# A new approach to non-Abelian hydrodynamics

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ABSTRACT: We present a new approach to describe hydrodynamics carrying non-Abelian macroscopic degrees of freedom. Based on the Kaluza-Klein compactification of a higher-dimensional neutral dissipative fluid on a manifold of non-Abelian isometry, we obtain a four-dimensional colored dissipative fluid coupled to Yang-Mills gauge field. We derive transport coefficients of resulting colored fluid, which feature non-Abelian character of color charges. In particular, we obtain color-specific terms in the gradient expansions and response quantities such as the conductivity matrix and the chemical potentials. We argue that our Kaluza-Klein approach provides a robust description of non-Abelian hydrodynamics, and discuss some links between this system and quark-gluon plasma and fluid/gravity duality.

*Denn die Menschen glauben an die Wahrheit dessen,*
*was ersichtlich stark geglaubt wird.*
*All truthful things are subject to interpretation.*
*Which interpretation prevails at a given time*
*is a function of power, not truth.*
*— Friedrich Nietzsche — 'The Will to Power'*

JHEP02(2017)122

# Contents

## 1 Introduction

Hydrodynamics has been an efficient approach for the description of strongly interacting state of matter. This boosted the research and application of hydrodynamics models, such as transport phenomena or hydrodynamic instabilities. One aspect in hydrodynamics that

has not been explored in detail yet is the dynamics of a colored fluid charged under non-Abelian Yang-Mills gauge fields, where the constituents of the fluid carry non-Abelian color charges and interact with non-Abelian vectors. Due to its non-Abelian nature, we expect that this system gives rise to a variety of physical phenomena richer than its Abelian counterpart, viz. Maxwell plasma. Nevertheless, the level of rigor in formulating the theoretical foundations of this model and the understanding of its ensuing physical properties are far lesser.

A robust description will contribute to the characterization of some important physical systems. For example, the quark-gluon plasma behaves as an almost perfect dense fluid carrying SU(3) color charge. However, the detailed microscopic understanding of the equilibration mechanisms after the heavy-ion collisions is still left to be an outstanding problem. A transient phase in the equilibration process is reached when the system is at local thermal equilibrium with yet non-equilibrated colored quark and gluon degrees of freedom (DOFs). Most of the analysis done so far is based on kinetic theory [1–8] and on the single-particle approach [9]. Integrating out momentum, one obtains a covariant color continuity equation which, together with the mechanical conservation laws of the fluid, constitute the main equations of the system. Still, the construction of the required collision terms which enter the Boltzmann equation is highly non-trivial and, except at weak coupling regime, there is no first-principles derivation. In addition, the applicability of kinetic theory is valid for not-so-far from equilibration situations. Consequently, we conclude that kinetic theory is a useful complementary tool, requiring prior knowledge of the structure of the hydrodynamic equations.

Alternative approaches include the Poisson bracket formulation [10] and the action principle [11, 12] of ideal fluid dynamics. In contrast, the study of dissipative effects, which constitute an integral part of hydrodynamics is well understood only at the level of the equations of motion (EOMs). The description of these effects at the level of an action requires placing the fluid on the Schwinger-Keldysh contour [13], which leads to certain additional supersymmetric DOFs [14, 15].

Another aspect that sheds light on the understanding of hydrodynamic structure is the duality between fluids and black holes [16–19]. This allowed us to discover previously neglected parity-breaking terms that were originated by quantum anomalies [20–22]. To study non-Abelian DOFs coupled to fluids, we need a new background of black hole with non-Abelian Yang-Mills hair [23–25]. However, in AdS/CFT correspondence, local symmetry in the bulk gravity is mapped to global symmetry in the boundary theory. Therefore, as the background field in the boundary theory is usually external and non-dynamical, we have no way of promoting non-Abelian global symmetries to gauge symmetries in the boundary theory. We note that some proposals have been put forward to modify the boundary conditions in such a way the the resulting boundary has dynamical fields [26]. However, these ideas have not been consistently embedded into the fluid/gravity duality and may present additional difficulties in the hydrodynamic formulation of non-Abelian fluids.

For these reasons, we view this state of affair at odds: self-gravitating hydrodynamics, whose gravitational interaction is also intrinsically nonlinear, has been rigorously investigated in various contexts of relativistic astrophysics of compact objects [27] and cosmology

of large-scale structures [28, 29]. We thus expect that non-Abelian hydrodynamics, at least at classical level, can also be rigorously formulated and investigated as much as the self-gravitating hydrodynamics. Such study would have a direct application to wider phenomena featuring non-Abelian DOFs such as the quark-gluon plasma [30] and the spintronics with strong spin-orbit coupling [31, 32].

In this work we propose a completely new approach to bypass all the above conceptual and technical difficulties. We start from a neutral and dissipative fluid coupled to Einstein gravity in $D$ dimensions, which we assume is completely characterized. The idea is to perform a Kaluza-Klein (KK) compactification [33, 34] of this system and obtain a fluid in $d = D - n$ dimensions whose constituents are charged under non-Abelian Yang-Mills fields, where $n$ is the dimension of the internal manifold. That is to say, we use KK dimensional reduction as a method to construct an ab initio description of non-Abelian hydrodynamics. The KK compactification mechanism endows the lower-dimensional system with a set of gauge fields, the so-called KK gauge fields. The compactification ansatz of internal manifold elucidate the resulting gauge symmetry of $d$-dimensional system. As we are interested in non-Abelian hydrodynamics, we will compactify on an SU(2) group manifold [35, 36]. Therefore, we will take $n = \dim(G) = 3$, where $G$ is the gauge group. We will perform this procedure on the EOMs of the starting higher-dimensional neutral fluid, which include dissipative terms.[1]

Our approach is based on the non-Abelian Kaluza-Klein compactification on a SU(2) group manifold, which we interpret as an internal manifold whose isometries generate the non-Abelian color symmetry in the physical system. Since we start with a fluid from the outset, the resulting theory is valid in the long-wavelength limit, coupled to new non-Abelian DOFs that the compactification generates.

KK compactification provided a robust tool for the understanding of the (hidden) structure and the dynamics of gravity-matter systems, which descends from a more fundamental theory such as string/M-theories. If we start with a fundamental theory in $D$ dimensions defined on a manifold $\mathcal{M}_D$, we can find a stable solution of its equations of motion of the form $\mathcal{M}_D = \mathcal{M}_d \times X_n$, where $d = (D - n)$, $\mathcal{M}_d$ is non-compact, reduced spacetime, and $X_n$ is a compact manifold of characteristic size $R$. At low energies, the compact space $X_n$ is not accessible by direct observations: it would take excitations of energy $E \sim 1/R$ to probe spacetime structures of a scale of order $R$. If $R$ is sufficiently small, this energy scale is gapped from the low-energy dynamics on $\mathcal{M}_d$. Nevertheless, the properties of $X_n$ will have important effects on the reduced theory. As emphasized, if $X_n$ is a manifold with isometry group $G$, then metric fluctuations along the Killing directions of $X_n$ generate Yang-Mills gauge fields with gauge group $G$, which will be present in the dynamics of the lower-dimensional theory.

From the viewpoint of KK theory, a novelty of our work is that we include energy-momentum tensor of dissipative fluid, sourcing the Einstein field equations. The procedure, however, must be self-consistent. A KK compactification is said to be consistent if all the

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[1]Heavy-ion collisions and other phenomenologically relevant phenomena occur off the equilibrium and consequently, dissipative effects are very important in their descriptions.
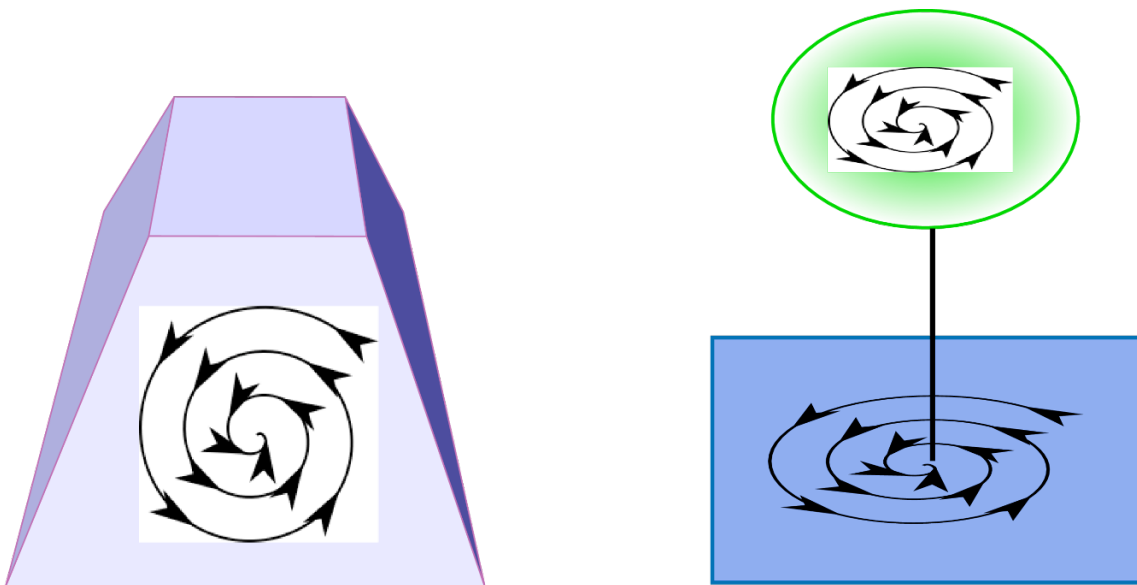
**Figure 1**. Our starting system is a $D$-dimensional dissipative fluid coupled to gravity (left). After KK compactification on a $n$-dimensional internal manifold with non-Abelian isometries, we obtain a $d$-dimensional dissipative fluid that, apart from being coupled to gravity, is charged under dynamic non-Abelian Yang-Mills gauge fields (right).

solutions of the $d$-dimensional theory satisfy the $D$-dimensional EOMs. In this work, we also present the necessary conditions to achieve a consistent reduction of fluid energy-momentum tensor.

Summarizing, the salient features of our approach are the following:

- We apply the KK method to a neutral fluid at the outset coupled to gravity, thus bypassing kinetic theory.

- The approach applies to dissipative fluids, for the compactification is at the level of equations of motion rather than action.

- The proposed KK method "generates" dynamical (non-)Abelian gauge fields which are self-consistently coupled to a charged/colored fluid.

- This mechanism provides an ab initio approach to (non-)Abelian hydrodynamics, distinct from gauge-gravity duality or fluid/gravity duality.

This paper is organized as follows. In section 2 we present the main results of our work: the dynamics of the system, its symmetries and its properties. In the following sections we explain the KK dimensional reduction and the method for obtaining our results. In particular, section 3 reviews the basics of relativistic hydrodynamics and provides the necessary set-up and notations for our calculations. In section 4, we review the dimensional reduction of the system Einstein-perfect fluid on a circle. This results in a fluid charged under a U(1) gauge field. In section 5 we do the KK compactification on an SU(2) group manifold of the Einstein-dissipative fluid system and study the conservation laws of the system. In section

6 we evaluate our energy-momentum tensor and identify all the dissipative coefficients of the $d$-dimensional fluid. In section 7 we explain the main properties of our system and discuss future directions we are currently investigating. Appendices provide the details of our computations.

## 2 Dissipative fluid dynamics with Yang-Mills charge

In this section, we recapitulate the dynamics and the main properties of a $d$-dimensional dissipative fluid that carries charges of non-Abelian Yang-Mills gauge group $G$.

We denote space-time indices by $\mu, \nu, \rho = 1, \ldots, d$, and the adjoint representation of the Yang-Mills group $G$ indices by $\alpha, \beta, \ldots = 1, \ldots, \dim(G)$.[2]

The energy-momentum tensor consists of two contributions by dissipative fluid and non-Abelian Yang-Mills gauge fields:

$$T_{\mu\nu}^{\text{total}} = T_{\mu\nu}^{\text{fluid}} + \frac{1}{2} Q_c^{-2}(x) \left( \mathbf{F}^\alpha{}_\mu{}^\rho \mathbf{F}^\alpha{}_{\rho\nu} - \frac{1}{2} \eta_{\mu\nu} (\mathbf{F}^\gamma)^2 \right), \tag{2.1}$$

where $\mathbf{F}^\alpha{}_{\mu\nu}$ is the non-Abelian field strength of the gauge field $\mathbf{A}^\alpha{}_\mu$, $Q_c$ refers to the coupling constant[3] and repeated color indices are summed over. The fluid energy-momentum tensor is further split to perfect fluid and dissipative parts,

$$T_{\mu\nu}^{\text{fluid}} = T_{\mu\nu}^{\text{perfect}} + T_{\mu\nu}^{\text{diss}}. \tag{2.2}$$

The perfect fluid contribution is given by

$$T_{\mu\nu}^{\text{perfect}}(x) = [\epsilon(x) + p(x)] \, u_\mu u_\nu + p(x) \, \eta_{\mu\nu}, \tag{2.3}$$

where $p$ is pressure, $\epsilon$ is energy density, $u_\mu$ is the velocity field and $\eta_{\mu\nu}$ is Minkowski metric.

As for the dissipative part $T_{\mu\nu}^{\text{diss}}$, in this description we will not choose any specific frame and will consider a generic energy-momentum tensor. Though we will make further explicit assumptions in section 6.4, we can generalize our results to any frame independent prescription.

The thermodynamic relation for the SU(2) charged perfect fluid after the KK compactification accounts for the chemical potentials $\boldsymbol{\mu}_\alpha^{\text{color}}$ associated to the color charges $\mathfrak{Q}_\alpha$,

$$\epsilon + p = \mathcal{T} s + \mathfrak{Q}^\alpha \boldsymbol{\mu}_\alpha^{\text{color}}. \tag{2.4}$$

$\mathcal{T}$ is the temperature and $s$ is the entropy density.

Let us specify the dynamics of the system. The first EOM corresponds to the fluid dynamics evolution. Inspired by the KK compactification of a fluid coupled to gravity (in particular Bianchi identities of Einstein equations, cf. section 5), we obtain the conservation of the total energy-momentum tensor,

$$\nabla^\mu T_{\mu\nu}^{\text{total}} = 0. \tag{2.5}$$

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[2]$\dim(G)$ will correspond to the dimension of the internal group manifold in the KK compactification.

[3]From the KK perspective $Q_c = Q_c(x)$ is a scalar quantity that corresponds to the dilaton-dependent gauge coupling, $Q_c(x) \propto e^{\phi(x)}$. When we set $\phi$ constant-valued, then $Q_c$ plays the role of a Yang-Mills coupling constant and it disappears from the EOMs of the gauge fields.

