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Research Article

The Waterfilling Game-Theoretical Framework for Distributed Wireless Network Information Flow

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We present a general game-theoretical framework for power allocation in the downlink of distributed wireless small-cell networks, where multiple access points (APs) or small base stations send independent coded network information to multiple mobile terminals (MTs) through orthogonal channels. In such a game-theoretical study, a central question is whether a Nash equilibrium (NE) exists, and if so, whether the network operates efficiently at the NE. For independent continuous fading channels, we prove that the probability of a unique NE existing in the game is equal to 1. Furthermore, we show that this power allocation problem can be studied as a potential game, and hence efficiently solved. In order to reach the NE, we propose a distributed waterfilling-based algorithm requiring very limited feedback. The convergence behavior of the proposed algorithm is discussed. Finally, numerical results are provided to investigate the price of anarchy or inefficiency of the NE.

1. Introduction

Recently, there has been an increasing interest for small-cell networks. In fact, they have been recognized as an effective and low-cost architecture to provide wireless data rate access to Internet users [1, 2]. These networks consist of numerous and densely deployed APs, known as outdoor femto cells or small-cells, connected to an existing backbone network with heterogeneous links, for example, fibers, ADSLs, and power lines. The general idea is to provide signal coverage and high data rates in dense environments, that is, areas with high user concentrations, by installing low-cost wireless access nodes and exploiting the existing heterogeneous wired infrastructures without a new high-cost cabling. In reality, the femto nodes may belong to different service providers eventually organized in coalitions to maximize their own revenues. In such a context, there is a critical trade-off between cooperation and competition among different providers who may share information and resources to maximize their own revenues. In order to enable both cooperation among providers and network scalability, the femto nodes need self-organizing mechanisms to perform communications and network control functions. Thus, distributed algorithms accounting for the revenues of different providers play a key role in this context.

In contrast to the legacy cell networks, in a small-cell network a user may be served by more than one femto node. This feature is strategic to cope with the heterogeneity of the core network. In fact, if a user were only connected to a single out-door femto-cell, it would suffer from low throughput from time to time due to the limited-backhaul capacity, despite the presence of a high-speed wireless link. As a result, users would access simultaneously to different femto-cells in order to aggregate the sum capacity of the backhaul links.

In this paper, we describe a small-cell network with N MTs served simultaneously by M femto nodes over N orthogonal channels, for example, FDMA, TDMA, and OFDM. For such a system, and we study the power allocation problem under the constraint of maximum transmit power at each femto node (The issue of load balancing [3] in the wired network, and how the different packets are split with respect to the backhaul capacity from a main decentralized scheduler, although important, is not investigated in this paper. We assume that perfect load balancing holds). This system is substantially different from the ones typically analyzed in literature. In fact, it does not reduce to a classical downlink of a cellular network modeled as a broadcast channel since there are several APs transmitting information simultaneously to the same MT. Nor does it reduce to N independent multiple access channels when considering each mobile as a receiver because of the power constraints at the APs. Finally, the considered system does not reduce to a multicellular or an adhoc network modeled as an interference channel since all the signals received at each MT carry useful information to be decoded. In this paper we assume that each signal of interest is decoded considering the remaining signals as interference. This scheme is susceptible to improvement by joint decoding of all the received signals. However, this decoding approach exceeds the scope of this paper.

In traditional wireless cellular networks, the power allocation is often implemented with centralized algorithms aiming at maximizing the sum of the Shannon transmission rate [4]. The maximization problem is solved by waterfilling algorithms [5–8] extended to multiuser contexts. The optimization is in general nonconvex but algorithms that reach local maximum are available [9–11]. Such a centralized power control scheme usually requires a unique shared resource allocation controller and complete channel state information (CSI) with consequent feedback and overhead. It is worth noting that this overload scales exponentially with the number of transmitters and receivers. Thus, such a fully centralized approach is not suitable for small-cell networks without centralized devices and with multiple service providers interested in their own revenues. Additionally, it is not scalable in dense networks.

Game theory [12] provides a possible analytical framework to develop decentralized and/or distributed algorithms for resource allocation in the context of interacting entities having eventually conflicting interests. Recently, noncooperative game theory and its analytical methodologies have been widely applied in wireless systems to solve communication control problems [13]. Distributed power allocation algorithms based on noncooperative games have been proposed for uplink single cell systems, that is, multiple access channels, and downlink multicellular networks or ad hoc networks, that is, interference channels. In [14], general results on potential games are provided and specialized to an uplink single-cell system with multiple access channel based on code division multiple access (CDMA). In [15], a digital subscriber line (DSL) is modeled as a multiple access system based on an OFDM scheme and an iterative waterfilling algorithm is proposed along the lines of the results in [16]. The classical uplink single-cell scenario is relaxed in [17] to include a jammer in the system and an iterative waterfilling algorithm is proposed.

In [16], power allocation on the interference channel is modeled as a noncooperative game, and the conditions for the existence and uniqueness of Nash equilibrium (NE) are established for a two-player version of the game. Similar conditions for the existence and uniqueness have been extended to the multiuser case in [18], where the authors focus on the practical design of distributed algorithms to compute the NE and propose an asynchronous iterative waterfilling algorithm for an interference channel. In [9], the so-called symmetric waterfilling game was studied. The authors assume that for a set of subchannels and receivers the channel gains from all transmitters are the same. The game is shown to have an infinite number of equilibria. The framework of the interference channel has been relaxed in [19] to include cognitive radio systems with transmitters and receivers equipped with multiple antennas, that is, multiple input multiple output (MIMO) systems. A distributive algorithm for the design of the beamformers at each secondary transmitter based on a noncooperative game is developped. Uniqueness and global stability of the Nash equilibrium are studied. Finally, it is worth to note that the DSL power allocation game in [15] is similar to our game from the mathematical point of view. However, it can be shown that with DSL crosstalk link channel coefficients the game in [15] is not a potential game. Therefore, in general, all the nice properties from potential games do not necessarily hold in their case.

In this paper, we adopt game-theoretical methodologies for power allocation problem in the downlink of small-cell networks (Note that a similar power allocation game can be considered for the uplink where MTs are the players taking decisions. However, it is impractical for MTs to have complete uplink CSI. Then, realistic models should take into account the assumption of knowledge reduction at the transmitters. The interested readers are referred to [20] for the framework of Bayesian games). We model femto cells of different operators as players who adaptively and rationally choose their transmission strategies, that is, their transmit power levels, with the aim of maximizing their own transmission sum-rates under maximum power constraints. We first consider the case where each femto cell decides its own power allocation based on the assumption of complete CSI. Later we remove this assumption, and we show that the same equilibrium can still be reached. In such a context it is important to characterize the NE set, for example, the existence and uniqueness of NE. This aspect plays a key role for the application of a distributed game-theoretical-based algorithm. In fact, the existence and uniqueness of an NE guarantees a predictable power allocation and the behavior of a self-organizing network. An answer to this relevant issue depends strongly on the channel fading statistics and the number of players of the investigated channel setting, as is apparent from the comparison of the results in [9–11]. We show that, for a quasi-static fading channel (a fading channel is quasi-static if it is constant during the transmission of a codeword but it may change from a codeword to the following one) with continuous probability density functions of the channel power attenuations, an NE exists and is unique with unit probability. Additionally, we point out that the considered game is a potential game and a simple decentralized algorithm based on the best-response algorithm can be readily proposed. However, a straightforward decentralized algorithm based on complete CSI would not be scalable since the required overhead would scale exponentially with the number of transmitters and receivers. Then, we propose a distributed iterative algorithm

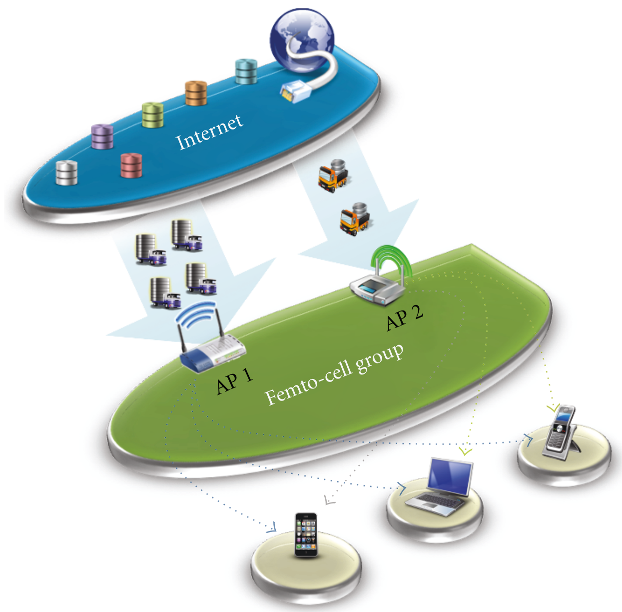


Figure 1: Illustration of femto-cell group with distributed network information flow.

