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Conformality of a differential with respect to Cheeger-Gromoll type metrics

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Abstract We investigate conformality of the differential of a mapping between Riemannian manifolds if the tangent bundles are equipped with a generalized metric of Cheeger-Gromoll type.

Keywords Conformal mappings · Cheeger-Gromoll type metrics · Second standard immersion

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1 Introduction and preliminaries

Generalized metrics of Cheeger-Gromoll type or (p,q)-metrics $h_{p,q}$, being a generalization of Sasaki metric h_S [5] and Cheeger-Gromoll metric h_{CG} [4], have been recently introduced by Benyounes et al. in [1] in the context of harmonic sections. In [2], the same authors studied the geometry of the tangent bundle equipped with this kind of metric. It is worth noticing that Munteanu in [7] investigated independently the geometry of tangent bundle equipped with a certain deformation of Cheeger-Gromoll metric other that in [1]. Yet in [6], Walczak and the first named author considered (p,q)-metrics in the context of Riemannian submersions and Gromov-Hausdorff topology.

In this paper we introduced (p,q,α)-metrics which are more general than (p,q)-metrics. (In contrast to [1] we do not assume that p,q and α are constant). We investigate relations between conformality of a map $\varphi:(M,g)\to(M',g')$ between Riemannian manifolds and its differential $\Phi=\varphi_*:(TM,h)\to(TM',h')$ between their tangent bundles equipped with (p,q,α)-metric h and (r,s,β)-metric h', respectively.

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Interesting enough, there is essential difference between the cases $\dim M = 2$ and $\dim M \geq 3$.

We prove that in the second case (Theorem 1) Φ is conformal if and only if φ is a homothety and totally geodesic immersion and some special relations between triples (p, q, α) and (r, s, β) hold. In this case Φ is also a homothety with the same dilatation as φ.

However, in the first case it may happen that Φ is conformal, although φ is not a totally geodesic immersion (Theorem 2). Then Φ is no longer a homothety. An example of such a map is given.

1.1 Cheeger-Gromoll type metrics

Consider a Riemannian manifold (M, g), and let $\pi : TM \to M$ be its tangent bundle. The Levi-Civita connection ∇ of g, gives a natural splitting $T(TM) = \mathcal{H} \oplus \mathcal{V}$ of the second tangent bundle $\pi_* : T(TM) \to TM$, where the *vertical* distribution $\mathcal{V}$ is the kernel of π_*, and the *horizontal* distribution is the kernel of, so called, connection map K. If $X, Z \in T_x M$ then by X^v_Z we denote the vertical lift of X to the point Z, i.e., X^v_Z is a tangent vector to the curve $t \mapsto Z + tX$ at $t = 0$. Every $A \in T_Z(TM)$ can be uniquely written as $A = \mathcal{H}A + \mathcal{V}A$, where $\mathcal{H}A \in \mathcal{H}_Z$ and $\mathcal{V}A \in \mathcal{V}_Z$ denote its horizontal and vertical part respectively. The vertical part of A is given by $(KA)^v_Z$.

Recall that K is a smooth $\mathbb{R}$-linear bundle morphism determined by the conditions:

- (K1) For every $Z \in TM$, $K : T_Z(TM) \to T_{\pi(Z)}M$ is the canonical isomorphism, i.e., $K(X^v_Z) = X$.
- (K2) For every vector field X on M and every $v \in T_x M$, $K(X_* v) = \nabla_v X$.

Notice that (K1) and (K2) imply the following properties

- (K3) For every Riemannian manifold (M', g') and every $X, Z \in T_x M$ and every map $\varphi : M \to M'$, $\varphi_{**} X^v_Z = (\varphi_* X)^v_{\varphi_*(Z)}$.
- (K4) For every curve γ in M and every vector field ξ along γ, $K(\dot{\xi}) = \nabla_{\dot{\gamma}} \xi$.

Let p, q, α be smooth functions on M. Assume q is non-negative and α is positive. Define (p, q, α)-metric $h = h_{p,q,\alpha}$ on TM as follows: For every $A, B \in T_Z(TM)$, $Z \in T_x M$,

$$h(A, B) = g(\pi_* A, \pi_* B) + \omega_\alpha(Z)^p \left(g(KA, KB) + q g(KA, Z) g(KB, Z)\right),$$

where $\omega_\alpha(Z) = (1 + \alpha g(Z, Z))^{-1}$. Here all functions p, q, α are evaluated at x. For any p, q, α, the Riemannian metric $h_{p,q,\alpha}$ is a special case of a metric considered in [7]. Notice that if p, q, α are constants and $\alpha = 1$ then $h_{p,q,\alpha}$ becomes a metric from [1]. In particular, $h_{0,0,1}$ (resp. $h_{1,1,1}$) is Sasaki metric h_S [5] (resp. Cheeger-Gromoll metric h_{CG} [4]).