The second EOM describes the dynamics of the non-Abelian Yang-Mills gauge fields and introduces a non-Abelian colored current.

$$(\mathcal{D}^\nu \mathbf{F}_{\mu\nu})^\alpha(x) = \mathbf{J}_\mu^\alpha(x)\,. \tag{2.6}$$

The quantity $\mathbf{J}_{\mu\alpha}^{\text{color}}(x)$

$$\mathbf{J}_\mu^{\alpha,\text{color}}(x) = Q_c\, \mathfrak{Q}^\alpha(x)\, u_\mu(x) + \mathbf{J}_\mu^{\alpha,\text{diss}}(x)\,, \tag{2.7}$$

allows us to define a covariantly conserved current[4]

$$\mathcal{J}^\alpha{}_\mu = Q_c^2\, \left[\mathbf{J}_\mu^\alpha - \mathcal{D}^\nu\left(Q_c^{-2}\right)\mathbf{F}^\alpha{}_{\mu\nu}\right]\,, \qquad (\mathcal{D}^\nu \mathcal{J}_\nu)^m = 0\,, \tag{2.8}$$

where $\mathfrak{Q}_\alpha(x)$ is the color charge density attached to the fluid.

Although the dissipative part contribution in $\mathbf{J}_{\mu\alpha}^{\text{diss}}(x)$ is frame-dependent, the color current $\mathbf{J}_{\mu\alpha}^{\text{color}}(x)$ is always covariantly conserved independent of the choice of frame. Further details for specific frame choices can be found in section 6.4.

The colored fluid must interact with the Yang-Mills gauge fields through the Lorentz force. In our formulation, for a the fluid characterized by the energy-momentum tensor $T_{\mu\nu}^{\text{fluid}}$, the Lorentz force naturally arises from the conservation of energy-momentum tensor current,

$$\nabla^\nu T_{\nu\mu}^{\text{fluid}}(x) = Q_c\, \mathbf{F}^{\alpha\,\nu}{}_\mu(x)\, \mathbf{J}_{\nu\alpha}^{\text{color}}(x)\,. \tag{2.9}$$

As the expression of energy-momentum tensor current is frame-dependent, departure from the Landau frame does not permit to read the transport coefficients associated with the dissipative effects from $T_{\mu\nu}^{\text{diss}}$. To correctly identify these coefficients, we need a frame-invariant formulation of the dissipative terms which is in agreement with the second law of thermodynamics, $\nabla^\mu \mathfrak{J}^s{}_\mu \geq 0$, where $\mathfrak{J}^s{}_\mu$ refers to the covariant entropy current. We adopt the following generalizations:

$$\left(P_\mu{}^\rho(x)P_\nu{}^\lambda(x) - \frac{1}{d-n}P_{\mu\nu}(x)P^{\rho\lambda}(x)\right)T_{\rho\lambda}^{\text{diss}}(x) = -2\eta(x)\,\sigma_{\mu\nu}(x)\,,$$

$$\frac{\partial p(x)}{\partial \mathfrak{Q}_m}u^\mu(x)\mathbf{J}^{\text{diss}}{}_{\mu m}(x) + \left(\frac{1}{d-n}P^{ab} - \frac{\partial p}{\partial \epsilon}u^\mu u^\nu\right)(x)T_{\mu\nu}^{\text{diss}}(x) = -\zeta(x)\theta(x)\,,$$

$$P_\mu{}^\nu(x)\left(\mathbf{J}^{\text{diss}}{}_{\nu m} + \frac{\mathfrak{Q}_m}{\epsilon+p}u^\rho T_{\rho\nu}^{\text{diss}}\right) - \boldsymbol{\kappa}_{mn}(x)\left(-P_\mu{}^\nu D_\nu\left(\frac{\boldsymbol{\mu}_n}{\mathcal{T}}\right) + \frac{1}{\mathcal{T}}\mathbf{F}^n{}_{\mu\nu}u^\nu\right) = 0\,, \tag{2.10}$$

where $P_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu$ is the projector to the hypersurface orthogonal to the fluid, $\boldsymbol{\kappa}_{mn}$ is the non-Abelian conductivity tensor, and $\theta \equiv \nabla_\mu u^\mu$, $\eta$, $\zeta$, $\sigma_{\mu\nu}$ are various dissipative coefficients. Covariant derivatives are defined in section 5.

This completes the summary of the equations that govern our system. It now remains to establish this set of EOMs and conservation laws. In this paper, we established them by starting from a higher-dimensional neutral fluid and then making a KK dimensional reduction. The idea was that we used the KK compactification as a guiding principle to

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[4]If $Q_c$ plays the role of a coupling constant, then the second term vanishes. We will explore the details of non-constant $Q_c$ and in section 5.

obtain expressions that preserve SU(2) covariance and the conservation laws, which arise upon recasting the higher-dimensional ones.

In the following sections, we explicitly show the calculations that lead to these equations.

## 3 Kaluza-Klein approach

Our goal is to construct non-Abelian hydrodynamics. It consists of two components: the colored matter fluid and the Yang-Mills gauge field. Constructing its hydrodynamics starting from a microscopic Yang-Mills-matter theory (such as QCD) is just a theoretical idea: it is not feasible nor shedding light on physics. As such, we look for a mesoscopic approach. The idea is to utilize the Kaluza-Klein compactification to construct both components of non-Abelian hydrodynamics simultaneously. Our starting point is a self-gravitating, dissipative and neutral fluid in a dynamic $D$-dimensional spacetime $\mathcal{M}_D(\widehat{g}_{MN})$, viz. a dissipative and neutral fluid coupled to the Einstein gravity, all in $D$ dimensions.[5] Our working assumption is that the $D$-dimensional matter is strongly interacting at the outset. While gravity is fundamentally weak, effective strength for the fluid depends on macroscopic conditions such as density and temperature.

### 3.1 Self-gravitating dissipative fluid

We will first characterize strongly interacting dissipative, neutral fluid in curved $D$-dimensional spacetime $\mathcal{M}_D(\widehat{g}_{MN})$. The hydrodynamic field variables of fluid consist of the velocity vector field $\widehat{u}^M(\widehat{x})$ and various other scalar fields. The velocity field is time-like, normalized[6]

$$\widehat{u}^M(\widehat{x})\widehat{u}^N(\widehat{x})\,\widehat{g}_{MN}(\widehat{x}) = -1\,, \tag{3.1}$$

such that it carries $(D-1)$ independent components. On the other hand, the number of independent scalar fields is set by the number of equations of state that we consider. For a perfect fluid, we will consider temperature $\widehat{T}(\widehat{x})$, pressure $\widehat{p}(\widehat{x})$, and energy density $\widehat{\epsilon}(\widehat{x})$ to be independent scalar variables. Likewise, for the dissipative coefficients, we take shear viscosity $\widehat{\eta}$, bulk viscosity $\widehat{\zeta}$, shear tensor $\widehat{\sigma}_{AB}$, and expansion scalar $\widehat{\theta}$ as the independent response variables associated with the $D$-dimensional neutral fluid.

The conservation laws and EOMs of the $D$-dimensional dissipative, neutral fluid follow from the conservation of energy-momentum tensor

$$\widehat{\nabla}^N \widehat{T}_{NM}^{\text{fluid}}(\widehat{x}) = 0. \tag{3.2}$$

In the long-wavelength limit, the energy-momentum tensor is given by a derivative expansion of hydrodynamic fields, which in our case consists of parity-even terms up to the first-order in gradients. It is given by two terms:

$$\widehat{T}_{MN}^{\text{fluid}}(\widehat{x}) = \widehat{T}_{MN}^{\text{perfect}}(\widehat{x}) + \widehat{T}_{MN}^{\text{diss}}(\widehat{x})\,, \tag{3.3}$$

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[5]We denote all $D$-dimensional variables as hatted quantities.

[6]We use the mostly plus signature.

where $\widehat{T}_{MN}^{\text{perfect}}$ is the perfect fluid part and $\widehat{T}_{MN}^{\text{diss}}$ contains the dissipative effects. In this work, we do not assume a priori an equation of state for the fluid, so we treat all hydrodynamic fields as being independent. For later treatment, we find it convenient to use the vielbein formalism. The vielbein $E_M{}^A$ is related to the metric as[7]

$$\widehat{g}_{MN}(\widehat{x}) = E_M{}^A(\widehat{x})E_N{}^B(\widehat{x})\eta_{AB}, \qquad \eta_{AB} = (-+\ldots+). \qquad (3.4)$$

Thus,

$$\widehat{T}_{MN}^{\text{fluid}}(\widehat{x}) = E_M{}^A(\widehat{x})E_N{}^B(\widehat{x})\left(\widehat{T}_{AB}^{\text{perfect}}(\widehat{x}) + \widehat{T}_{AB}^{\text{diss}}(\widehat{x})\right). \qquad (3.5)$$

At zeroth-order in the gradient expansion, the fluid is perfect, so

$$\widehat{T}_{MN}^{\text{perfect}}(\widehat{x}) = [\widehat{\epsilon}(\widehat{x}) + \widehat{p}(\widehat{x})]\widehat{u}_M(\widehat{x})\widehat{u}_N(\widehat{x}) + \widehat{p}(\widehat{x})\widehat{g}_{MN}(\widehat{x}). \qquad (3.6)$$

To study the dissipative part of energy-momentum tensor, it is necessary to specify the hydrodynamic frame. This dependence on the hydrodynamic frame arises as a consequence that the macroscopic variables that characterize the fluid do not have unique microscopic definitions. This permits us to have some freedom to select a convenient frame and therefore redefine them in a simple manner. A convenient choice to fix this arbitrariness utilizes the projection of the dissipative part in the energy-momentum tensor to the hypersurface orthogonal to the velocity vector,

$$\widehat{u}^M \widehat{T}_{MN}^{\text{diss}} = 0. \qquad (3.7)$$

This is referred to as the Landau frame. In this frame, the most general form of dissipative part of energy-momentum tensor is given by

$$\widehat{T}_{AB}^{\text{diss}}(\widehat{x}) = -2\widehat{\eta}(\widehat{x})\,\widehat{\sigma}_{AB}(\widehat{x}) - \widehat{\zeta}(\widehat{x})\,\widehat{P}_{AB}(\widehat{x})\,\widehat{\theta}(\widehat{x}), \qquad (3.8)$$

where $\widehat{\eta}$ is the shear viscosity and $\widehat{\zeta}$ is the bulk viscosity of the $D$-dimensional neutral fluid. We also denote the projection tensor to the hypersurface orthogonal to the velocity vector as $\widehat{P}_{AB}$, the shear tensor as $\widehat{\sigma}_{AB}$, and the expansion scalar as $\widehat{\theta}$. They are defined as follows:

$$\begin{aligned} \widehat{P}_{AB}(\widehat{x}) &= \widehat{\eta}_{AB} + \widehat{u}_A(\widehat{x})\widehat{u}_B(\widehat{x}), \\ \widehat{\sigma}_{AB}(\widehat{x}) &= \widehat{P}_{(A}{}^C(\widehat{x})\widehat{P}_{B)}{}^D(\widehat{x})\widehat{\mathfrak{D}}_C\widehat{u}_D(\widehat{x}) - \frac{1}{\widehat{p}(x)}\widehat{\theta}(\widehat{x})\,\widehat{P}_{AB}(\widehat{x}), \\ \widehat{\theta}(\widehat{x}) &= \widehat{\mathfrak{D}}_A\widehat{u}^A(\widehat{x}), \end{aligned} \qquad (3.9)$$

where $\mathfrak{D}_A = E_A{}^M\partial_M + \widehat{\omega}_A(\widehat{x})$ is the Lorentz covariant derivative, and $\widehat{\omega}_A$ is the spin connection acting on the tangent frame.

We minimally couple this $D$-dimensional neutral, dissipative fluid to the $D$-dimensional gravity, whose metric field is given by $\widehat{g}_{MN}$. The system is described by the $D$-dimensional Einstein field equations sourced by the fluid,

$$\widehat{R}_{MN}(\widehat{x}) - \frac{1}{2}\widehat{g}_{MN}(\widehat{x})\widehat{R}(\widehat{x}) = 8\pi G_D\left[\widehat{T}_{MN}^{\text{perfect}}(\widehat{x}) + \widehat{T}_{MN}^{\text{diss}}(\widehat{x})\right]. \qquad (3.10)$$

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[7]We can straightforwardly incorporate fermionic DOFs, such as a supersymmetric fluid interacting with supergravity, as our scheme utilizes the vielbein formalism approach.

Two remarks are in order. First, it is important to stress that the energy-momentum tensor sourcing the Einstein's equation includes both ideal and dissipative parts. Second, our approach admits straightforward extension to any higher orders in the gradient expansion. This is an interesting program we leave for future development.

Before we dwell into details of computations, in the next section, we overview the main aspects of the KK compactification of this system.

## 3.2  Non-Abelian Kaluza-Klein reduction

Our goal is to construct self-consistent non-Abelian hydrodynamics using the approach of the dimensional reduction in KK theory. In this section, we sketch the main aspects of the KK compactification approach and the guidelines of our developments.

We start with the $D$-dimensional Einstein-neutral fluid system given by eq. (3.10) and dimensionally reduce it on $n$-dimensional compact space $X_n$. We can effectively split the gravitational DOFs in $D$ dimensions into gravitational and additional DOFs in the $d = (D-n)$-dimensional reduced spacetime. The additional DOFs are scalar fields that characterize the size and shape of $X_n$ and, if the manifold admits Killing symmetries, vector fields with gauge symmetries. Likewise, we can split the fluid energy-momentum tensor in $D$ dimension into fluid's energy-momentum tensor and some vector currents in $d$-dimensional, reduced spacetime. Depending on the properties of Killing vectors on $X_n$, these vector currents can be either Abelian or non-Abelian. In this treatment, one must only keep a consistent truncation of light modes, setting the massive modes to zero. Consistency requires that heavy modes that are dropped are not sourced by the light modes one keeps.

Note that we are performing the KK dimensional reduction for both the gravity in the left-hand side of (3.10) and the fluid in the right-hand side. As for the gravity, it is known that the KK dimensional reductions that involve Abelian isometries are always guaranteed to be consistent, as the heavy and light modes do not mix each other. It is also known that, for some internal spaces (maximally symmetric spaces and group manifolds), dimensional reductions that involve non-Abelian isometries are consistent as well. As for the matter, KK compactification of a fluid without gravity (and hence, without dynamical gauge fields coupled to the fluid) on $n$-dimensional torus $X_n = \mathbb{T}^n$ is straightforward, as was recently studied in [37]. The reduction leads to a fluid carrying $U(1)^n$ "global" charges, and to relations between $D$-dimensional heat transport coefficients and $d$-dimensional, reduced charge transport coefficients. The results are in agreement with results known independently, so it suggests that the KK reduction that involves Abelian isometries is consistent for the fluid as well.

Consider next the KK dimensional reduction of Einstein-fluid system on a group manifold $X_n = G$ [35] of dimension $n = \dim(G)$ and of curvature scale $R$. The group manifold $G$ is describable in terms of the Maurer-Cartan one-forms $\sigma^m$. These one-forms are invariant under left multiplications by a group element $g \in G$. Thus, this left multiplication is an isometry of metric $g(G)$ of the $n$-dimensional internal space. So, in $d$-dimensional reduced spacetime, the gauge symmetries include the diffeomorphisms of spacetime and the massless fields of the $d$-dimensional, system will be the metric $g_{\mu\nu}$ and the non-Abelian Yang-Mills gauge fields with gauge group $G$. Likewise, in $d$-dimensional reduced spacetime, the neu-

tral fluid we started with becomes a fluid carrying $G$ 'global' charges. The Einstein-fluid equation then gauges this global charges to $G$ 'color' charges so that the fluid is minimally coupled to the non-Abelian gauge field. This is the main reason why we reduce the higher-dimensional Einstein-fluid system on group manifolds: the reduction naturally lead to 'color' charges and couples the $G$-colored fluid to dynamical $G$-color Yang-Mills fields. The reduction will translate the $D$-dimensional conservation laws into the $d$-dimensional, reduced conservation of both energy-momentum tensor and non-Abelian vector currents.

