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Estimates for the Poisson kernel and the evolution kernel on the Heisenberg group

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Abstract. We obtain an upper estimate for the Poisson kernel for the class of second-order left invariant differential operators on the semi-direct product of the $2n+1$-dimensional Heisenberg group $\mathcal{H}_n$ and an Abelian group $A=\mathbb{R}^k$. We also give an upper estimate for the transition probabilities of the evolution on $\mathcal{H}_n$ driven by the Brownian motion (with drift) in $\mathbb{R}^k$.

1. Introduction

1.1. Poisson kernel on higher rank NA groups

Let S be a semi-direct product $S=N\rtimes A$ where N is a connected and simply connected nilpotent Lie group and A is isomorphic with $\mathbb{R}^k$. For $g\in S$ we let $x(g)=x$ and $a(g)=a$ denote the components of g in this product so that $g=(x,a)$.

In what follows we identify the group A, its Lie algebra $\mathfrak{a}$, and $\mathfrak{a}^*$, the space of linear forms on $\mathfrak{a}$, with the Euclidean space $\mathbb{R}^k$ endowed with the usual scalar product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$. For the vector $x\in\mathbb{R}^k$ we write $x^2=x\cdot x=\langle x,x\rangle=\sum_{i=1}^k x_i^2$. By $\|\cdot\|_\infty$, we denote the maximum norm $\|x\|_\infty=\max_{1\le i\le k}|x_i|$.

We assume that there is a basis $X_1,\dots,X_m$ for $\mathfrak{n}$ that diagonalizes the A-action. Let $\lambda_1,\dots,\lambda_m\in\mathfrak{a}^*=\mathbb{R}^k$ be the corresponding roots, i.e., for every $H\in\mathfrak{a}$, $[H,X_j]=\lambda_j(H)X_j$, $j=1,\dots,m$. As in [4] we assume that there is an element $H_o\in\mathbb{R}^k$ such that $\lambda_j(H_o)>0$ for $1\le j\le m$.

Let, for $\alpha=(\alpha_1,\dots,\alpha_k)\in\mathbb{R}^k$ and real d_j's,

$$\mathcal{L}_\alpha=\sum_{j=1}^r\left(\mathrm{e}^{2\lambda_j(a)}X_j^2+d_j\mathrm{e}^{\lambda_j(a)}X_j\right)+\Delta_\alpha,\quad\text{where }\Delta_\alpha=\sum_{i=1}^k\left(\partial_{a_i}^2-2\alpha_i\partial_{a_i}\right),\qquad(1.1)$$

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and $X_1, \dots, X_r$ satisfy Hörmander condition, i.e., they generate the Lie algebra $\mathfrak{n}$ of N. Then, $\mathcal{L}_\alpha$ is a left invariant differential operator on S. Define

$$\rho_0 = \sum_{j=1}^{m} \lambda_j \quad \text{and} \quad \text{set } \chi(g) = \det(\mathrm{Ad}(g)) = \mathrm{e}^{\rho_0 \cdot a}.$$

Let $A^+ = \mathrm{Int}\{a \in \mathbb{R}^k : \lambda_j(a) \geq 0 \text{ for } 1 \leq j \leq r\}$. If $\alpha \in A^+$ then there exists a *Poisson kernel* ν for $\mathcal{L}_\alpha$ [4]. That is, there is a C^∞ function ν on N such that every bounded $\mathcal{L}_\alpha$-harmonic function F on S may be written as a Poisson integral against a bounded function f on $S/A = N$,

$$F(g) = \int_{S/A} f(gx)\nu(x)\mathrm{d}x = \int_N f(x)\check{\nu}^a(x^{-1}n)\mathrm{d}x,$$

where $\check{\nu}^a(x) = \nu(a^{-1}x^{-1}a)\chi(a)^{-1}$. Conversely the Poisson integral of any $f \in L^\infty(N)$ is a bounded $\mathcal{L}_\alpha$-harmonic function.

For $t \in \mathbb{R}^+$ and $\rho \in A^+$, let $\delta_t^\rho = \mathrm{Ad}((\log t)\rho)|_N$. Then, $t \mapsto \delta_t^\rho$ is a one parameter group of automorphisms of N for which the corresponding eigenvalues on $\mathfrak{n}$ are all positive. It is known [10] that then N has δ_t^ρ-homogeneous norm: a non-negative continuous function $|\cdot|_\rho$ on N such that $|n|_\rho = 0$ if and only if $n = e$ and $|\delta_t^\rho n|_\rho = t|n|_\rho$. For many years the best pointwise estimate in higher rank available in the literature was

$$\nu(x) \leq C_\rho(1 + |x|_\rho)^{-\varepsilon}$$

for some $\varepsilon > 0$, where $\rho \in A^+$ ([4,5]). This estimate was significantly improved by the authors in [12,13]. A simplified version of [12, Theorem 1.2] says

THEOREM 1.1. ([12, Theorem 1.2]) *For every given $\rho \in A^+$, there exist positive constants C and c (c is explicitly computable) such that the following estimate holds*

$$\nu(x) \leq C(1 + |x|_\rho)^{-c\rho_0(\rho)\gamma(\alpha)},$$

where $\gamma(\alpha) = 2\min_{1\leq j\leq r} \frac{\lambda_j(\alpha)}{\lambda_j^2}$.

1.2. Statements of the main results

The estimate given in Theorem 1.1 is not optimal. In this work we consider the case where $N = \mathcal{H}_n$, the $2n+1$-dimensional Heisenberg group, which we realize as $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with the Lie group multiplication given by

$$(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 \cdot y_2).$$

In this realization

$$(x_1, y_1, z_1) = (0, y_1, z_1)(x_1, 0, 0).$$

Hence $\mathcal{H}_n$ decomposes as a semi-direct product of $\mathbb{R}^{2n}$ and $\mathbb{R}^n$ and the corresponding Lie algebra $\mathfrak{h}_n$ is spanned by the left invariant vector fields

$$X_j = \partial_{x_j}, \quad Y_j = \partial_{y_j} + x_j \partial_z, \quad Z = \partial_z, \tag{1.2}$$

where $1 \leq j \leq n$. Let $A = \mathbb{R}^k$ and let $\xi_{1,j}, \xi_{2,j}, \xi_3 \in \left(\mathbb{R}^k\right)^*$, $1 \leq j \leq n$, be such that

$$\xi_{1,j} + \xi_{2,j} = \xi_3$$

independently of j. For $x \in \mathbb{R}^n$, $a \in \mathbb{R}^k$, and $i = 1, 2$, we set

$$e^{\xi_i(a)} x = \left(e^{\xi_{i,1}(a)} x_1, e^{\xi_{i,2}(a)} x_2, \ldots, e^{\xi_{i,n}(a)} x_n\right).$$

We then define an A action on $\mathcal{H}_n$ by automorphisms of $\mathcal{H}_n$ by

$$a(x, y, z)a^{-1} = (e^{\xi_1(a)} x, e^{\xi_2(a)} y, e^{\xi_3(a)} z), \tag{1.3}$$

We then let $S = \mathcal{H}_n \rtimes A$.

Let $\overline{X}_j, \overline{Y}_j$, and $\overline{Z}$ be, respectively, X_j, Y_j, and Z considered as left invariant vector fields on S. Then,

$$\overline{X}_j = e^{\xi_{1,j}(a)} X_j, \quad \overline{Y}_j = e^{\xi_{2,j}(a)} Y_j, \quad \overline{Z} = e^{\xi_3(a)} Z.$$

We set $\mathcal{L}_\alpha = \sum_{j=1}^n \left(\overline{X}_j^2 + \overline{Y}_j^2\right) + \overline{Z}^2 + \Delta_\alpha$, where Δ_α is as in (1.1). Then,

$$\mathcal{L}_\alpha = \sum_{j=1}^n \left(e^{2\xi_{1,j}(a)} X_j^2 + e^{2\xi_{2,j}(a)} Y_j^2\right) + e^{2\xi_3(a)} Z^2 + \Delta_\alpha. \tag{1.4}$$

We assume also that $\xi_{i,j}(\alpha) > 0$, $1 \leq j \leq n$, $\ i = 1, 2$.

THEOREM 1.2. *For every $\rho \in A^+$ and $\varepsilon > 0$ there exists a constant $C = C_{\rho,\varepsilon} > 0$ such that*

$$\nu(x, y, z) \leq C(1 + |(x, y, z)|_\rho)^{-\gamma},$$

where

$$\gamma = \frac{1}{2} \min_j \xi_{1,j}(\rho) \min_j (\xi_{1,j}(\alpha)/\xi_{1,j}^2) + \frac{1}{2} \min_j (\xi_{1,j}(\rho), \xi_{2,j}(\rho)) \min_j (\xi_{1,j}(\alpha)/\xi_{1,j}^2, \xi_{2,j}(\alpha)/\xi_{2,j}^2, \xi_3(\alpha)/\xi_3^2)$$

for $\|(y, z)\| \geq \varepsilon$ and $\|x\|_\infty \geq \varepsilon$,

$$\gamma = \min_j \xi_{1,j}(\rho) \min_j (\xi_{1,j}(\alpha)/\xi_{1,j}^2)$$

for $\|x\|_\infty \geq \varepsilon$, and

$$\gamma = \min_j (\xi_{1,j}(\rho), \xi_{2,j}(\rho)) \min_j (\xi_{1,j}(\alpha)/\xi_{1,j}^2, \xi_{2,j}(\alpha)/\xi_{2,j}^2, \xi_3(\alpha)/\xi_3^2)$$

for $\|(y, z)\| \geq \varepsilon$.

REMARK. *Our estimates in the rank 1 case are considerably weaker than ones available in the literature. (See [3,7,8,16], for example.) However, in the higher rank case the estimates given in Theorem 1.2 are much better than those from Theorem 1.1. See the example in Sect. 5.2 on p. 25.*

The proof of Theorem 1.2 requires both analytic and probabilistic techniques. Some of them were introduced in [4] and used in [6–8,12]. In particular, we use the following *skew-product formula* for the semigroup T_t generated by $\mathcal{L}_\alpha$, on a general NA group,

$$T_t f(x,a) = \mathbf{E}_a U^\sigma(0,t) f(x,\sigma_t), \tag{1.5}$$

where the expectation is taken with respect to the diffusion σ_t on $\mathbb{R}^k$ generated by Δ_α, i.e., the Brownian motion with drift, and $U^\sigma(s,t)$ is the evolution generated by L^{σ_t} where $\sigma \in C([0,\infty),\mathbb{R}^k)$ and, for $a \in \mathbb{R}^k$,

$$L^a = \sum_{j=1}^{n} \left(\mathrm{e}^{2\xi_{1,j}(a)} X_j^2 + \mathrm{e}^{2\xi_{2,j}(a)} Y_j^2 \right) + \mathrm{e}^{2\xi_3(a)} Z^2. \tag{1.6}$$

Thus, $U^\sigma(s,t)$ is the (unique) family of bounded (on appropriate space of functions on $\mathcal{H}_n$) convolution operators $U^\sigma(s,t)f = f * P^\sigma(t,s)$, with smooth kernels (transition probabilities) $P^\sigma(t,s)$, which have some properties generalizing semigroup property (see p. 8).

