infty}a_n^p<+\infty$, where $C_p=1-(1-1/p)^{p-1}$ for $p\geq 2$ and $C_p=1-1/p$ for $1<p\leq 2$.

Xu *et al.* [14] gave a further improvement of the inequality (1.2):

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^p < \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty}\left(1-\frac{Z_p}{2(n-1)^{1-1/p}}\right)a_n^p,$$

where $Z_p = p-1-\frac{(p-1)^2}{p}2^{1/p}$ for $1 < p \leq 2$ and $Z_p = 1-(\frac{p-1}{p})^{p-1}2^{\frac{p-1}{p}}$ for $p > 2$.

For the special parameter $p = 5/4$, Deng *et al.* [15] established

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^{5/4} \leq 5^{5/4}\sum_{n=1}^{\infty}\left(1-\frac{1}{10(n^{1/5}+\eta_{5/4})}\right)a_n^{5/4},$$

where $\eta_{5/4} = 5^{5/4}/[10(5^{5/4} - (\sum_{n=1}^{\infty}\frac{1}{n^{5/4}}(\sum_{m=1}^{n}m^{-4/5})^{1/4})) - 1] = 0.46\cdots$.

In [16], Long and Linh discussed Hardy's inequality with the parameter $p < 0$, and proved that

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^p < \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty}a_n^p \tag{1.3}$$

for $p \leq -1$ and

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^p < \frac{2^{1-p}}{1-p}\sum_{n=1}^{\infty}a_n^p$$

for $-1 < p < 0$ if $a_n > 0$ ($n = 1,2,\ldots$) with $\sum_{n=1}^{\infty}a_n^p < +\infty$.

It is the aim of this paper to present an improvement of inequality (1.3) for the parameter $p \leq -1$. Our main result is Theorem 1.1.

Theorem 1.1 *Let* $p \leq -1$, $d(p) = (1+(2^{-1/p}-1)p)/[8(1+(2^{-1/p}-1)p)+2]$ *and* $a_n > 0$ ($n = 1,2,\ldots$) *with* $\sum_{n=1}^{\infty}a_n^p < +\infty$, *then*

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty}\left(1-\frac{d(p)}{(n-1/2)^{1-1/p}}\right)a_n^p.$$

2 Lemmas

In order to establish our main result we need several lemmas, which we present in this section.

Lemma 2.1 (see [17, Corollary 1.3]) *Suppose that* $a, b \in \mathbb{R}$ *with* $a < b$, $f : [a,b]^n \to \mathbb{R}$ *has continuous partial derivatives and*

$$D_m = \left\{x = (x_1, x_2, \ldots, x_n) \middle| a \leq \min_{1\leq k\leq n}\{x_k\} < x_m = \max_{1\leq k\leq n}\{x_k\} \leq b\right\}, \quad m = 1,2,\ldots,n.$$

If $\frac{\partial f(x)}{\partial x_m} > 0$ *holds for all* $x = (x_1, x_2, \ldots, x_n) \in D_m$ *and* $m = 1,2,\ldots,n$, *then*

$$f(x_1, x_2, \ldots, x_n) \geq f(x_{\min}, x_{\min}, \ldots, x_{\min})$$

for all $x_m \in [a,b]$ ($m = 1,2,\ldots,n$), *where* $x_{\min} = \min_{1\leq k\leq n}\{x_k\}$.

Lemma 2.2 *Let $n \in \mathbb{R}$ be a positive natural number and $r \in \mathbb{R}$ with $r \geq 1$. Then*

$$\sum_{k=1}^{n}\left(k-\frac{1}{2}\right)^{1/r} \geq \frac{r}{r+1}\left(n^{1+1/r}-1\right)+2^{-1/r}. \tag{2.1}$$

Proof We use mathematical induction to prove inequality (2.1). We clearly see that inequality (2.1) becomes equality for $n = 1$. We assume that inequality (2.1) holds for $n = i$ ($i \in \mathbb{N}$, $i \geq 1$), namely

$$\sum_{k=1}^{i}\left(k-\frac{1}{2}\right)^{1/r} \geq \frac{r}{r+1}\left(i^{1+1/r}-1\right)+2^{-1/r}.$$