From the KK compactification, we obtain the system of colored fluid interacting with Yang-Mills theory. Nevertheless, the reduction also will bring in additional DOFs. Depending on the physical situations we are interested in, one may keep them as part of the system or truncate them out. For the formulation of non-Abelian hydrodynamics, we will only keep the non-Abelian gauge field dynamics but none others such as the gravitational dynamics. That is, we will decouple the gravitational DOFs and consider non-Abelian hydrodynamics on $d$-dimensional Ricci-flat spacetime. Such decoupling can be achieved if, for instance, one takes in $D$ dimensions nontrivial cosmological constant and $n$-form field strength and the Freund-Rubin ansatz. With fine-tuning of the cosmological constant and taking $G_D$ to zero while keeping $R^{n+2}/G_D$ held fixed, one can decouple the gravity while keeping nontrivial Yang-Mills gauge dynamics in Ricci-flat $d$-dimensional spacetime. We will also need to truncate the dilaton (that parametrizes the volume of $G$) and other scalar fields that emerge by setting them to be constant-valued. Varying them, however, would result in change of the $d$-dimensional equations of state.

Let us stress that the above approach we propose relies on neither kinetic theory nor Lagrangian formulations. In this regard, our approach offers an ab initio derivation of the non-Abelian hydrodynamics modulo well-motivated assumption that a neutral fluid coupled to Einstein field equations is self-consistent in $D$ dimensions.

Finally, let us comment on a technical caveat related to the Yang-Mills gauge group. In our approach, the KK dimensional reduction is done on the EOMs. This bears some consequences in the possible choices of the group manifold $X_n = G$. In particular, dimensional reduction of the EOMs allows for gauge groups whose structure constants are traceful, i.e., $f_{mn}{}^n \neq 0$ (cf. [38]).

## 4 Charged fluid coupled to Maxwell theory

As a step to introduce the technicalities that KK theory requires and build intuitions therein, we first consider the KK reduction of Einstein-fluid system on a group manifold with Abelian isometries. Thus, we choose the internal manifold to be a $n$-torus, $X_n = \mathrm{U}(1)^n$. For simplicity, we will take the internal manifold isotropic, $R_1 = R_2 = \cdots = R_n = R$, and we will restrict ourselves to a perfect fluid, leaving incorporation of the dissipative effects to next section.

Consider the KK reduction of a perfect fluid given by eq. (3.10) on a $\mathbb{S}^1$ internal circle of radius $R$, where $\widehat{T}_{MN}^{\text{fluid}} = \widehat{T}_{MN}^{\text{perfect}}$. We will show that the KK reduction gives rise to a charged perfect fluid interacting with Maxwell electromagnetism.

## 4.1 Reduction on Abelian group manifold

For the KK reduction on a circle, let us assume the following ansatz for the vielbein $E_M{}^A(\widehat{x})$ in eq. (3.4) as

$$E_M{}^A(\widehat{x}) = \begin{pmatrix} e^{\alpha\phi(x)}e_\mu{}^a(x) & e^{\beta\phi(x)}A_\mu(x) \\ 0 & e^{\beta\phi(x)} \end{pmatrix}. \tag{4.1}$$

Curved indices of the $D$-dimensional spacetime will be split as $M = \{\mu, z\}$ whereas we will denote flat indices as $A = \{a, \underline{z}\}$. We will also assume that all the fields that appear in the ansatz only depend on the $d$-dimensional coordinates $x^\mu$ of $\mathcal{M}_d$.[8] The dilaton $\phi(x)$, which measures the size of $X_n$, is weighed by the reduction-specific coefficients

$$\alpha^2 = \frac{n}{2(d+n-2)(d-2)} \qquad \text{and} \qquad \beta = -\frac{(d-2)\alpha}{n}. \tag{4.2}$$

Though in this section we evaluate $n = 1$, we will keep $n$ generic.

Let us start by substituting the compactification ansatz into the $D = (d+1)$-dimensional Einstein field equations

$$\widehat{G}_{MN}(x) \equiv \widehat{R}_{MN}(x) - \frac{1}{2}\widehat{g}_{MN}(x)\widehat{R}(x) = \widehat{T}_{MN}^{\text{fluid}}(x), \tag{4.3}$$

and recast the differential equations. The components $\widehat{G}_{\mu\nu}$, $\widehat{G}_{\mu z}$, and $\widehat{G}_{zz}$ give the $d$-dimensional gravitational, gauge, and dilaton field equations, respectively. Though we do not specify the structure of fluid energy-momentum tensor $\widehat{T}_{MN}^{\text{fluid}}$, we will return to it after analyzing the component equations.

The $\widehat{G}_{\mu z}$ components imply the Maxwell equations coupled to a current.

$$\nabla^\nu \left(Q_e^{-2}(x)F_{\mu\nu}\right) = Q_e^{-2}(x)J_\mu^{\text{e}}(x), \tag{4.4}$$

where $Q_e$ is the dilaton-dependent gauge coupling,

$$Q_e(x) \equiv e^{(d-1)\alpha\,\phi(x)}, \tag{4.5}$$

and the current is given by

$$J_\mu^{\text{e}}(x) \equiv 2e^{(\beta+2\alpha)\phi(x)}Q_e(x)\,e_\mu{}^a(x)\widehat{T}_{a\underline{z}}^{\text{fluid}}(x). \tag{4.6}$$

Hence, the $\widehat{G}_{\mu z}$ components of Einstein equations automatically define the electromagnetic dynamics of the system, including the current $J_\mu^{\text{e}}(x)$ of the fluid. Thus, the fluid becomes charged whenever it has non-vanishing flow around $\mathbb{S}^1$. Being proportional to $T_{a\underline{z}}^{\text{fluid}}(x)$, the electric current $J_\mu^{\text{e}}(x)$ will be proportional to the reduced velocity field $u_\mu(x)$. The dilaton field that measures the size of $\mathbb{S}^1$ has the effect of spacetime-dependent unit of electric charge, $Q_e(x)$. As discussed in the previous section, we take the KK reduction as an ab initio approach for deriving consistent hydrodynamic equations. As such, we will eventually set the dilaton to be constant-valued.

---

[8]In section 5, we will assume some dependence on the internal coordinates, which will yield to non-Abelian gauge symmetry upon reduction.

This same pattern to the other components of (3.10). From the $\widehat{G}_{\mu\nu}$ components, we obtain the $d$-dimensional Einstein equations sourced by the charged fluid, the U(1) gauge field and the dilaton:

$$G_{\mu\nu}(x) \equiv R_{\mu\nu}(x) - \frac{1}{2}g_{\mu\nu}(x)R(x) = T_{\mu\nu}^{\mathrm{total}}(x)\,, \tag{4.7}$$

where the right-hand side defines the total energy-momentum tensor of the $d$-dimensional system

$$T_{\mu\nu}^{\mathrm{total}}(x) = T_{\mu\nu}^{\mathrm{fluid}}(x) + \left(\frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{4}g_{\mu\nu}(\partial\phi)^2\right)(x) + \frac{1}{2}Q_e^{-2}(x)\left(F_{\mu\nu}^2 - \frac{1}{4}F^2 g_{\mu\nu}\right)(x)\,. \tag{4.8}$$

The last two terms are contributions of dilaton field and Maxwell field, while the first term is the energy-momentum tensor of charged fluid, defined by

$$T_{\mu\nu}^{\mathrm{fluid}}(x) \equiv e^{2\alpha\phi(x)}e_\mu{}^a(x)e_\nu{}^b(x)\widehat{T}_{ab}^{\mathrm{fluid}}(x)\,. \tag{4.9}$$

Finally, let us consider the $\widehat{G}_{zz}$ component. We obtain the $d$-dimensional dilation field equation, sourced by both the fluid and the Maxwell gauge field,

$$\Box\phi(x) = 2\alpha\,D(x)\,, \tag{4.10}$$

Again, the right-hand side of the equation defines the dilatation current,

$$D(x) \equiv -\frac{d-1}{4}Q_e^{-2}(x)F_{\mu\nu}^2(x) + (d-1)\,e^{2\alpha\phi(x)}\,\widehat{T}_{\underline{zz}}^{\mathrm{fluid}}(x) - e^{2\alpha\phi(x)}\,\widehat{T}^{\mathrm{fluid}}(x)\,. \tag{4.11}$$

where $\widehat{T}^{\mathrm{fluid}} := \widehat{T}^{\mathrm{fluid}}{}_M{}^M$ is the trace of the $D = (d+1)$-dimensional energy-momentum tensor.

The Einstein tensor in the defining equation eq. (4.3) obeys the Bianchi identity, from which conservation laws of various currents we identified above are derived. The conservation laws on current $J_\mu^{\mathrm{el}}$ and total energy-momentum tensor $T_{\mu\nu}^{\mathrm{total}}$ result relevant for the Maxwell-plasma system. For the charge current, covariant divergence of eq. (4.4) gives

$$\nabla^\mu\left[\nabla^\nu\left(Q_e^{-2}F_{\mu\nu}\right)\right] = \nabla^{[\mu}\nabla^{\nu]}\left(Q_e^{-2}(x)\right)F_{\mu\nu} = 0\,, \tag{4.12}$$

where we have used the torsion-free condition for $d$-dimensional spacetime. This implies

$$\nabla^\mu(e^{(\beta-(d-3)\alpha)\phi}e_\mu{}^a\widehat{T}_{a\underline{z}})(x) = 0 \qquad \to \qquad \nabla^\mu(Q_e(x)J_\mu^{\mathrm{e}}(x)) = 0\,, \tag{4.13}$$

which results the to the conservation law of electric current $J_\mu^{\mathrm{e}}$, generalized by the dilaton field.

From the Bianchi identity of Einstein tensor in eq. (4.7) we obtain

$$\nabla^\mu T_{\mu\nu}^{\mathrm{total}}(x) = 0\,. \tag{4.14}$$

This implies that variations in the fluid energy-momentum tensor are balanced by the change of the Maxwell energy-momentum tensor and the dilaton.

$$\nabla^\mu T_{\mu\nu}^{\mathrm{fluid}}(x) = -\nabla^\mu\left(Q_e^{-2}\left[F_{\mu\nu}^2 - \frac{1}{4}F^2 g_{\mu\nu}\right]\right)(x)\,. \tag{4.15}$$

On-shell, this conservation is equivalent to

$$\nabla^\mu T^{\text{fluid}}_{\mu\nu} + (e^{[\beta-(d-3)\alpha]\phi} e_\mu{}^a \widehat{T}^{\text{fluid}}_{a\underline{z}}) F_\nu{}^\mu + e^{2\alpha\phi}[(d-1)\alpha \widehat{T}^{\text{fluid}}_{\underline{z}\underline{z}} - \alpha\widehat{T}^{\text{fluid}}]\nabla_\nu\phi = 0\,. \quad (4.16)$$

We interpret this as the generalization of the Lorentz force equation of Maxwell-plasma under the presence of the dilaton field. Once again, the role of the KK approach is just a tool to facilitate the ab initio derivation of charged fluid interacting with Maxwell theory. Therefore, setting the dilaton to be constant-valued we obtain the standard form of the Lorentz force equation:

$$\nabla^\mu T^{\text{fluid}}_{\mu\nu}(x) = Q_e(x) F^\mu{}_\nu(x) J^{\text{el}}_\mu(x)\,. \quad (4.17)$$

## 4.2 Abelian reduction of energy-momentum tensor

So far, we have not made any assumption on the energy-momentum tensor $\widehat{T}^{\text{fluid}}_{MN}$ of the neutral fluid we started from. We now study $\widehat{T}^{\text{fluid}}_{MN}$ under a well-motivated ansatz for the higher-dimensional velocity field $\widehat{u}(\widehat{x})$ and the other scalar quantities. To gain better intuition about physics, we will restrict the $D$-dimensional neutral fluid to a perfect fluid. In section 5, we will consider the dissipative contributions.

The $D$-dimensional velocity field $\widehat{u}^M$ has $(D-1)$ independent components, as it is conveniently normalized by eq. (3.1):

$$\widehat{u}^M(\widehat{x})\widehat{u}^N(\widehat{x})\widehat{g}_{MN}(\widehat{x}) = -1\,. \quad (4.18)$$

The ansatz that we will assume for the velocity field is:

$$\begin{aligned}
\widehat{u}_a &= u_a(x)\cosh\varphi(x)\,, \\
\widehat{u}_{\underline{z}} &= \sinh\varphi(x)\,,
\end{aligned} \quad (4.19)$$

where $u_a(x)$ is the velocity field of charged fluid in $d$ dimensions, which is normalized as $u^a(x)u^b(x)\eta_{ab} = -1$. The scalar field $\varphi(x)$ parametrizes the degree of freedom associated with the internal component of the velocity, $\widehat{u}_{\underline{z}}$. Substituting the ansatze for the vielbein eq. (4.1) and the velocity fields eq. (4.19) into the energy-momentum tensor, we will obtain the defining variables of the $d$-dimensional fluid in terms of the $D$-dimensional ones. That is to say, we find that the energy-momentum tensor in $d$ dimensions is

$$\begin{aligned}
T^{\text{perfect}}_{\mu\nu}(x) &= e_\mu{}^a(x)e_\nu{}^b(x)\widehat{T}_{ab}(x) \\
&= (\epsilon+p)\,u_\mu(x)u_\nu(x) + p\,g_{\mu\nu}(x)\,,
\end{aligned} \quad (4.20)$$

where the energy density $\epsilon(x)$ and the pressure $p(x)$ are given by

$$\epsilon(x) = e^{2\alpha\phi(x)}\left(\widehat{\epsilon}(x)\cosh^2\varphi(x) + \widehat{p}(x)\sinh^2\varphi(x)\right)\,, \qquad p(x) = e^{2\alpha\phi(x)}\widehat{p}(x)\,. \quad (4.21)$$

By substituting the velocity ansatz eq. (4.19) into eq. (4.6) we obtain the electric current

$$J^{\text{e}}_\mu(x) \equiv 2\left(\epsilon(x) + p(x)\right)e^{[\beta+(d+1)\alpha]\phi(x)}\tanh\varphi(x)\,u_\mu(x)\,. \quad (4.22)$$

As anticipated, the charge current is proportional to the velocity field $u_\mu$. Again, let us analyze the case for which the dilaton field is constant. Then, the energy-momentum conservation, eq. (4.20) leads to

$$\nabla^\mu T_{\mu\nu}^{\text{perfect}}(x) = Q_e(x) F^\mu{}_\nu(x) J_\mu^{\text{e}}(x) \,. \tag{4.23}$$

This is precisely the Lorentz force equation we have directly derived from the reduction of the Einstein-fluid system in the last section.