1.2 Conformal mappings and metrics

Recall that a map $\varphi : (M, g) \to (M', g')$ between Riemannian manifolds is *conformal* if $\varphi^* g' = \lambda g$ for some positive function λ on M. The function λ is called a *dilatation*. A conformal mapping with constant dilatation is called a *homothety*. If $\dim M < \dim M'$ a conformal mapping φ is often called *weakly conformal*.

Let λ be a strictly positive C^∞-function on M, and $g^\lambda = \lambda g$. The Levi-Civita connections ∇ and ∇^λ of g and g^λ are related as follows: $\nabla^\lambda = \nabla + S^{g,\lambda}_M$ where $S = S^{g,\lambda}_M$ is a symmetric

(1, 2)-tensor field given by (compare [3], p. 64):

$$S(X,Y)=\frac{1}{2\lambda}\left((X\lambda)Y+(Y\lambda)X-g(X,Y)\operatorname{grad}\lambda\right),\quad X,Y\in\Gamma(M,TM).$$

Suppose $\varphi : M \to M'$ is an immersion, e.g., conformal mapping. Then for every $x \in M$ we may choose an open neighbourhood U_x of x such that $L_{x'} = \varphi(U_x)$ $(x' = \varphi(x))$ is a regular submanifold of M'. Let $j : L_{x'} \to M'$ be the inclusion map, and let $\bar{g} = j^*g'$ be the induced metric tensor on $L_{x'}$. Moreover, let Π denote the second fundamental form of $L_{x'}$. We say that the immersion $\varphi : (M, g) \to (M', g')$ is *totally geodesic* if for every $x \in M$, $L_{x'}$ is a totally geodesic submanifold of (M', g'). One can prove the following

Lemma 1 *Suppose $\varphi : (M, g) \to (M', g')$ is a conformal mapping with a dilatation λ. Choose $x \in M$ and put $x' = \varphi(x)$. Let $\gamma : (-\epsilon, \epsilon) \to L_{x'}$ be a curve in M and ξ be a vector field along γ. Put $\gamma' = \varphi \circ \gamma$ and $\xi' = \varphi_*\xi$. Then*

$$\bar{\nabla}_{\dot{\gamma}'}\xi' = \varphi_*\nabla_{\dot{\gamma}}\xi + \varphi_*S(\dot{\gamma}, \xi),$$

where $S = S^{g,\lambda}_M$, and ∇ and $\bar{\nabla}$ are the Levi-Civita connections of g and $\bar{g}$ respectively.

Adopt the notations from Lemma 1. Put $Z = \xi(0)$, $Z' = \xi'(0)$, $v = \dot{\gamma}(0)$ and $v' = \dot{\gamma}'(0)$. Suppose that a vector $A \in T_Z(TM)$ is tangent to the curve ξ (it is convenient to think of vector fields along curves as of curves in the tangent bundle), i.e., $A = \dot{\xi}(0)$. Next, let K and K' denote connection maps induced from ∇ and ∇' respectively. Moreover put $\Phi = \varphi_* : TM \to TM'$. As a direct consequence of Lemma 1, the equation $\nabla' = \nabla + \Pi$ and properties of connection map we get

Lemma 2 *The vectors $K(A)$ and $K'(\Phi_*A)$ are related as follows:*

$$K'(\Phi_*A) = \varphi_*K(A) + \varphi_*S(v, Z) + \Pi(v', Z').$$

In particular, if A is horizontal then $K'(\Phi_*A) = \varphi_*S(v, Z) + \Pi(v', Z')$.

Remark 1 If $\dim M = \dim M'$ then the term Π is omitted.

Let $U = U_x$ and $L' = L_{x'}$ and let $\pi' : TL' \to L'$ be a natural projection. Since $\varphi : U \to L'$ is a conformal diffeomorphism, so is its inverse. In particular $\Phi : TU \to TL'$ is a diffeomorphism. Therefore we have

Corollary 1 *Take $A' \in T_{Z'}(TL')$ such that $v' = {\pi'}_*A$. Then*

$$K(\Phi_*^{-1}A') = \varphi_*^{-1}K'(A') + \varphi_*^{-1}S'(v', Z') - \varphi_*^{-1}\Pi(v', Z'),$$

or equivalently

$$K(\Phi_*^{-1}A') = \varphi_*^{-1}\bar{K}(A') + \varphi_*^{-1}S'(v', Z'),$$

where $S' = S^{\bar{g},\mu}_{L'}$ with $\mu = (1/\lambda) \circ \varphi^{-1}$, and $\bar{K}$ is the connection map induced from $\bar{\nabla}$.