In order to get estimates for the Poisson kernel it is necessary to have estimates for $P^\sigma(t,0)(x)$. The best general result we are aware of in the literature is Theorem 1.3 below. See [6,7] and [12].

Let

$$A^\sigma(s,t) = \sum_{\substack{j=1,\dots,k \\ d=1,2}} \int_s^t \mathrm{e}^{d\lambda_j(\sigma_u)} \mathrm{d}u. \tag{1.7}$$

THEOREM 1.3. *Let $K \subset N$ be closed and $e \notin K$, where e is the identity element of N. Then, there exist constants C_1, C_2, and ν such that for every $x \in K$ and for every t,*

$$P^\sigma(t,0)(x) \le C_1 \left(\int_0^t \chi(\sigma_u)^{2/\nu} \mathrm{d}u \right)^{-\nu/2} \exp\left(\frac{\tau(x)}{4} - \frac{\tau(x)^2}{C_2 A^\sigma(0,t)} \right),$$

where τ is a subadditive norm which is smooth on $N \setminus \{e\}$.

It is clear that this estimate is not optimal; it follows from formula (2.6) below, for example, that if N is Abelian, a similar estimate holds without the $\frac{\tau(x)}{4}$ term. In the rank-one case the presence of this term does not cause a problem; it is enough to consider x is in a compact set. In the higher rank case, this term does create difficulties. Our second main result, Theorem 1.4 below, which plays the crucial role in the proof of Theorem 1.2, is an estimate for $P^\sigma(t,0)$ on $\mathcal{H}_n$ which does not contain such a term. We conjecture that a similar result holds general for nilpotent groups N.

In order to state this result, let

$$\begin{aligned} A^{\sigma}_{k,j}(s,t) &= \int_s^t \mathrm{e}^{2\xi_{k,j}(\sigma(u))}\mathrm{d}u, \quad A^{\sigma}_{k}(s,t) = \sum_{1\le j\le n} A^{\sigma}_{k,j}(s,t), \quad k=1,2, \\ A^{\sigma}_{3}(s,t) &= \int_s^t \mathrm{e}^{2\xi_{3}(\sigma(u))}\mathrm{d}u, \quad A^{\sigma}_{k,\Pi}(s,t) = \prod_{j=1}^{n} A^{\sigma}_{k,j}(s,t), \quad k=1,2. \end{aligned} \tag{1.8}$$

We also set

$$\begin{aligned} &A_0 = A^{\sigma}_{1,\Pi}(0,t)^{\frac{1}{2}} A^{\sigma}_{2,\Pi}(0,t)^{\frac{1}{2}} A^{\sigma}_{3}(0,t)^{\frac{1}{2}}, \quad A_1 = A^{\sigma}_{1}(0,t), \\ &A_2 = A^{\sigma}_{2}(0,t) + A^{\sigma}_{3}(0,t), \quad A_3 = A^{\sigma}_{1}(0,t) + A^{\sigma}_{2}(0,t) + A^{\sigma}_{3}(0,t). \end{aligned} \tag{1.9}$$

Finally we let

$$\phi(m) = \left(\frac{\|m\|^{1/2}}{\|m\|^{1/2}+1}\right)^2. \tag{1.10}$$

THEOREM 1.4. *There are positive constants C and D such that for all x, y and z,*

$$\begin{aligned} &P^{\sigma}(t,0)(x,y,z) \\ &\le C A_0^{-1}\left(\|(y,z)\|^{1/2} + 1 + A_1^{1/2}\right)\exp\left(-D\frac{\|x\|_\infty^2}{A_1} - D\frac{\|(y,z)\|}{A_3}\phi(y,z)\right). \end{aligned}$$

The proof of Theorem 1.4 is based on our third main result, Corollary 3.5, that allows us to decompose the diffusion defined by $P^{\sigma}(t,s)$ on a Lie group N which can be expressed as an appropriate semi-direct product of two subgroups, into vertical and horizontal components, in much the same way that formula (1.5) decomposes the diffusion defined by $\mathcal{L}_\alpha$ on S.

2. Preliminaries

2.1. Exponential functionals of Brownian motion

Let b_s, $s \ge 0$, be the Brownian motion on $\mathbb{R}$ starting from $a \in \mathbb{R}$ and normalized so that $\mathbf{E}b_s = a$ and $\operatorname{Var} b_s = 2s$.

For $d > 0$ and $\mu > 0$ we define the following exponential functional

$$I_{d,\mu} = \int_0^\infty \mathrm{e}^{d(b_s-\mu s)}\mathrm{d}s. \tag{2.1}$$

THEOREM 2.1. (Dufresne, [9]) *Let $b_0 = 0$. Then, the functional $I_{2,\mu}$ is distributed as $(4\gamma_{\mu/2})^{-1}$, where $\gamma_{\mu/2}$ denotes a gamma random variable with parameter $\mu/2$, i.e., $\gamma_{\mu/2}$ has a density $(1/\Gamma(\mu/2))x^{\frac{\mu}{2}-1}\mathrm{e}^{-x}\mathbf{1}_{[0,+\infty)}(x)$, where Γ is the gamma function.*

As a corollary from Theorem 2.1, by scaling the Brownian motion and changing the variable, we get the following

THEOREM 2.2. *Let* $b_0 = a$. *Then,*

$$\mathbf{E}_a f(I_{d,\mu}) = c_{d,\mu} \mathrm{e}^{\mu a} \int_0^\infty f(x) x^{-\mu/d} \exp\left(-\frac{\mathrm{e}^{\mathrm{d}a}}{\mathrm{d}^2 x}\right) \frac{\mathrm{d}x}{x}.$$

The following lemma follows from Theorem 2.2. (See [12, Lemma 5.4] for details.)

LEMMA 2.3. *Let* $\sigma_u = b_u - 2\alpha u$ *be the k-dimensional Brownian motion with a drift. Let* $d > 0$, *and let* $\ell \in (\mathbb{R}^k)^*$ *be such that* $\ell(\alpha) > 0$. *Then,*

$$\mathbf{E}_a f\left(\int_0^\infty \mathrm{e}^{\mathrm{d}\ell(\sigma_u)} \mathrm{d}u\right) = c_{d,\ell,\alpha} \mathrm{e}^{\gamma \ell(a)} \int_0^\infty f(u) u^{-\gamma/d} \exp\left(-\frac{\mathrm{e}^{\mathrm{d}\ell(a)}}{2\mathrm{d}^2 \ell^2 u}\right) \frac{\mathrm{d}u}{u},$$

where $\gamma = 2\ell(\alpha)/\ell^2$.

2.2. Some probabilistic lemmas

If b_t is the Brownian motion starting from $x \in \mathbb{R}$, then the corresponding Wiener measure on the space $C([0,\infty), \mathbb{R})$ is denoted by $\mathbf{W}_x$. The following lemma follows from formula 1.1.4 on p. 125 in [1].

LEMMA 2.4. *There exists a constant* $c > 0$ *such that for all* $x \leq y$,

$$\mathbf{W}_x\left(\sup_{0<s<t} |b_s| \geq y\right) \leq c \mathrm{e}^{-(y-x)^2/4t}.$$

The following two equalities follow easily from the reflection principle for the Brownian motion [11].

LEMMA 2.5. *If* $x > a > 0$, *then*

$$\mathbf{W}_0\left(\sup_{u\in[0,t]} b_u \geq a \text{ and } b_t \leq x\right) = 2\mathbf{W}_0(b_t > a) - \mathbf{W}_0(b_t > x),$$

whereas if $x < a$ *with* $a > 0$, *then*

$$\mathbf{W}_0\left(\sup_{u\in[0,t]} b_u \geq a \text{ and } b_t \leq x\right) = \mathbf{W}_0(b_t > 2a - x).$$

Note that $\mathbf{W}_0(b_t > a) = 1 - \Phi(a/\sqrt{t})$, where $\Phi(x) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^x \mathrm{e}^{-u^2/4} \mathrm{d}u$. As a corollary from Lemma 2.5 we get the following.

LEMMA 2.6. *For* $a \geq 0$, $x, y \in \mathbb{R}$ *with* $x < y$, *and* $t > 0$, *let*

$$R_1 = \{-a \leq x < y \leq a\}, \quad R_2 = \{x < y < -a\},$$
$$R_3 = \{a < x < y\}, \quad R_4 = \{0 < x < a < y\}.$$

Then,

$$\mathbf{W}_0\left(\sup_{u\in[0,t]}|b_u|\geq a \text{ and } b_t\in[x,y]\right)$$
$$\leq\begin{cases}2\Phi\left(\frac{2a-x}{\sqrt{t}}\right)-2\Phi\left(\frac{2a-y}{\sqrt{t}}\right)+2\Phi\left(\frac{2a+y}{\sqrt{t}}\right)-2\Phi\left(\frac{2a+x}{\sqrt{t}}\right), & \text{on } R_1,\\ 2\Phi\left(\frac{2a-x}{\sqrt{t}}\right)-2\Phi\left(\frac{2a-y}{\sqrt{t}}\right)+\Phi\left(\frac{-x}{\sqrt{t}}\right)-\Phi\left(\frac{-y}{\sqrt{t}}\right), & \text{on } R_2,\\ \Phi\left(\frac{y}{\sqrt{t}}\right)-\Phi\left(\frac{x}{\sqrt{t}}\right)+2\Phi\left(\frac{2a+y}{\sqrt{t}}\right)-2\Phi\left(\frac{2a+x}{\sqrt{t}}\right), & \text{on } R_3,\\ 2\left(1-\Phi\left(\frac{a}{\sqrt{t}}\right)\right)-\Phi\left(\frac{y}{\sqrt{t}}\right)-\Phi\left(\frac{2a-x}{\sqrt{t}}\right)+\Phi\left(\frac{2a+x}{\sqrt{t}}\right)-\Phi\left(\frac{2a+y}{\sqrt{t}}\right), & \text{on } R_4.\end{cases} \tag{2.2}$$

COROLLARY 2.7. *Assume that* $a>|n|+\delta$, $\delta>0$, *and* $0<\varepsilon/2<\delta$. *Then,*

$$\varepsilon^{-1}\mathbf{W}_0\left(\sup_{u\in[0,t]}|b_u|\geq a \quad \text{and} \quad b_t\in[n-\varepsilon/2,n+\varepsilon/2]\right)$$
$$\leq\frac{1}{\sqrt{\pi t}}\left(\mathrm{e}^{-(2a-n)^2/(4t)}+\mathrm{e}^{-(2a+n)^2/(4t)}\right).$$

COROLLARY 2.8. *Assume that* $a\geq 0$. *Then,*

$$\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbf{W}_0\left(\sup_{u\in[0,t]}|b_u|\geq a \text{ and } b_t\in[n-\varepsilon/2,n+\varepsilon/2]\right)$$
$$\leq\begin{cases}\frac{2}{\sqrt{\pi t}}\mathrm{e}^{-(2a-|n|)^2/(4t)} & |n|<a,\\ \frac{2}{\sqrt{\pi t}}\mathrm{e}^{-n^2/(4t)} & 0\leq a\leq |n|.\end{cases}$$

2.3. Disintegration of the diffusion into vertical and horizontal components—skew-product formula

2.3.1. Vertical component

Let $\mathcal{L}_\alpha$ be defined by (1.4). The process σ_t in $\mathbb{R}^k$ generated by the operator Δ_α, i.e., the Brownian motion with drift -2α, is called a *vertical component* of the diffusion generated by $\mathcal{L}_\alpha$.