One can straightforwardly generalize the above construction by taking the internal space $X_n$ to be an $n$-torus $\mathbb{T}^n$. It will give rise to a fluid charged under $n$ independent Abelian electromagnetic fields with $\text{U}(1)^n$ gauge symmetry.

After analyzing the system of a fluid charged under Abelian gauge fields, we will address the case for which the gauge symmetry is non-Abelian. To carry out this problem, the internal manifold will be a group manifold whose isometry group is non-Abelian. We will choose $\text{SU}(2)$ for simplicity but the procedure applies to any other gauge group.

## 5 Colored fluid coupled to Yang-Mills theory

We now construct non-Abelian hydrodynamics of Yang-Mills plasma. Here, our goal is to derive ab initio the EOMs of a dissipative fluid carrying non-Abelian $\text{SU}(2)$ charges and interacting with Yang-Mills theory. To do so, our idea is again to start with an Einstein-fluid system in $D$ dimensions eq. (3.10) and perform a KK dimensional reduction on a $\text{SU}(2)$ group manifold [35] (for a review, cf. [38–40]). After the reduction, we will find an $\text{SU}(2)$ colored fluid interacting with $\text{SU}(2)$ Yang-Mills theory in $d$ dimensions. As $\text{SU}(2)$ group manifold is three-dimensional, our setup corresponds to $n = 3$ and hence $D = d + 3$. Nevertheless, this method can be applied to any group manifold $G$, having thus a colored fluid interacting with Yang-Mills theory of gauge group $G$.

### 5.1 Compactification on $\text{SU}(2)$ group manifold

Let us consider the following KK ansatz for the $D$-dimensional vielbein:

$$E_M{}^A = \begin{pmatrix} e^{\alpha\phi} \widehat{e}_\mu{}^a & e^{\beta\phi} \widehat{A}_\mu{}^p \widehat{e}_p{}^\alpha \\ 0 & g^{-1} e^{\beta\phi} \widehat{e}_m{}^\alpha \end{pmatrix} \,, \tag{5.1}$$

where

$$\begin{aligned}
\widehat{e}_\mu{}^a(\widehat{x}) &= e_\mu{}^a(x) \,, \\
\widehat{e}_m{}^\alpha(\widehat{x}) &= \mathfrak{u}_m{}^n(y) \mathcal{V}_n{}^\alpha(x) \,, \\
\widehat{A}_\mu{}^m(\widehat{x}) &= (\mathfrak{u}^{-1})_n{}^m(y) A_\mu{}^n(x) \,.
\end{aligned} \tag{5.2}$$

and $g$ is a coupling constant $g$. As in the Abelian reduction, we split the curved manifold indices as $M = (\mu, m)$ where $\mu = 1 \ldots d$ and $m = 1 \ldots n$, tangent space indices as $A = \{a, \alpha\}$, where $a = 1 \ldots d$ and $\alpha = 1 \ldots n$, and local coordinates as $\widehat{x}^M = (x^\mu, y^m)$. For the $\text{SU}(2)$ case, $n = 3$. These algebra-valued indices can be freely lowered and raised without loss of generality.

In various Weyl factors, the dilaton field $\phi(x)$ is weighed with the coefficients

$$\alpha^2 = \frac{n}{2(d+n-2)(d-2)} \qquad \text{and} \qquad \beta = -\frac{(d-2)\alpha}{n} \, . \tag{5.3}$$

The matrix $\mathfrak{u}_m{}^n(y)$ in eq. (5.2) is a twist field that carries the information of the SU(2) group manifold. After the reduction, this information is encoded in the $d$-dimensional system through the structure constants,

$$f_{mn}{}^p := -2(\mathfrak{u}^{-1})_m{}^s(y)(\mathfrak{u}^{-1})_n{}^t(y)\partial_{[s}\mathfrak{u}_{t]}{}^p(y) \, . \tag{5.4}$$

Though the twist matrix field $\mathfrak{u}_m{}^n(y)$ varies over the group manifold (hence depends on the internal coordinates $y$), the combination on the r.h.s. of this equation needs to be constant-valued in order for them to be the structure constants of the Lie algebra associated with the group manifold.

The ansatz can be explicitly expressed in terms of the Maurer-Cartan one-forms $\sigma^m$ of the SU(2) group manifold by combining the fields as

$$\widehat{E}^a(x) = e^{\alpha\phi(x)}e^a(x) \, ,$$
$$\widehat{E}^\alpha(x) = g^{-1}e^{\beta\phi(x)}\mathcal{V}_m{}^\alpha(x)(\boldsymbol{\sigma}^m - g\mathbf{A}^m(x)) \, , \tag{5.5}$$

where $\boldsymbol{\sigma}^m \equiv u_n{}^m \mathrm{d}x^n$ is the left-invariant one-form of $G$, satisfying the Maurer-Cartan equation

$$\mathrm{d}\boldsymbol{\sigma}^m + \frac{1}{2}f_{np}{}^m\boldsymbol{\sigma}^n \wedge \boldsymbol{\sigma}^p = 0 \, , \tag{5.6}$$

and thus $f_{np}{}^m$ are the structure constants of the isometry group $G$ of the internal manifold.

Before carrying out the non-Abelian reduction on the group manifold $G$, we introduce new notations for the physical variables in $d$ dimensions. We shall build from the scalar vielbein $\mathcal{V}$ two scalar metrics

$$\mathbf{M}_{mn} = \mathcal{V}_m{}^\alpha\mathcal{V}_n{}^\beta\delta_{\alpha\beta} \qquad \text{and} \qquad \mathbb{M}^{\alpha\beta} \equiv \mathcal{V}_m{}^\alpha\mathcal{V}_n{}^\beta\delta^{mn}, \tag{5.7}$$

which are SU(2) invariant and SU(2) covariant, respectively. We denote the trace as $\mathbb{M} \equiv \mathbb{M}_\alpha{}^\alpha$. We define the covariant derivatives $D_\mu(\mathbf{A})$ and $\mathcal{D}_\mu(\mathcal{V})$ as

$$D_\mu\mathcal{V}_m{}^\alpha \equiv \partial_\mu\mathcal{V}_m{}^\alpha - g\mathbf{A}_{\mu m}{}^n\mathcal{V}_n{}^\alpha \, ,$$
$$\mathcal{D}_\mu\mathcal{V}_m{}^\alpha \equiv D_\mu\mathcal{V}_m{}^\alpha + \mathbb{Q}_\mu{}^\alpha{}_\beta\mathcal{V}_m{}^\beta \, , \tag{5.8}$$

where the elementary gauge field used in $D_\mu$ is given by

$$\mathbf{A}_{\mu m}{}^n(x) \equiv A^p{}_\mu(x)f_{pm}{}^n \tag{5.9}$$

and the composite gauge fields used in $\mathcal{D}_\mu$ are built from the scalar vielbein as

$$\mathbb{P}_{a\,\alpha\beta}(x) \equiv e_a{}^\mu\mathbb{P}_{\mu\,\alpha\beta} = \frac{1}{2}\left(\mathcal{V}_\alpha{}^m D_a(\mathbf{A})\mathcal{V}_m{}^\beta + \mathcal{V}_\beta{}^m D_a(\mathbf{A})\mathcal{V}_m{}^\alpha\right) \, ,$$
$$\mathbb{Q}_{a\,\alpha\beta}(x) \equiv e_a{}^\mu\mathbb{Q}_{\mu\,\alpha\beta} = \frac{1}{2}\left(\mathcal{V}_\alpha{}^m D_a(\mathbf{A})\mathcal{V}_m{}^\beta - \mathcal{V}_\beta{}^m D_a(\mathbf{A})\mathcal{V}_m{}^\alpha\right) \, . \tag{5.10}$$

The distinction is that, while $D_\mu$ is the ordinary gauge covariant derivative, $\mathcal{D}_\mu$ accounts for quantities that are adjoined by the scalar vielbein $\mathcal{V}_m{}^\alpha$. Finally, the Yang-Mills field strength two-form $\mathbf{F}^m$ of $\mathbf{A}^m$ is defined as

$$\mathbf{F}^m \equiv d\mathbf{A}^m + \frac{1}{2} g f_{np}{}^m \mathbf{A}^n \wedge \mathbf{A}^p. \tag{5.11}$$

This field strength typically appears dressed up by the scalar fields, so we also denote the tangent space (both in internal and spacetime manifolds) field strength two-form as $\mathbf{F}^\alpha{}_{ab} \equiv \mathcal{V}_m{}^\alpha \mathbf{F}^m{}_{ab}$.

## 5.2 Field equations for Yang-Mills plasma

To obtain the EOMs of the $d$-dimensional system we will substitute the ansatz eq. (5.1) into Einstein equations and recast the resulting expressions.[9]

Let us start with the EOMs for the SU(2) gauge fields. They descend from the $\widehat{G}_{\mu n}$ components in eq. (3.10). Working in the tangent space we obtain

$$\mathcal{D}^b(Q_c^{-2}(x)\mathbf{F}^\beta{}_{ab}(x)) + Q_c^{-2}(x)\mathbb{P}^{b\beta}{}_\gamma(x)\,\mathbf{F}^\gamma{}_{ab}(x) = Q_d^{-2}(x)\mathbf{J}_a^\beta(x), \tag{5.12}$$

where

$$Q_c(x) := e^{\frac{1}{3}\alpha(d+1)\phi(x)} \tag{5.13}$$

is the dilaton-dependent gauge coupling, and

$$\mathbf{J}_{a\beta}(x) = 2\left[g\,Q_c^2(x)\,\epsilon_{\beta\gamma\delta}\mathbb{P}_a{}^\gamma{}_\lambda(x)\,\mathbb{M}^{\delta\lambda}(x) - Q_c(x)\left(e^{2\alpha\phi(x)}\widehat{T}_{a\beta}^{\text{fluid}}(x)\right)\right]. \tag{5.14}$$

is the color current. For covariantly constant scalars, eq. (5.12) is reduced to

$$\mathcal{D}^b(\mathbf{F}^\beta{}_{ab})(x) = \mathbf{J}_a^\beta(x), \tag{5.15}$$

which is the standard form of the Yang-Mills field equations coupled to color current.

The Einstein field equations descend from the $\widehat{G}_{\mu\nu}$ components:

$$G_{\mu\nu}(x) = R_{\mu\nu}(x) - \frac{1}{2}g_{\mu\nu}(x)R(x) = T_{\mu\nu}^{\text{total}}(x), \tag{5.16}$$

where $T_{\mu\nu}^{\text{total}} := e_\mu{}^a e_\nu{}^b T_{ab}^{\text{total}}$ is the total energy-momentum tensor, with

$$T_{ab}^{\text{total}}(x) = e^{2\alpha\phi(x)}\widehat{T}_{ab}^{\text{fluid}}(x) + \frac{1}{2}Q_c^{-2}(x)\left(\eta^{cd}\mathbf{F}^\alpha{}_{ac}(x)\mathbf{F}^\alpha{}_{bd}(x) - \frac{1}{2}\eta_{ab}(\mathbf{F}^\gamma)^2(x)\right)$$
$$+ \frac{1}{2}\left(\partial_a\phi(x)\partial_b\phi(x) - \frac{1}{2}\eta_{ab}(\partial\phi)^2(x)\right) + \left(\mathbb{P}_{a\beta\gamma}(x)\mathbb{P}_{b\beta\gamma}(x) - \frac{1}{2}\mathbb{P}^2(x)\eta_{ab}\right)$$
$$- g^2 Q_c^2(x)\left(\mathbb{M}^{\gamma\delta}(x)\mathbb{M}^{\gamma\delta}(x) - \frac{1}{2}\mathbb{M}^2(x)\right)\eta_{ab}. \tag{5.17}$$

From the first line, we read off the energy-momentum tensor $T_{ab}^{\text{fluid}}$ of the colored fluid:

$$T_{ab}^{\text{fluid}}(x) = e^{2\alpha\phi(x)}\widehat{T}_{ab}^{\text{fluid}}(x). \tag{5.18}$$

---

[9]Details of the calculations for extracting the equations of motion are relegated to appendix A.

Other field equations also yield relevant information on currents and their conservation laws. The equation of motion for dilaton field is obtained from the trace of eq. (3.10), $\widehat{G}_m{}^m$:

$$\Box\phi(x) \;=\; \frac{1}{2\alpha}D(x)\,, \tag{5.19}$$

where $D(x)$ is the dilation current

$$
\begin{aligned}
D(x) = &-\frac{1}{2(d-2)}Q_c^{-2}(x)(\mathbf{F}^\alpha)^2(x) + \frac{2g^2}{d-2}Q_c^2(x)\left(\mathbb{M}^{\alpha\beta}(x)\mathbb{M}^{\alpha\beta}(x) - \frac{1}{2}\mathbb{M}^2(x)\right) \\
&-\frac{2}{d-2}e^{2\alpha\phi(x)}\left(\frac{3}{d+1}\widehat{T}^{\text{fluid}}(x) - \widehat{T}^{\text{fluid}}_{\alpha\beta}(x)\delta^{\alpha\beta}\right).
\end{aligned} \tag{5.20}
$$

The first line is the contribution of SU(2) gauge fields and scalar fields, whereas the second one is the contribution of colored fluid. As we can check, there is no non-linear contribution of the dilaton field itself apart from the Weyl factors.

The equation of motion for the algebra-valued scalar fields $\mathcal{V}_m{}^\alpha(x)$ is given by a linear combination of the $\widehat{G}_{mn}$ components and the trace $\widehat{G}_m{}^m$:

$$\mathcal{D}^a(\mathbb{Q})\mathbb{P}_{a\alpha\beta} = \mathbb{J}_{\alpha\beta}\,, \tag{5.21}$$

where

$$
\begin{aligned}
\mathbb{J}^{\alpha\beta}(x) = &\frac{1}{2}Q_c^{-2}(x)\left[\mathbf{F}^\alpha{}_{ab}(x)\mathbf{F}^\beta{}_{cd}(x)\eta^{ac}\eta^{bd} - \frac{1}{3}(\mathbf{F}^\gamma)^2\delta^{\alpha\beta}\right] + 2e^{2\alpha\phi}\left[\frac{1}{3}\widehat{T}^{\text{fluid}}\delta_{\alpha\beta} - \widehat{T}^{\text{fluid}}_{\alpha\beta}\right] \\
&+4g^2\,Q_c^2(x)\left[\mathbb{M}^{\alpha\gamma}\mathbb{M}^{\beta\gamma} - \frac{1}{2}\mathbb{M}^{\alpha\beta}\mathbb{M} - \frac{1}{3}\left(\mathbb{M}^{\gamma\delta}\mathbb{M}^{\gamma\delta} - \frac{1}{2}\mathbb{M}^2\right)\delta_{\alpha\beta}\right].
\end{aligned} \tag{5.22}
$$

The first line of this expression is the contribution of SU(2) gauge fields and colored fluid, while the last line corresponds to the contribution of algebra-valued scalar fields.

## 5.3 Conservation laws

The non-Abelian reduction of the Einstein-fluid system has led to a Yang-Mills plasma, consisting of colored fluid interacting with non-Abelian gauge fields (and also coupled to gravity, dilaton and algebra-valued scalar fields). In this section, we will further investigate the conservation laws of the system.