Then for $n = i + 1$ we have

$$\begin{aligned}\sum_{k=1}^{i+1}\left(k-\frac{1}{2}\right)^{1/r} &= \sum_{k=1}^{i}\left(k-\frac{1}{2}\right)^{1/r}+\left(i+\frac{1}{2}\right)^{1/r}\\ &\geq \frac{r}{r+1}\left(i^{1+1/r}-1\right)+2^{-1/r}+\left(i+\frac{1}{2}\right)^{1/r}\\ &= \frac{r}{r+1}\left[(i+1)^{1+1/r}-1\right]+2^{-1/r}+\left(i+\frac{1}{2}\right)^{1/r}-\int_{i}^{i+1}x^{1/r}\,dx. \end{aligned} \tag{2.2}$$

Note that $x^{1/r}$ ($r \geq 1$) is concave on $(0, +\infty)$, therefore Hermite-Hadamard's inequality implies that

$$\left(i+\frac{1}{2}\right)^{1/r} \geq \int_{i}^{i+1}x^{1/r}\,dx. \tag{2.3}$$

From (2.2) and (2.3) we know that inequality (2.1) holds for $n = i + 1$. □

Remark 2.1 The inequality

$$2^{-1/r} \geq \frac{r}{r+1} \tag{2.4}$$

holds for all $r \geq 1$ with equality if and only if $r = 1$.

Proof We clearly see that inequality (2.4) becomes equality for $r = 1$.

If $r > 1$, then it is well known that the function $(1 + 1/r)^r$ is strictly increasing on $(1, +\infty)$, so we get

$$\left(1+\frac{1}{r}\right)^{r} > 2. \tag{2.5}$$

Therefore, inequality (2.4) follows from (2.5). □

Lemma 2.3 *The inequality*

$$\frac{(3r+1)2^{-1-1/r}}{r+1} > \left(2^{-1/r}-\frac{r}{r+1}\right)\frac{3r^2+1}{r} \tag{2.6}$$

holds for all $r \geq 1$.

Proof Let $r \geq 1$, then we clearly see that

$$(6\log 2 - 3)r^2 + r + 2\log 2 - 2 \geq 4(2\log 2 - 1) > 0. \tag{2.7}$$

Inequality (2.7) leads to

$$e^{(3r^2-r+2)/(6r^2+2)} < 2. \tag{2.8}$$

It follows from the well-known inequality $(1+x)^{1/x} < e$ $(x > 0)$ that

$$e > \left(1 + \frac{3r^2 - r + 2}{6r^3 + 2r}\right)^{(6r^3+2r)/(3r^2-r+2)}. \tag{2.9}$$

From (2.8) and (2.9) we have

$$2^{-1/r} < \frac{6r^3 + 2r}{6r^3 + 3r^2 + r + 2}. \tag{2.10}$$

Therefore, inequality (2.6) follows easily from (2.10). □

Lemma 2.4 *Let* $r \geq 1$ *and*

$$f(x) = \frac{x^r}{(\frac{r}{r+1}x^{1+1/r} + 2^{-1/r} - \frac{r}{r+1})^{r+1}}. \tag{2.11}$$

Then f *is convex on* $[1/2, +\infty)$.

Proof From (2.11) we have

$$\begin{aligned} f'(x) &= \frac{-\frac{2r+1}{r+1}x^{r+1/r} + (2^{-1/r} - \frac{r}{r+1})rx^{r-1}}{(\frac{r}{r+1}x^{1+1/r} + 2^{-1/r} - \frac{r}{r+1})^{r+2}}, \\ f''(x) &= \frac{\frac{(3r+1)(2r+1)}{(r+1)^2}x^{2+2/r} - (2^{-1/r} - \frac{r}{r+1})\frac{(3r^2+1)(2r+1)}{r(r+1)}x^{1+1/r} + r(r-1)(2^{-1/r} - \frac{r}{r+1})^2}{(\frac{r}{r+1}x^{1+1/r} + 2^{-1/r} - \frac{r}{r+1})^{r+3}}x^{r-2}. \end{aligned} \tag{2.12}$$

It follows from Lemma 2.3 and (2.12) that

$$f''(x) \geq \frac{\frac{(3r+1)(2r+1)}{(r+1)^2}2^{-1-1/r} - (2^{-1/r} - \frac{r}{r+1})\frac{(3r^2+1)(2r+1)}{r(r+1)}}{(\frac{r}{r+1}x^{1+1/r} + 2^{-1/r} - \frac{r}{r+1})^{r+3}}x^{r+1/r-1} > 0 \tag{2.13}$$

for all $x \in [1/2, +\infty)$.