2.3.2. Horizontal component

Let $C_\infty(\mathcal{H}_n)$ be the space of continuous functions f on $\mathcal{H}_n$ for which $\lim_{x\to\infty} f(x)$ exists. For $\mathcal{X}\in\mathfrak{h}_n$, we let $\tilde{\mathcal{X}}$ denote the corresponding right-invariant vector field. For a multi-index $I=(i_1,\dots,i_m)$, $i_j\in\mathbb{Z}^+$ and a basis $\mathcal{X}_1,\dots,\mathcal{X}_m$ of the Lie algebra $\mathfrak{h}_n$ we write $\mathcal{X}^I=\mathcal{X}_1^{i_1}\dots\mathcal{X}_m^{i_m}$. For $k,l=0,1,2,\dots,\infty$ we define

$$C^{(k,l)}(\mathcal{H}_n)=\left\{f:\tilde{\mathcal{X}}^I\mathcal{X}^J f\in C_\infty(\mathcal{H}_n) \text{ for every} |I|<k+1 \text{ and } |J|<l+1\right\}$$

and

$$\|f\|^0_{(k,l)} = \sup_{|I|=k,|J|=l} \|\tilde{\mathcal{X}}^I \mathcal{X}^J f\|_\infty, \quad \|f\|_{(k,l)} = \sup_{|I|\le k,|J|\le l} \|\tilde{\mathcal{X}}^I \mathcal{X}^J f\|_\infty. \tag{2.3}$$

In particular, $C^{(0,k)}(\mathcal{H}_n)$ is a Banach space with the norm $\|f\|_{0,k}$.

For a continuous function $\sigma : [0,\infty) \to \mathbb{R}^k$, we consider the operator L^{σ_t} where L^a is as in (1.6). Let $\{U^\sigma(s,t) : 0 \le s \le t\}$ be the (unique) family of bounded operators on $C_\infty(\mathcal{H}_n)$ which satisfies

i) $U^\sigma(s,s) = \mathrm{Id}$, for all $s \ge 0$,
ii) $\lim_{h\to 0} U^\sigma(s,s+h)f = f$ in $C_\infty(\mathcal{H}_n)$,
iii) $U^\sigma(s,r)U^\sigma(r,t) = U^\sigma(s,t), 0 \le s \le r \le t$,
iv) $\partial_s U^\sigma(s,t)f = -L^{\sigma_s}U^\sigma(s,t)f$ for every $f \in C^{(0,2)}(\mathcal{H}_n)$,
v) $\partial_t U^\sigma(s,t)f = U^\sigma(s,t)L^{\sigma_t}f$ for every $f \in C^{(0,2)}(\mathcal{H}_n)$,
vi) $U^\sigma(s,t) : C^{(0,2)}(\mathcal{H}_n) \to C^{(0,2)}(\mathcal{H}_n)$.

The operator $U^\sigma(s,t)$ is a convolution operator with a probability measure with a smooth density, i.e., $U^\sigma(s,t)f = f * P^\sigma(t,s)$. In particular, $U^\sigma(s,t)$ is left invariant. By iii), $P^\sigma(t,r) * P^\sigma(r,s) = P^\sigma(t,s)$ for $t > r > s$. Existence of $U^\sigma(s,t)$ follows from [15]. Notice that from above properties it follows that

vii) $U^{\sigma\circ\theta_u}(s,t) = U^\sigma(s+u,t+u)$, where $\sigma\circ\theta_u(s) = \sigma_{s+u}$ is the shift operator.

A stochastic process (evolution) in $\mathcal{H}_n$ corresponding to transition probabilities $P^\sigma(t,s)$ is called a *horizontal component* of the diffusion generated by $\mathcal{L}_\alpha$.

2.3.3. Skew-product formula

Let $U^\sigma(s,t)$ and $P^\sigma(t,s)$ be as in Sect. 2.3.2. For $f \in C_c(N\times\mathbb{R}^k)$ and $t \ge 0$, we put

$$T_t f(x,a) = \mathbf{E}_a U^\sigma(0,t)f(x,\sigma_t) = \mathbf{E}_a f *_N P^\sigma(t,0)(x,\sigma_t), \tag{2.4}$$

where the expectation is taken with respect to the distribution of the process σ_t (Brownian motion with drift) in $\mathbb{R}^k$ with the generator Δ_α. The operator $U^\sigma(0,t)$ acts on the first variable of the function f (as a convolution operator).

THEOREM 2.9. *The family T_t defined in* (2.4) *is the semigroup of operators generated by $\mathcal{L}_\alpha$. That is, $\partial_t T_t f = \mathcal{L}_\alpha T_t f$ and* $\lim_{t\to 0} T_t f = f$.

We refer to formula (2.4) as the *skew-product formula*. By now the proof of the above statement is standard and it goes along the lines of [7] with obvious changes. (In Sect. 3 below a more general skew-product formula is proved.)

2.4. Evolution equation in $\mathbb{R}^n$

Let

$$L^t = \frac{1}{2}\sum_{i,j=1}^n a_{ij}(t)\partial_i\partial_j + \sum_{j=1}^n b_j(t)\partial_j \tag{2.5}$$

be a differential operator on $C^\infty(\mathbb{R}^n)$, where $\partial_i = \partial_{x_i}$ and $a(t) = [a_{ij}(t)]$ is a symmetric, positive definite matrix and the a_{ij} and b_j belong to $C([0,\infty),\mathbb{R})$. For $s \le t$, let $U(s,t)$ be the unique family of operators on $C_\infty(\mathbb{R}^n)$ satisfying conditions (i)-(vi) on page 8 where L^{σ_t} is replaced by L^t. Our goal in this section is to compute the corresponding convolution kernel $P(s,t)$.

Let

$$A_{ij}(s,t) = \int_s^t a_{ij}(u)\mathrm{d}u \equiv A_{i,j}, \quad B_j(s,t) = \int_s^t b_j(u)\mathrm{d}u \equiv B_j.$$

PROPOSITION 2.10. *Let* $A = [A_{ij}]$ *and* $B = (B_1, B_2, \ldots, B_n)^t$. *Then,*

$$P(t,s)(x) = (2\pi)^{-\frac{n}{2}}(\det A)^{-\frac{1}{2}}\mathrm{e}^{-\frac{1}{2}(A^{-1}(x-B))\cdot(x-B)}. \tag{2.6}$$

Proof. For $f_o \in C_\infty(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$, we write $f(x,t) = f_o * P(t,s)(x)$. We note that for $t > s$, $\partial_t f(x,t) = L^t f(x,t)$ and $f(x,s) = f_o(x)$. We form the Fourier transform concluding $\partial_t \hat{f}(\xi,t) = \left(-\frac{1}{2}\sum_{i,j=1}^n a_{ij}\xi_i\xi_j + \sum_{j=1}^n i b_j \xi_j\right)\hat{f}(\xi,t)$. Solving the above equation and forming the inverse transformation we get the proposition. □

2.5. Poisson kernel

Let μ_t be the semigroup of probability measures on $S = \mathcal{H}_n \rtimes \mathbb{R}^k$ generated by $\mathcal{L}_\alpha$. It is known [5,8] that $\lim_{t\to\infty}(\pi(\check{\mu}_t), f) = (\nu, f)$, where π denotes the projection from S onto $\mathcal{H}_n$, and $(\check{\mu}, f) = (\mu, \check{f})$, $\check{f}(x) = f(x^{-1})$. Let $a \in \mathbb{R}^k$ and let μ be a measure on $\mathcal{H}_n$. We define $(\mu^a, f) = (\mu, f\circ \mathrm{Ad}(a))$. For $a \in \mathbb{R}^k$, we have

$$\nu^a(h) = \nu(a^{-1}ha)\chi(a)^{-1}, \quad h \in \mathcal{H}_n, \tag{2.7}$$

where $\chi(b) = \mathrm{e}^{\rho_0\cdot b}$ and $\rho_0 = \sum_{j=1}^n(\xi_{1,j}+\xi_{2,j}) + \xi_3 = (n+1)\xi_3$. It is an easy calculation to check that

$$\lim_{t\to\infty}(\pi(\check{\mu}_t)^a, f) = (\nu^a, f). \tag{2.8}$$

The next lemma follows from Theorem 2.9. (See [12, Lemma 4.1] for the details.)

LEMMA 2.11. *We have* $(\pi(\check{\mu}_t)^a, f) = (\mathbf{E}_a\check{P}^\sigma(t,0), f)$.

By (2.8) and Lemma 2.11 it follows that

$$(\nu^a, f) = \lim_{t\to\infty}(\pi(\check{\mu}_t)^a, f) = \lim_{t\to\infty}(\mathbf{E}_a\check{P}^\sigma(t,0), f). \tag{2.9}$$

3. Split groups and the skew-product formula

Assume that

$$S = N \rtimes A, \quad N = M \rtimes V, \quad S_o = V \rtimes A,$$

where N is nilpotent, M is normal in N, V is a subgroup of N normalized by A, and $A = \mathbb{R}^k$. We denote the general element of these groups by:

$$g = (m, v, a) = (m, x), \quad g \in S, \quad m \in M, \quad v \in V, \quad x \in S_o.$$

Let $\{X_1, \dots, X_n\}$ and $\{Y_1, \dots, Y_m\}$ be Jordan–Hölder bases for $\mathfrak{v}$ and $\mathfrak{m}$, respectively, where $\{X_1, \dots, X_{n_o}\}$ and $\{Y_1, \dots, Y_{m_o}\}$ generate $\mathfrak{v}$ and $\mathfrak{m}$, respectively. We assume also that the X_i and Y_i are eigenvectors for the ad_H-action, $H \in \mathfrak{a}$.