Likewise in section 4 for Maxwell plasma, we have not made any assumption on gravity and scalar fields so far. Nevertheless, in order to study the conservation of the simplest model for Yang-Mills plasma, we will truncate the system so that the $d$-dimensional metric is flat and scalar fields are covariantly constant. Such truncations will impose some constraints on the corresponding field equations of $\phi$ and $\mathcal{V}_m{}^\alpha$, namely, eqs. (5.20) and (5.22). For this truncation to be consistent, we would need to solve these constraints. They will in turn impose some conditions on the $d$-dimensional Einstein equations[10] and the Yang-Mills field equations through Weyl factors and scalar potentials. In this section, we will

---

[10]As for gravity, we can decouple the DOFs associated to the metric by taking the limit $G_D \to 0$ for the Einstein equations.

simply consider the simplest consistent solution of these scalar fields, but will not explore the arena of possible non-trivial solutions. Nevertheless, it should be interesting to look into the implications of such nontrivial solutions (and their stability) in the context of fluid/gravity duality. It will also be important to understand to what extent these solutions constrain the values of the transport coefficients and other quantities that characterize the lower-dimensional fluid.

Firstly, let us analyze the color currents of the system and their conservation laws. The SU(2) Yang-Mills field equation eq. (5.12) can be recast:

$$D^b \left( Q_c^{-2} \mathbb{M}_{mn} \mathbf{F}^n{}_{ab} \right) = \mathbf{J}_{ma} \,, \tag{5.23}$$

where

$$\mathbf{J}_{ma} = 2g\epsilon_{\sigma\gamma\delta} \mathbb{M}^{\delta\lambda} \mathbb{P}_{a\gamma\lambda} \mathcal{V}_m{}^\sigma - 2e^{-\frac{1}{3}\alpha(d-5)\phi} \widehat{T}^{\text{fluid}}_{a\gamma} \mathcal{V}_m{}^\gamma \,. \tag{5.24}$$

This allows to define a current $\mathcal{J}^m$ which is covariantly conserved, $D^\mu \mathcal{J}^m{}_\mu = 0$. Its expression is given by eq. (5.14) (see appendices for calculation):

$$\mathcal{J}^m{}_a = Q_c^2 \, \mathbb{M}^{mn} \left[ \mathbf{J}_{na} - D^b \left( Q_c^{-2} \mathbb{M}_{np} \right) F^p{}_{ab} \right] \,. \tag{5.25}$$

The interpretation is clear: the first term is the color current sourced by the algebra-valued scalar fields, while the second term is the color current sourced by the colored fluid itself. Being the non-Abelian counterpart of the U(1) charged current, the second term is proportional to the off-diagonal block of the energy-momentum tensor, $\widehat{T}^{\text{fluid}}_{a\beta}$. This block is non-zero if the $D$-dimensional fluid flows on the group manifold, so $\mathbf{J}^{\text{color}}_{ma}$ is proportional to the internal velocity fields $u_a$.

Secondly, let us analyze the heat current of the Yang-Mills plasma and their conservation laws. We already discussed that the Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$ of the $d$-dimensional Einstein equation, eq. (5.16) leads to the conservation of the total energy-momentum tensor

$$\nabla^\mu T^{\text{total}}_{\mu\nu} = 0 \,. \tag{5.26}$$

We would like to obtain the relations that this condition imposes among the $d$-dimensional degrees of freedom. Applying a covariant divergence on the total energy-momentum tensor eq. (5.17) and substituting the field equations of the Yang-Mills fields and scalar fields, we are left with an expression that involves first derivatives of the scalar fields and components of the energy-momentum tensor $\widehat{T}^{\text{fluid}}_{MN}$.[11] This expression is the non-Abelian generalization of the Lorentz force, which involves not only the Yang-Mills field strength but also the algebra-valued scalar fields. Nevertheless, if we set these scalar fields to be covariantly constant, $D_a \mathcal{V}_m{}^\alpha = D_a \varphi = 0$, we obtain

$$D^a T^{\text{fluid}}_{ab} + 2Q_c^{-1}(x)e^{2\alpha\phi(x)} \widehat{T}_{c\alpha} \mathcal{V}_n{}^\alpha \mathbf{F}^n{}_{bc} = e^{2\alpha\phi} \left( D^a \widehat{T}^{\text{fluid}}_{ab} + 2Q_c^{-1}(x) \widehat{T}_{c\alpha} \mathbf{F}^\alpha{}_{bc} \right) = 0 \,. \tag{5.27}$$

I.e., we get the standard expression of Lorentz force for Yang-Mills plasma:

$$D^a T^{\text{fluid}}_{ab}(x) = Q_c(x) \mathbf{F}^{\alpha\,a}{}_b(x) \mathbf{J}^{\text{color}}_{\alpha\,a}(x). \tag{5.28}$$

After doing the KK reduction of gravity sourced by a generic fluid $\widehat{T}^{\text{fluid}}_{MN}$, we are going to evaluate $\widehat{T}^{\text{fluid}}_{MN} = (\widehat{T}^{\text{perfect}} + \widehat{T}^{\text{diss}})_{MN}$ and study in detail the resulting $d$-dimensional fluid.

_____

[11]We relegate details of the calculation to appendix A.

## 6 Colored fluid from non-Abelian reduction

In this section, we will implement the KK compactification of the fluid energy-momentum tensor to construct the colored fluid and read off its defining variables.

### 6.1 Non-Abelian reduction of fluid

The energy-momentum tensor and the defining variable of the $d$-dimensional fluid will be read off after inserting the compactification ansatze for the vielbein and the rest of the expressions into the EOMs of the $D$-dimensional system.

For the non-Abelian reduction of the velocity fields $\widehat{u}^A$, we will assume an ansatz such that none of its components depend on the coordinates of the internal group manifold $G$. We can parametrize them as follows

$$
\begin{aligned}
\widehat{u}^a &= u^a(x)\cosh\varphi(x)\,, \\
\widehat{u}^\alpha &= \mathbf{n}^\alpha(x)\sinh\varphi(x)\,,
\end{aligned}
\tag{6.1}
$$

where

$$
u^a u^b \eta_{ab} = -1 \qquad \text{and} \qquad \mathbf{n}^\alpha \mathbf{n}^\beta \delta_{\alpha\beta} = 1. \tag{6.2}
$$

The $d$-dimensional velocity has $(d-1)$ independent components, and the $n$-dimensional unit vector $\mathbf{n}$ has $(n-1)$ independent components. In total, along with $\varphi$, there are $(d-1)+(n-1)+1 = D-1$ independent components. The angular variable $\varphi$ measures the relative magnitude between the external and "internal" velocity fields. The unit vector $u^a$ is the boost in external spacetime, while the unit vector $\mathbf{n}$ is the boost in the internal group manifold. They all fluctuate in external spacetime.

With this ansatz, we will now study the $d$-dimensional energy-momentum tensor of the fluid, eq. (3.3).

### 6.2 Perfect colored fluid

Firstly, we are going to characterize the colored perfect fluid in $d$ dimensions. This will allow us to identify its thermodynamic and scalar quantities in terms of quantities in $D$ dimensions.

The energy-momentum tensor of the $d$-dimensional perfect colored fluid is given by

$$
T_{ab}^{\text{perfect}}(x) = [\epsilon(x) + p(x)]\, u_a(x)u_b(x) + p(x)\,\eta_{ab}\,, \tag{6.3}
$$

where, using eq. (5.18), the quantities are related to the $D$-dimensional ones as

$$
\begin{aligned}
p(x) &= e^{2\alpha\phi(x)}\widehat{p}(x)\,, \\
\epsilon(x) &= e^{2\alpha\phi(x)}\left[\cosh^2\varphi(x)\widehat{\epsilon}(x) + \sinh^2\varphi(x)\widehat{p}(x)\right]\,.
\end{aligned}
\tag{6.4}
$$

From this, we find the speed of sound, $c_s$, in the perfect colored fluid as

$$
c_s^2 \equiv \frac{\partial p}{\partial \epsilon} = \frac{1}{\cosh^2\varphi(x)(\widehat{c_s}^{-2} - 1) + 1}\,, \qquad \text{where} \qquad \widehat{c}_s^2 = \frac{\partial \widehat{p}}{\partial \widehat{\epsilon}}\,. \tag{6.5}
$$

The faster the fluid is boosted inside the group manifold, the slower the sound speed of the colored fluid.

The boost inside the group manifold generates the color current. From the current $\mathbf{J}_{ma}^{\text{color}}$, eq. (5.25), we have

$$\mathbf{J}_{ma}^{\text{color}}(x) = Q_c(x)\mathfrak{Q}_m(x)\,u_a(x)\,.\tag{6.6}$$

Here, $\mathfrak{Q}_m(\text{x})$ is the color charge density attached to the fluid, which is defined as

$$\mathfrak{Q}_m(x) = 2(\epsilon(x) + p(x))\mathcal{V}_m{}^\alpha(x)\mathbf{n}_\alpha(x)\tanh\varphi(x)\,.\tag{6.7}$$

## 6.3 Entropy current

The $D$-dimensional neutral fluid has entropy density $\widehat{s}$, so the entropy current is given by

$$\widehat{\mathfrak{J}^s}_A = \widehat{s}\widehat{u}_A,\tag{6.8}$$

In the perfect fluid limit, the entropy current is covariantly conserved

$$\widehat{\nabla}^M\widehat{\mathfrak{J}^s}_M = 0\,.\tag{6.9}$$

From the ansatz eq. (4.19), the entropy in $d$ dimensions is given by

$$s = e^{2\alpha\phi}\,\widehat{s}\,\cosh\varphi\,,\tag{6.10}$$

and the entropy current in $d$ dimensions is given by

$$\mathfrak{J}_\alpha^s = s(x)\,\mathbf{n}_\alpha(x)\tanh\varphi(x),\qquad \mathfrak{J}_\mu^s(x) = s(x)\,u_\mu(x).\tag{6.11}$$

The conservation law eq. (6.9) is reduced to

$$\nabla^\mu\mathfrak{J}^s_\mu = 0\,.\tag{6.12}$$

where we have used the spin connection components of appendix A.

The neutral perfect fluid in $D$ dimensions satisfies the thermodynamic relation

$$\widehat{\epsilon} + \widehat{p} = \widehat{\mathcal{T}}\widehat{s}\,,\tag{6.13}$$

where $\widehat{\mathcal{T}}$ is the temperature. After the reduction, the $d$-dimensional fluid is colored, so its thermodynamic relation must account for the chemical potentials $\boldsymbol{\mu}_m^{\text{color}}$ associated to the charges $\mathfrak{Q}_m$ in the form

$$\epsilon + p = \mathcal{T}s + \mathfrak{Q}^m\boldsymbol{\mu}_m^{\text{color}}\,.\tag{6.14}$$

Requiring this Euler relation to hold in $d$ dimensions, we obtain that the $d$-dimensional temperature and chemical potentials are given by

$$\mathcal{T}(x) = \widehat{\mathcal{T}}(x)\,\frac{1}{\cosh\varphi(x)}\,,$$
$$\boldsymbol{\mu}_m^{\text{color}}(x) = \mathbf{n}_\alpha(x)\mathcal{V}^\alpha{}_m(x)\tanh\varphi(x)\,.\tag{6.15}$$

So far, we have described the $d$-dimensional perfect fluid carrying non-Abelian SU(2) charges and given all its defining quantities in terms of the $D$-dimensional neutral fluid parameters. These results are in full agreement with the ones obtained for the Abelian case in section 4. Built upon these consistency checks, we are going to consider dissipative effects of the fluid in the next section.

### 6.4 Non-Abelian dissipative fluid

We are going to extend our previous analysis by considering the dissipative part of energy-momentum tensor, $\widehat{T}_{MN}^{\text{diss}}$. This piece is given by

$$\widehat{T}_{AB}^{\text{diss}} = -2\widehat{\eta}\widehat{\sigma}_{AB} - \widehat{\zeta}\widehat{P}_{AB}\widehat{\theta}. \tag{6.16}$$

The correction of first-order in derivatives in $\widehat{T}_{AB}^{\text{diss}}$ will generate terms of first-order derivatives of the components of velocity fields $\widehat{u}_A$. Being velocity fields, these terms play the same role as second-order derivative of ordinary fields. Therefore, we will eliminate the derivatives by using their equations of motions, namely, the conservation laws.

In particular, if we consider eqs. (5.23) and (5.27), we obtain

$$u^\mu(x)\widehat{\nabla}_\mu\varphi(x) = c_s^2(x)\,\theta(x)\tanh\varphi(x), \tag{6.17}$$

where $\theta(x) \equiv \nabla_\mu u^\mu(x)$. Moreover, $\theta(x)$ is related to $\widehat{\theta} \equiv \widehat{\nabla}_M\widehat{u}^M(x)$ by

$$\widehat{\theta}(x) = \cosh\varphi(x)\left(\theta + \mathbf{n}^\alpha u^\mu\widehat{\nabla}_\mu\mathbf{n}_\alpha\right)(x), \tag{6.18}$$

so that when substituting, we have

$$\widehat{\theta}(x) = \cosh^3\varphi(x)\left(\frac{c_s^2(x)}{\widehat{c}_s^2(x)}\right)\theta(x). \tag{6.19}$$

In addition, the $d$-dimensional acceleration $a_\mu \equiv u^\nu\widehat{\nabla}_\nu u_\mu$ is given by

$$a_\mu = \frac{\text{sech}^2\varphi(x)}{e^{2\alpha\phi(x)}\widehat{\epsilon}(x) + p(x)}\widehat{\nabla}_\mu\left(\frac{1}{p(x)}\right) + n\theta(x)c_s^2(x)u_\mu(x), \tag{6.20}$$

where

$$\widehat{\nabla}_\mu\left(\frac{1}{p(x)}\right) = \frac{e^{2\alpha\phi(x)}\widehat{\epsilon}(x) + p(x)}{2p^2(x)}\sinh 2\varphi(x)\widehat{\nabla}_\mu\varphi(x). \tag{6.21}$$

With these results, we can estimate the $d$-dimensional coefficients associated with the dissipative terms. For the $D$-dimensional neutral fluid, the shear and bulk viscosities can be read off from $\widehat{T}_{AB}^{\text{diss}}$. This occurs due to the fact that the fluid is described in the Landau frame, i.e.,

$$\widehat{u}^A\widehat{T}_{AB}^{\text{diss}} = 0. \tag{6.22}$$

Upon the non-Abelian KK dimensional reduction, the rearrangement of DOFs into $d$-dimensional Lorentz covariant representations implies that the reduced ones do not satisfy the Landau frame condition. In particular, we obtain

$$u^a(x)\widehat{T}_{ab}^{\text{diss}}(x) + \frac{1}{\cosh\varphi(x)}\widehat{u}^\alpha(x)\widehat{T}_{\alpha b}^{\text{diss}}(x) = 0, \tag{6.23}$$

which straightforwardly leads to $u^a T_{ab}^{\text{diss}} \neq 0$.