Therefore, Lemma 2.4 follows from inequality (2.13). □

Lemma 2.5 *Let* $r \geq 1$, $0 \leq t \leq 4$ *and* $c = (r + 1 - 2^{1/r}r)/[8(r + 1 - 2^{1/r}r) + 2]$, *then*

$$(r+1)\left(2^{-1/r} - \frac{r}{r+1}\right)(1 - ct)t \geq 1 - \left[1 + \left(\frac{1+r}{2^{1/r}r} - 1\right)t\right]^{-r}. \tag{2.14}$$

Proof If $r = 1$, then we clearly see that inequality (2.14) becomes equality. Next, we assume that $r > 1$. Let

$$f(t) = (r+1)\left(2^{-1/r} - \frac{r}{r+1}\right)(1-ct)t - 1 + \left[1 + \left(\frac{1+r}{2^{1/r}r} - 1\right)t\right]^{-r}. \tag{2.15}$$

Then simple computations lead to

$$f(0) = 0, \tag{2.16}$$

$$f'(t) = \frac{(r+1)(2^{-1/r} - \frac{r}{r+1})}{[1 + (\frac{r+1}{2^{1/r}r} - 1)t]^{r+1}}\left\{(1-2ct)\left[1 + \left(\frac{r+1}{2^{1/r}r} - 1\right)t\right]^{r+1} - 1\right\}. \tag{2.17}$$

Note that

$$1 - 2ct \geq 1 - 8c = \frac{1}{4(r+1-2^{1/r}r)+1} > 0, \tag{2.18}$$

$$\left[1 + \left(\frac{r+1}{2^{1/r}r} - 1\right)t\right]^{r+1} \geq 1 + (r+1)\left(\frac{r+1}{2^{1/r}r} - 1\right)t. \tag{2.19}$$

It follows from Remark 2.1 and (2.17)-(2.19) that

$$f'(t) \geq \frac{(r+1)(2^{-1/r} - \frac{r}{r+1})}{[1 + (\frac{r+1}{2^{1/r}r} - 1)t]^{r+1}}\left\{(1-2ct)\left[1 + (r+1)\left(\frac{r+1}{2^{1/r}r} - 1\right)t\right] - 1\right\}. \tag{2.20}$$

Let

$$g(t) = (1-2ct)\left[1 + (r+1)\left(\frac{r+1}{2^{1/r}r} - 1\right)t\right] - 1. \tag{2.21}$$

Then from $g(0) = 0$ and $g(4) = 4(r+1-2^{1/r}r)^2/[2^{1/r}r(4(r+1-2^{1/r}r)+1)] \geq 0$ together with the fact that $g(t)$ is a concave parabola we know that

$$g(t) \geq 0 \tag{2.22}$$

for $t \in [0,4]$.

Therefore, Lemma 2.5 follows easily from (2.15) and (2.16) together with (2.20)-(2.22). □

Lemma 2.6 *Let* $r \geq 1$, $c = (r+1-2^{1/r}r)/[8(r+1-2^{1/r}r)+2]$, N *is a positive natural number,* $a_k > 0$ $(k = 1,2,\ldots,N)$ *and* $B_N = \min_{1\leq k\leq N}\{(k-1/2)^{1/r}a_k\}$, *then*

$$\begin{aligned}
&\left(\frac{r+1}{r}\right)^r \sum_{n=1}^{N}\left(1 - \frac{c}{(n-1/2)^{1+1/r}}\right)a_n^r - \sum_{n=1}^{N}\left(\frac{n}{\sum_{k=1}^{n} 1/a_k}\right)^r \\
&\quad \geq B_N^r\left[\left(\frac{r+1}{r}\right)^r \sum_{n=1}^{N}\left(1 - \frac{c}{(n-1/2)^{1+1/r}}\right)\frac{1}{n-1/2} \right. \\
&\qquad \left. - \sum_{n=1}^{N}\left(\frac{n}{\sum_{k=1}^{n}(k-1/2)^{1/r}}\right)^r\right].
\end{aligned} \tag{2.23}$$