Let

$$L_M = \sum_{i=1}^{n_o} \left(X_i^2 + c_i X_i \right), \quad L_V = \sum_{i=1}^{m_o} \left(Y_i^2 + d_i Y_i \right), \quad L_N = L_V + L_M,$$

where $c_i, d_i \in \mathbb{R}$, and

$$D_o = \Delta_\alpha + L_V, \quad D = \Delta_\alpha + L_N$$

considered as elements of the universal enveloping algebra $\mathfrak{A}(\mathfrak{s})$ where $\alpha \in \mathbb{R}^k$ and Δ_α is as in (1.1). For $g \in S$ and $X \in \mathfrak{A}(\mathfrak{s})$ we let $X^g = \mathrm{Ad}(g)X$.

We consider the diffusion defined by D_o on S_o as the vertical component and that defined by L_M on M as the horizontal. Explicitly, for any topological space X we let $\Omega_X^r = C([r, \infty), X)$ and $\Omega_X = \Omega_X^0$. Then, for $\tau \in \Omega_{S_o}$ the operator $L_M^t = L_M^{\tau(t)}$, considered as a left invariant operator on $C^\infty(M)$ produces an operator $U_M^\tau(s, t)$ on $C_\infty(M)$, $0 \le s \le t$ as on p. 8. We write

$$U_M^\tau(s, t) f(x) = \int_M K_{t,s}^{M,\tau}(y, x) f(y) \mathrm{d}y,$$

where dy is Haar measure on M. The equality $U_M^\tau(s, r) U_M^\tau(r, t) = U_M^\tau(s, t)$, $0 \le s < r < t$ is equivalent with the Chapman–Kolmogorov equation, [2, (15.8), p. 320], $\int_M K_{t,r}^{M,\tau}(y, z) K_{r,s}^{M,\tau}(z, x) dz = K_{t,s}^{M,\tau}(y, x)$. For each $r \ge 0$ and $m \in M$, there is a corresponding Markov process with state space Ω_M^r and a probability measure $\mathbf{W}_{m;r}^{M,\tau}$. We omit r from the notation when it is 0. In particular, for $t_n > t_{n-1} > \dots > t_1 > r$ and the function $f(\tau) = h(\tau(t_n), \tau(t_{n-1}), \dots, \tau(t_1))$,

$$\int_{\Omega_M^r} f(\tau) \mathrm{d}\mathbf{W}_{m;r}^{M,\tau}(\tau)$$
$$= \int_{M^n} K_{t_n,t_{n-1}}^{M,\tau}(x_n, x_{n-1}) \dots K_{t_1,r}^{M,\tau}(x_1, m) h(x_n, x_{n-1}, \dots, x_1) \mathrm{d}x_n \dots \mathrm{d}x_1. \quad (3.1)$$

Similarly, we denote the respective transition kernels for D_o, D, Δ_α on S_o, S and $\mathbb{R}^k$ by $K_{t,s}^o$, $K_{t,s}$, and $K_{t,s}^A$, respectively. The corresponding operators are $U^o(s, t) = \mathrm{e}^{(t-s)D_o}$, $U(s, t) = \mathrm{e}^{(t-s)D}$, and $U^A(s, t) = \mathrm{e}^{(t-s)\Delta_\alpha}$. We denote the corresponding measures on $\Omega_{S_o}^r$, Ω_S^r and Ω_A^r by $\mathbf{W}_{x;r}^{S_o}$, $\mathbf{W}_{m,x;r}^S$, and $\mathbf{W}_{a;r}^A$, respectively, where $x = (v, a) \in S_o$ and $m \in M$.

The following proposition is an extension of Theorem 2.9 to the case where S_o is non-abelian. The proof follows [7].

PROPOSITION 3.1. *For* $f \in C_\infty(S)$,

$$U(0,t)f(m,x) = \int_{\Omega_M} \left(U_M^\tau(0,t)f\right)(m,\tau(t))\mathrm{d}\mathbf{W}_x^{S_o}(\tau) \equiv T_{0,t}f(m,x).$$

In the proof of Proposition 3.1 we will need the following lemma.

LEMMA 3.2.

$$T_{0,t}f(m,x) = \int_0^t U^o(0,t-u)|_y \left[L_M^y|_m T_{0,u}f(m,y)\right](x)\mathrm{d}u + (U^o(0,t)f)(m,x),$$

where the subscript indicates the variable on which the operator operates.

Proof.

$$\begin{aligned}
&(T_{0,t}f - U^o(0,t)f)(m,x)\\
&= \int U_M^\tau(0,t)f(m,\tau(t))\mathrm{d}\mathbf{W}_x^{S_o}(\tau) - \int U_M^\tau(t,t)f(m,\tau(t))\mathrm{d}\mathbf{W}_x^{S_o}(\tau)\\
&= \int U_M^\tau(t-u,t)\big|_{u=0}^{u=t} f(m,\tau(t))\mathrm{d}\mathbf{W}_x^{S_o}(\tau)\\
&= -\int\int_0^t \partial_u U_M^\tau(t-u,t)f(m,\tau(t))\mathrm{d}u\mathrm{d}\mathbf{W}_x^{S_o}(\tau)\\
&= \int_0^t\int L_M^{\tau(t-u)}U_M^\tau(t-u,t)f(m,\tau(t))\mathrm{d}\mathbf{W}_x^{S_o}(\tau)\mathrm{d}u\\
&= \int_0^t\int L_M^{\tau\circ\theta_{t-u}(0)}U_M^{\tau\circ\theta_{t-u}}(0,u)f(m,\tau\circ\theta_{t-u}(u))\mathrm{d}\mathbf{W}_x^{S_o}(\tau)\mathrm{d}u\\
&= \int_0^t\int_{S_o}\int L_M^y U_M^\tau(0,u)f(m,\tau(u))\mathrm{d}\mathbf{W}_y^{S_o}(\tau)K_{(t-u)}^o(x,y)\mathrm{d}y\mathrm{d}u\\
&= \int_0^t U^o(0,t-u)|_y[L_M^y|_m T_u f(m,y)](x)\mathrm{d}u.
\end{aligned}$$

□

Proof of Proposition 3.1. From Lemma 3.2

$$\begin{aligned}
\partial_t T_{0,t}(f)(m,x) &= \partial_t \int_0^t U^o(0,t-u)|_y \left[L_M^y|_m T_{0,u}f(m,y)\right](x)\mathrm{d}u\\
&\quad + \partial_t(U^o(0,t)f)(m,x)\\
&= L_M^x|_m T_{0,t}f(m,x) + D^o(T_{0,t}f - U^o(0,t)f)(m,x)\\
&\quad + D^oU^o(0,t)f(m,x)\\
&= \overline{L}_M T_{0,t}f(m,x) + D^oT_{0,t}f(m,x),
\end{aligned}$$

where $\overline{L}_M$ is L_M considered as a left invariant operator on S. This proves Proposition 3.1. □

We may express $\mathbf{W}^S_{m,x}$ in terms of $\mathbf{W}^{S_o}_x$ and $\mathbf{W}^\tau_m$. Recall that for $\tau \in \Omega_{S_o}$, $L^t_M = L^{\tau(t)}_M$.

Let $f \in C(S)$. Then, by Proposition 3.1, we have[1]

$$\begin{aligned}\int_{\Omega_S} f(\tau(t))\mathrm{d}\mathbf{W}^S_{m,x}(\tau) &= \int_{M\times S_o} K_{t,0}(m,x;l,y)f(l,y)\mathrm{d}l\mathrm{d}y\\ &= \int_{\Omega_{S_o}}\left(\int_M K^{M,\eta}_{t,0}(m;l)f(l,\eta(t))\mathrm{d}l\right)\mathrm{d}\mathbf{W}^{S_o}_x(\eta)\\ &= \int_{\Omega_{S_o}}\int_{\Omega_M} f(\mu(t),\eta(t))\mathrm{d}\mathbf{W}^{M,\eta}_m(\mu)\mathrm{d}\mathbf{W}^{S_o}_x(\eta). \quad (3.2)\end{aligned}$$

Note that $(\mu,\eta) \in \Omega_S$. This suggests the following:

THEOREM 3.3.

$$\int_{\Omega_S} f(\tau)\mathrm{d}\mathbf{W}^S_{m,x}(\tau) = \int_{\Omega_{S_o}}\int_{\Omega_M} f(\mu,\eta)\mathrm{d}\mathbf{W}^{M,\eta}_m(\mu)\mathrm{d}\mathbf{W}^{S_o}_x(\eta). \quad (3.3)$$

Hence,

$$\mathbf{W}^S_{m,x}(\mu,\eta) = \mathbf{W}^{M,\eta}_m(\mu)\mathbf{W}^{S_o}_x(\eta).$$

Proof. We have the following proposition, where $\mathbf{W}_{w;s}$ is the measure corresponding to any general Markov process $\xi(t)$ on Ω^s_X (with $\xi(s) = w$). This is a restatement and generalization of Lemma 4.1.4, p. 189 from [14].

PROPOSITION 3.4. *Suppose that for* $s < t$,

$$f(\tau) = h(\tau|_{[s,t]}, \tau|_{[t,\infty)}).$$

Then,

$$\int_{\Omega^s_X} f(\tau)\mathrm{d}\mathbf{W}_{w,s}(\tau) = \int_{\Omega^s_X}\int_{\Omega^t_X} h(\tilde\psi,\psi)\mathrm{d}\mathbf{W}_{\tilde\psi(t),t}(\psi)\mathrm{d}\mathbf{W}_{w,s}(\tilde\psi).$$

Let $g = (m,x)$. The right-hand side of (3.3) defines a measure on Ω_S which we temporarily denote $\tilde{\mathbf{W}}^S_g$. The sequence of equalities (3.2) prove that $\mathbf{W}^S_g(f) = \tilde{\mathbf{W}}^S_g(f)$ for $f(\tau) = h(\tau(t))$.

Suppose that

$$f(\tau) = h(\tau(t_1), \tau(t_2)).$$

[1] dl denotes *right*-invariant Haar measure on S_o. Hence dldm is right-invariant Haar measure on S. Expressing densities with respect to right-invariant measure is not a problem as long as we do not write our kernels as convolutions. It has the convenience that the measures split in the semi-direct product decomposition.