Corollary 2 *Suppose $\varphi : (M, g) \to (M', g')$ is a conformal mapping and M is connected. Φ_* maps horizontal vectors onto horizontal vectors if and only if φ is a totally geodesic homothety.*

Proof ($\Rightarrow$) If Φ_* maps horizontal vectors onto horizontal vectors then by Lemma 2, $\varphi_* S + \Pi$ vanishes identically. Since $\varphi_* S$ and Π are always orthogonal and a conformal mapping is an immersion it follows that S and Π vanish identically. Applying the definition of S with $X = Y = \operatorname{grad}\lambda$, we get that $\operatorname{grad}\lambda$ is the zero vector field. Consequently, λ is constant and therefore φ is a homothety. Since Π vanishes, φ is totally geodesic.

($\Leftarrow$) Obvious. □

1.3 Algebraic lemmas

Suppose two finite dimensional real vector spaces V and W equipped with inner products $\langle , \rangle_V$ and $\langle , \rangle_W$ are given. Let $B : V \times V \to W$ be a symmetric, bilinear form on V. Moreover, let $C \geq 0$. Consider a condition

$$\langle B(X, Z), B(Y, Z)\rangle_W = C\langle X, Y\rangle_V \langle Z, Z\rangle_V. \tag{1}$$

for every $X, Y, Z \in V$. Then, if X, Y are orthogonal

$$\langle B(X, X), B(Y, Y)\rangle_W = -C\langle X, X\rangle_V \langle Y, Y\rangle_V. \tag{2}$$

Lemma 3 *Assume that B satisfies the condition* (1). *If* $\dim V \geq 3$ *then* $C = 0$. *In particular, B vanishes.*

Proof Suppose that $C \neq 0$. Take an orthonormal pair X, Y. Let $\xi = B(X, X)$ and $\zeta = B(Y, Y)$. By (1) we have

$$\langle \xi, \xi\rangle = \langle \zeta, \zeta\rangle = C > 0. \tag{3}$$

In particular, $\xi \neq 0$ and $\zeta \neq 0$. Applying (2) we see that

$$\langle \xi, \zeta\rangle = -C.$$

Using above and (3) one can obtain that $\xi = -\zeta$. Next, since $\dim V \geq 3$ we may find $Z \in V$ such that X, Y, Z is an orthonormal triple. Let $\eta = B(Z, Z)$. Then, by above $\xi = -\zeta = \eta = -\xi$, which contradicts the fact that $\xi \neq 0$. □

Notice that the assumption $\dim V \geq 3$ is essential. Namely we have

Lemma 4 (*a*) *A symmetric bilinear form* $B : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ *satisfies* (1) *if and only if there exists an angle θ such that*

$$B(X, Y) = \pm\sqrt{C}e^{i\theta} XY \quad \text{or} \quad B(X, Y) = \pm\sqrt{C}e^{i\theta}\bar{X}\bar{Y}, \tag{4}$$

where we identify $\mathbb{R}^2$ with $\mathbb{C}$.
(*b*) *If* $\dim V = 2$ *and a non-zero symmetric bilinear form* $B : V \times V \to W$ *satisfies the condition* (1) *then there exists a* 2*-dimensional subspace U of W such that the image of B is equal to U and with respect to orthonormal bases of V and U, B is of the form* (4).

Proof (a) Elementary exercise. (b) Take an orthonormal basis X, Y of V. Since $B \neq 0$, we have $C \neq 0$. Consequently, $\xi = B(X, X)$, $\zeta = B(Y, Y)$ and $\eta = B(X, Y)$ are nonzero vectors of W of length $\sqrt{C}$. By (1) we see that $\langle \xi, \eta\rangle = \langle \zeta, \eta\rangle = 0$. Moreover, $\langle \xi, \zeta\rangle = -C$, by (2). It follows that $\xi = -\zeta$. Consequently, the image U of B is a two-dimensional subspace spanned by ξ, η.

Now taking orthonormal bases of V and U, e.g., X, Y and $\xi/\sqrt{C}, \eta/\sqrt{C}$, we reduce (b) to (a). $\square$

2 Conformality of a differential

In this section all manifolds are connected. Let (M, g) and (M', g') be Riemannian manifolds of dimensions m and m', respectively. We assume that $m, m' \geq 2$. Denote by ∇ and ∇' the Levi-Civita connections of g and g', respectively. Equip their tangent bundles $\pi : TM \to M$ and $\pi' : TM' \to M'$ with (p, q, α)-metric h and (r, s, β)-metric h', respectively.

Let $\varphi : M \to M'$ and $\Phi = \varphi_* : TM \to TM'$. Put $g = \langle , \rangle$ and $g' = \langle , \rangle'$. Denote by $|\cdot|$ and $|\cdot|'$ the norms induced by g and g', respectively. Moreover, denote by $\|\cdot\|$ and $\|\cdot\|'$ the norms induced by h and h', respectively.