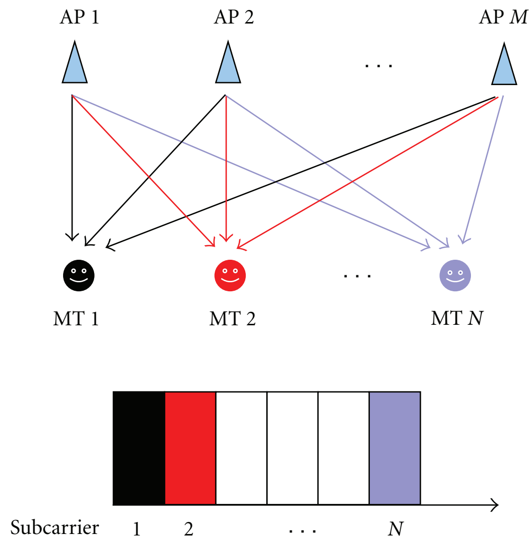


Figure 2: The multiuser OFDM model.

which requires the transmission of the total received power at each MT at each iteration step. With this distributed algorithm, the overhead scales only linearly with the number of receivers. The convergence rate of the proposed algorithm is analyzed. The price of anarchy is also investigated by numerical analysis.

The paper is organized as follows. In Section 2, we introduce the system model and formulate the problem. In Section 3, we study the existence and uniqueness of NE and characterize the NE set. In Section 4, we show that the game at hand is a potential game. Based on the property of potential games and observations on the required information, we propose a distributed algorithm converging to the NE. We investigate the convergence issue. Numerical analysis of the price of anarchy and the convergence rate are provided in Section 5. Section 6 concludes the paper by summarizing the main results and insights on the system behaviour acquired in this work.

2. System Model and Problem Statement

2.1. MultiSource MultiDestination System Model. We consider a wireless system in downlink with M noncooperative APs simultaneously sending information to N MTs over N orthogonal channels, for example, different time slots, frequency bands, or groups of subcarriers in time division multiple access (TDMA), frequency division multiple access (FDMA), or OFDM systems, respectively, as shown in Figure 2. Each channel is preassigned to a different MT by a scheduler and each MT receives signals only on the assigned channel. Without loss of generality, throughout this paper we assign channel n to MT n, for $n = 1, \ldots, N$. This implies that both the MT set and the channel set share the same index in our model. Note that the system model at hand does not reduce to a classical multiple access channel, a broadcast channel, or an interference channel [6].

We assume that the channels are block fading (in different scientific communities these channels are also referred to as quasi-static fading or delay constrained channels), that is, the fading coefficients are constant during the transmission of a codeword or block. Within a given transmission block, let $\mathbf{G} \in \mathbb{R}_{++}^{M\times N}$ be the channel gain matrix whose (m, n) entry is $g_{m,n}$, the channel gain of the link from AP m to MT n on the preassigned channel n. The matrix $\mathbf{G}$ is random with independent entries. We further assume that the distribution function of each positive entry $g_{m,n}$ is a *continuous* function.

By assuming that the MTs use low-complexity single-user decoders [6], the signal-to-interference-plus-noise-ratio (SINR) of the signal from AP m received at MT n is given by

$$\gamma_{m,n} = \frac{g_{m,n}p_{m,n}}{\sigma^2 + \sum_{j=1,j\neq m}^{M} g_{j,n}p_{j,n}}, \tag{1}$$

where $p_{m,n}$ is the power transmitted from AP m on subchannel n, and σ^2 is the variance of the white Gaussian noise. For AP m, write the maximum achievable sum-rate as [6]

$$R_m = \sum_{n=1}^{N} \log(1 + \gamma_{m,n}), \quad \forall m, \tag{2}$$

and the power constraint as

$$\sum_{n=1}^{N} p_{m,n} \leq P_m^{\max}, \quad \forall m, \tag{3}$$

where $P_m^{\max}$ is maximum transmit power of AP m and $P_m^{\max} > 0$, for all m.

2.2. Power Allocation as a NonCooperative Game. Here, we introduce the power allocation problem as a noncooperative strategic game. Because of the competitive nature of the APs, belonging in general to different service providers, AP m aims to maximize its own transmission rate R_m (2) by choosing its transmit power vector $\mathbf{p}_m \triangleq [p_{m,1}, \ldots, p_{m,N}]^{\mathrm{T}}$, subject to its power constraint (3). Denote by vector $\mathbf{p} = [\mathbf{p}_1^{\mathrm{T}}, \ldots, \mathbf{p}_M^{\mathrm{T}}]^{\mathrm{T}}$ the outcome of the game in terms of transmit power levels of all M APs on the N channels. We can completely describe this noncooperative power allocation game as

$$\mathcal{G} \triangleq [\mathcal{M}, \{\mathcal{P}_m\}_{m\in\mathcal{M}}, \{u_m\}_{m\in\mathcal{M}}], \tag{4}$$

where the elements of the game are

(i) *Player set:* $\mathcal{M} = \{1, \ldots, M\}$;

(ii) *Strategy set:* $\{\mathcal{P}_1, \ldots, \mathcal{P}_M\}$, where the strategy set of player m is

$$\mathcal{P}_m = \left\{ \mathbf{p}_m : p_{m,n} \geq 0,\ \forall n,\ \sum_{n=1}^{N} p_{m,n} \leq P_m^{\max} \right\}; \tag{5}$$

(iii) *Utility or payoff function set:* $\{u_1, \ldots, u_M\}$, with

$$u_m(\mathbf{p}_m, \mathbf{p}_{-m}) = \sum_{n=1}^{N} \log\left(1 + \frac{g_{m,n} p_{m,n}}{\sigma^2 + \sum_{j\neq m} g_{j,n} p_{j,n}}\right) = R_m, \tag{6}$$

where $\mathbf{p}_{-m}$ denotes the power vector of length $(M-1)N$ consisting of elements of $\mathbf{p}$ other than the mth element, that is,

$$\mathbf{p}_{-m} = \left[\mathbf{p}_1^{\mathrm{T}}, \ldots, \mathbf{p}_{m-1}^{\mathrm{T}}, \mathbf{p}_{m+1}^{\mathrm{T}}, \ldots, \mathbf{p}_M^{\mathrm{T}}\right]^{\mathrm{T}}. \tag{7}$$

In such a noncooperative game setting, each player m acts selfishly, aiming to maximize its own payoff, given other players' strategies and regardless of the impact of its strategy may have on other players and thus on the overall performance. The process of such selfish behaviors usually results in *Nash equilibrium*, a common solution concept for noncooperative games [21].

Definition 1. A power strategy profile $\mathbf{p}^{\star}$ is a Nash equilibrium If, for every $m \in \mathcal{M}$,

$$u_m(\mathbf{p}_m^{\star}, \mathbf{p}_{-m}^{\star}) \geq u_m(\mathbf{p}_m, \mathbf{p}_{-m}^{\star}), \tag{8}$$

for all $\mathbf{p}_m \in \mathcal{P}_m$.

From the previous definition, it is clear that an NE simply represents a particular "steady" state of a system, in the sense that, once reached, no player has any motivation to unilaterally deviate from it. The powers allocated in our system correspond to an NE.

3. Characterization of Nash Equilibrium Set

In many cases, an NE results from learning and evolution processes of all the game participants. Therefore, it is fundamental to predict and characterize the set of such points from the system design perspective of wireless networks. In the rest of the paper, we focus on characterizing the set of NEs. The following questions are addressed one by one.

(i) Does an NE exist in our game?

(ii) Is the NE unique or there exist multiple NE points?

(iii) How to reach an NE if it exists?

(iv) How does the system perform at NE?

Throughout this section we investigate the existence and uniqueness of a Nash equilibrium.

It is known that in general an NE point does not necessarily exist. In the following theorem we establish the existence of a Nash equilibrium in our game.

Theorem 1. *A Nash equilibrium exists in game $\mathcal{G}$.*

Proof. Since $\mathcal{P}_m$ is convex, closed, and bounded for each m; $u_m(\mathbf{p}_m, \mathbf{p}_{-m})$ is continuous in both $\mathbf{p}_m$ and $\mathbf{p}_{-m}$; and $u_m(\mathbf{p}_m, \mathbf{p}_{-m})$ is concave in $\mathbf{p}_m$ for any set $\mathbf{p}_{-m}$, at least one Nash equilibrium point exists for $\mathcal{G}$ [12, 22]. □

Once existence is established, it is natural to consider the characterization of the equilibrium set. The uniqueness of an equilibrium is a rare but desirable property, if we wish to predict the network behavior. In fact, many game problems have more than one NE [12]. As an example of games with infinite NEs, we could consider a special case of our game $\mathcal{G}$, namely, the *symmetric waterfilling game* [9] where the channel coefficients are assumed to be symmetric. Then, in general, our game $\mathcal{G}$ does not have a unique NE. But with the assumption of independent and identically distributed (*i.i.d.*) continuous entries in $\mathbf{G}$, we will show that the probability of having a unique NE is equal to 1.