Then, with $\tau = (\mu, \eta)$, $w = (m, x)$ and $0 < t_1 < t_2$,

$$\begin{aligned}\int_{\Omega_S} h(\tau(t_1), \tau(t_2))\mathrm{d}\mathbf{W}^S_w(\tau) &= \int_{\Omega_S}\int_{\Omega_S^{t_1}} h(\tilde\tau(t_1), \tau(t_2))\mathrm{d}\mathbf{W}^S_{\tilde\tau(t_1),t_1}(\tau)\mathrm{d}\mathbf{W}^S_w(\tilde\tau)\\ &= \int_{\Omega_{S_o}}\int_{\Omega_M}\int_{\Omega_S^{t_1}} h\left(\tilde\mu(t_1), \tilde\eta(t_1), \tau(t_2)\right)\mathrm{d}\mathbf{W}^S_{\tilde\tau(t_1),t_1}(\tau)\mathrm{d}\mathbf{W}^{M,\tilde\eta}_m(\tilde\mu)\mathrm{d}\mathbf{W}^{S_o}_x(\tilde\eta)\\ &= \int_{\Omega_{S_o}}\int_{\Omega_{S_o}^{t_1}}\int_{\Omega_M}\int_{\Omega_M^{t_1}} h(\tilde\mu(t_1), \tilde\eta(t_1), \mu(t_2), \eta(t_2))\\ &\quad\times \mathrm{d}\mathbf{W}^{M,\eta}_{\tilde\mu(t_1),t_1}(\mu)\mathrm{d}\mathbf{W}^{M,\tilde\eta}_m(\tilde\mu)\mathrm{d}\mathbf{W}^{S_o}_{\tilde\eta(t_1),t_1}(\eta)\mathrm{d}\mathbf{W}^{S_o}_x(\tilde\eta).\end{aligned} \tag{3.4}$$

We wish to combine (3.4) into a single η integral. We write (3.4) as

$$\int_{\Omega_S} h(\tau(t_1), \tau(t_2))\mathrm{d}\mathbf{W}^S_w(\tau) = \int_{\Omega_{S_o}}\int_{\Omega_{S_o}^{t_1}} H(\tilde\eta, \eta)\mathrm{d}\mathbf{W}^{S_o}_{\tilde\eta(t_1),t_1}(\eta)\mathrm{d}\mathbf{W}^{S_o}_x(\tilde\eta),$$

where

$$H(\tilde\eta, \eta) = \int_{\Omega_M}\int_{\Omega_M^{t_1}} h(\tilde\mu(t_1), \tilde\eta(t_1), \mu(t_2), \eta(t_2))\mathrm{d}\mathbf{W}^{M,\eta}_{\tilde\mu(t_1),t_1}(\mu)\mathrm{d}\mathbf{W}^{M,\tilde\eta}_m(\tilde\mu).$$

As a function of η, h depends only on $\eta|_{[t_1,\infty)}$. The $\tilde\eta$ dependence is also not problem since

$$\begin{aligned}&\int_{\Omega_M} h(\tilde\mu(t_1), \tilde\eta(t_1), \mu(t_2), \eta(t_2))\mathrm{d}\mathbf{W}^{M,\tilde\eta}_m(\tilde\mu)\\ &\quad= \int_{S_o} h(y, \tilde\eta(t_1), \mu(t_2), \eta(t_2))K^{M,\tilde\eta}_{t_1,0}(m, y)\mathrm{d}y\end{aligned}$$

which depends on $\tilde\eta|_{[0,t_1]}$. Hence

$$\begin{aligned}\int_{\Omega_S} h(\tau(t_1), \tau(t_2))\mathrm{d}\mathbf{W}^S_w(\tau) &= \int_{\Omega_{S_o}}\int_{\Omega_{S_o}^{t_1}} H(\tilde\eta, \eta)\mathrm{d}\mathbf{W}^{S_o}_{\tilde\eta(t_1),t_1}(\eta)\mathrm{d}\mathbf{W}^{S_o}_x(\tilde\eta)\\ &= \int_{\Omega_{S_o}} H(\eta, \eta)\mathrm{d}\mathbf{W}^{S_o}_w(\eta)\\ &= \int_{\Omega_{S_o}}\int_{\Omega_M}\int_{\Omega_M^{t_1}} h(\eta(t_1), \tilde\mu(t_1), \eta(t_2), \mu(t_2))\mathrm{d}\mathbf{W}^{M,\eta}_{\tilde\mu(t_1),t_1}(\mu)\mathrm{d}\mathbf{W}^{M,\eta}_m(\tilde\mu)\mathrm{d}\mathbf{W}^{S_o}_w(\eta)\\ &= \int_{\Omega_{S_o}}\int_{\Omega_M} h(\eta(t_1), \mu(t_1), \eta(t_2), \mu(t_2))\mathrm{d}\mathbf{W}^{M,\eta}_m(\mu)\mathrm{d}\mathbf{W}^{S_o}_w(\eta)\end{aligned}$$

as desired. The general case follows similarly. □

For $\sigma \in \Omega^A$, and $(m, x) \in M \times V$, let $\mathbf{W}^{V,\sigma}_x$, $\mathbf{W}^{N,\sigma}_{m,x}$ be the measures on Ω^N and Ω^V, respectively, defined similarly to the definition of $\mathbf{W}^{M,\eta,\sigma}_m$.

COROLLARY 3.5. *For a.e.* σ *with respect to* $\mathbf{W}^A_a$ *and* $(\mu, \gamma) \in \Omega^M \times \Omega^V$,

$$\mathbf{W}^{N,\sigma}_{m,x}(\mu, \gamma) = \mathbf{W}^{M,\gamma,\sigma}_m(\mu)\mathbf{W}^{V,\sigma}_x(\gamma).$$

Proof. Theorem 3.3 implies:

$$\mathbf{W}^{S}_{m,x,a}(\mu,\gamma,\sigma)=\mathbf{W}^{M,\gamma,\sigma}_{m}(\mu)\mathbf{W}^{S_o}_{x,a}(\gamma,\sigma)=\mathbf{W}^{M,\gamma,\sigma}_{m}(\mu)\mathbf{W}^{V,\sigma}_{x}(\gamma)\mathbf{W}^{A}_{a}(\sigma).$$

On the other hand, Theorem 3.3 also implies

$$\mathbf{W}^{S}_{m,x,a}(\mu,\gamma,\sigma)=\mathbf{W}^{N,\sigma}_{m,x}(\mu,\gamma)\mathbf{W}^{A}_{a}(\sigma)$$

which proves the corollary. □

The following result is the analog of the skew-product formula (2.4).

COROLLARY 3.6. *For a.e.* σ *with respect to* $\mathbf{W}^A_a$

$$\begin{aligned}&\int_N K^{N,\sigma}_{t,0}(m,x;m_1,x_1)f(m_1,x_1)\,\mathrm{d}m_1\mathrm{d}x_1\\&\quad=\int_M K^{M,\gamma,\sigma}_{t,0}(m,m_1)f(m_1,\gamma_t)\,\mathrm{d}m_1\,\mathrm{d}\mathbf{W}^{V,\sigma}_{x}(\gamma).\end{aligned}$$

Proof. This is immediate from Corollary 3.5 and (3.1) with $n=1$. □

4. Upper estimate for P^σ

Let notation be as in Sect. 1.2. Then, $\mathfrak{h}_n$ is split by the subalgebras $\mathfrak{v}$ and $\mathfrak{m}$ spanned by $\{X_1,\dots,X_n\}$ and $\{Y_1,\dots,Y_n,Z\}$, respectively, i.e.,

$$\mathcal{H}_n=M\rtimes V,$$

where M and V are the corresponding Lie groups, which we identify with $\mathbb{R}^{n+1}$ and $\mathbb{R}^n$, respectively. Let $L_M=\sum_{j=1}^n\overline{Y}_j^2+\overline{Z}^2$, i.e.,

$$L_M=\sum_{j=1}^n\left(\mathrm{e}^{2\xi_{2,j}(a)}\partial^2_{y_j}+2\mathrm{e}^{2\xi_{2,j}(a)}x_j\partial_{y_j}\partial_z+\mathrm{e}^{2\xi_{2,j}(a)}x_j^2\partial_z^2\right)+\mathrm{e}^{2\xi_3(a)}\partial_z^2$$

and

$$L_V=\sum_{j=1}^n\overline{X}_j^2=\sum_{j=1}^n\mathrm{e}^{2\xi_{1,j}(a)}\partial^2_{x_j}.$$

We replace a by $\sigma(t)$, $t\ge 0$, where $\sigma\in C([0,+\infty),\mathbb{R}^k)$ is a continuous function on the half-line with values in $A=\mathbb{R}^k$, and x by $\eta(t)$, $t\ge 0$, where $\eta=(\eta_1,\dots,\eta_n)$ is a continuous path in $V=\mathbb{R}^n$, i.e., belongs to $C([0,\infty),\mathbb{R}^n)$, and get time-dependent operators

$$\begin{aligned}L^{\sigma,\eta}_M=&\sum_{j=1}^n\left(\mathrm{e}^{2\xi_{2,j}(\sigma(t))}\partial^2_{y_j}+2\mathrm{e}^{2\xi_{2,j}(\sigma(t))}\eta_j(t)\partial_{y_j}\partial_z+\mathrm{e}^{2\xi_{2,j}(\sigma(t))}\eta_j(t)^2\partial_z^2\right)\\&+\mathrm{e}^{2\xi_3(\sigma(t))}\partial_z^2\end{aligned}$$

and

$$L_V^\sigma = \sum_{j=1}^{n} e^{2\xi_{1,j}(\sigma(t))}\partial_{x_j}^2.$$

Then, the matrix $a_{L_V^\sigma}$ from (2.5) for L_V^σ is $a^\sigma = 2m_1^\sigma$ where, for $j = 1, 2$,

$$m_j^\sigma = \mathrm{diag}[e^{2\xi_{j,1}(\sigma)}, \ldots, e^{2\xi_{j,n}(\sigma)}] \tag{4.1}$$

(the off diagonal entries are all 0) while the matrix for $L_M^{\sigma,\eta}$ is

$$a_{L_M^{\sigma,\eta}} = a^{\sigma,\eta} = 2\begin{bmatrix} m_2^\sigma & b^{\sigma,\eta} \\ (b^{\sigma,\eta})^t & d^{\sigma,\eta} \end{bmatrix},$$

where

$$\begin{aligned} b^{\sigma,\eta} &= (e^{2\xi_{2,1}(\sigma)}\eta_1, e^{2\xi_{2,2}(\sigma)}\eta_2, \ldots, e^{2\xi_{2,n}(\sigma)}\eta_n)^t, \\ d^{\sigma,\eta} &= \sum_{j=1}^{n} e^{2\xi_{2,j}(\sigma)}\eta_j^2 + e^{2\xi_3(\sigma)} = \langle b^{\sigma,\eta}, \eta\rangle + e^{2\xi_3(\sigma)}. \end{aligned} \tag{4.2}$$

Let b_t be the 1-dimensional Brownian motion normalized so that

$$\mathbf{W}_x(b_t \in \mathrm{d}y) = p_t(x, \mathrm{d}y) = (4\pi t)^{-1/2} e^{-(x-y)^2/4t}\mathrm{d}y.$$

Then, by (2.6),

$$K_{t,s}^{V,\sigma}(x, \mathrm{d}z) = \prod_{1\le j\le n} p_{\int_s^t e^{2\xi_{1,j}(\sigma u)}\mathrm{d}u}(x_j, \mathrm{d}z_j). \tag{4.3}$$

Thus the process $\eta(t)$ generated by L_V^σ has coordinates $\eta_j(t)$ which are independent Brownian motions with time changed according to the clock governed by σ.