On account of the frame-dependent structure of the energy-momentum tensor, departure from the Landau frame means that we cannot read off the $d$-dimensional transport coefficients associated with the dissipative terms from $T_{\mu\nu}^{\text{diss}}$. To correctly identify these

coefficients, we need a frame-invariant formulation of the dissipative terms. In addition, according to the second law of thermodynamics, it has to be guaranteed that the entropy current $\mathfrak{J}^s{}_a$ satisfies $\nabla^\mu \mathfrak{J}^s{}_\mu \geq 0$. Such frame-invariant description was developed in [41] for a fluid charged under an Abelian gauge field $A_\mu$. Here, we generalize this result to account for non-Abelian symmetry.

Using the frame-invariant approach as a guiding principle and also based on the gauge covariance of SU(2) algebra-valued quantities, we formulate the following expressions for the transport coefficients in the presence of non-Abelian gauge fields $\mathbf{A}^m{}_\mu$:

$$\left( P_a{}^c(x) P_b{}^d(x) - \frac{1}{d-n} P_{ab}(x) P^{cd}(x) \right) T^{\mathrm{diss}}_{cd}(x) = -2\eta(x)\,\sigma_{ab}(x)\,,$$

$$\frac{\partial p(x)}{\partial \mathfrak{Q}_m} u^a(x) \mathbf{J}^{\mathrm{diss}}{}_{am}(x) + \left( \frac{1}{d-n} P^{ab} - \frac{\partial p}{\partial \epsilon} u^a u^b \right)(x) T^{\mathrm{diss}}_{ab}(x) = -\zeta(x)\theta(x)\,,$$

$$P_a{}^b(x) \left( \mathbf{J}^{\mathrm{diss}}{}_{bm} + \frac{\mathfrak{Q}_m}{\epsilon + p} u^c T^{\mathrm{diss}}_{cb} \right) - \boldsymbol{\kappa}_{mn}(x) \left( -P_a{}^b D_b\left( \frac{\boldsymbol{\mu}_n}{\mathcal{T}} \right) + \frac{1}{\mathcal{T}} \mathbf{F}^n{}_{ab} u^b \right) = 0\,, \qquad (6.24)$$

where $\mathbf{J}^{\mathrm{diss}}{}_{am}$ follows from eq. (5.25) using $\widehat{T}_{MN} = \widehat{T}^{\mathrm{diss}}_{MN}$, $\boldsymbol{\kappa}_{mn}$ is the non-Abelian conductivity tensor, and $\eta$, $\zeta$, $\sigma$ are the $d$-dimensional dissipative coefficients.

At this stage, in order to obtain the effective dissipative coefficients, we need to substitute the expressions that we obtained for $\mathbf{J}^{\mathrm{diss}}_{am}$ and $\widehat{T}^{\mathrm{diss}}_{ab}$ and work out these three equations.[12] From them, we read off the following expressions:

$$\eta(x) = e^{2\alpha\phi} \widehat{\eta}(x) \cosh\varphi(x)\,,$$

$$\boldsymbol{\kappa}_{mn}(x) = e^{2\alpha\phi} \widehat{\eta}(x)\mathcal{T}(x)\cosh\varphi(x) \left( \delta_{mn} - \frac{\sinh^4\varphi}{\cosh^2\varphi} \mathcal{V}_m{}^\alpha \mathbf{n}_\alpha \mathcal{V}_n{}^\beta \mathbf{n}_\beta \right)(x)\,. \qquad (6.25)$$

$$\zeta(x) = 2e^{2\alpha\phi(x)} \widehat{\eta}(x)\,\cosh\varphi \left[ \frac{1}{d-n} + c_s^4 \left( 1 - \cosh^4\varphi \frac{1}{p}\frac{\partial\widehat{p}}{\partial\widehat{\epsilon}} + e^{2\alpha\phi(x)}\cosh^5\varphi\,\widehat{\zeta}\left( \frac{\partial\widehat{p}}{\partial\widehat{\epsilon}} \right)^2 \right) \right]\,.$$

It is important to stress that when getting rid of any dependence on the scalar fields $\varphi$, we recover the $d$-dimensional quantities multiplied by the dilaton factor $e^{2\alpha\phi}$, which parametrizes the volume of the internal manifold. On the other hand, it is worth to mention that the non-Abelian behavior of the conductivity matrix arises from the dependence of the scalar vielbein $\mathcal{V}_m{}^\alpha$.

The analysis in this section demonstrates that the non-Abelian KK dimensional reduction is an ab initio and efficient method for deriving the structure and dynamics of Yang-Mills plasma. Moreover, the construction that leads to eq. (6.25) gives a hydrodynamic frame-independent transport. We see from eq. (6.25) that, apart from viscosities, we have the non-Abelian conductivity matrix $\boldsymbol{\kappa}_{mn}$, which is directly connected to the non-Abelian degrees of freedom in the system. We remark that a similar quantity was obtained in the context of the fluid/gravity duality [24].

Now that we have clearly formulated non-Abelian hydrodynamics, we can study various related issues. Understanding conductivity is a major challenge in recent approaches to holographic superfluids. One can show that, at the phase transition, a set of SU(2) currents

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[12]In ref. [37], this calculation was performed for a neutral fluid compactified on a torus.

can be used as an order parameter [42]. Moreover, it was observed in [25] that employing a non-Abelian gauge transformation allows one to obtain a finite conductivity without breaking translational symmetry.

On the other hand, this theory results in a very suitable and robust framework where to study the quark-gluon plasma. In this respect, one important phenomenon of this system is the study of the relaxation time. This is the time at which the non-Abelian character of the plasma is relaxed, thus becoming purely Abelian. This is a known property that has not been theoretically understood neither for quark-gluon plasma nor for spintronics systems.[13] Since our construction can describe the dissipative part of non-Abelian hydrodynamics, we expect it to be useful in elucidating the relaxation mechanism of the color current.

## 7 Outlooks

In this work, we have proposed a new approach for constructing non-Abelian hydrodynamics, consisting of colored fluid interacting with Yang-Mills theory. Based on non-Abelian KK dimensional reduction, the geometric systematics of proposed approach enables one to understand the properties of Yang-Mills plasma even in strongly coupled, non-perturbative regime.

We presented an ab initio approach for constructing hydrodynamics charged under both Maxwell and Yang-Mills plasma. With the non-Abelian KK reduction, we compactified the Einstein-fluid equations on a group manifold. The only working assumption is that we started with the most general dissipative, neutral fluid coupled to Einstein equation. After the reduction, we obtained Yang-Mills plasma equations for a dissipative, colored fluid interacting non-Abelian gauge fields. Though having done the reduction on $\mathbb{S}^1$ and SU(2) group manifold, this procedure can be applied to any type of group manifold. Our approach is not restricted by symmetries that are only symmetries of the Lagrangian. Hence, the KK reduction approach seems to be a robust and covariant method to naturally obtain hydrodynamics coupled to (non-)Abelian gauge fields. The method straightforwardly extends to dissipative hydrodynamics coupled to gravity and a specific form of dilaton scalar field, which would also bear applications to early universe cosmology, formation of large-scale structure or compact objects, and colored turbulence.

We studied the conservation laws of colored fluid and obtained a non-Abelian covariantly conserved current $\mathbf{J}_{am}$, which is proportional to the fluid velocity field, as predicted by [30]. In addition, truncating the scalar fields coming from the gravity sector to constant values, we obtained the equation for non-Abelian Lorentz force.

We showed that the reduction procedure does not preserve the hydrodynamic frames. As a consequence, the effective transport coefficients could not be straightforwardly read off from the reduced system. We proposed a frame-independent formulation of dissipative fluids for the non-Abelian gauge fields that is thermodynamically valid and generalizes the one given in [41]. With this construction, we identified the $d$-dimensional dissipative susceptibilities that characterize the effective fluid in terms of the $D$-dimensional ones. In

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[13]It is worth to mention that our system can be coupled to additional fermionic degrees of freedom, as we are using the vielbein formalism.

particular, we have obtained a conductivity matrix whose non-Abelian nature is given by the scalar vielbein $\mathcal{V}_m{}^\alpha$.

The Yang-Mills plasma equations we obtained were in complete agreement with the equations of Maxwell plasma derived in section 4. If we set the structure constants $f_{mn}{}^p = 0$, we could check that these equations were reduced to the equations for charged fluid coupled to U(1)$^3$ Abelian gauge fields. The results of this section could also be straightforwardly extended to other, higher-dimensional group manifold $G$. We claimed that, for fixed $d$, the large-$D$ limit should be taken seriously as it corresponds to the limit for which rank($G$) gets large, revealing a new perspective to the planar limit of Yang-Mills plasma. Results on this aspect will be relegated to a separate publication.

We believe the proposed approach marks significant advances toward the understanding of the evolution of nuclear matter after a heavy-ion collision. Hydrodynamics with non-Abelian degrees of freedom that have not thermalized is a transient phase and the lack of a first-principle derivation of the equations that govern its evolution has been a major obstacle for further developments.

Having now the ab initio construction of fluid and field equations, we can utilize complementary methods such as kinetic theory or gauge/gravity duality to shed more light of this regime. Gravitational solutions with Abelian gauge fields have recently been studied [19, 43, 44]. Therefore, we provide a robust formulation of non-Abelian hydrodynamics where to test fluid/gravity duality beyond Abelian fluids.

In addition to a phenomenological description of quark-gluon plasma, recent formulation of fluid dynamics in terms of fluid/gravity duality has increased the interest in the analysis of fluids coupled to Yang-Mills fields. In this picture, fluid is a field theory dual to a black hole in higher-dimensional, asymptotically anti-de Sitter spacetime (see [45] for a review). It would be interesting to further explore the physics of black holes with non-Abelian and dilatonic hairs using the non-Abelian Kaluza-Klein reduction [46].

## Acknowledgments

# A Einstein equations on a group manifold

In this appendix, we elaborate technical details of the non-Abelian Kaluza-Klein compactification on a group manifold. We also explain the convention used in this work.

We will consider that our starting system is defined on a $D$ dimensional manifold $\mathcal{M}_D(\widehat{g})$ with coordinates $\widehat{x}^M$, for $M = 1, \ldots, D$. For the tangent spacetime description we introduce a vielbein $E_M{}^A$, where $A = 1, \ldots, D$, which satisfies

$$\widehat{g}_{MN}(\widehat{x}) = E_M{}^A(\widehat{x})E_N{}^B(\widehat{x})\eta_{AB}, \qquad \eta_{AB} = (- + \ldots +). \tag{A.1}$$

## A.1 General ansatz

We will perform a KK dimensional reduction. To do so, we will assume that $\mathcal{M}_D(\widehat{g}) = \mathcal{M}_d(g) \times X_n(\boldsymbol{M})$. $\mathcal{M}_d(\widehat{g})$ is the $d$-dimensional external spacetime manifold on which our resulting system will live whereas $X_n(\boldsymbol{M})$ is the $n$-dimensional internal manifold. The coordinates are split as $\widehat{x}^M = \{x^\mu, y^m\}$, where $\mu = 1, \ldots, d$ and $m = 1, \ldots, n$. Despite the scalar matrix $M_{mn}$ will parametrize the fluctuations of the internal manifold, the final $d$-dimensional system cannot have any functional dependence on $X_n$. The

We start with the reduction ansatz for the vielbein expressed in terms of the Maurer-Cartan one-forms:

$$\begin{aligned}
\widehat{E}^a(x,y) &= e^{\alpha\phi(x)}e^a(x), \\
\widehat{E}^\alpha(x,y) &= g^{-1}e^{\beta\phi(x)}\mathcal{V}_m{}^\alpha(x)(\boldsymbol{\sigma}^m - g\mathbf{A}^m(x)),
\end{aligned} \tag{A.2}$$

where $\boldsymbol{\sigma}^m \equiv \mathfrak{u}_n{}^m(y)\mathrm{d}y^n$ are the twist matrices, which will depend on the group manifold coordinates $y$. Here, $g$ is a gauge coupling parameter.

We will compute various geometric quantities. The spin-connection is defined as

$$\widehat{\omega}_{C,AB} = -\widehat{\Omega}_{CA,B} + \widehat{\Omega}_{AB,C} - \widehat{\Omega}_{BC,A} \tag{A.3}$$

where

$$\widehat{\Omega}_{AB,C} = \frac{1}{2}\left(E_A{}^M E_B{}^N - E_B{}^M E_A{}^N\right)\partial_N E_M{}^D\widehat{\eta}_{DC}. \tag{A.4}$$

Substituting the vielbein ansatz, we obtain the following expressions:

$$\begin{aligned}
\widehat{\omega}_{c,ab} &= e^{-\alpha\phi}\left[\omega_{c,ab} + 2\alpha\eta_{c[a}\partial_{b]}\phi\right], \\
\widehat{\omega}_{c,a\beta} &= -\frac{1}{2}e^{(-2\alpha+\beta)\phi}\mathbf{F}_{\mu\nu}{}^n e_c{}^\mu e_a{}^\nu\mathcal{V}_{n\beta}, \\
\widehat{\omega}_{c,\alpha\beta} &= +\frac{1}{2}e^{-(\alpha+\beta)\phi}\left[\mathcal{V}_\alpha{}^m e_c{}^\mu D_\mu(e^{\beta\phi}\mathcal{V}_{m\beta}) - \mathcal{V}_\beta{}^m e_c{}^\mu D_\mu(e^{\beta\phi}\mathcal{V}_{m\alpha})\right], \\
\widehat{\omega}_{\gamma,ab} &= +\frac{1}{2}e^{(-2\alpha+\beta)\phi}\mathbf{F}_{\mu\nu}{}^m e_a{}^\mu e_b{}^\nu\mathcal{V}_{m\gamma}, \\
\widehat{\omega}_{\gamma,a\beta} &= -\frac{1}{2}e^{-(\alpha+\beta)\phi}e_a{}^\mu\mathcal{V}_\beta{}^m\mathcal{V}_\gamma{}^n D_\mu(e^{\beta\phi}\boldsymbol{M}_{mn}), \\
\widehat{\omega}_{\gamma,\alpha\beta} &= +\frac{g}{2}e^{-\beta\phi}f_{mn}{}^p\left[\mathcal{V}_\gamma{}^n\mathcal{V}_\alpha{}^m\mathcal{V}_{p\beta} + \mathcal{V}_\alpha{}^m\mathcal{V}_\beta{}^n\mathcal{V}_{p\gamma} - \mathcal{V}_\beta{}^m\mathcal{V}_\gamma{}^n\mathcal{V}_{p\alpha}\right],
\end{aligned} \tag{A.5}$$

where

$$
\begin{aligned}
\boldsymbol{M}_{mn} &= \mathcal{V}_m{}^\alpha \mathcal{V}_n{}^\beta \delta_{\alpha\beta}\,, \\
\mathbf{F}^m{}_{\mu\nu} &\equiv \partial_\mu \mathbf{A}^m{}_\nu - \partial_\nu \mathbf{A}^m{}_\mu + g f_{np}{}^m \mathbf{A}^n{}_\mu \mathbf{A}^p{}_\nu\,, \\
D_\mu \mathcal{V}_m{}^\alpha &\equiv \partial_\mu \mathcal{V}_m{}^\alpha - g f_{nm}{}^p \mathbf{A}^n{}_\mu \mathcal{V}_p{}^\alpha
\end{aligned}
\tag{A.6}
$$

and

$$
f_{mn}{}^p = -(\mathfrak{u}^{-1})_m{}^s (\mathfrak{u}^{-1})_n{}^t (\partial_s \mathfrak{u}_t{}^p - \partial_t \mathfrak{u}_s{}^p)\,.
\tag{A.7}
$$