For any player m, given all other players' strategy profile $\mathbf{p}_{-m}$, the *best-response* power strategy $\mathbf{p}_m$ can be found by solving the following maximization problem:

$$\begin{aligned} \max_{\mathbf{p}_m} \quad & u_m(\mathbf{p}_m, \mathbf{p}_{-m}) \\ \text{s.t.} \quad & \sum_{n=1}^{N} p_{m,n} \leq P_m^{\max} \\ & p_{m,n} \geq 0, \quad \forall n \end{aligned} \tag{9}$$

which is a convex optimization problem, since the objective function u_m is concave in $\mathbf{p}_m$ and the constraint set is convex. Therefore, the Karush-Kuhn-Tucker (KKT) conditions for optimization are sufficient and necessary for the optimality [5]. The KKT conditions are derived from the Lagrangian for each player m,

$$\begin{aligned} \mathcal{L}_m(\mathbf{p}, \boldsymbol{\lambda}, \boldsymbol{\nu}) = & \sum_{n=1}^{N} \log\left(1 + \frac{g_{m,n} p_{m,n}}{\sigma^2 + \sum_{j\neq m} g_{j,n} p_{j,n}}\right) \\ & - \lambda_m \left(\sum_{n=1}^{N} p_{m,n} - P_m^{\max}\right) + \sum_{n=1}^{N} \nu_{m,n} p_{m,n} \end{aligned} \tag{10}$$

and are given by

$$\frac{g_{m,n}}{\sigma^2 + \sum_{j=1}^{M} g_{j,n} p_{j,n}} - \lambda_m + \nu_{m,n} = 0, \quad \forall n, \tag{11}$$

$$\lambda_m \left(\sum_{n=1}^{N} p_{m,n} - P_m^{\max} \right) = 0, \tag{12}$$

$$\nu_{m,n} p_{m,n} = 0, \quad \forall n, \tag{13}$$

where $\lambda_m \geq 0$, $\nu_{m,n} \geq 0$, for all m and for all n are dual variables associated with the power constraint and transmit power positivity, respectively. The solution to (11)–(13) is known as waterfilling [6]:

$$p_{m,n} = \left(\frac{1}{\lambda_m} - \frac{\sigma^2 + \sum_{j \neq m} g_{j,n} p_{j,n}}{g_{m,n}} \right)^+, \quad \forall n, \tag{14}$$

where $(x)^+ \triangleq \max\{0, x\}$ and λ_m satisfies

$$\sum_{n=1}^{N} \left(\frac{1}{\lambda_m} - \frac{\sigma^2 + \sum_{j \neq m} g_{j,n} p_{j,n}}{g_{m,n}} \right)^+ = P_m^{\max}. \tag{15}$$

In order to analyze the equilibrium set, we establish necessary and sufficient conditions for a point being an NE in the game $\mathcal{G}$.

Theorem 2. *A power strategy profile* $\{\mathbf{p}_1^\star, \ldots, \mathbf{p}_M^\star\}$ *is a Nash equilibrium of the game* $\mathcal{G}$ ***if and only if*** *each player's power* $\mathbf{p}_m^\star$ *is the single-player waterfilling result* (9) *while treating other players' signals as noise. The corresponding necessary and sufficient conditions are:*

$$\frac{g_{m,n}}{\sigma^2 + \sum_{j=1}^{M} g_{j,n} p_{j,n}} - \lambda_m + \nu_{m,n} = 0, \quad \forall m \ \forall n, \tag{16}$$

$$\lambda_m \left(\sum_{n=1}^{N} p_{m,n} - P_m^{\max} \right) = 0, \quad \forall m, \tag{17}$$

$$\nu_{m,n} p_{m,n} = 0, \quad \forall m \quad \forall n. \tag{18}$$

The proof can be found in Appendix A.

From (16), it is easy to verify that necessarily $\lambda_m > 0$, since $\nu_{m,n} \geq 0$ and $g_{m,n} > 0$, for all m and for all n. Also, from (17), we have

$$\sum_{n=1}^{N} p_{m,n} = P_m^{\max}, \quad \forall m. \tag{19}$$

This equation implies that, at the NE, all APs transmit at their maximum power by conveniently distributing the power over all the orthogonal channels.

However, it is still difficult to find an analytical solution from (16)–(18), since the system consisting of (14) and (15) is nonlinear. To simplify this problem, we could consider *linear* equations instead of nonlinear ones. The following lemma provides a key step in this direction.

Lemma 1. *For any realization of channel matrix* $\mathbf{G}$*, there exist unique values of the Lagrange dual variables* $\boldsymbol{\lambda}$ *and* $\boldsymbol{\nu}$ *for any Nash equilibrium of the game* $\mathcal{G}$*. Furthermore, there is a unique vector* $\mathbf{s} = [s_1, \ldots, s_n]^T$ *such that any vector* $\mathbf{p}$ *corresponding to a Nash equilibrium satisfies*

$$\sum_{m=1}^{M} g_{m,n} p_{m,n} \triangleq s_n, \quad \forall n. \tag{20}$$

The proof can be found in Appendix B.

Now, let $\mathbf{Z}$ be the following $(M+N) \times MN$ matrix:

$$\mathbf{Z} = \begin{bmatrix} \mathbf{I}_M & \mathbf{I}_M & \cdots & \mathbf{I}_M \\ \mathbf{g}_1^{\mathrm{T}} & 0_M^{\mathrm{T}} & \cdots & 0_M^{\mathrm{T}} \\ 0_M^{\mathrm{T}} & \mathbf{g}_2^{\mathrm{T}} & \cdots & 0_M^{\mathrm{T}} \\ \vdots & \vdots & \ddots & \vdots \\ 0_M^{\mathrm{T}} & 0_M^{\mathrm{T}} & \cdots & \mathbf{g}_N^{\mathrm{T}} \end{bmatrix}_{(M+N)\times MN}, \tag{21}$$

where $\mathbf{g}_n$ is the nth column of $\mathbf{G}$, $\mathbf{I}_M$ is the $M \times M$ identity matrix, and 0_M is the zero vector of length M. Let $\mathbf{c}$ be the following vector of length $M+N$:

$$\mathbf{c} = \left[P_1^{\max} \ \ P_2^{\max} \cdots P_m^{\max} \ \ s_1 \ \ s_2 \cdots s_N \right]^{\mathrm{T}}. \tag{22}$$

Then, (19) and (20) can be written in the form of *linear* matrix equation

$$\mathbf{Zp} = \mathbf{c}. \tag{23}$$

Define the following sets:

$$\mathcal{X} \triangleq \{(m,n) : \nu_{m,n} = 0\}, \qquad \mathcal{N} \triangleq \{n : \exists m \text{ such that } (m,n) \in \mathcal{X}\}, \tag{24}$$

and denote by $|\mathcal{X}|$ and $|\mathcal{N}|$ their cardinalities. From (18), if an index $(m,n) \notin \mathcal{X}$ we must have $p_{m,n} = 0$. Without loss of generality, we assume that $\mathcal{N} = \{1, \ldots, \widetilde{N}\}$ for $\widetilde{N} \leq N$. Let $\widetilde{\mathbf{Z}}$ be the $(M+\widetilde{N}) \times M\widetilde{N}$ matrix formed from the first $M+\widetilde{N}$ rows and first $M\widetilde{N}$ columns of $\mathbf{Z}$, $\mathbf{p}$ is formed from the first $M\widetilde{N}$ elements of $\mathbf{p}$, and $\widetilde{\mathbf{c}}$ is formed from the first $M+\widetilde{N}$ elements of $\mathbf{c}$. Then, any NE solution must satisfy

$$\widetilde{\mathbf{Z}}\mathbf{p} = \widetilde{\mathbf{c}}. \tag{25}$$

Let $\widehat{\mathbf{Z}}$ be the $(\mathrm{M} + \widetilde{N}) \times |\mathcal{X}|$ matrix formed from the columns of $\widetilde{\mathbf{Z}}$ that correspond to the elements of $\mathcal{X}$. Similarly, let $\mathbf{p}$ be the vector of length $|\mathcal{X}|$ with entries $p_{m,n}$ such that $(m,n) \in \mathcal{X}$ (same order as they were in $\mathbf{p}$). Then, any NE solution satisfies

$$\widehat{\mathbf{Z}}\mathbf{p} = \widetilde{\mathbf{c}}. \tag{26}$$

Lemma 2. *For any realization of a random* $M \times N$ *channel gain matrix* $\mathbf{G}$ *with i.i.d. continuous entries, if* $M\widetilde{N} > M + \widetilde{N}$*, the probability that* $|\mathcal{X}| \leq M + \widetilde{N}$ *is equal to* 1.