Let

$$A^{\sigma,\eta}(s,t) = \int_s^t a^{\sigma,\eta}(u)\,\mathrm{d}u.$$

For an $n \times n$ invertible matrix A we set

$$B(A)(x) = \frac{1}{2}A^{-1}x\cdot x \quad \text{and} \quad \mathcal{D}(A) = (2\pi)^{-\frac{n}{2}}(\det A)^{-\frac{1}{2}}.$$

In this notation, again by (2.6),

$$K_{t,s}^{M,\sigma,\eta}(m^1, m^2) = \mathcal{D}(A^{\sigma,\eta}(s,t))e^{-B(A^{\sigma,\eta}(t,s))(m^1-m^2)}, \quad m^1, m^2 \in M = \mathbb{R}^{n+1}.$$

LEMMA 4.1. *Let A be a symmetric, positive semi-definite matrix. Then,*

$$B(A)(x) \ge \|x\|^2/(2\|A\|).$$

Proof. Let $A = C^2$ where $C^t = C$. Since $\|A\| = \max_i \lambda_i$ where $\lambda_i \geq 0$ are the eigenvalues of A, $\|C\| = \|A\|^{\frac{1}{2}}$. Then,

$$2B(A)(x) = \|C^{-1}x\|^2 = \|C\|^{-2}\|C\|^2\, \|C^{-1}x\|^2 \geq \|C\|^{-2}\|x\|^2 = \|A\|^{-1}\|x\|^2.$$

□

Now, we want to estimate the operator norm of $A^{\sigma,\eta}$.

LEMMA 4.2. *There is a $C > 0$ such that*

$$\|A^{\sigma,\eta}(0,t)\| \leq C A_2^{\sigma}(0,t)(1 + \Lambda^{\eta}(0,t))^2 + C A_3^{\sigma}(0,t) \tag{4.4}$$

where A_i is as in (1.8) and $\Lambda^{\eta}(s,t) = \sup_{s\leq u\leq t} \|\eta(u)\|_{\infty}$.

Proof. We have, with the notation introduced in (4.1) and (4.2),

$$\begin{aligned} a^{\sigma,\eta} &= 2\begin{bmatrix} m_2^{\sigma} & b^{\sigma,\eta} \\ (b^{\sigma,\eta})^t & d^{\sigma,\eta} \end{bmatrix} = 2\begin{bmatrix} m_2^{\sigma} & b^{\sigma,\eta} \\ (b^{\sigma,\eta})^t & \langle b^{\sigma,\eta}, \eta\rangle \end{bmatrix} + 2\begin{bmatrix} 0 & 0 \\ 0 & \mathrm{e}^{2\xi_3(\sigma)} \end{bmatrix} \\ &= 2\begin{bmatrix} m_2^{\sigma/2} & 0 \\ (b^{\sigma/2,\eta})^t & 0 \end{bmatrix}\begin{bmatrix} m_2^{\sigma/2} & b^{\sigma/2,\eta} \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 0 \\ 0 & \mathrm{e}^{2\xi_3(\sigma)} \end{bmatrix}. \end{aligned} \tag{4.5}$$

There is a constant $D = D_r$ such that for all $r \times r$ matrices A,

$$\|A\| \leq D \max_{1\leq i,j\leq r} |a_{ij}| \leq D \sum_{1\leq i,j\leq n} |a_{ij}|.$$

The norm of the first and the second matrix on the right-hand side of (4.5) is dominated by a multiple of

$$\sum_{j=1}^{n} \mathrm{e}^{\xi_{2,j}(\sigma)} + \sum_{j=1}^{n} \mathrm{e}^{\xi_{2,j}(\sigma)}|\eta_j| \leq (1 + \|\eta\|_{\infty}) \sum_{j=1}^{n} \mathrm{e}^{\xi_{2,j}(\sigma)},$$

where $\|\cdot\|_{\infty}$ denotes the ℓ^{∞}-norm on $\mathbb{R}^n$. Hence

$$\|a^{\sigma,\eta}(t)\| \leq C\left((1 + \|\eta(t)\|_{\infty})^2 \left(\sum_{j=1}^{n} \mathrm{e}^{\xi_{2,j}(\sigma(t))}\right)^2 + \mathrm{e}^{2\xi_3(\sigma(t))}\right),$$

and so

$$\begin{aligned} \|A^{\sigma,\eta}(0,t)\| &\leq \int_0^t \|a^{\sigma,\eta}(u)\| \mathrm{d}u \\ &\leq C\int_0^t \left((1 + \|\eta(u)\|_{\infty})^2 \left(\sum_{j=1}^{n} \mathrm{e}^{\xi_{2,j}(\sigma(t))}\right)^2 + \mathrm{e}^{2\xi_3(\sigma(u))}\right) \mathrm{d}u. \end{aligned} \tag{4.6}$$

But, in $L^2(0,t)$,

$$\int_0^t \left(\sum_{j=1}^n \mathrm{e}^{\xi_{2,j}(\sigma(u))}\right)^2 \mathrm{d}u = \Big\| \sum_{j=1}^n \mathrm{e}^{\xi_{2,j}(\sigma(\cdot))}\Big\|_2^2 \le \left(\sum_{j=1}^n \Big\|\mathrm{e}^{\xi_{2,j}(\sigma(\cdot))}\Big\|_2\right)^2$$

$$\le n \sum_{j=1}^n \Big\|\mathrm{e}^{\xi_{2,j}(\sigma(\cdot))}\Big\|_2^2 = n A_2^\sigma(0,t).$$

The lemma follows. □

LEMMA 4.3. *There exists a constant $C > 0$ such that*

$$\mathcal{D}(A^{\sigma,\eta}(s,t)) \le C A_{2,\Pi}^\sigma(s,t)^{-\frac{1}{2}} A_3^\sigma(s,t)^{-\frac{1}{2}},$$

where notation is as in (1.8).

Proof. We introduce the integrals of the objects defined in (4.1) and (4.2),

$$M_j^\sigma(s,t) = \int_s^t m_j^\sigma(u)\mathrm{d}u, \quad B^{\sigma,\eta}(s,t) = \int_s^t b^{\sigma,\eta}(u)\mathrm{d}u,$$

and

$$D^{\sigma,\eta}(s,t) = \int_s^t d^{\sigma,\eta}(u)\mathrm{d}u.$$

By (4.5) we get

$$\det A^{\sigma,\eta}(s,t) = 2^{n+1} \det \begin{bmatrix} M_2^\sigma(s,t) & B^{\sigma,\eta}(s,t) \\ (B^{\sigma,\eta}(s,t))^t & D^{\sigma,\eta}(s,t) \end{bmatrix}$$

$$= 2^{n+1} A_{2,\Pi}^\sigma(s,t) \sum_j \left(\int_s^t \mathrm{e}^{2\xi_{2,j}(\sigma(u))} \eta_j^2(u)\mathrm{d}u - \frac{(\int_s^t \mathrm{e}^{2\xi_{2,j}(\sigma(u))}\eta_j(u)\mathrm{d}u)^2}{\int_s^t \mathrm{e}^{2\xi_{2,j}(\sigma(u))}\mathrm{d}u} \right)$$

$$+ 2^{n+1} A_{2,\Pi}^\sigma(s,t) A_3^\sigma(s,t).$$

By the Cauchy–Schwarz inequality the expression under the $\sum_j$ is non-negative. Thus, we get $\det A^{\sigma,\eta}(s,t) \ge 2^{n+1} A_{2,\Pi}^\sigma(s,t) A_3^\sigma(s,t)$. □

Next, we estimate the evolution kernel on $\mathcal{H}_n$ generated by L^{σ_t},

$$P^\sigma(t,0)(m,v) = K^\sigma(t,0)(0,0;m,v).$$

PROPOSITION 4.4. *There are positive constants C and D such that*

$$A_{1,\Pi}^\sigma(0,t)^{\frac{1}{2}} A_{2,\Pi}^\sigma(0,t)^{\frac{1}{2}} A_3^\sigma(0,t)^{\frac{1}{2}} P^\sigma(0,t)(m,v)$$

$$\le C(\|m\|^{\frac{1}{2}}+1) \exp\left(-\frac{D\|v\|_\infty^2}{A_1^\sigma(0,t)} - \frac{D\|m\|^2}{(A_2^\sigma(0,t)+A_3^\sigma(0,t))(\|m\|^{\frac{1}{2}}+\|v\|_\infty+2)^2} \right)$$

$$+ C A_1^\sigma(0,t)^{1/2} \exp\left(-\frac{\|m\| + \|v\|_\infty^2}{2A_1^\sigma(0,t)} \right).$$

Proof. We allow the constants C and D to change from line to line. By Lemma 4.1 and Lemma 4.3,

$$\begin{aligned} K_{t,s}^{M,\sigma,\eta}(m^1,m^2) &= \mathcal{D}(A^{\sigma,\eta}(s,t))\mathrm{e}^{-B(A^{\sigma,\eta}(s,t))(m^1-m^2)} \\ &\le C A_{2,\Pi}^{\sigma}(s,t)^{-\frac{1}{2}} A_3^{\sigma}(s,t)^{-\frac{1}{2}} \mathrm{e}^{-\frac{\|m^1-m^2\|^2}{2\|A^{\sigma,\eta}(s,t)\|}}. \end{aligned} \tag{4.7}$$

From Corollary 3.6, (4.7), and (4.3), for $m^i \in M$ and $v^1 \in V$,

$$\begin{aligned} &\int K^{\sigma}(t,0)(m^1,v^1;m^2,y)\psi(y)\,\mathrm{d}y = \int K_{t,0}^{M,\sigma,\eta}(m^1,m^2)\psi(\eta(t))\,\mathrm{d}\mathbf{W}_{v^1}^{V,\sigma}(\eta) \\ &\le \mathcal{D}(A^{\sigma,\eta}(0,t)) \int \mathrm{e}^{-\frac{\|m^2-m^1\|^2}{2\|A^{\eta,\sigma}(0,t)\|}} \psi(\eta(t))\,\mathrm{d}\mathbf{W}_{v^1}^{V,\sigma}(\eta) \\ &\le C A_{2,\Pi}^{\sigma}(0,t)^{-\frac{1}{2}} A_3^{\sigma}(0,t)^{-\frac{1}{2}} \\ &\quad \times \int \mathrm{e}^{-\frac{\|m^2-m^1\|^2}{2\|A^{\sigma,\eta}(s,t)\|}} \psi\left(\eta_1(A_{1,1}^{\sigma}(0,t)),\ldots,\eta_n(A_{1,n}^{\sigma}(0,t))\right)\,\mathrm{d}\mathbf{W}_{v^1}(\eta). \end{aligned} \tag{4.8}$$

Then, (4.8) and Lemma 4.2 imply

$$\begin{aligned} &\left(A_{2,\Pi}^{\sigma}(0,t)^{-\frac{1}{2}} A_3^{\sigma}(0,t)^{-\frac{1}{2}}\right)^{-1} \int P^{\sigma}(t,0)(m,y)\psi(y)\mathrm{d}y \\ &\qquad \le C \int \mathrm{e}^{-\frac{D\|m\|^2}{A_2^{\sigma}(0,t)(1+\Lambda^{\eta}(0,t))^2 + A_3^{\sigma}(0,t)}} \psi(\eta(t))\,\mathrm{d}\mathbf{W}_0^{V,\sigma}(\eta). \end{aligned} \tag{4.9}$$

For $v \in \mathbb{R}^n$ given and $\varepsilon > 0$, let $\psi_\varepsilon(\cdot) = \varepsilon^{-n}\mathbf{1}_{B_\varepsilon(v)}(\cdot)$, where $B_\varepsilon(v) = \prod_{j=1}^n B_\varepsilon^1(v_j)$ and $B_\varepsilon^1(v_j) = [v_j - \varepsilon/2, v_j + \varepsilon/2]$. We will estimate (4.9) with ψ_ε in place of ψ as ε tends to zero.