In the paper we use the following notation: If φ (resp. Φ) is conformal mapping then its dilatation will be always denoted by λ (resp. Λ).

2.1 Technical lemmas

Lemma 5 *Suppose that φ and Φ are conformal mappings. Then for any $Z \in T_xM$ and $x' = \varphi(x)$,*

$$\Lambda(Z) = \lambda(x)\frac{\left(1 + \alpha(x)|Z|^2\right)^{p(x)}}{\left(1 + \lambda(x)\beta(x')|Z|^2\right)^{r(x')}}, \tag{5}$$

$$q(x) = \lambda(x)s(x'). \tag{6}$$

Proof Let $X, Z \in T_xM$. Applying conformality of Φ and (K3) we have: $\|(\varphi_*X)^v_{\varphi_*Z}\|'^2 = \|\Phi_*X^v_Z\|'^2 = \Lambda(Z)\|X^v_Z\|^2$. Using now the definitions of h and h' and conformality of φ, one can easily get

$$\lambda(x)\frac{|X|^2 + \lambda(x)s(x')\langle X, Z\rangle^2}{\left(1 + \lambda(x)\beta(x')|Z|^2\right)^{r(x')}} = \Lambda(Z)\frac{|X|^2 + q(x)\langle X, Z\rangle^2}{\left(1 + \alpha(x)|Z|^2\right)^{p(x)}}. \tag{7}$$

Taking nonzero vector X orthogonal to Z, (7) becomes (5). Next, let $Z \neq 0$. Putting $X = Z$ in (7) and comparing the result with (5) we get (6). $\square$

Lemma 6 *If Φ is conformal then so is φ. Moreover $\lambda(x) = \Lambda(0_x)$, $x \in M$.*

Proof Let $x \in M$ and $Z = 0_x \in T_xM$. By (K3) and conformality of Φ, for every $X, Y \in T_xM$

$$\begin{aligned}\langle \varphi_*X, \varphi_*Y\rangle' &= h'\left((\varphi_*X)^v_{\varphi_*Z}, (\varphi_*Y)^v_{\varphi_*Z}\right)\\ &= \Lambda(Z)h\left(X^v_Z, Y^v_Z\right) = \Lambda(Z)\langle X, Y\rangle.\end{aligned}$$

$\square$

Lemma 7 *Adopt the notation from Sect. 1.2. Suppose that φ and Φ are conformal mappings. Then*

(a) *φ is a homothety.*

(b) *For every $x \in M$ one of the following conditions holds:*

$$p(x) = r(x') = 0, \tag{8}$$
$$p(x) = r(x') \neq 0 \quad \text{and} \quad \lambda\beta(x') = \alpha(x), \tag{9}$$
$$p(x) = r(x') = 1 \quad \text{and} \quad \lambda\beta(x') \neq \alpha(x), \tag{10}$$
$$p(x) = 1 \quad \text{and} \quad r(x') = 0. \tag{11}$$

(c) *If for every $x \in M$ either (8) or (9) holds then Φ is also a homothety with the dilatation $\Lambda = \lambda$. Moreover, φ is totally geodesic.*

(d) *If (10) or (11) holds globally then for every $v, w, Z \in T_x M$*

$$\langle \Pi(\varphi_* v, \varphi_* Z), \Pi(\varphi_* w, \varphi_* Z)\rangle' = C\langle v, w\rangle\langle Z, Z\rangle, \tag{12}$$

where $C = \lambda(\alpha(x) - \lambda\beta(x')) \neq 0$ in the case (10), and $C = \lambda\alpha(x) \neq 0$ in the case (11). In particular, φ is not totally geodesic.

Proof Assume that $v \neq 0$. Take vectors $A \in T_Z(TM)$ and $A' \in T_{Z'}(TL')$ as in Lemma 1 and Corollary 1. Moreover we may assume that A and A' are horizontal with respect to ∇ and $\bar{\nabla}$, respectively, i.e., $K(A) = 0$ and $\bar{K}(A') = 0$.