Lemma 3. *(1) If* $M\widetilde{N} > M + \widetilde{N}$ *and* $|\mathcal{X}| \leq M + \widetilde{N}$*, the probability that* $\operatorname{rank}(\widehat{\mathbf{Z}}) = |\mathcal{X}|$ *is equal to* 1.

(2) If $M\widetilde{N} \leq M + \widetilde{N}$*, the probability that* $\operatorname{rank}(\widetilde{\mathbf{Z}}) = M\widetilde{N}$ *is equal to* 1.

The proofs of Lemmas 2 and 3 can be found in Appendices C and D, respectively.

Based on Lemmas 1, 2, and 3, we derive the following theorem.

Theorem 3. *For any realization of a random $M \times N$ channel gain matrix* $\mathbf{G}$ *with statistically independent continuous entries, the probability that a unique Nash equilibrium exists in the game $\mathcal{G}$ is equal to* 1.

The proof can be found in Appendix E.

Thus, from Theorems 1 and 3, we have established the existence and uniqueness of NE in our game $\mathcal{G}$.

4. Distributed Power Allocation and Its Convergence to the Nash Equilibrium

An equilibrium has practical interests only if it is reachable from nonequilibria states. In fact, there is no reason to expect a system to operate initially at equilibrium. The convergence of an algorithm to an equilibrium is in general a very hard problem usually related to the specific algorithm and requiring the analysis of synchronous or asynchronous update mechanisms (for power allocation algorithms in interference channels see [18, 23]).

4.1. Potential Game Approach. Fortunately, our game $\mathcal{G}$ can be studied as a potential game (The notation of potential games was firstly used for games in strategic form by Rosenthal (1973) [24], and later generalized and summarized by Monderer (1996) [25]). Potential games are known to have appealing properties for the convergence of the best-response or greedy algorithms to the equilibrium. All the potential games admit a *potential function*. This potential function is a unique global function that all the players optimize when they optimize their own utility functions. Thus, the set of pure Nash equilibria can be found by simply locating the local optima of the potential function. Such games have received increasing attention recently in wireless networks [14, 26, 27], since the existence of potential function enables the design of fully distributed algorithms for resource allocation problems. In fact, there are various notions of potential games such as exact potential, weighted potential, ordinal potential, generalized ordinal potential, pseudo potential, and so forth. These potential games could possess slightly different properties for the existence and convergence of NE. Here, we consider only the exact potential games, since they are closely related to our game. Exact potential games are defined in the following statement.

Definition 2. A strategic game $\mathcal{G}$ is an exact potential game if there exists a function $v : \mathcal{P} \mapsto \mathbb{R}$ satisfying

$$\begin{aligned} &v(\mathbf{p}_m, \mathbf{p}_{-m}) - v(\mathbf{q}_m, \mathbf{p}_{-m}) \\ &\quad = u_m(\mathbf{p}_m, \mathbf{p}_{-m}) - u_m(\mathbf{q}_m, \mathbf{p}_{-m}), \quad \forall m, \end{aligned} \tag{27}$$

for all $(\mathbf{p}_m, \mathbf{p}_{-m})$, $(\mathbf{q}_m, \mathbf{p}_{-m}) \in \mathcal{P}$. The function v is referred to as exact potential of the game.

Equation (27) implies that the NE of the original game $\mathcal{G}$ must coincide with the NE of the potential game, which is defined as a new game with v as an identical utility function for all the players. Therefore, we can transform the noncooperative strategic game $\mathcal{G}$ into a potential game, if we can find a potential function that quantifies the variation in terms of utility due to unilateral perturbation of each player's strategy, as indicated in (27).

Taking inspiration from the result derived in the single channel case [14], we have the following lemma.

Lemma 4. *The game $\mathcal{G}$ is an exact potential game with the following potential function:*

$$\begin{aligned} v^{\star}(\mathbf{p}_m, \mathbf{p}_{-m}) &= \sum_{n=1}^{N} \log\left(\sigma^2 + \sum_{m=1}^{M} g_{m,n} p_{m,n}\right) \\ &= \sum_{n=1}^{N} \log\left[g_{m,n} p_{m,n} + \underbrace{\left(\sigma^2 + \sum_{j \neq m} g_{j,n} p_{j,n}\right)}_{\text{aggregate interference + noise}} \right]. \end{aligned} \tag{28}$$

Proof. From (28) and (6), we observe that the first derivatives of $v^{\star}$ and u_m are equal, that is,

$$\frac{\partial v^{\star}}{\partial \mathbf{p}_m} = \frac{\partial u_m}{\partial \mathbf{p}_m} = \sum_{n=1}^{N} \frac{g_{m,n}}{\sigma^2 + \sum_{j=1}^{N} g_{j,n} p_{j,n}}, \quad \forall m \tag{29}$$

which implies that the property of exact potential (28) is satisfied. This completes the proof. □

We denote by $\zeta_{m,n}$ the term $(\sigma^2 + \sum_{j \neq m} g_{j,n} p_{j,n})$ which stands for the aggregate interference plus noise of user m on subchannel n. In order to find user m's single-user best-response in the potential game, one needs to solve the following maximization problem:

$$\begin{aligned} \max_{\mathbf{p}_m} \; v^{\star}(\mathbf{p}_m, \mathbf{p}_{-m}) \Longleftrightarrow \max_{\mathbf{p}_m} \quad & \sum_{n=1}^{N} \log(\zeta_{m,n} + g_{m,n} p_{m,n}) \\ \text{s.t.} \quad & \sum_{n=1}^{N} p_{m,n} \leq P_m^{\max} \\ & p_{m,n} \geq 0, \quad \forall n. \end{aligned} \tag{30}$$

Note that the problem (30) can be solved as a convex optimization, when the private channel gain $g_m = \{g_{m,1}, \ldots, g_{m,N}\}$ and the aggregate interference plus noise $\zeta_m = \{\zeta_{m,1}, \ldots, \zeta_{m,N}\}$ are both known to player m. It is easy to verify that this single-user best-response is the same waterfilling solution expressed in (14), due to the property of potential function.

4.2. Distributed Algorithm and Convergence Property. Note that if each AP has complete CSI, that is, knowledge of the channel gain matrix $\mathbf{G}$, defined as in Section 2, the uniqueness of the NE guaranties that each AP can determine independently the power allocation at the NE in a decentralized manner. In order to acquire information about the whole matrix $\mathbf{G}$ at each AP, a feedback channel is usually needed to transmit the channel estimations from MTs to APs. With this information, each AP can solve locally the system of equations (16)–(18) or perform locally a best-response algorithm based on the repeated maximization of problem (30) by starting from a random point $\mathbf{p}_{-m} \in \prod_{j\neq m}\mathcal{P}_j$. However, the structure of problem (30) suggests an alternative distributed approach to reduce eventually the signalling on the feedback channel. In fact, the repeated optimization of problem (30) can be performed in a distributed way by feeding back at each AP m only the private channel gain g_m and the aggregate interference plus noise ζ_m. Nevertheless, note that such a distributed implementation of the algorithm would lead to a transition phase where the APs are not transmitting at an equilibrium point. In our numerical results, we ignore the cost of feedback, and we focus on analyzing the theoretical upper-bound.

The above discussion yields a simple algorithm based on the iterative waterfilling [28] detailed in the following.

In this algorithm, we assume that the same game could be myopically played repeatedly: in each round, every myopic player (a myopic player has no memory of past game-rounds) chooses its best-response according to the single-player waterfilling that depends on the current state of the game. The following theorem shows the convergence and optimality of the algorithm.

Theorem 4. *The DPIWF algorithm converges to a unique Nash equilibrium of the noncooperative game $\mathcal{G}$.*

The proof can be found in Appendix F.

A more general discussion about the convergence and stability properties of potential games can be found in [25, 29]. In [25], it shows that every bounded potential game (a game is called a bounded game if the payoff functions are bounded) has the *approximate finite improvement property* (AFIP), that is, for every $\epsilon > 0$, every ϵ-improvement path is finite. Then, it is obvious that every such finite improvement path of the exact potential games terminates in an ϵ-equilibrium point (an ϵ-equilibrium is a strategy profile that approximately satisfies the condition of Nash equilibrium). In other words, the *sequential best-response* (players move in turn and always choose a best-response) converges to the ϵ-equilibrium independent of the initial point. Note that this is a very flexible condition for the convergence, since *order of playing can be deterministic or random and need not to be synchronized.* It is one of the most interesting properties of the potential games, especially in order to distributively find the equilibrium in self-organizing systems. In [29], it shows that potential games are characterized by strong stability properties (Lyapunov stable, see its definition in Theorem 5.34 of [29]). Also note that if the game has a unique NE, then it is globally stable.