Proof Let $a_k = b_k/(k-1/2)^{1/r}$ $(k = 1,2,\ldots,N)$, then $B_N = \min_{1\le k\le N}\{b_k\}$ and inequality (2.23) becomes

$$\begin{aligned}
&\left(\frac{r+1}{r}\right)^r \sum_{n=1}^{N}\left(1-\frac{c}{(n-1/2)^{1+1/r}}\right)\frac{b_n^r}{n-1/2} - \sum_{n=1}^{N}\left(\frac{n}{\sum_{k=1}^{n}\frac{(k-1/2)^{1/r}}{b_k}}\right)^r \\
&\quad \ge B_N^r\left[\left(\frac{r+1}{r}\right)^r \sum_{n=1}^{N}\left(1-\frac{c}{(n-1/2)^{1+1/r}}\right)\frac{1}{n-1/2}\right. \\
&\qquad \left. - \sum_{n=1}^{N}\left(\frac{n}{\sum_{k=1}^{n}(k-1/2)^{1/r}}\right)^r\right].
\end{aligned} \tag{2.24}$$

Let $D_m = \{\mathbf{b} = (b_1,b_2,\ldots,b_N) | b_m = \max_{1\le k\le N}\{b_k\} > \min_{1\le k\le N}\{b_k\}\}$ $(m = 1,2,\ldots,N)$, and

$$\begin{aligned}
f(b_1,b_2,\ldots,b_N) &= \left(\frac{r+1}{r}\right)^r \sum_{n=1}^{N}\left(1-\frac{c}{(n-1/2)^{1+1/r}}\right)\frac{b_n^r}{n-1/2} \\
&\quad - \sum_{n=1}^{N}\left(\frac{n}{\sum_{k=1}^{n}\frac{(k-1/2)^{1/r}}{b_k}}\right)^r.
\end{aligned} \tag{2.25}$$

Then for any $\mathbf{b} \in D_m$ $(m = 1,2,\ldots,N)$ we have

$$\begin{aligned}
\frac{\partial f(\mathbf{b})}{\partial b_m} &= \left(1-\frac{c}{(m-1/2)^{1+1/r}}\right)\frac{(r+1)^r b_m^{r-1}}{(m-1/2)r^{r-1}} \\
&\quad - \frac{r(m-1/2)^{1/r}}{b_m^2}\sum_{n=m}^{N}\frac{n^r}{(\sum_{k=1}^{n}\frac{(k-1/2)^{1/r}}{b_k})^{r+1}} \\
&> \left(1-\frac{c}{(m-1/2)^{1+1/r}}\right)\frac{(r+1)^r b_m^{r-1}}{(m-1/2)r^{r-1}} \\
&\quad - r(m-1/2)^{1/r} b_m^{r-1}\sum_{n=m}^{+\infty}\frac{n^r}{(\sum_{k=1}^{n}(k-1/2)^{1/r})^{r+1}}.
\end{aligned} \tag{2.26}$$

From Lemma 2.2 and (2.26) one has

$$\begin{aligned}
\frac{1}{r(m-1/2)^{1/r} b_m^{r-1}}\frac{\partial f(\mathbf{b})}{\partial b_m} &> \left(\frac{r+1}{r}\right)^r\left(1-\frac{c}{(m-1/2)^{1+1/r}}\right)\frac{1}{(m-1/2)^{1+1/r}} \\
&\quad - \sum_{n=m}^{+\infty}\frac{n^r}{(\frac{r}{r+1}n^{1+1/r} + 2^{-1/r} - \frac{r}{r+1})^{r+1}}.
\end{aligned} \tag{2.27}$$

It clearly follows from Lemma 2.4 and the Hermite-Hadamard inequality that

$$\int_{m-1/2}^{m+1/2}\frac{x^r}{(\frac{r}{r+1}x^{1+1/r} + 2^{-1/r} - \frac{r}{r+1})^{r+1}} \ge \frac{m^r}{(\frac{r}{r+1}m^{1+1/r} + 2^{-1/r} - \frac{r}{r+1})^{r+1}}$$

and

$$\int_{m-1/2}^{+\infty}\frac{x^r}{(\frac{r}{r+1}x^{1+1/r} + 2^{-1/r} - \frac{r}{r+1})^{r+1}} \ge \sum_{n=m}^{+\infty}\frac{n^r}{(\frac{r}{r+1}n^{1+1/r} + 2^{-1/r} - \frac{r}{r+1})^{r+1}}. \tag{2.28}$$