Put $U = U_x$ and $L' = L_{x'}$. Let $J : TL' \to TM'$ be the inclusion map. Put $\bar{h} = J^* h'$. Since $\varphi : (U, g) \to (L', \bar{g})$ and $\Phi : (TU, h) \to (TL, \bar{h})$ are conformal diffeomorphisms, so are $\varphi^{-1} : (L', \bar{g}) \to (U, g)$ and $\Phi^{-1} : (TL', \bar{h}) \to (TU, h)$. Dilatations of φ^{-1} and Φ^{-1} are equal to $\mu = (1/\lambda) \circ \varphi^{-1}$ and $\hat{\mu} = (1/\Lambda) \circ \Phi^{-1}$, respectively. Thus we have

$$\|\Phi_* A\|'^2 = \Lambda(Z)\|A\|^2,$$
$$\|\Phi_*^{-1} A'\|^2 = \hat{\mu}(Z')\|A'\|'^2.$$

Put for a while $S = S(v, Z)$, $S' = S'(v', Z')$ and $\Pi' = \Pi(v', Z')$. Since $\pi_* A = v$, $\pi'_* A' = v'$, $K(A) = 0$, $K'(A') = \bar{K}(A') + \Pi' = \Pi'$ and Z' is orthogonal to Π', we have

$$\|A\|^2 = |v|^2,$$
$$\|A'\|'^2 = |v'|'^2 + \omega_\beta(Z')^r|\Pi'|'^2$$

Next applying Lemma 1 and Corollary 1, the equalities $\pi'_* \Phi_* A = v'$ and $\pi_* \Phi_*^{-1} A' = v$, and the fact that $\varphi_* S$ is orthogonal to Π' we get

$$\|\Phi_* A\|'^2 = |v'|'^2 + \omega_\beta(Z')^r \left(\lambda(x)|S|^2 + s\lambda^2(x)\langle S, Z\rangle^2 + |\Pi'|'^2\right)$$
$$\|\Phi_*^{-1} A'\|^2 = |v|^2 + \omega_\alpha(Z)^p \left(\mu(x')|S'|'^2 + q\mu^2(x')\langle S', Z'\rangle'^2\right)$$

Combining now above equalities and using the definitions of μ and $\hat{\mu}$ we get

$$\Lambda|v|^2 = |v'|'^2 + \omega_\beta(Z')^r \left(\lambda|S|^2 + \lambda^2\langle S, Z\rangle^2 + |\Pi'|'^2\right), \tag{13}$$
$$\frac{1}{\Lambda}\left(|v'|'^2 + \omega_\beta(Z')^r|\Pi'|'^2\right) = |v|^2 + \omega_\alpha(Z)^p \left(\frac{1}{\lambda}|S'|'^2 + \frac{1}{\lambda^2}\langle S', Z'\rangle'^2\right), \tag{14}$$

where $\lambda = \lambda(x)$ and $\Lambda = \Lambda(Z)$. Multiplying equations (13) and (14) side by side we conclude that

$$0 = \omega_\beta(Z')^r \lambda|v|^2|S|^2 + \text{non-negative expression.}$$

Since $v \neq 0$, $S(v, Z) = 0$. Since $x \in M$, $v, Z \in T_x M$ were arbitrary the tensor field S vanishes identically. Therefore, λ is a constant function and thus φ is a homothety. Hence (a) is proved.

Substituting $S = 0$ in (13) we get

$$|\Pi(v', Z')|'^2 = \frac{\Lambda(Z) - \lambda}{\omega_\beta(Z')^r}|v|^2.$$

Applying Lemma 5 we get

$$|\Pi(\varphi_* v, \varphi_* Z)|'^2 = \lambda((1 + \alpha|Z|^2)^p - (1 + \lambda\beta|Z|^2)^r)|v|^2. \tag{15}$$

Using the facts that the map $(v, Z) \mapsto |\Pi(\varphi_* v, \varphi_* Z)|'^2$ is non negative and symmetric with respect to v, Z, we conclude (b).

If (8) or (9) holds then by (5) it follows that $\Lambda = \lambda$. Moreover, in these cases (15) becomes $|\Pi(\varphi_* v, \varphi_* Z)|'^2 = 0$. This proves (C).

If (10) or (11) holds then it is an elementary computation to check that Π satisfies (12), proving (d). □

Lemma 8 *Suppose that* $\dim M \geq 3$ *or* $\dim M' \leq \dim M + 1$. *Then under the assumptions of Lemma 7 we have: φ is totally geodesic homothety, Φ is a homothety and its dilatation Λ is equal to λ.*

Proof It suffices to show that under the assumptions the conditions (10) and (11) cannot hold. Then the assertion follows from Lemma 7 (a) and (c). Suppose that (10) or (11) holds. Then by Lemma 7 (d) it follows that the symmetric bilinear form $B : T_x M \times T_x M \to T_{x'} M'$ given by $B(v, w) = \Pi(\varphi_* v, \varphi_* w)$ satisfies the condition (1) with $C \neq 0$. If $\dim M \geq 3$ then we have a contradiction with Lemma 3, if $\dim M' \leq \dim M + 1$ then we have a contradiction with Lemma 4. □

2.2 Main results

We begin with some definitions. Suppose $\bar{M}$ is a submanifold of a Riemannian manifold (M', g'). Suppose that a real-valued non-negative function C on $\bar{M}$ is given. We say that $\bar{M}$ is *optimal with a coefficient* C if for every $x' \in \bar{M}$ the second fundamental form Π of $\bar{M}$ at x' satisfies (1) with the constant $C(x')$ that is

$$\langle \Pi(u, w), \Pi(v, w) \rangle = C(x')\langle u, v \rangle \langle w, w \rangle, \quad u, v, w \in T_{x'}\bar{M}.$$

In particular, every totally geodesic submanifold is optimal with the coefficient 0. By Lemma 3 and Lemma 4 it follows that if $\dim \bar{M} \geq 3$ or $\operatorname{codim} \bar{M} \leq 1$ then each optimal submanifold is totally geodesic.