In the *simultaneous best-response* algorithm all the players choose their best-responses simultaneously at each iteration. It is not difficult to verify that, in the general case, it does not necessarily converge, due to the "ping-pong" effect generated by myopic players. However, [30] has shown that for infinite pseudopotential games, a general class of games including also exact potential games, with convex strategy space and single-valued best-response (games with strictly multiconcave potential, concave in each players' unilateral deviation, have single-valued best-response), the sequence of simultaneous best-responses, reminiscent of fictitious play, also converges to the equilibrium.

It is interesting to note that for many practical systems with finite transmit power states, similar results still hold for the convergence of the sequential best-response. The only difference is that, in the finite case, the existence of exact potential function implies the *finite improvement property* (FIP), and therefore, the sequential best-response converges to the exact NE instead of an ϵ-equilibrium.

Although the final convergence of the DPIWF algorithm is proved, one may wonder whether the optimum of the potential function (28) coincides with the optimum social welfare, that is, the optimal total information rate transmitted in the network. We discuss the price of anarchy in the following section.

5. Numerical Evaluation

In this part, numerical results are provided to validate our theoretical claims and assess the price of anarchy, that is, the performance loss in terms of the transmit sum-rate of all APs in the network due to a noncooperative game compared to the maximum social welfare. We denote this transmit sum-rate in the network as the actual total network rate, and defined it as

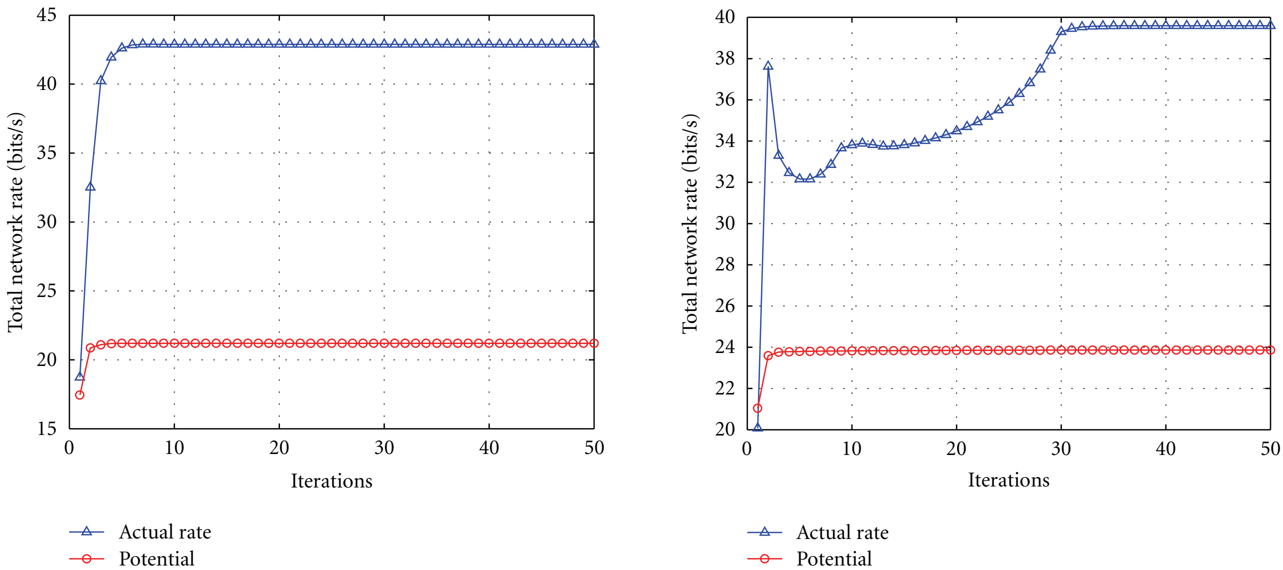
$$u(\mathbf{p}) = \sum_{m=1}^{M} \mathrm{u}_m(\mathbf{p}). \tag{31}$$

We consider frequency-selective fading channels with channel matrix $\mathbf{G}$ of size $M \times N$, where M is the total number of transmitters (players) and N is the total number of receivers. We assume that the Rayleigh fading channel gain $g_{m,n}$ are *i.i.d.* among players and channels. The maximum power constraint for each player m is assumed to be identical and normalized to $\overline{P}_m = 1$.

In Figure 3, we show the convergence behaviors of potential function and the actual total network rate, shortly referred to as "actual rate", by using the proposed DPIWF algorithm for a random channel realization. We set the number of transmitters to $M = 10$ and the number of receivers to $N = 10$. As expected, in both Figures 3(a) and 3(b) the potential function converges rapidly (at the 4th iteration). In Figure 3(a), the actual rate converges slightly slower (at the 6th iteration) and maintains a monotonically increasing slope. However, in Figure 3(b), the actual rate finally converges, but unfortunately it does not increase monotonically and it converges only at the 34th iteration with a convergence rate much slower than the potential function. Note that we use this example to show that a "defective" convergence may happen during the iteration steps.

initialize $t = 0,\ p_{m,n}^{(0)} = 0, \forall m \ \ \forall n$
repeat
  $t = t + 1$
  for $m = 1$ to M **do**
    for $n = 1$ to N **do**
      $\zeta_{m,n}^{(t)} = \sigma^2 + \sum_{j \neq m} g_{j,n} p_{j,n}^{(t)}$
    end for
    $[p_{m,1}^{(t+1)}, \ldots, p_{m,N}^{(t+1)}] = \arg\max_{\mathbf{p}_m \geq 0 \sum_n p_{m,n} \leq \overline{P}_m} \sum_n \log(\zeta_{m,n}^{(t)} + g_{m,n} p_{m,n})$
  end for
until convergence

Algorithm 1: DPIWF algorithm.



(a) An example of "ideal" convergence

(b) An example of "defective" convergence

Figure 3: Convergence and performance of potential function and actual total network rat.

In order to measure the performance efficiency of distributed networks operating at the unique NE, we provide here the optimal centralized approach as a target upper-bound for the total network rate. We ignore the performance loss caused by the necessary uplink and downlink signalling transmission. The total network rate maximization problem can be formulated as

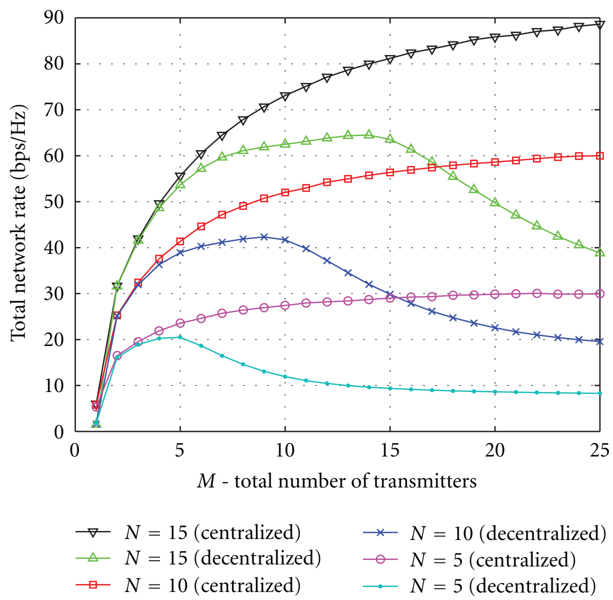
$$\begin{aligned} \max_{\mathbf{p}} \quad & u(p) \\ \text{s.t.} \quad & \sum_n p_{m,n} \leq \overline{P}_m, \quad \forall m \\ & p_{m,n} \geq 0, \quad \forall m \ \forall n. \end{aligned} \tag{32}$$

The optimization problem (32) is difficult to solve since the objective function is *nonconvex* in $\mathbf{p}$. However, a relaxation of this optimization problem [11] can be considered as a geometric programming problem [31]. As well known, a geometric programming can be transformed into a convex optimization problem and then solved in an efficient way. A low-complexity algorithm was proposed in [11] to solve the dual problem by updating dual variables through a gradient descent. Note that the algorithm always converges, but may converges to a local maximum point in a few cases. We use this algorithm in our simulations.

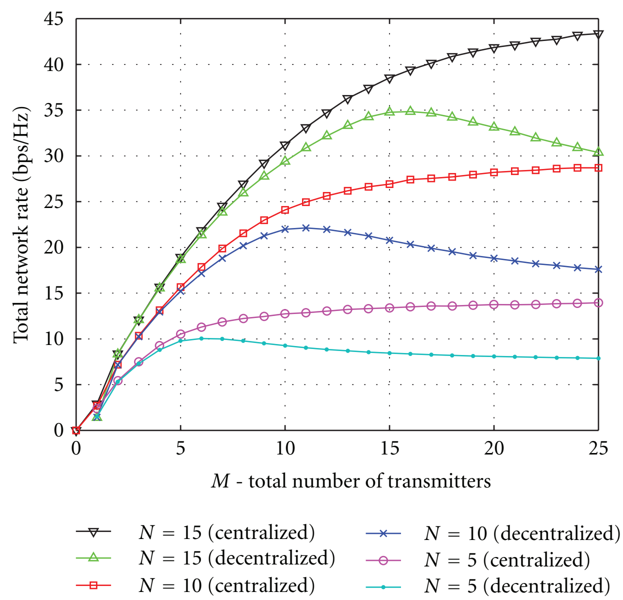
In the following part, we address two main practical questions through numerical results.