Note that

$$\int_{m-1/2}^{+\infty}\frac{x^r}{(\frac{r}{r+1}x^{1+1/r}+2^{-1/r}-\frac{r}{r+1})^{r+1}}$$
$$=\left(\frac{1+r}{r}\right)^r\frac{2^{1/r}}{r+1-2^{1/r}r}\left\{1-\left[1+\left(\frac{r+1}{2^{1/r}r}-1\right)(m-1/2)^{-1-1/r}\right]^{-r}\right\}, \tag{2.29}$$

$$0<(m-1/2)^{-1-1/r}\le 2^{1+1/r}\le 4. \tag{2.30}$$

From Lemma 2.5 and (2.30) one has

$$\left(\frac{r+1}{r}\right)^r\left(1-\frac{c}{(m-1/2)^{1+1/r}}\right)\frac{1}{(m-1/2)^{1+1/r}}$$
$$\ge\left(\frac{1+r}{r}\right)^r\frac{2^{1/r}}{r+1-2^{1/r}r}\left\{1-\left[1+\left(\frac{r+1}{2^{1/r}r}-1\right)(m-1/2)^{-1-1/r}\right]^{-r}\right\}. \tag{2.31}$$

Inequalities (2.27), (2.28), and (2.31) together with (2.29) lead to the conclusion that

$$\frac{\partial f(\mathbf{b})}{\partial b_m}>0 \tag{2.32}$$

for any $\mathbf{b}=(b_1,b_2,\ldots,b_N)\in D_m$ and $m=1,2,\ldots,N$.

It follows from Lemma 2.1 and (2.32) that

$$f(b_1,b_2,\ldots,b_N)\ge f(B_N,B_N,\ldots,B_N). \tag{2.33}$$

Therefore, inequality (2.24) follows from (2.25) and (2.33). □

Lemma 2.7 *Let* $r\ge 1$, $c=(r+1-2^{1/r}r)/[8(r+1-2^{1/r}r)+2]$, *then*

$$\left(\frac{r+1}{r}\right)\left(1-2^{1+1/r}c\right)>1. \tag{2.34}$$

Proof We clearly see that inequality (2.34) holds for $r=1$. Next, we assume that $r>1$, let $t=2^{1+1/r}$, then $0<t<4$ and Lemma 2.5 leads to

$$\left(\frac{r+1}{r}\right)^r\left(1-2^{1+1/r}c\right)-1\ge\frac{(\frac{r+1}{r})^r}{2(r+1-2^{1/r}r)}\left[1-\left(1+\frac{2(r+1-2^{1/r}r)}{r}\right)^{-r}\right]-1$$
$$\ge\frac{(\frac{r+1}{r})^r}{2(r+1-2^{1/r}r)}\frac{2(r+1-2^{1/r}r)}{1+2(r+1-2^{1/r}r)}-1$$
$$=\frac{(\frac{r+1}{r})^r}{1+2(r+1-2^{1/r}r)}-1. \tag{2.35}$$

Note that

$$r+1-2^{1/r}r<1-\log 2 \tag{2.36}$$

for all $r\ge 1$. In fact, let $x\ge 1$ and

$$f(x)=x-2^{1/x}x+1. \tag{2.37}$$

Then

$$f'(x) = 1 + \left(\frac{\log 2}{x} - 1\right)2^{1/x}, \tag{2.38}$$

$$f''(x) = -\frac{(\log 2)^2}{x^3}2^{1/x} < 0. \tag{2.39}$$

It follows from (2.38) and (2.39) that

$$f'(x) > \lim_{x\to+\infty}\left[1 + \left(\frac{\log 2}{x} - 1\right)2^{1/x}\right] = 0. \tag{2.40}$$

Equation (2.37) and inequality (2.40) lead to the conclusion that

$$f(x) < \lim_{x\to+\infty}\left(x - 2^{1/x}x + 1\right) = 1 - \log 2. \tag{2.41}$$

From (2.35) and (2.36) together with the fact that $[(r+1)/r]^r \geq 2$ we have

$$\left(\frac{r+1}{r}\right)^r\left(1 - 2^{1+1/r}c\right) - 1 > \frac{2}{1 + 2(1 - \log 2)} - 1 = \frac{2\log 2 - 1}{3 - 2\log 2} > 0. \tag{2.42}$$