We will calculate the components of the Ricci tensor $\widehat{R}_{AB} = \widehat{R}_{ACBD}\widehat{\eta}^{CD}$ and the scalar curvature $\widehat{R} = \widehat{R}_{AB}\widehat{\eta}^{AB}$ by substituting the components of the spin connection $\widehat{\omega}_{MAB}$ into the expression for the Riemann tensor,

$$
\widehat{R}_{MNAB} = \partial_M \widehat{\omega}_{NAB} - \partial_N \widehat{\omega}_{MAB} + \widehat{\omega}_{MA}{}^E \widehat{\omega}_{NEB} - \widehat{\omega}_{NA}{}^E \widehat{\omega}_{MEB}\,.
\tag{A.8}
$$

## A.2 SU(2) group manifold

In what follows, we restrict to the SU(2) group manifold, so that $f_{mn}{}^p$ will be the SU(2) structure constants, $f_{mnp} = \epsilon_{mnp}$. In this case, the components of the spin connection are given by [47]

$$
\begin{aligned}
\widehat{\omega}_{ab} &= \omega_{ab} + 2\alpha e^{-\alpha\phi}\eta_{c[a}D_{b]}\phi \widehat{e}^c + \frac{1}{2}e^{(-2\alpha+\beta)\phi}\mathbf{F}^\beta{}_{ab}\widehat{e}^\beta\,, \\
\widehat{\omega}_{a\beta} &= -e^{-\alpha\phi}\mathbb{P}_{a\beta\gamma}\widehat{e}^\gamma - \beta e^{-\alpha\phi}D_a\phi \widehat{e}^\beta + \frac{1}{2}e^{(-2\alpha+\beta)\phi}\mathbf{F}^\beta{}_{ab}\widehat{e}^b\,, \\
\widehat{\omega}_{\alpha\beta} &= e^{-\alpha\phi}\mathbb{Q}_{a\alpha\beta}\widehat{e}^a + \frac{g}{2}e^{-\beta\phi}(\mathbb{M}^{\gamma\delta}\epsilon_{\alpha\beta\delta} + \mathbb{M}^{\beta\delta}\epsilon_{\alpha\gamma\delta} - \mathbb{M}^{\alpha\delta}\epsilon_{\beta\gamma\delta})\widehat{e}^\gamma\,.
\end{aligned}
\tag{A.9}
$$

where

$$
\mathbf{F}^\alpha{}_{ab} \equiv \mathcal{V}_m{}^\alpha \mathbf{F}^m{}_{ab}\,,
\tag{A.10}
$$

$M_{\alpha\beta}$ is the SU(2) covariant scalar matrix

$$
\mathbb{M}^{\alpha\beta} \equiv \mathcal{V}_m{}^\alpha \mathcal{V}_n{}^\beta \delta^{mn}\,,
\tag{A.11}
$$

and

$$
\mathbb{P}_{a\,\alpha\beta} \equiv \frac{1}{2}[\mathcal{V}_\alpha{}^m D_a \mathcal{V}_m{}^\beta + \mathcal{V}_\beta{}^m D_a \mathcal{V}_m{}^\alpha]\,, \quad \mathbb{Q}_{a\,\alpha\beta} \equiv \frac{1}{2}[\mathcal{V}_\alpha{}^m D_a \mathcal{V}_m{}^\beta - \mathcal{V}_\beta{}^m D_a \mathcal{V}_m{}^\alpha]\,.
\tag{A.12}
$$

The Ricci tensor components are

$$
\begin{aligned}
\widehat{R}_{ab} &= e^{-2\alpha\phi}\left[R_{ab} - \frac{1}{2}\partial_a\phi\partial_b\phi - \mathbb{P}_{a\,\alpha\beta}\mathbb{P}_{b\,\alpha\beta} - \alpha\Box\phi\eta_{ab} - \frac{1}{2}e^{-\frac{2}{3}\alpha(n+1)\phi}\mathbf{F}^\alpha{}_{ac}\mathbf{F}^\alpha{}_{bd}\eta^{cd}\right]\,, \\
\widehat{R}_{a\beta} &= -\frac{1}{2}e^{\frac{1}{3}\alpha(n-5)\phi}\left[\mathcal{D}^b(e^{-\frac{2}{3}\alpha(n+1)\phi}\mathbf{F}^\beta{}_{ab}) + e^{-\frac{2}{3}\alpha(n+1)\phi}\mathbf{F}^\gamma{}_{ab}\mathbb{P}^b{}_{\beta\gamma} - 2g\epsilon_{\beta\gamma\delta}\mathbb{M}^{\delta\lambda}\mathbb{P}_{a\gamma\lambda}\right]\,, \\
\widehat{R}_{\alpha\beta} &= -\frac{1}{2}e^{-2\alpha\phi}\left[\mathcal{D}_a\mathbb{P}^a{}_{\alpha\beta} - \frac{2}{3}\alpha(d-2)\Box\phi\delta_{\alpha\beta} - \frac{1}{2}e^{-\frac{2}{3}\alpha(d+1)\phi}\mathbf{F}^\alpha{}_{ab}\mathbf{F}^\beta{}_{cd}\eta^{ac}\eta^{bd}\right. \\
&\qquad\left. -4g^2 e^{\frac{2}{3}\alpha(d+1)\phi}\left(\mathbb{M}^{\alpha\gamma}\mathbb{M}^{\beta\gamma} - \frac{1}{2}\mathbb{M}^{\alpha\beta}\mathbb{M}\right) + 2g^2 e^{\frac{2}{3}\alpha(d+1)\phi}\left(\mathbb{M}^{\gamma\delta}\mathbb{M}^{\gamma\delta} - \frac{1}{2}\mathbb{M}^2\right)\delta_{\alpha\beta}\right]\,,
\end{aligned}
\tag{A.13}
$$

where

$$
\mathbb{M} \equiv \mathbb{M}_{\alpha\alpha} \qquad \text{and} \qquad \mathcal{D}_a\mathcal{V}_m{}^\alpha = D_a\mathcal{V}_m{}^\alpha + \mathbb{Q}_{a\alpha\beta}\mathcal{V}_m{}^\beta\,.
\tag{A.14}
$$

### A.3 Equations of motion

Our starting point is the $D$-dimensional Einstein-fluid equation

$$\mathcal{G}_{MN} \equiv \widehat{R}_{MN} - \left( \widehat{T}_{MN} + \frac{1}{D-2} g_{MN} \widehat{T} \right) = 0 \,. \tag{A.15}$$

We will analyze the tensor $\mathcal{G}_{AB} = E_A{}^M E_B{}^N \mathcal{G}_{MN}$:

$$\mathcal{G}_{AB} \equiv \widehat{R}_{AB} - \left( \widehat{T}_{AB} + \frac{1}{D-2} \eta_{AB} \widehat{T} \right) = 0 \,. \tag{A.16}$$

Here, we analyze each components of the Einstein-fluid equation. We begin with the internal components, $\mathcal{G}_{mn}$:

$$\mathcal{G}_{mn} = g^{-2} e^{2\beta\phi} \mathfrak{u}_m{}^p \mathfrak{u}_n{}^q \mathcal{V}_p{}^\alpha \mathcal{V}_q{}^\beta \mathcal{G}_{\alpha\beta} \,. \tag{A.17}$$

On one hand, this equation has to be satisfied for any scalar fields $\mathcal{V}_m{}^\alpha$. As the twist matrices $\mathfrak{u}_m{}^n$ depend on internal coordinates, we have that $\mathcal{G}_{\alpha\beta} = 0$, where $\mathcal{G}_{\alpha\beta}$ is given by

$$\begin{aligned}
\mathcal{G}_{\alpha\beta} = -\frac{1}{2} e^{-2\alpha\phi} &\left[ \mathcal{D}_a \mathbb{P}^a{}_{\alpha\beta} - \frac{2}{3}\alpha(d-2)\square\phi\,\delta_{\alpha\beta} - \frac{1}{2} e^{-\frac{2}{3}\alpha(d+1)\phi} \mathbf{F}^\alpha{}_{ab} \mathbf{F}^\beta{}_{cd} \eta^{ac} \eta^{bd} \right.\\
&\left. - 4g^2 e^{\frac{2}{3}\alpha(d+1)\phi} \left( \mathbb{M}^{\alpha\gamma} \mathbb{M}^{\beta\gamma} - \frac{1}{2}\mathbb{M}^{\alpha\beta}\mathbb{M} \right) + 2g^2 e^{\frac{2}{3}\alpha(d+1)\phi} \left( \mathbb{M}^{\gamma\delta} \mathbb{M}^{\gamma\delta} - \frac{1}{2}\mathbb{M}^2 \right) \delta_{\alpha\beta} \right]\\
&+ \frac{1}{d+1}\delta_{\alpha\beta} \widehat{T}^{\text{fluid}} - \widehat{T}^{\text{fluid}}_{\alpha\beta} \,,
\end{aligned} \tag{A.18}$$

Solving the trace part, $\mathcal{G}_{\alpha\beta}\delta^{\alpha\beta} = 0$, where

$$\begin{aligned}
\mathcal{G}_{\alpha\beta}\delta^{\alpha\beta} = e^{-2\alpha\phi}(d-2) &\left[ \alpha\square\phi + \frac{1}{4(d-2)} e^{-\frac{2}{3}\alpha(d+1)\phi} (\mathbf{F}^\alpha)^2 \right.\\
&\left. - \frac{g^2}{d-2} e^{\frac{2}{3}\alpha(d+1)\phi} \left( \mathbb{M}^{\alpha\beta} \mathbb{M}^{\alpha\beta} - \frac{1}{2}\mathbb{M}^2 \right) + \frac{1}{d-2} e^{2\alpha\phi} \left( \frac{3}{d+1}\widehat{T}^{\text{fluid}} - \widehat{T}^{\text{fluid}}_{\alpha\beta}\delta^{\alpha\beta} \right) \right] \,,
\end{aligned} \tag{A.19}$$

we solve for $\square\phi$ and substitute back to eq. (A.18). We then obtain from $\mathcal{G}_{\alpha\beta} = 0$ that

$$\begin{aligned}
\mathcal{D}^a \mathbb{P}_{a\alpha\beta} = \frac{1}{2} e^{-\frac{2}{3}\alpha(d+1)\phi} &\left[ \mathbf{F}^\alpha{}_{ab} \mathbf{F}^\beta{}_{cd} \eta^{ac} \eta^{bd} - \frac{1}{3}(\mathbf{F}^\gamma)^2 \delta_{\alpha\beta} \right]\\
&+ 4g^2 e^{\frac{2}{3}\alpha(d+1)\phi} \left[ \mathbb{M}^{\alpha\gamma}\mathbb{M}^{\beta\gamma} - \frac{1}{2}\mathbb{M}^{\alpha\beta}\mathbb{M} - \frac{1}{3}\left( \mathbb{M}^{\gamma\delta}\mathbb{M}^{\gamma\delta} - \frac{1}{2}\mathbb{M}^2 \right) \delta_{\alpha\beta} \right]\\
&+ 2e^{2\alpha\phi} \left[ \frac{1}{d+1}\widehat{T}^{\text{fluid}}\eta_{\alpha\beta} - \widehat{T}^{\text{fluid}}_{\alpha\beta} - \frac{1}{3}\left( \frac{3}{d+1}\widehat{T}^{\text{fluid}} - \widehat{T}^{\text{fluid}}_{\gamma\delta}\delta^{\gamma\delta} \right) \delta_{\alpha\beta} \right] \,. \tag{A.20}
\end{aligned}$$

Let us consider now the field equations $\mathcal{G}_{\mu n} = 0$. As

$$\mathcal{G}_{\mu n} = E_\mu{}^A E_n{}^\beta \mathcal{G}_{A\beta} = E_\mu{}^a E_n{}^\beta \mathcal{G}_{a\beta} = g^{-1} e^{(\alpha+\beta)\phi} \mathfrak{u}_m{}^p e_\mu{}^a \mathcal{V}_p{}^\beta \mathcal{G}_{a\beta} \,, \tag{A.21}$$

it follows that $\mathcal{G}_{\mu n} = 0$ is equivalent to $\mathcal{G}_{a\beta} = 0$.

We now study the field equations $\mathcal{G}_{a\beta} = 0$. We have

$$\mathcal{D}^b(e^{-\frac{2}{3}\alpha(d+1)\phi}\mathbf{F}^\beta{}_{ab}) + e^{-\frac{2}{3}\alpha(d+1)\phi}\mathbf{F}^\gamma{}_{ab}\mathbb{P}^b{}_{\beta\gamma} - 2g\epsilon_{\beta\gamma\delta}\mathbb{M}^{\delta\lambda}\mathbb{P}_{a\gamma\lambda} + 2e^{-\frac{1}{3}\alpha(d-5)\phi}\widehat{T}^{\text{fluid}}_{a\beta} = 0\,. \tag{A.22}$$

Finally, let us consider the $d$-dimensional components of the Einstein field equations, $\mathcal{G}_{\mu\nu} = 0$. Using the equations for other components, this equation implies

$$\mathcal{G}_{\mu\nu} = E_\mu{}^A E_\nu{}^B \mathcal{G}_{AB} = e^{2\alpha\phi}e_\mu{}^a e_\nu{}^b \mathcal{G}_{ab} = 0\,. \tag{A.23}$$

Therefore the resulting equation is $\mathcal{G}_{ab} = 0$, where

$$\begin{aligned}
\mathcal{G}_{ab} = R_{ab} &- \frac{1}{2}\partial_a\phi\partial_b\phi - \mathbb{P}_{a\beta\gamma}\mathbb{P}_{b\beta\gamma} - \frac{1}{2}e^{-\frac{2}{3}\alpha(d+1)\phi}\left[\mathbf{F}^\gamma{}_{ac}\mathbf{F}^\gamma{}_{bd}\eta^{cd} - \frac{1}{2(d-2)}(\mathbf{F}^\gamma)^2\eta_{ab}\right] \\
&- \frac{g^2}{d-2}e^{\frac{2}{3}\alpha(d+1)\phi}\left(\mathbb{M}^{\gamma\delta}\mathbb{M}^{\gamma\delta} - \frac{1}{2}\mathbb{M}^2\right)\eta_{ab} - e^{2\alpha\phi}\left[\widehat{T}^{\text{fluid}}_{ab} - \frac{1}{d-2}\widehat{T}^{\text{fluid}}_{cd}\eta^{cd}\eta_{ab}\right]\,.
\end{aligned} \tag{A.24}$$

These are the Einstein equations for the $d$-dimensional system, which can be equivalently rewritten as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T^{\text{total}}_{\mu\nu}\,, \tag{A.25}$$

where $T^{\text{total}}_{\mu\nu} = e_\mu{}^a e_\nu{}^b T^{\text{total}}_{ab}$ is

$$\begin{aligned}
T^{\text{total}}_{ab} = \frac{1}{2}&\left(\partial_a\phi\partial_b\phi - \frac{1}{2}(\partial\phi)^2\eta_{ab}\right) + \mathbb{P}_{a\beta\gamma}\mathbb{P}_{b\beta\gamma} - \frac{1}{2}\mathbb{P}^2\eta_{ab} \\
&+ e^{-\frac{2}{3}\alpha(d+1)\phi}\left[\frac{1}{2}\mathbf{F}^\alpha{}_{ac}\mathbf{F}^\alpha{}_{bd}\eta^{cd} - \frac{1}{4}(\mathbf{F}^\gamma)^2\eta_{ab}\right] \\
&- g^2 e^{\frac{2}{3}\alpha(d+1)\phi}\left(\mathbb{M}^{\gamma\delta}\mathbb{M}^{\gamma\delta} - \frac{1}{2}\mathbb{M}^2\right)\eta_{ab} + e^{2\alpha\phi}\widehat{T}^{\text{fluid}}_{ab}\,.
\end{aligned} \tag{A.26}$$

From this expression, we also see that the energy-momentum tensor of non-Abelian hydrodynamics $T^{\text{fluid}}_{ab}$ is given by

$$T^{\text{fluid}}_{ab} = e^{2\alpha\phi}\widehat{T}^{\text{fluid}}_{ab}\,. \tag{A.27}$$

# B Conservation laws

In this section we will calculate the conservation laws of the $d$-dimensional theory, namely the current conservation and the Lorentz force. Despite of not making any assumption on the scalar fields, after obtaining the most general expressions we will study the cases for which scalar fields are covariantly constant,

$$D_\mu \mathcal{V}_m{}^a = D_\mu\phi = 0\,,$$

in order to make contact with the conservation laws considered in hydrodynamics, where no degrees of freedom associated to scalar fields take place.