(1) How does the network performance behave in average at the unique NE in comparison to the global optimal solution or global welfare? More precisely, we are interested in comparing the *average* total network rate instead of the *instantaneous* total network rate. We denote by $\overline{u}(M,N)$ the average total network rate for a M transmitters and N receivers system, that is,

$$\overline{u}(M,N) = \mathbb{E}_{\mathbf{G}}\left[\sum_{m=1}^{M}\sum_{n=1}^{N}\log\left(1 + \frac{p_{m,n}g_{m,n}}{\sigma^2 + \sum_{j \neq m} p_{j,n}g_{j,n}}\right)\right], \tag{33}$$

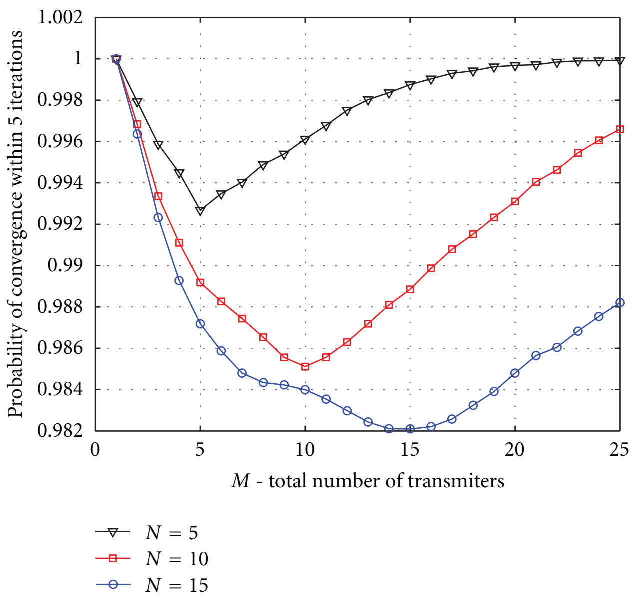


(a) $\sigma^2 = 0.1$

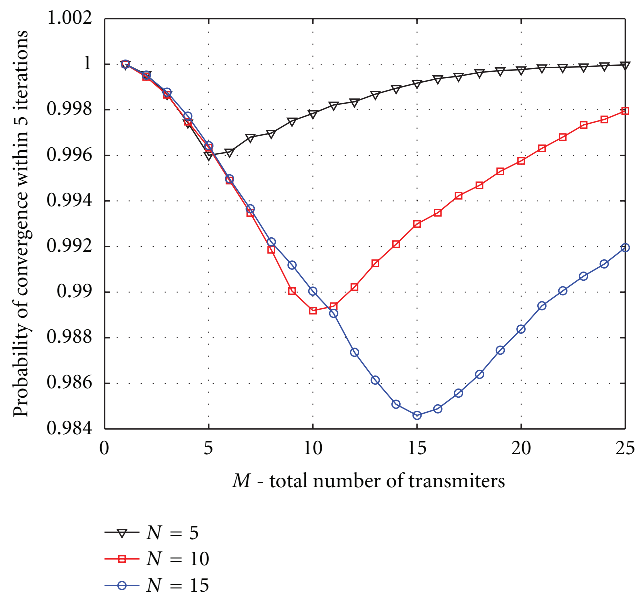


(b) $\sigma^2 = 1$

FIGURE 4: Average total network rate, decentralized versus centralized optimality.



(a) $\sigma^2 = 0.1$



(b) $\sigma^2 = 1$

FIGURE 5: Probability of convergence within 5 iterations.

(2) What about the convergence behavior for the actual total network rate when using DPIWF algorithm? Does it converge as rapidly as in Figure 3(a) for the most of the cases?

Let us consider the first question. In Figure 4, we compare the average total network rate of both *decentralized* and *centralized* networks for two different channel noise levels $\sigma^2 = 0.1$ and 1, respectively. The plots are obtained through Monte-Carlo simulations over 10^4 realizations for the channel gain matrix $\mathbf{G}$. Figures 4(a) and 4(b) show the total network rate as a function of the number of transmitters M for different number of receivers N. More specifically, $N = 5, 10, 15$. We note that in both Figures 4(a) and 4(b), the centralized optimal approach always outperforms the decentralized noncooperative algorithm. Additionally, for a fixed number of transmitters N, when we increase the number of receivers M, the performance loss of decentralized systems compared to the centralized social welfare becomes greater and greater. This phenomenon can be intuitively understood as follows: *when there is a great number of* **selfish** *players, the hostile competition turns the multiuser communication system into an interference-limited environment, where interference significantly degrade the performance efficiency.*

In Figure 4, we also note that for a fixed N the average performance of centralized systems is an increasing function of M, and the average performance of decentralized systems corresponding to NE reaches a maximum and then decreases flatting out. For the typical values of N, that is, $N = 5, 10, 15$, in Figure 4(a), when $\sigma^2 = 0.1$ the average performance of decentralized systems are maximized at $M = 4, 9, 14$, respectively; in Figure 4(b), when $\sigma^2 = 1$ the average performance of decentralized systems are maximized at $M = 6, 11, 16$, respectively. This comparison simply shows that different noise variance (in general channel condition) have a different impact on the decentralized system performance. This observation is fundamental for improving the spectral efficiency of a distributed multiuser small cell networks: *For a given area, that is, a given number of receivers N and given channel conditions, there exists an optimal number of access points, denoted as $M^\star$, to be installed in the network.* Roughly speaking, when $M > M^\star$, the system is *saturated* due to the increasing competition for the shared limited resources; when $M < M^\star$, the system operates in a *unsaturated* state, since system resources are not fully exploited.

Let us now consider the second question. In Figure 5, we show the probability of convergence to the NE within 5 iterations for $\sigma^2 = 0.1$ and 1, respectively. To be more precise, we say that the algorithm converges at the fifth iteration if the total network rate exceeds 99% of the rate at the NE. We find that the probability of convergence is satisfactory. It is greater than 0.982 in all cases and tends to 1 when $M \gg N$ and $M \ll N$. Another interesting observation is that the minimal convergence probability always occurs when $M = N$, regardless of the noise value σ^2.

6. Conclusions and Future Works

In this paper, we study the power allocation problem in the wireless small-cell networks as a strategic noncooperative game. Each transmitter (AP) is modeled as a player in the game who decides, in a distributed way, how to allocate its total power through several independent fading channels. We studied the existence and uniqueness of NE. Under the condition of independent continuous fading channels, we showed that the probability of having a unique equilibrium is equal to 1. The game at hand is shown to be a potential game. A distributed algorithm requiring very limited feedback has been proposed based on the potential game analysis. The convergence and stability issues have been addressed. Numerical studies have shown that the DPIWF algorithm can converge rapidly within 5 iterations with very high probability.

Appendices

A. Proof of Theorem 2

Proof. We prove the necessary and sufficient parts separately.

(1) *Proof of necessary condition (the only if part)*. From the definition of NE (Definition 1), if a power set $\{\mathbf{p}_m\}$ is an NE, it must satisfy all the best-response conditions in (8) simultaneously. Suppose a situation that all the players' power except player m's power reaches the NE point: $\{p_1^\star, \dots, p_{m-1}^\star, p_m, p_{m+1}^\star, \dots, p_M^\star\}$. In this case when all other players' powers are fixed, as shown in (9), the best-response of player m is to set its power according to (14). This is exactly given by the single-player waterfilling treating all other players' signals as noise.

(2) *Proof of sufficient condition (the if part)*. From convex optimization theory [5], we know that the KKT conditions of the convex optimization problem are necessary and sufficient conditions for optimality. Therefore, we can say that a power strategy $\mathbf{p}_m$ satisfies the best-response condition if and only if it satisfies the single-player KKT conditions (11)–(13). Then collectively, we say a set $\{\mathbf{p}_m\}$ satisfies all the best-response conditions simultaneously if and only if it satisfies (16)–(18). From Definition 1, if a set $\{\mathbf{p}_m\}$ satisfies all the best-response conditions, it must be an NE.