Therefore, inequality (2.34) follows from (2.42). □

Lemma 2.8 *Let* $r \geq 1$, $c = (r + 1 - 2^{1/r}r)/[8(r + 1 - 2^{1/r}r) + 2]$, N *is a positive natural number,* $a_k > 0$ $(k = 1, 2, \ldots, N)$ *and* $B_N = \min_{1\leq k\leq N}\{(k - 1/2)^{1/r}a_k\}$, *then*

$$\begin{aligned}&\left(\frac{r+1}{r}\right)^r\sum_{n=1}^{N}\left(1 - \frac{c}{(n-1/2)^{1+1/r}}\right)a_n^r - \sum_{n=1}^{N}\left(\frac{n}{\sum_{k=1}^{n}1/a_k}\right)^r\\&\quad\geq 2B_N^r\left[\left(\frac{r+1}{r}\right)^r\left(1 - 2^{1+1/r}c\right) - 1\right].\end{aligned} \tag{2.43}$$

Proof Let $m \in \{1, 2, \ldots, N\}$, $f(0) = 0$ and

$$f(m) = \left(\frac{r+1}{r}\right)^r\sum_{n=1}^{m}\left(1 - \frac{c}{(n-1/2)^{1+1/r}}\right)\frac{1}{n - 1/2} - \sum_{n=1}^{m}\left(\frac{n}{\sum_{k=1}^{n}(k-1/2)^{1/r}}\right)^r. \tag{2.44}$$

Then

$$f(1) = 2\left[\left(\frac{1+r}{r}\right)^r\left(1 - 2^{1+1/r}c\right) - 1\right], \tag{2.45}$$

$$f(m) - f(m-1) = \frac{(\frac{1+r}{r})^r}{m - 1/2}\left(1 - \frac{c}{(m-1/2)^{1+1/r}}\right) - \left(\frac{m}{\sum_{k=1}^{m}(k-1/2)^{1/r}}\right)^r. \tag{2.46}$$

It follows from Lemma 2.2 and (2.46) together with Remark 2.1 that

$$\begin{aligned}&f(m) - f(m-1)\\&\quad\geq \frac{(\frac{1+r}{r})^r}{m - 1/2}\left(1 - \frac{c}{(m-1/2)^{1+1/r}}\right) - \left(\frac{m}{\frac{r}{r+1}(m^{1+1/r} - 1) + 2^{-1/r}}\right)^r\end{aligned}$$

$$\begin{aligned}&\geq \frac{(\frac{1+r}{r})^r}{m-1/2}\left(1-\frac{c}{(m-1/2)^{1+1/r}}\right)-\left(\frac{m}{\frac{r}{r+1}m^{1+1/r}}\right)^r\\&=\frac{(\frac{1+r}{r})^r[(4(r+1-2^{1/r}r)+1)(m-1/2)^{1+1/r}-m(r+1-2^{1/r}r)]}{m(m-1/2)^{2+1/r}[8(r+1-2^{1/r}r)+2]}.\end{aligned}\tag{2.47}$$

Let

$$g(t)=\left[4\left(r+1-2^{1/r}r\right)+1\right](t-1/2)^{1+1/r}-\left(r+1-2^{1/r}r\right)t.\tag{2.48}$$

Then

$$\begin{aligned}g(1)&=\left[4\left(r+1-2^{1/r}r\right)+1\right]2^{-1-1/r}-\left(r+1-2^{1/r}r\right)\\&\quad>\left(2^{1-1/r}-1\right)\left(r+1-2^{1/r}r\right)\geq 0,\end{aligned}\tag{2.49}$$

$$\begin{aligned}g'(t)&=\left(1+\frac{1}{r}\right)\left[4\left(r+1-2^{1/r}r\right)+1\right](t-1/2)^{1/r}-\left(r+1-2^{1/r}r\right)\\&\quad>\left(2^{2-1/r}-1\right)\left(r+1-2^{1/r}r\right)\geq 0\end{aligned}\tag{2.50}$$

for $t\geq 1$.

From (2.47)-(2.50) we get

$$f(1)<f(2)<\cdots<f(N-1)<f(N).\tag{2.51}$$

Therefore, Lemma 2.8 follows easily from Lemma 2.6, (2.44), (2.45), and (2.51). □

3 Proof of Theorem 1.1

Let $r=-p$, $c=c(r)=d(-r)$ and $b_n=1/a_n$ $(n=1,2,\ldots)$, then $r\geq 1$, $c=(r+1-2^{1/r}r)/[8(r+1-2^{1/r}r)+2]$, $b_n>0$ and $\sum_{n=1}^{\infty}b_n^r<+\infty$.