## B.1 Current conservation

Current conservation follows from the consistency condition of the EOMs. Before applying a covariant derivative $\mathcal{D}_a$ on (A.22), we first rewrite the equation of motion for gauge field as

$$\left[ D^b \left( e^{-\frac{2}{3}\alpha(d+1)\phi} \mathbb{M}_{nm} \mathbf{F}^m{}_{ab} \right) - 2g\epsilon_{\sigma\gamma\delta} \mathbb{M}^{\delta\lambda} \mathbb{P}_{a\gamma\lambda} \mathcal{V}_n{}^\sigma + 2e^{-\frac{1}{3}\alpha(d-5)\phi} \widehat{T}^{\text{fluid}}_{a\gamma} \mathcal{V}_n{}^\gamma \right] \mathcal{V}_\beta{}^n = 0 \,. \tag{B.1}$$

As $\mathcal{V}_\beta{}^n$ is non-degenerate in general, without loss of generality, we can assume the equation of motion to be

$$D^b \left( e^{-\frac{2}{3}\alpha(d+1)\phi} \mathbb{M}_{nm} \mathbf{F}^m{}_{ab} \right) = \mathbf{J}_{ma} \,, \tag{B.2}$$

where

$$\mathbf{J}_{ma} = 2g\epsilon_{\sigma\gamma\delta} \mathbb{M}^{\delta\lambda} \mathbb{P}_{a\gamma\lambda} \mathcal{V}_m{}^\sigma - 2e^{-\frac{1}{3}\alpha(d-5)\phi} \widehat{T}^{\text{fluid}}_{a\gamma} \mathcal{V}_m{}^\gamma \,. \tag{B.3}$$

We can expand the covariant derivative and multiply by $(Q_c^2 \mathbb{M})^{-1}$,

$$D^b F^m{}_{ab} = e^{\frac{2}{3}\alpha(d+1)\phi} \mathbb{M}^{mn} \left[ \mathbf{J}_{na} - D^b \left( e^{\frac{2}{3}\alpha(d+1)\phi} \mathbb{M}_{np} \right) F^p{}_{ab} \right] \,. \tag{B.4}$$

Then if we apply another covariant derivative $D_a$, the l.h.s. vanishes and we find that the current $\mathcal{J}_m$ that is covariantly conserved, $D^a \mathcal{J}_{ma} = 0$, is given by

$$\mathcal{J}^m{}_a = e^{\frac{2}{3}\alpha(d+1)\phi} \mathbb{M}^{mn} \left[ \mathbf{J}_{na} - D^b \left( e^{-\frac{2}{3}\alpha(d+1)\phi} \mathbb{M}_{np} \right) F^p{}_{ab} \right] \,. \tag{B.5}$$

If we set the scalar fields $D_a \mathcal{V}_m{}^\beta = 0$, then $\mathbb{P}_{a\beta\gamma} = 0$ and the color current will be purely associated to the off-diagonal components of the $D$-dimensional fluid energy-momentum tensor.

## B.2 Lorentz force

To study the Lorentz force, we will make use of the Bianchi identity of the Einstein tensor

$$\nabla^\mu \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = \nabla^\mu T^{\text{total}}_{\mu\nu} = 0 \,. \tag{B.6}$$

Upon vielbein compatibility, this is equivalent to

$$\mathfrak{D}^a T^{\text{total}}_{ab} = 0 \,, \tag{B.7}$$

where $\mathfrak{D}_a = e_a{}^\mu (\partial_\mu + \omega_\mu)$, where $\omega$ is the $d$-dimensional spin connection. Explicitly,

$$\begin{aligned}
\mathfrak{D}^a T^{\text{total}}_{ab} = & \; \mathfrak{D}^a \left( e^{2\alpha\phi} \widehat{T}^{\text{fluid}}_{ab} \right) + \frac{1}{2} \left( D^a \partial_a \phi \partial_b \phi + \partial_a \phi D^a \partial_b \phi - \partial_c \phi D^a \partial_c \phi \eta_{ab} \right) + \mathcal{D}^a \mathbb{P}_{a\beta\gamma} \mathbb{P}_{b\beta\gamma} \\
& + \mathbb{P}_{a\beta\gamma} \mathcal{D}^a \mathbb{P}_{b\beta\gamma} - \mathcal{D}^a \mathbb{P}_{c\beta\gamma} \mathbb{P}_{c\beta\gamma} \eta_{ab} + \frac{1}{2} \mathfrak{D}^a \left( e^{\frac{2}{3}\alpha(d+1)\phi} \left[ \mathbf{F}^\alpha{}_{ac} \mathbf{F}^\alpha{}_{bd} \eta^{cd} - \frac{1}{2} (\mathbf{F}^\gamma)^2 \eta_{ab} \right] \right) \\
& + \frac{1}{2} e^{\frac{2}{3}\alpha(d+1)\phi} \mathcal{D}^a \left[ \mathbf{F}^\alpha{}_{ac} \mathbf{F}^\alpha{}_{bd} \eta^{cd} - \frac{1}{2} (\mathbf{F}^\gamma)^2 \eta_{ab} \right] \\
& - g^2 \mathfrak{D}^a \left[ e^{\frac{2}{3}\alpha(d+1)\phi} \left( \mathbb{M}^{\gamma\delta} \mathbb{M}^{\gamma\delta} - \frac{1}{2} \mathbb{M}^2 \right) \eta_{ab} \right] \,.
\end{aligned} \tag{B.8}$$

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Let us analyze various terms separately.

$$\frac{1}{2}\left(\mathfrak{D}^a\partial_a\phi\partial_b\phi + \partial_a\phi D^a\partial_b\phi - \partial_c\phi D^a\partial_c\phi\eta_{ab}\right) = \alpha\partial_b\phi\left[-\frac{1}{8(d-2)}e^{-\frac{2}{3}\alpha(d+1)\phi}(\mathbf{F}^\gamma)^2\right.$$
$$+\frac{1}{2(d-2)}g^2e^{\frac{2}{3}\alpha(d+1)\phi}$$
$$\left.-\frac{1}{2(d-2)}e^{2\alpha\phi}\left(\frac{3}{d+1}\widehat{T} - \widehat{T}_{\lambda\sigma}\delta^{\lambda\sigma}\right)\right],$$
$$\tag{B.9}$$

and

$$\mathcal{D}^a\mathbb{P}_{a\beta\gamma}\mathbb{P}_{b\beta\gamma} + \frac{1}{2}\mathcal{D}_{[a}\mathbb{P}_{b]\beta\gamma}\mathbb{P}_{a\beta\gamma} = \frac{1}{2}e^{-\frac{2}{3}\alpha(d+1)\phi}\left[\mathbf{F}^\beta\mathbf{F}^\gamma - \frac{1}{3}(\mathbf{F}^\sigma)^2\delta_{\beta\gamma}\right]\mathbb{P}_{b\beta\gamma}$$
$$+4g^2e^{\frac{2}{3}\alpha(d+1)\phi}\left[\mathbb{M}^{\beta\lambda}\mathbb{M}^{\gamma\lambda} - \frac{1}{2}\mathbb{M}^{\beta\gamma}\mathbb{M} - \frac{1}{3}\delta_{\beta\gamma}V(\mathbb{M})\right]\mathbb{P}_{b\beta\gamma}$$
$$-2\mathbb{P}_{[a|\beta\lambda}\mathcal{V}_\lambda{}^m D_{|b]}\mathcal{V}_m{}^\gamma\mathbb{P}_{a\beta\gamma} - \frac{g}{2}\mathcal{V}_\beta{}^m f_{nm}{}^p\mathbf{F}^n{}_{ab}\mathcal{V}_p{}^\gamma\mathbb{P}_{a\beta\gamma}.$$
$$\tag{B.10}$$

Using the Bianchi identity $D\mathbf{F}^m = 0$ and the above equations of motion, we have

$$\frac{1}{2}\mathcal{D}^a\left\{e^{-\frac{2}{3}\alpha(d+1)\phi}\left[\mathbf{F}^\alpha{}_{ac}\mathbf{F}^\alpha{}_{bd}\eta^{cd} - \frac{1}{2}(\mathbf{F}^\gamma)^2\eta_{ab}\right]\right\}$$
$$= \frac{1}{2}\mathcal{D}^a\left(e^{-\frac{2}{3}\alpha(d+1)\phi}\mathbb{M}_{mn}\right)\left[\mathbf{F}^m{}_{ac}\mathbf{F}^n{}_{bd}\eta^{cd} - \frac{1}{2}\mathbf{F}^m\mathbf{F}^n\eta_{ab}\right] - \frac{1}{2}\mathcal{D}^a\left(e^{-\frac{2}{3}\alpha(d+1)\phi}\right)\mathbf{F}^\alpha{}_{ac}\mathbf{F}^\alpha{}_{bc}$$
$$+ e^{-\frac{2}{3}\alpha(d+1)\phi}\mathbf{F}^\alpha{}_{cd}\mathbb{F}^\beta{}_{bc}\mathbb{P}^d{}_{\alpha\beta} - g\epsilon_{\alpha\gamma\delta}\mathbb{M}^{\delta\lambda}\mathbb{P}_{c\gamma\lambda}\mathbf{F}^\alpha{}_{bc} + 2e^{-\frac{1}{3}\alpha(d-5)\phi}\widehat{T}_{c\alpha}\mathcal{V}_n{}^\alpha\mathbf{F}^n{}_{bc}.$$
$$\tag{B.11}$$

Summing up all the terms, we have

$$\mathfrak{D}^a T^{\text{total}}_{ab} = \mathfrak{D}^a T^{\text{fluid}}_{ab} + \alpha\partial_b\phi\left[-\frac{1}{8(d-2)}e^{-\frac{2}{3}\alpha(d+1)\phi}(\mathbf{F}^\gamma)^2 + \frac{1}{2(d-2)}g^2e^{\frac{2}{3}\alpha(d+1)\phi}\right.$$
$$\left.-\frac{1}{2(d-2)}e^{2\alpha\phi}\left(\frac{3}{d+1}\widehat{T}^{\text{fluid}} - \widehat{T}^{\text{fluid}}_{\lambda\sigma}\delta^{\lambda\sigma}\right)\right]$$
$$+\frac{1}{2}e^{-\frac{2}{3}\alpha(d+1)\phi}\left[\mathbf{F}^\beta\mathbf{F}^\gamma - \frac{1}{3}(\mathbf{F}^\sigma)^2\delta_{\beta\gamma}\right]\mathbb{P}_{b\beta\gamma}$$
$$+4g^2e^{\frac{2}{3}\alpha(d+1)\phi}\left[\mathbb{M}^{\beta\lambda}\mathbb{M}^{\gamma\lambda} - \frac{1}{2}\mathbb{M}^{\beta\gamma}\mathbb{M} - \frac{1}{3}\delta_{\beta\gamma}V\right]\mathbb{P}_{b\beta\gamma}$$
$$-2\mathbb{P}_{[a|\beta\lambda}\mathcal{V}_\lambda{}^m D_{|b]}\mathcal{V}_m{}^\gamma\mathbb{P}_{a\beta\gamma} - \frac{g}{2}\mathcal{V}_\beta{}^m f_{nm}{}^p\mathbf{F}^n{}_{ab}\mathcal{V}_p{}^\gamma\mathbb{P}_{a\beta\gamma}$$
$$+\frac{1}{2}\mathcal{D}^a\left(e^{-\frac{2}{3}\alpha(d+1)\phi}\mathbb{M}_{mn}\right)\left[\mathbf{F}^m{}_{ac}\mathbf{F}^n{}_{bd}\eta^{cd} - \frac{1}{2}\mathbf{F}^m\mathbf{F}^n\eta_{ab}\right]$$
$$-\frac{1}{2}\mathcal{D}^a\left(e^{-\frac{2}{3}\alpha(d+1)\phi}\right)\mathbf{F}^\alpha{}_{ac}\mathbf{F}^\alpha{}_{bc} + e^{-\frac{2}{3}\alpha(d+1)\phi}\mathbf{F}^\alpha{}_{cd}\mathbf{F}^\beta{}_{bc}\mathbb{P}^d{}_{\alpha\beta}$$
$$-g\epsilon_{\alpha\gamma\delta}\mathbb{M}^{\delta\lambda}\mathbb{P}_{c\gamma\lambda}\mathbf{F}^\alpha{}_{bc} + 2e^{-\frac{1}{3}\alpha(d-5)\phi}\widehat{T}_{c\alpha}\mathcal{V}_n{}^\alpha F^n{}_{bc} - g^2 D^a\left[e^{\frac{2}{3}\alpha(d+1)\phi}V(\mathbb{M})\eta_{ab}\right].$$
$$\tag{B.12}$$

If we assume flat Minkowski in $d$ dimensions and that scalar fields are covariantly constant,

$$D_a \mathcal{V}_m{}^\alpha = D_a \phi = 0 \,, \tag{B.13}$$

this expression reduces to

$$D^a T_{ab}^{\text{fluid}} + 2e^{-\frac{1}{3}\alpha(d-5)\phi} \widehat{T}_{c\alpha} \mathcal{V}_n{}^\alpha \mathbf{F}^n{}_{bc} = e^{2\alpha\phi} \left( D^a \widehat{T}_{ab}^{\text{fluid}} + 2e^{-\frac{1}{3}\alpha(d+1)} \widehat{T}_{c\alpha}^{\text{fluid}} \mathbf{F}^\alpha{}_{bc} \right) = 0 \,. \tag{B.14}$$

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