Remark 2 Observe that if $\varphi : M \to \bar{M}$ is a conformal diffeomorphism such that (12) holds then $\bar{M}$ is optimal with the coefficient $C/(\lambda \circ \varphi^{-1})^2$.

Proposition 1 *Suppose* $\dim \bar{M} = 2$. *Denote by κ' and $\bar{\kappa}$ the sectional curvatures of M' and $\bar{M}$, and let $\sigma = T_{x'}\bar{M}$. If $\bar{M}$ is optimal submanifold of M' with a coefficient C then $\bar{M}$ is minimal submanifold of M' and $\bar{\kappa}(\sigma) = \kappa'(\sigma) - 2C(x')$. In particular, if C is constant and M' is a space of constant curvature then so is $\bar{M}$.*

Proof The fact that $\bar{M}$ is minimal follows immediately from Lemma 4: it suffices to calculate the trace of the bilinear form given by (4). The second statement follows from (1), (2) and the Gauss Equation. □

Suppose now that two Riemannian manifolds (M, g) and (M', g') are given and $\dim M \leq \dim M'$. Equip their tangent bundles TM and TM' with (p, q, α)-metric h and (r, s, β)-metric h' respectively. Suppose next that the functions $p, q, r, s, \alpha, \beta$ are constant, and $\varphi : M \to M'$ is an imbedding (injective immersion). Let $\bar{M} = \varphi(M)$.

Theorem 1 *Let* $\dim M \geq 3$ *or* $\dim M' \leq \dim M + 1$.

- (I) *Suppose that φ is a conformal mapping with a dilatation λ. Then $\Phi = \varphi_* : TM \to TM'$ is conformal if and only if $q = \lambda(s \circ \varphi^{-1})$, φ is a homothety, $\bar{M}$ is totally geodesic and for every $x \in M (x' = \varphi(x))$ one of the conditions (8) or (9) holds.*
- (II) *If Φ is a conformal mapping then φ and Φ are homotheties and $\Lambda = \lambda$.*

Theorem 2 *Let* $\dim M = 2$ *and* $\dim M' \geq \dim M + 2$.

- (III) *Suppose that φ is a conformal mapping with a dilatation λ. Then $\Phi = \varphi_* : TM \to TM'$ is conformal if and only if φ is a homothety, $q = \lambda(s \circ \varphi^{-1})$, $\bar{M}$ is optimal with the coefficient*

$$C = \frac{1}{\lambda}\left((p\alpha) \circ \varphi^{-1} - \lambda\beta r\right)$$

and for every $x \in M (x' = \varphi(x))$ one of the properties (8)–(11) is satisfied.
- (IV) *Suppose Φ is a conformal mapping. Then φ a homothety and*
  - *(IV1) If for every $x \in M$ one of the conditions (8) or (9) holds then Φ is also a homothety and $\Lambda = \lambda$.*
  - *(IV2) If one of the conditions (10) or (11) holds globally then φ is a minimal immersion and for every plane $\sigma = \varphi_*(T_x M)$, $x \in M$, the Gauss curvature $\kappa(\sigma)$ of M is*

$$\kappa(\sigma) = \lambda\kappa'(\sigma) - 2C(\varphi(x))\lambda, \tag{16}$$

where $\kappa'(\sigma)$ is the Gauss curvature of M'. Moreover, Φ is not a homothety. Its dilatation Λ is

$$\Lambda(Z) = \lambda \frac{1 + \alpha(x)g(Z, Z)}{1 + \lambda\beta(x')r(x')g(Z, Z)}, \quad Z \in T_x M, x' = \varphi(x). \tag{17}$$

Proof of Theorem 1 and Theorem 2 (I, III $\Rightarrow$) Suppose φ and Φ are conformal mappings. By Lemma 7, φ is a homothety and for every $x \in M$ $(x' = \varphi(x))$ one of the conditions (8)-(11) holds. By Lemma 5, $q = \lambda(s \circ \varphi^{-1})$. If $\dim M \geq 3$ or $\dim M' \leq \dim M + 1$, by Lemma 7 (d) and Lemma 8 conditons (10) and (11) cannot hold. Therefore Lemma 7 (c) implies that $\bar{M}$ is totally geodesic. Moreover, if (10) or (11) is satisfied then by Lemma 7 (d) and Remark 2, $\bar{M}$ is optimal with the coefficient $C = (1/\lambda)((p\alpha) \circ \varphi^{-1} - \lambda\beta r)$.