This completes the proof. □

B. Proof of Lemma 1

Proof. Consider an NE $\mathbf{p} \in \mathbb{R}^{KN\times 1}$. Theorem 2 yields the following equation:

$$\phi(\mathbf{p}) + \boldsymbol{\nu} - \boldsymbol{\lambda} = \mathbf{0}, \tag{B.1}$$

where

$$\phi(\mathbf{p}) = \begin{bmatrix} \dfrac{g_{1,1}}{\sigma^2 + \sum_j p_{j,1} g_{j,1}} \\ \dfrac{g_{1,2}}{\sigma^2 + \sum_j p_{j,1} g_{j,1}} \\ \vdots \\ \dfrac{g_{K,N}}{\sigma^2 + \sum_j p_{j,N} g_{j,N}} \end{bmatrix}_{KN\times 1},$$

$$\boldsymbol{\nu} = \begin{bmatrix} \nu_{1,1} \\ \nu_{1,2} \\ \vdots \\ \nu_{K,N} \end{bmatrix}_{KN\times 1},$$

$$\boldsymbol{\lambda} = \begin{bmatrix} (\lambda_1)_{N\times 1} \\ (\lambda_2)_{N\times 1} \\ \vdots \\ (\lambda_K)_{N\times 1} \end{bmatrix}_{KN\times 1}. \tag{B.2}$$

Now, let us assume that there exist two different Nash equilibria, for example, $\mathbf{p}^0, \mathbf{p}^1$ $(\mathbf{p}^0 \neq \mathbf{p}^1)$. Then, the following equation must also hold:

$$\underbrace{(\mathbf{p}^1 - \mathbf{p}^0)^{\mathrm{T}} (\mathbf{p}^0 - \mathbf{p}^1)^{\mathrm{T}}}_{\alpha^{\mathrm{T}}} \left(\underbrace{\begin{matrix} \phi(\mathbf{p}^0) \\ \phi(\mathbf{p}^1) \end{matrix}}_{\beta} + \underbrace{\begin{matrix} \nu^0 - \lambda^0 \\ \nu^1 - \lambda^1 \end{matrix}}_{\gamma} \right) = 0. \tag{B.3}$$

Equation (B.3) implies that

$$\begin{aligned}
\alpha^{\mathrm{T}}\beta &= (\mathbf{p}^1 - \mathbf{p}^0)^{\mathrm{T}}\phi(\mathbf{p}^0) + (\mathbf{p}^0 - \mathbf{p}^1)^{\mathrm{T}}\phi(\mathbf{p}^1) \\
&= \sum_{n=1}^{N}\sum_{k=1}^{K}\Bigg[\left(p_{k,n}^1 - p_{k,n}^0\right)\frac{g_{k,n}}{\sigma^2 + \sum_{j=1}^{K} p_{j,n}^0 g_{j,n}} \\
&\qquad + \left(p_{k,n}^0 - p_{k,n}^1\right)\frac{g_{k,n}}{\sigma^2 + \sum_{j=1}^{K} p_{j,n}^1 g_{j,n}}\Bigg] \\
&= \sum_{n=1}^{N}\sum_{k=1}^{K}\frac{g_{k,n}\left(p_{k,n}^0 - p_{k,n}^1\right)\sum_{j=1}^{K}\left[g_{j,n}\left(p_{j,n}^0 - p_{j,n}^1\right)\right]}{\left(\sigma^2 + \sum_{j=1}^{K} p_{j,n}^0 g_{j,n}\right)\left(\sigma^2 + \sum_{j=1}^{K} p_{j,n}^1 g_{j,n}\right)} \\
&= \sum_{n=1}^{N}\frac{\left[\sum_{j=1}^{K} g_{j,n}\left(p_{j,n}^0 - p_{j,n}^1\right)\right]^2}{\left(\sigma^2 + \sum_{j=1}^{K} p_{j,n}^0 g_{j,n}\right)\left(\sigma^2 + \sum_{j=1}^{K} p_{j,n}^1 g_{j,n}\right)} \geq 0, \\
\alpha^{\mathrm{T}}\gamma &= (\mathbf{p}^1 - \mathbf{p}^0)^{\mathrm{T}}(\nu^0 - \lambda^0) + (\mathbf{p}^0 - \mathbf{p}^1)^{\mathrm{T}}(\nu^1 - \lambda^1) \\
&= \sum_{n=1}^{N}\sum_{k=1}^{K}\Big[\left(p_{k,n}^1 - p_{k,n}^0\right)\left(\nu_{k,n}^0 - \lambda_k^0\right) \\
&\qquad + \left(p_{k,n}^0 - p_{k,n}^1\right)\left(\nu_{k,n}^1 - \lambda_k^1\right)\Big] \\
&= \sum_{k=1}^{K}\left[\underbrace{\left(\sum_{n=1}^{N} p_{k,n}^1 - \sum_{n=1}^{N} p_{k,n}^0\right)}_{\overline{P}_k - \overline{P}_k = 0}\left(\lambda_k^1 - \lambda_k^0\right)\right] \\
&\quad + \sum_{n=1}^{N}\sum_{k=1}^{K}\left(p_{k,n}^0\nu_{k,n}^1 + p_{k,n}^1\nu_{k,n}^0\right) \\
&= \sum_{n=1}^{N}\sum_{k=1}^{K}\left(p_{k,n}^0\nu_{k,n}^1 + p_{k,n}^1\nu_{k,n}^0\right) \geq 0.
\end{aligned} \tag{B.4}$$

From the previous expressions, it is easy to see that (B.3) holds if and only if we have $\alpha^{\mathrm{T}}\beta = 0$ and $\alpha^{\mathrm{T}}\gamma = 0$. These conditions are equivalent to the following:

$$\begin{aligned}
&\sum_{k=1}^{K} g_{k,n}p_{k,n}^0 - \sum_{k=1}^{K} g_{k,n}p_{k,n}^1 = 0, \quad \forall n, \\
&p_{k,n}^0\nu_{k,n}^1 = p_{k,n}^1\nu_{k,n}^0 = 0, \quad \forall n\ \forall k.
\end{aligned} \tag{B.5}$$

First, from (B.5), we observe that the value of s_n, with $s_n = \sum_k g_{k,n}p_{k,n}$, is fixed for any NE point. Second, for a specific positive power coefficient, for example, $p_{k^*,n^*}^0 > 0$, we must have $\nu_{k^*,n^*}^0 = 0$ due to (13). Therefore, from (25) we must also have $\nu_{k^*,n^*}^1 = 0$. This implies $\lambda_{k^*}^1 = \lambda_{k^*}^0$ because of (11). Finally, since s_n is fixed for any NE point, we obtain $\nu_{k^*,n}^0 = \nu_{k^*,n}^1$, for all n. The same proof holds for any other index k^*. □

C. Proof of Lemma 2

Proof. When $\nu_{m,n} = 0$, from (11) we have

$$\lambda_m - g_{m,n}d_n = 0, \quad \forall (m,n) \in \mathcal{X}, \tag{C.1}$$

where $d_n \triangleq 1/(\sigma^2 + s_n)$. From Lemma 1, we know that all the Nash equilibria must satisfy (C.1), with the same λ_m and d_n. In (C.1), the number of independent linear equations is $|\mathcal{X}|$, while the number of unknown parameters is $M + \widetilde{N}$, since the remaining d_n, $n \notin \mathcal{N}$ are known to be $d_n = 1/\sigma^2$. It is well known that the set of solutions to a system of linear equations is empty, if the number of independent equations is larger than the number of variables [32]. Since each random entry $g_{m,n}$ is independently distributed with continuous distribution function, it is obvious that, with probability 1, the equations of the system (C.1) are independent from each other. Therefore, we must have $|\mathcal{X}| \leq M + \widetilde{N}$. □

D. Proof of Lemma 3

Proof. We only give the proof for the case as $M\widetilde{N} > M + \widetilde{N}$. The case as $M\widetilde{N} \leq M + \widetilde{N}$ can be proved in a similar way. Matrix $\widehat{\mathbf{Z}}$ can be transformed into a 2×2 block matrices, by applying some elementary column and row operations, as follows:

$$\begin{aligned}
\widehat{\mathbf{Z}} &\xrightarrow{\text{column}} \begin{bmatrix} \mathbf{I}_\tau & \mathbf{A}_{\tau\times\xi_2} \\ \mathbf{B}_{\xi_1\times\tau} & \mathrm{C}_{\xi_1\times\xi_2} \end{bmatrix} \xrightarrow{\text{column}} \begin{bmatrix} \mathbf{I}_\tau & \mathbf{0}_{\tau\times\xi_2} \\ \mathbf{B}_{\xi_1\times\tau} & \widehat{\mathbf{C}}_{\xi_1\times\xi_2} \end{bmatrix} \\
&\xrightarrow{\text{row}} \begin{bmatrix} \mathbf{I}_\tau & \mathbf{0}_{\tau\times\xi_2} \\ \mathbf{0}_{\xi_1\times\tau} & \widehat{\mathbf{C}}_{\xi_1\times\xi_2} \end{bmatrix},
\end{aligned} \tag{D.1}$$

where $\tau = \min\{M, \widetilde{N}\}$, $\xi_1 = M + \widetilde{N} - \tau \geq \tau$, and $\xi_2 = |\mathcal{X}| - \tau$. $\widehat{\mathbf{C}}$ is a $\xi_1 \times \xi_2$ matrix, where each column contains one or two random variables, and each row contains at least one random variable. Again we can transform $\widehat{\mathbf{C}}$ in row echelon form, denoted as $\widehat{\mathbf{C}}_r$. Note that the rank of $\widehat{\mathbf{C}}_r$ is ξ_2 with probability 1, since each leading coefficient of a row is a random variable or the linear combination of two *i.i.d.* random variables. So, with probability 0, any leading coefficient takes the value of 0. Therefore, we have rank$(\widehat{\mathbf{Z}}) = \tau + \xi_2 = |\mathcal{X}|$ with probability 1. □

E. Proof of Theorem 3

Proof. If $M\widetilde{N} > M + \widetilde{N}$, we have from Lemma 2 that, with probability 1, rank$(\widehat{\mathbf{Z}}) = |\mathcal{X}|$. Any NE must satisfy (26); assume that two different power strategies $\mathbf{p}$ and $\mathbf{p}'$ are both solutions to (26). Then $\widehat{\mathbf{Z}}(\mathbf{p} - \mathbf{p}') = 0$. Since the rank of $\widehat{\mathbf{Z}}$ is equal to the number of its columns, the rank-nullity theorem [32] implies $\mathbf{p} - \mathbf{p}' = 0$. Then, if the NE exists it is unique.