Let $\mathbf{E}_v^{\eta}$ denote expectation with respect to $\mathrm{d}\mathbf{W}_v^{V,\sigma}(\eta)$. For $k = 1, 2, \ldots$, define the sets of paths in V,

$$\mathcal{A}_k = \left\{ \eta : k-1 \le \Lambda^{\eta}(0,t) = \sup_{0\le u\le t} \|\eta(u)\|_\infty < k \right\}.$$

The integral on the right in (4.9) can be written as an infinite sum and estimated as follows

$$\begin{aligned} &\sum_{k=1}^{\infty} \mathbf{E}_0^{\eta} \exp\left(-\frac{D\|m\|^2}{A_2^{\sigma}(0,t)(1+\Lambda^{\eta}(0,t))^2 + A_3^{\sigma}(0,t)}\right) \psi_\varepsilon(\eta(t))\mathbf{1}_{\mathcal{A}_k}(\eta) \\ &\quad \le \sum_{k=1}^{\infty} \exp\left(-\frac{D\|m\|^2}{4(A_2^{\sigma}(0,t) + A_3^{\sigma}(0,t))k^2}\right) \mathbf{E}_0^{\eta}\psi_\varepsilon(\eta(t))\mathbf{1}_{\mathcal{A}_k}(\eta). \end{aligned} \tag{4.10}$$

To simplify notation we introduce

$$\begin{aligned} c_k &= \exp\left(-\frac{D\|m\|^2}{4(A_2^{\sigma}(0,t) + A_3^{\sigma}(0,t))k^2}\right), \\ \mathcal{E}_k(\varepsilon) &= \mathbf{E}_0^{\eta}\psi_\varepsilon(\eta(t))\mathbf{1}_{\mathcal{A}_k}(\eta) = \varepsilon^{-n}\mathbf{W}_0^{V,\sigma}\,(\eta \in \mathcal{A}_k \text{ and } \eta(t) \in B_\varepsilon(v)). \end{aligned}$$

Let $v \neq 0$ and choose $\varepsilon/2 < \|v\|_\infty$. If $\eta \in \mathcal{A}_k$, then $\|\eta(t)\|_\infty \geq \|v\|_\infty - \varepsilon/2$. Hence, $\mathcal{E}_k = 0$ for $k < \|v\|_\infty - \varepsilon/2$.

Let, for $k = 1, 2, \ldots,$

$$\Lambda^{\eta_j}(0,t) = \sup_{0 \leq u \leq t} |\eta_j(u)| \text{ and } \mathcal{A}_k^j = \{\eta : k-1 \leq \Lambda^{\eta_j}(0,t) < k\}.$$

Since the coordinates $\eta_j(t)$ of $\eta(t)$ are independent (Brownian motions with time changed—see p. 15 after (4.3)), we can estimate (recall that $\mathbf{W}_0$ is the law of a classical Brownian motion),

$$\begin{aligned}
\mathcal{E}_k(\varepsilon) &\leq \varepsilon^{-n} \sum_{j=1}^n \mathbf{W}_0^{V,\sigma}\left(\eta \in \mathcal{A}_k^j \text{ and } \eta(t) \in B_\varepsilon(v)\right) \\
&= \varepsilon^{-n} \sum_{j=1}^n \mathbf{W}_0^{V,\sigma}\left(\eta \in \mathcal{A}_k^j \text{ and } \eta_j(t) \in B^1_\varepsilon(v_j)\right) \mathbf{W}_0^{V,\sigma}\left(\eta_i(t) \in B^1_\varepsilon(v_i) \text{ for } i \neq j\right) \\
&= \sum_{j=1}^n \varepsilon^{-1} \mathbf{W}_0\left(\eta \in \mathcal{A}_k^j \text{ and } \eta_j(A^\sigma_{1,j}(0,t)) \in B^1_\varepsilon(v_j)\right) \\
&\quad \times \prod_{i \neq j}\left(\varepsilon^{-1}\mathbf{W}_0\left(\eta_i(A^\sigma_{1,i}(0,t)) \in B^1_\varepsilon(v_i)\right)\right). \qquad (4.11)
\end{aligned}$$

LEMMA 4.5. *Assume that* $a > \|v\|_\infty + \delta$, $\delta > 0$, *and* $0 < \varepsilon/2 < \delta$. *Then,*

$$\begin{aligned}
&\varepsilon^{-n}\mathbf{W}_0^{V,\sigma}\left(\sup_{u \in [0,t]} \|\eta(u)\|_\infty \geq a \text{ and } \eta(t) \in B_\varepsilon(v)\right) \\
&\quad \leq A^\sigma_{1,\Pi}(0,t)^{-1/2} \sum_{j=1}^n \left(\mathrm{e}^{-(2a-v_j)^2/2A^\sigma_{1,j}(0,t)} + \mathrm{e}^{-(2a+v_j)^2/2A^\sigma_{1,j}(0,t)}\right).
\end{aligned}$$

Proof. Reasoning as in (4.11) we see that the left side of the above inequality is bounded by

$$\begin{aligned}
&C \sum_{j=1}^n \left(\prod_{i \neq j} A^\sigma_{1,i}(0,t)\right)^{-1/2} \\
&\quad \times \varepsilon^{-1}\mathbf{W}_0\left(\sup_{u \in [0, A^\sigma_{1,j}(0,t)/2]} |\eta_j(u)| \geq a \text{ and } \eta_j(A^\sigma_{1,j}(0,t)) \in B^1_\varepsilon(v_j)\right).
\end{aligned}$$

By our assumption it follows that for every j, $a > |v_j| + \delta$. Hence, the result follows by Corollary 2.7. □

LEMMA 4.6.

$$A^\sigma_{2,\Pi}(0,t)^{\frac{1}{2}} A^\sigma_3(0,t)^{\frac{1}{2}} P^\sigma(0,t)(m,v) \leq CI, \text{ where } I = \limsup_{\varepsilon \to 0^+} \sum_{k \geq \|v\|_\infty} c_k \mathcal{E}_k(\varepsilon).$$

Furthermore, the sum converges uniformly in ε.

Proof. The inequality follows by letting ε tend to 0 in (4.10). The uniform convergence follows from Lemma 4.5. □

Let n_o be the smallest natural number such that $n_o \geq \|v\|_\infty$.

LEMMA 4.7. *We have the following estimates*

$$\limsup_{\varepsilon\to 0^+} \mathcal{E}_{n_o}(\varepsilon) \leq C A^\sigma_{1,\Pi}(0,t)^{-1/2} \mathrm{e}^{-\|v\|^2_\infty/2A^\sigma_1(0,t)},$$

while for $k \geq n_o + 1$,

$$\limsup_{\varepsilon\to 0^+} \mathcal{E}_k(\varepsilon) \leq C A^\sigma_{1,\Pi}(0,t)^{-1/2} \exp\left(-\frac{(2(k-1)-\|v\|_\infty)^2}{2A^\sigma_1(0,t)}\right).$$

Proof. Consider $\mathcal{E}_{n_0}$. Let $j \in \{1,\ldots,n\}$ be fixed. Suppose first that $|v_j| < n_o - 1$. Then, using Corollary 2.8, the jth term in (4.11) (with $k = n_o$) can be dominated by a multiple of $A^\sigma_{1,\Pi}(0,t)^{-1/2}\mathrm{e}^{-\frac{(2(n_o-1)-|v_j|)^2}{2A^\sigma_{1,j}(0,t)}} \prod_{i\neq j} \mathrm{e}^{-\frac{|v_i|^2}{2A^\sigma_{1,i}(0,t)}}$. Notice that $|v_j|$ cannot be equal to $\|v\|_\infty$. Thus, we are done in this case.

Now suppose that $|v_j| \geq n_o - 1$. Then, using Corollary 2.8 again, we dominate the jth term in (4.11) by $C A^\sigma_{1,\Pi}(0,t)^{-1/2}\mathrm{e}^{-\frac{|v_j|^2}{2A^\sigma_{1,j}(0,t)}} \prod_{i\neq j} \mathrm{e}^{-\frac{|v_i|^2}{2A^\sigma_{1,i}(0,t)}}$. The result for $\mathcal{E}_0$ follows.

Now we consider $\mathcal{E}_k$. Since $k \geq n_o + 1$ it follows that $k - 1 \geq |v_j|$ for every j. Therefore, by Corollary 2.8 the jth term in (4.11) is estimated by

$$C A^\sigma_{1,\Pi}(0,t)^{-1/2}\mathrm{e}^{-\frac{(2(k-1)-|v_j|)^2}{2A^\sigma_{1,j}(0,t)}} \prod_{i\neq j} \mathrm{e}^{-\frac{|v_i|^2}{2A^\sigma_{1,i}(0,t)}}.$$

□

Next, we estimate $I = \limsup_{\varepsilon\to 0^+} \sum_{k\geq\|v\|_\infty} c_k \mathcal{E}_k(\varepsilon)$. From Lemma 4.7,

$$\begin{aligned}
&A^\sigma_{1,\Pi}(0,t)^{1/2} I \\
&= A^\sigma_{1,\Pi}(0,t)^{1/2} \limsup_{\varepsilon\to 0^+}\left(c_{n_o}\mathcal{E}_{n_o}(\varepsilon) + \sum_{k\geq n_o+1} c_k \mathcal{E}_k(\varepsilon)\right) \\
&\leq C \exp\left(-\frac{\|v\|^2_\infty}{2A^\sigma_1(0,t)} - \frac{D\|m\|^2}{4(A^\sigma_2(0,t)+A^\sigma_3(0,t))n_o^2}\right) \\
&\quad + \sum_{k=n_o+1}^{\infty} \exp\left(-\frac{D\|m\|^2}{4(A^\sigma_2(0,t)+A^\sigma_3(0,t))k^2} - \frac{(2(k-1)-\|v\|_\infty)^2}{2A^\sigma_1(0,t)}\right).
\end{aligned} \tag{4.12}$$