(I, III $\Leftarrow$) Taking horizontal (resp. vertical) $A \in T_Z TM$, computing $h'(\Phi_* A, \Phi_* A)$ and applying relations between $p, q, r, s, \alpha, \beta$ and λ one can conclude that Φ is conformal.

(II) Suppose Φ is conformal. By Lemma 6, φ is also conformal. Therefore (I) and Lemma 5 imply that Φ and φ are homotheties and $\Lambda = \lambda$.

(IV) As above we conclude that φ is conformal.

(IV1) It is a consequence of Lemma 7 (c).

(IV2) Suppose for every $x \in M$ one of the conditions (10) or (11) holds. By Proposition 1, (I) and the fact that the curvature under the action of a homothety with dilatation λ is scaled by $1/\lambda$, φ is a minimal immersion and (16) holds. Conditions (10), (11) and equation (5) imply (17). □

As a direct consequence of of Theorem 1 we obtain

Corollary 3 *Suppose* $\dim M \geq 3$ *or* $\dim M' \leq \dim M + 1$. *Let* $\varphi : (M, g) \to (M', g')$ *be an imbedding. Then we have:*

(a) $\Phi : (TM, h_S) \to (TM', h'_S)$ *is conformal if and only if* φ *is totally geodesic homothety.*

(b) $\Phi : (TM, h_{CG}) \to (TM', h'_{CG})$ *is conformal if and only if* φ *is totally geodesic isometric imbedding.*

(c) $\Phi : (TM, h_{CG}) \to (TM', h'_S)$ *is never conformal.*

2.3 An example to Theorem 2

It is important to show that there is essential difference between Theorems 1 and 2. To do this we give an example of 2-dimensional manifold M, 4-dimensional manifold M' and an immersion $\varphi : M \to M'$ such that $\bar{M} = \varphi(M)$ is optimal but not totally geodesic.

Let $\Sigma^d(\rho)$ denote Euclidean d-dimensional sphere of radius ρ centred at the origin in $\mathbb{R}^{d+1}$. Recall (see [3], Chapter 4 §5 page 139) that the *second standard immersion* of $\Sigma^2(1)$ it is a map $\varphi : \Sigma^2(1) \to \Sigma^4(1/\sqrt{3})$ defined as follows: Consider harmonic homogeneous polynomials u_i, $i = 1, \dots, 5$, in $\mathbb{R}^3$ given by

$$u_1 = x_2 x_3, \quad u_2 = x_1 x_3, \quad u_3 = x_1 x_2,$$
$$u_4 = \frac{1}{2}\left(x_1^2 - x_2^2\right), \quad u_5 = \frac{\sqrt{3}}{6}\left(x_1^2 + x_2^2 - 2x_3^2\right),$$

and let $u = (u_1, \dots, u_5)$. We define φ to be the restriction $u|\Sigma^2(1)$. Then $\varphi : \Sigma^2(1) \to \Sigma^4(1/\sqrt{3})$ is an isometric immersion (but not imbedding). Nevertheless, $\bar{M} = \varphi(\Sigma^2(1))$ is a minimal submanifold of $\Sigma^4(1/\sqrt{3})$. We show that $\bar{M}$ is optimal with the constant coefficient $C = 1$.

Lemma 9 *Suppose* (M, g) *is a Riemannian manifold and* $u : M \to \mathbb{R}^{d+1}$. *Assume that the image* $\bar{M} = u(M)$ *is contained in* $\Sigma = \Sigma^d(\rho)$ *and* $u : M \to \Sigma$ *is an imbedding. Denote by* Π *and* $\bar{\Pi}$ *the second fundamental form of* $\bar{M}$ *in* Σ *and* $\bar{M}$ *in* $\mathbb{R}^{d+1}$, *respectively. Then for every* $x' \in \Sigma$ *and for every basis* (e_i) *of* $T_{x'}\bar{M}$

$$\langle \Pi(e_i, e_k), \Pi(e_j, e_l)\rangle = \langle \bar{\Pi}(e_i, e_k), \bar{\Pi}(e_j, e_l)\rangle - \frac{1}{\rho^2}\langle e_i, e_k\rangle\langle e_j, e_l\rangle,$$

where $\langle , \rangle$ *is the canonical inner product in* $\mathbb{R}^{d+1}$.

Proof Elementary exercise. □

Proposition 2 *If* φ *is the second standard immersion then the submanifold* $\bar{M} = \varphi(\Sigma^2(1))$ *is optimal with the constant coefficient* $C = 1$.