If $M\widetilde{N} \leq M + \widetilde{N}$, we have from Lemma 3 that, with probability 1, there is at most one solution to (25). Since any NE must satisfy (25) and we know that there is at least one NE solution, we conclude that NE is unique. □

F. Proof of Theorem 4

Proof. We prove this theorem in two steps.

(1) *Algorithm convergence*. It is easy to see that the potential function $v^\star(\mathbf{P})$ is nondecreasing within each round of the single-player waterfilling. Moreover, since each player's transmit power is limited by a maximum but finite power constraint, there must exist an upper-bound for the potential function $v^\star(\mathbf{P})$. This confirms the convergence.

(2) *Converge to NE*. From the discussions above, we directly have that the KKT condition of the potential game coincide with the KKT condition of the original OFDM game $\mathcal{G}$, due to the property of potential function (27). By using the sufficient part of Theorem 2, we know that if each player's power allocation $\mathbf{p}_m$ is given by the single-player waterfilling while treating other players's signals as noise, the set $\{\mathbf{p}_m\}$ must be an NE of the original game $\mathcal{G}$. Therefore, we can conclude that if the algorithm DPIWF converges (through the process of iterating single-player waterfilling), it converges to an NE point.

This completes the proof. □

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References

[1] M. Debbah, "Mobile flexible networks: the challenges ahead," in *Proceedings of International Conference on Advanced Technologies for Communications (ATC '08)*, pp. 3–7, Hanoi, Vietnam, October 2008, invited paper.

[2] http://iosc-workshop.homeip.net/.

[3] S. Kandula, D. Katabi, S. Sinha, and A. Berger, "Dynamic load balancing without packet reordering," *ACM Journal of Computer and Communication Review*, vol. 37, no. 2, pp. 51–62, 2007.

[4] C. E. Shannon, "A mathematical theory of communication," *Bell System Technical Journal*, vol. 27, pp. 379–423, 623–656, 1948.

[5] S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, Cambridge, UK, 2004.

[6] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, John Wiley & Sons, New York, NY, USA, 1991.

[7] A. J. Goldsmith and P. P. Varaiya, "Capacity of fading channels with channel side information," *IEEE Transactions on Information Theory*, vol. 43, no. 6, pp. 1986–1992, 1997.

[8] D. Tse and P. Viswanath, *Fundamentals of Wireless Communication*, Cambridge University Press, Cambridge, UK, 2005.

[9] E. Altman, K. Avrachenkov, and A. Garnaev, "Closed form solutions for symmetric water filling games," in *Proceedings of the 27th IEEE Communications Society Conference on Computer Communications (INFOCOM '08)*, pp. 673–681, Phoenix, Ariz, USA, April 2008.

[10] K. B. Song, S. T. Chung, G. Ginis, and J. M. Cioffi, "Dynamic spectrum management for next-generation DSL systems," *IEEE Communications Magazine*, vol. 40, no. 10, pp. 101–109, 2002.

[11] J. Papandriopoulos and J. S. Evans, "Low-complexity distributed algorithms for spectrum balancing in multi-user DSL networks," in *Proceedings of the IEEE International Conference on Communications (ICC '06)*, pp. 3270–3275, July 2006.

[12] D. Fudenberg and J. Tirole, *Game Theory*, MIT Press, Cambridge, Mass, USA, 1991.

[13] S. Lasaulce, M. Debbah, and E. Altman, "Methodologies for analyzing equilibria in wireless games: a look at pure, mixed, and correlated equilibria," *IEEE Signal Processing Magazine*, vol. 26, no. 5, pp. 41–52, 2009.

[14] G. Scutari, S. Barbarossa, and D. P. Palomar, "Potential games: a framework for vector power control problems with coupled constraints," in *Proceedings of IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP '06)*, pp. 241–244, Toulouse, France, May 2006.

[15] Z.-Q. Luo and J.-S. Pang, "Analysis of iterative waterfilling algorithm for multiuser power control in digital subscriber lines," *EURASIP Journal on Applied Signal Processing*, vol. 2006, Article ID 24012, 10 pages, 2006.

[16] W. Yu, G. Ginis, and J. M. Cioffi, "Distributed multiuser power control for digital subscriber lines," *IEEE Journal on Selected Areas in Communications*, vol. 20, no. 5, pp. 1105–1115, 2002.

[17] R. H. Gohary, Y. Huang, Z.-Q. Luo, and J.-S. Pang, "A generalized iterative water-filling algorithm for distributed power control in the presence of a Jammer," *IEEE Transactions on Signal Processing*, vol. 57, no. 7, pp. 2660–2674, 2009.

[18] G. Scutari, D. P. Palomar, and S. Barbarossa, "Asynchronous iIterative water-filling for Gaussian frequency-selective interference channels," *IEEE Transactions on Information Theory*, vol. 54, no. 7, pp. 2868–2878, 2008.

[19] G. Scutari and D. P. Palomar, "MIMO cognitive radio: a game theoretical approach," *IEEE Transactions on Signal Processing*, vol. 58, no. 2, pp. 761–780, 2010.

[20] G. He, M. Debbah, and E. Altman, "A Bayesian game-theoretic approach for distributed resource allocation in fading multiple access channels," *EURASIP Journal on Wireless Communications and Networking*, vol. 2010, Article ID 391684, 10 pages, 2010.

[21] N. John, "Equilibrium points in N-person games," *Proceedings of the National Academy of Sciences*, vol. 36, no. 1, pp. 48–49, 1950.

[22] J. B. Rosen, "Existence and uniqueness of equilibrium points for concave N-person games," *Econometrica*, vol. 33, pp. 520–534, 1965.

[23] G. Scutari, D. P. Palomar, and S. Barbarossa, "Simultaneous iterative water-filling for Gaussian frequency-selective interference channels," in *Proceedings of IEEE International Symposium on Information Theory (ISIT '06)*, pp. 600–604, July 2006.

[24] R. W. Rosenthal, "A class of games possessing pure-strategy Nash equilibria," *International Journal of Game Theory*, vol. 2, no. 1, pp. 65–67, 1973.

[25] D. Monderer and L. S. Shapley, "Potential games," *Games and Economic Behavior*, vol. 14, no. 1, pp. 124–143, 1996.

[26] J. O. Neel, J. H. Reed, and R. P. Gilles, "Convergence of cognitive radio networks," in *Proceedings of IEEE Wireless Communications and Networking Conference (WCNC '04)*, vol. 4, pp. 2250–2255, Atlanta, Ga, USA, March 2004.

[27] T. Heikkinen, "A potential game approach to distributed power control and scheduling," *Computer Networks*, vol. 50, no. 13, pp. 2295–2311, 2006.

[28] W. Yu, *Competition and cooperation in multi-user communication environements*, Ph.D. thesis, Stanford University, June 2002, http://www.comm.utoronto.ca/~weiyu/thesis_2.pdf.

[29] J. Neel, *Analysis and design of cognitive radio networks and distributed radio resource management algorithms*, PhD dissertation, September 2006, http://scholar.lib.vt.edu/theses/available/etd-12082006-141855/.

[30] P. Dubey, O. Haimanko, and A. Zapechelnyuk, "Strategic complements and substitutes, and potential games," *Games and Economic Behavior*, vol. 54, no. 1, pp. 77–94, 2006.

[31] M. Chiang, "Geometric programming for communication systems," *Foundations and Trends in Communications and Information Theory*, vol. 2, no. 1-2, pp. 1–153, 2005.

[32] C. D. Meyer, *Matrix Analysis and Applied Linear Algebra*, SIAM, Philadelphia, Pa, USA, 2000.