It follows from Lemmas 2.7 and 2.8 that one has

$$\sum_{n=1}^{N}\left(\frac{n}{\sum_{k=1}^{n}1/b_k}\right)^r\leq\left(\frac{r+1}{r}\right)^r\sum_{n=1}^{N}\left(1-\frac{c}{(n-1/2)^{1+1/r}}\right)b_n^r.\tag{3.1}$$

Letting $n\to+\infty$, (3.1) leads to

$$\sum_{n=1}^{\infty}\left(\frac{n}{\sum_{k=1}^{n}1/b_k}\right)^r\leq\left(\frac{r+1}{r}\right)^r\sum_{n=1}^{\infty}\left(1-\frac{c}{(n-1/2)^{1+1/r}}\right)b_n^r.\tag{3.2}$$

Therefore, Theorem 1.1 follows immediately from (3.2) and $r=-p$ together with $b_n=1/a_n$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Y-MC provided the main idea and carried out the proof of Lemmas 2.1 and 2.2. QX carried out the proof of Lemmas 2.3-2.5 and Theorem 1.1. X-MZ carried out the proof of Lemmas 2.6-2.8. All authors read and approved the final manuscript.

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References

1. Hardy, GH, Littlewood, JE, Pólya, G: Inequalities. Cambridge University Press, Cambridge (1952)
2. Beesack, PR: Hardy's inequality and its extensions. Pac. J. Math. **11**, 39-61 (1961)
3. Mitrinović, DS, Pečarić, JE, Fink, AM: Inequalities Involving Functions and Their Integrals and Derivatives. Kluwer Academic, Dordrecht (1991)
4. Yang, B-C, Zeng, Z-H, Debnath, L: On new generalizations of Hardy's integral inequality. J. Math. Anal. Appl. **217**(1), 321-327 (1998)
5. Chen, C-P, Qi, F: Generalization of Hardy's inequality. Proc. Jangjeon Math. Soc. **7**(1), 57-61 (2004)
6. Pachpatte, BG: Mathematical Inequalities. Elsevier, Amsterdam (2005)
7. Liu, H-P, Zhu, L: New strengthened Carleman's inequality and Hardy's inequality. J. Inequal. Appl. **2007**, Article ID 84104 (2007)
8. Persson, L-E, Oguntuase, JA: Refinement of Hardy's Inequality for "All" p. Banach and Function Spaces II pp. 129-144. Yokohama Publ., Yokohama (2008)
9. Lü, Z-X, Gao, Y-C, Wei, Y-X: Note on the Carleman's inequality and Hardy's inequality. Comput. Math. Appl. **59**(1), 94-97 (2010)
10. Liu, J-Z, Zhang, X-D, Jiang, B: Some generalizations and improvements of discrete Hardy's inequality. Comput. Math. Appl. **63**(3), 601-607 (2012)
11. Yang, B-C, Zhu, Y-H: An improvement on Hardy's inequality. Acta Sci. Natur. Univ. Sunyatseni **37**(1), 41-44 (1998) (in Chinese)
12. Huang, Q-L: An improvement for Hardy's inequality in an interval. Acta Sci. Natur. Univ. Sunyatseni **39**(3), 20-24 (2000) (in Chinese)
13. Wen, J-J, Zhang, R-X: A strengthened improvement of Hardy's inequality. Math. Pract. Theory **32**(3), 476-482 (2002) (in Chinese)
14. Xu, Q, Zhou, M-X, Zhang, X-M: On a strengthened version of Hardy's inequality. J. Inequal. Appl. **2012**, 300 (2012)
15. Deng, Y-P, Wu, S-H, He, D: A sharpened version of Hardy's inequality for parameter $p = 5/4$. J. Inequal. Appl. **2013**, 63 (2013)
16. Long, NT, Linh, NVD: The Carleman's inequality for a negative power number. J. Math. Anal. Appl. **259**(1), 219-225 (2001)
17. Zhang, X-M, Chu, Y-M: A new method to study analytic inequalities. J. Inequal. Appl. **2010**, Article ID 698012 (2010)

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