Proof Let $M=\Sigma^2(1)$ and $\Sigma=\Sigma^4(1/\sqrt{3})$. Adopt the notations from Lemma 9. Denote by $\tilde{\varphi}$ the restriction of φ to lower half sphere $\Sigma^2_-(1)$. Since $\varphi(\Sigma^2(1))$ coincides with the closure of $\tilde{\varphi}(\Sigma^2_-(1))$, it suffices to prove that $\bar{M}_-=\tilde{\varphi}(\Sigma^2_-(1))$ is optimal with the coefficient one.

Let f be the stereographic projection $\Sigma^2(1)\to\mathbb{R}^2$ from the north pole. Put $\psi=f\circ\tilde{\varphi}^{-1}$. Fix $x'=\psi^{-1}(t)$, where $t=(t_1;t_2)\in\mathbb{R}^2$, $|t|<1$. Let $e_i=(\partial/\partial\psi_i)(x')$. Put $t^2=t_1^2+t_2^2$ and $t^4=(t^2)^2$. Since φ is an isometric immersion and f is the stereographic projection we have

$$\langle e_i,e_j\rangle=\frac{4\delta_{ij}}{(t^2+1)^2},\quad i,j=1,2,$$

where δ_{ij} is the Kronecker symbol. In the light of Lemma 9 (with $\rho=1/\sqrt{3}$), to finish the proof it suffices to show that

$$\begin{aligned}\langle\bar{\Pi}(e_i,e_k),\bar{\Pi}(e_j,e_k)\rangle&=(3\delta_{ik}+1)\frac{16\delta_{ij}}{(t^2+1)^4}\\&=(3\delta_{ik}+1)\langle e_i,e_j\rangle^2,\quad i,j,k=1,2.\end{aligned}\tag{18}$$

After elementary but laborious calculations we get

$$\begin{aligned}\bar{\Pi}(e_1,e_1)=\frac{4}{(t^2+1)^4}&4t_2\left(1-t^2-2t_1^2\right);8t_1\left(1-t_1^2\right);4t_1t_2\left(t_1^2-t_2^2-3\right);\\&t^4-8t_1^2t_2^2+6t_2^2-6t_1^2+1;\sqrt{3}\left(t^4-2t^2-4t_1^2+1\right)\end{aligned}$$

and

$$\begin{aligned}\bar{\Pi}(e_2,e_2)=\frac{4}{(t^2+1)^4}&8t_2\left(1-t_2^2\right);4t_1\left(1-t^2-2t_2^2\right);4t_1t_2\left(t_2^2-t_1^2-3\right);\\&6t_2^2-6t_1^2+8t_1^2t_2^2-t^4-1;\sqrt{3}\left(t^4-2t^2-4t_2^2+1\right)\end{aligned}$$

and

$$\begin{aligned}\bar{\Pi}(e_1,e_2)=\frac{4}{(t^2+1)^4}&2t_1\left(t^2-4t_2^2+1\right);2t_2\left(t^2-4t_1^2+1\right);\\&8t_1^2t_2^2-t^4+1;4t_1t_2\left(t_1^2-t_2^2\right);-4\sqrt{3}t_1t_2.\end{aligned}$$

Now one can check that (18) holds. □

Now let $\mathbb{R}P^2$ denote the real 2-dimensional projective space. We treat $\mathbb{R}P^2$ as a Riemannian manifold whose metric g is given by the standard two-sheeted covering map $\hat{\pi}:\Sigma^2(1)\to\mathbb{R}P^2$. Put $\hat{\varphi}(\hat{x})=\varphi(x)$ if $\hat{x}=\hat{\pi}(x)$. Since $\varphi(x)=\varphi(-x)$, the map $\hat{\varphi}$ is well defined. Moreover, $\hat{\varphi}:\mathbb{R}P^2\to\Sigma^4(1/\sqrt{3})$ is an imbedding. It is called *the first standard imbedding* of $\mathbb{R}P^2$ into $\Sigma^4(1/\sqrt{3})$.

Take constants $q,\alpha>0$. Suppose Cheeger-Gromoll type metrics h and h' on $T(\mathbb{R}P^2)$ and $T(\Sigma^4(1/\sqrt{3}))$ are given. If

(1) $h=h_{1,q,\alpha+1}$ and $h'={h'}_{1,q,\alpha}$, or
(2) $h=h_{1,q,1}$ and $h'={h'}_{0,q,1}$

then $\hat{\varphi}_*$ is a conformal mapping, but not a homothety. Its dilatation is $\Lambda(Z)=(1+(\alpha+1)g(Z,Z))/(1+\alpha g(Z,Z))$ in the case of (1) and $\Lambda(Z)=1+g(Z,Z)$ in the case of (2).

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