For a, b non-negative $a + b \geq \sqrt{a^2+b^2}$. Also, for $k \geq n_o + 1$,

$$\begin{aligned}
(k-1) + (k-1) - \|v\|_\infty &\geq n_o + (k - 1 - \|v\|_\infty),\\
k - 1 - \|v\|_\infty &\geq n_o - \|v\|_\infty \geq 0.
\end{aligned}$$

Hence the summation in the last line of (4.12) is bounded by

$$\sum_{k=n_o+1}^{\infty} \exp\left(-\frac{D\|m\|^2}{4(A_2^\sigma(0,t)+A_3^\sigma(0,t))k^2} - \frac{(n_o+(k-1)-\|v\|_\infty)^2}{2A_1^\sigma(0,t)}\right)$$
$$\leq e^{-\frac{n_o^2}{2A_1^\sigma(0,t)}} \sum_{k=n_o+1}^{\infty} \exp\left(-\frac{D\|m\|^2}{4(A_2^\sigma(0,t)+A_3^\sigma(0,t))k^2} - \frac{(k-1-\|v\|_\infty)^2}{2A_1^\sigma(0,t)}\right). \tag{4.13}$$

We split the sum in (4.13) into two parts: $n_o+1 \leq k \leq n_o+\|m\|^{\frac{1}{2}}$ and $k > n_o+\|m\|^{\frac{1}{2}}$, and estimate the corresponding parts by the following two terms:

$$\|m\|^{\frac{1}{2}} e^{-\frac{n_o^2}{2A_1^\sigma(0,t)}} \exp\left(-\frac{D\|m\|^2}{4(A_2^\sigma(0,t)+A_3^\sigma(0,t))(\|m\|^{\frac{1}{2}}+\|v\|_\infty+2)^2}\right)$$

and

$$e^{-\frac{n_o^2}{2A_1^\sigma(0,t)}} \sum_{k\geq n_o+\|m\|^{\frac{1}{2}}+1} \exp\left(-\frac{D\|m\|^2}{4(A_2^\sigma(0,t)+A_3^\sigma(0,t))k^2} - \frac{(k-1-n_o)^2}{2A_1^\sigma(0,t)}\right).$$

The above expression is bounded by

$$e^{-\frac{n_o^2}{2A_1^\sigma(0,t)}} \int_{\|m\|^{\frac{1}{2}}}^{\infty} e^{-\frac{r^2}{2A_1^\sigma(0,t)}}\, \mathrm{d}r \leq \sqrt{2}A_1^\sigma(0,t)^{1/2} e^{-\frac{\|v\|_\infty^2}{2A_1^\sigma(0,t)} - \frac{\|m\|}{2A_1^\sigma(0,t)}}.$$

Proposition 4.4 follows. □

As a corollary we get the following result, where the notation is as in (1.9) and (1.10).

COROLLARY 4.8. *There are positive constants C and D such that in the region $\|v\|_\infty \leq \|m\|^{\frac{1}{2}}$,*

$$A_0 P^\sigma(0,t)(m,v) \leq C(\|m\|^{1/2}+1)\exp\left(-D\frac{\|v\|_\infty^2}{A_1} - D\frac{\|m\|}{A_2}\phi(m)\right)$$
$$+CA^{1/2}\exp\left(-D\frac{\|m\|+\|v\|_\infty^2}{A_1}\right)$$

while in the region $\|v\|_\infty \geq \|m\|^{\frac{1}{2}}$,

$$A_0 P^\sigma(0,t)(m,v) \leq C(\|m\|^{1/2}+1+A^{1/2})\exp\left(-D\frac{\|m\|+\|v\|_\infty^2}{A_1}\right).$$

Theorem 1.4 follows immediately from this corollary along with the observations that $\phi(m) \leq 1$, $A_1 \leq A_3$ and $A_2 \leq A_3$.

5. Upper estimate for the Poisson kernel

Let $\nu(h) = \nu(x, y, z), x, y \in \mathbb{R}^n, z \in \mathbb{R}$, be the Poisson kernel on $\mathcal{H}_n$ for the operator $\mathcal{L}_\alpha$ in (1.4). Then, from (1.2)

$$\mathcal{L}_\alpha = \sum_{j=1}^{n} \left(e^{2\xi_{1,j}(a)} \partial_{x_j}^2 + e^{2\xi_{2,j}(a)} \partial_{y_j}^2 + 2e^{2\xi_{2,j}(a)} x_j \partial_{y_j} \partial_z + e^{2\xi_{2,j}(a)} x_j^2 \partial_z^2 \right) + e^{2\xi_3(a)} \partial_z^2 + \Delta_\alpha .$$

Recall that we assume that $\xi_{i,j}(\alpha) > 0$. Hence α belongs to the positive Weyl chamber A^+. The operator Δ_α generates the Brownian motion $\sigma(u)$ with drift -2α, i.e., $\sigma(u) = b(u) - 2\alpha u$, where $b(u)$ is the k-dimensional standard Brownian motion normalized so that $\operatorname{Var} b_u = 2u$.

Let ν^a be as in (2.7). Recall that $h = (x, y, z) = (m, v)$ with $v = x$ and $(y, z) = m$.

THEOREM 5.1. *For all compact subsets $K \not\ni e$ of $\mathcal{H}_n$, all $\rho \in A^+$, and all $\varepsilon > 0$ there exists a constant $C = C(K, \rho, \varepsilon) > 0$ such that for all $s < 0$,*

$$\nu^{s\rho}(h) \le C e^{-\rho_0(s\rho)} e^{s \min_{1\le j\le n} \xi_{1,j}(\rho) \min_{1\le j\le n}(\xi_{1,j}(\alpha)/\xi_{1,j}^2)/2} \times e^{s \min_{1\le j\le n}(\xi_{1,j}(\rho), \xi_{2,j}(\rho)) \min_j(\xi_{1,j}(\alpha)/\xi_{1,j}^2, \xi_{2,j}(\alpha)/\xi_{2,j}^2, \xi_3/\xi_3^2)/2} \quad \text{if } h \in K \cap \{\phi(m) \ge \varepsilon,\ \|v\|_\infty \ge \varepsilon\}, \tag{5.1}$$

$$\nu^{s\rho}(h) \le C e^{-\rho_0(s\rho)} \times e^{s \min_{1\le j\le n} \xi_{1,j}(\rho) \min_{1\le j\le n}(\xi_{1,j}(\alpha)/\xi_{1,j}^2)} \quad \text{if } h \in K \cap \{\|v\|_\infty \ge \varepsilon\} \tag{5.2}$$

and

$$\nu^{s\rho}(h) \le C e^{-\rho_0(s\rho)} \times e^{s \min_{1\le j\le n}(\xi_{1,j}(\rho), \xi_{2,j}(\rho)) \min_{1\le j\le n}(\xi_{1,j}(\alpha)/\xi_{1,j}^2, \xi_{2,j}(\alpha)/\xi_{2,j}^2, \xi_3/\xi_3^2)} \quad \text{if } h \in K \cap \{\phi(m) \ge \varepsilon\}, \tag{5.3}$$

where $\phi(m)$ is defined in (1.10).

Proof. First we consider elements $h = (m, v)$ from the set $K_1 = K \cap \{(m, v) : \phi(m) \ge \varepsilon\}$. Let A_j be defined as in (1.9) but with $t = \infty$. By Theorem 1.4, we have

$$\nu^{s\rho} \le C \mathbf{E}_{s\rho} A_0^{-1} \exp\left(-\frac{D}{A_1} - \frac{D}{A_3}\right) + C \mathbf{E}_{s\rho} A_0^{-1} A_1^{1/2} \exp\left(-\frac{D}{A_1} - \frac{D}{A_3}\right). \tag{5.4}$$

We estimate the first expectation on the right.

$$\mathbf{E}_{s\rho} A_0^{-1} \exp\left(-\frac{D}{A_1} - \frac{D}{A_3}\right) \le \left(\mathbf{E}_{s\rho}(A_0^{-1})^2\right)^{1/2} \left(\mathbf{E}_{s\rho} \exp\left(-\frac{2D}{A_1} - \frac{2D}{A_3}\right)\right)^{1/2} \le \left(\mathbf{E}_{s\rho}(A_0^{-1})^2\right)^{1/2} \left(\mathbf{E}_{s\rho} \exp\left(-\frac{4D}{A_1}\right)\right)^{1/4} \left(\mathbf{E}_{s\rho} \exp\left(-\frac{4D}{A_3}\right)\right)^{1/4}. \tag{5.5}$$

We estimate the first expectation above. By the Cauchy–Schwarz inequality we get,

$$\begin{aligned}\mathbf{E}_{s\rho}(A_0^{-1})^2 &= \mathbf{E}_{s\rho}(A_{1,\Pi}^{\sigma})^{-1}(A_{2,\Pi}^{\sigma})^{-1}(A_3^{\sigma})^{-1}\\ &= \mathrm{e}^{-2\rho_0(s\rho)}\mathbf{E}_0(A_{1,\Pi}^{\sigma})^{-1}(A_{2,\Pi}^{\sigma})^{-1}(A_3^{\sigma})^{-1}\\ &\le \mathrm{e}^{-2\rho_0(s\rho)}(\mathbf{E}_0(A_{1,\Pi}^{\sigma})^{-4})^{1/4}(\mathbf{E}_0(A_{2,\Pi}^{\sigma})^{-4})^{1/4}(\mathbf{E}_0(A_3^{\sigma})^{-2})^{1/2}. \end{aligned} \tag{5.6}$$

The expectation $\mathbf{E}_0(A_3^{\sigma})^{-2}$ is finite by Lemma 2.3. Since by Lemma 2.3, for $k = 1, 2$ and $j = 1, \ldots, n$, and all $d > 0$, the expected values $\mathbf{E}_0(A_{k,j}^{\sigma})^{-d}$ are also finite, we can apply the Cauchy–Schwarz inequality $n-1$ times to each of the remaining expectation in (5.6) and conclude their finiteness.

Now we consider $\mathbf{E}_{s\rho}\exp(-4D_1/A_1)$ and $\mathbf{E}_{s\rho}\exp(-4D_2/A_3)$ from (5.5).

Clearly,

$$\mathbf{E}_{s\rho}\exp(-4D_1/A_1) \le \mathbf{E}_0\exp(-4D_1/(M(s\rho)A_1), \tag{5.7}$$

where $M(s\rho) = \max_{1\le j\le n}\mathrm{e}^{2\xi_{1,j}(s\rho)} = \mathrm{e}^{2s\min_{1\le j\le n}\xi_{1,j}(\rho)}$.

Proceeding exactly in the same way as in the proof of [12, Lemma 6.2] we show that (5.7) is bounded by

$$CM(s\rho)^{\min_{1\le j\le n}(\xi_{1,j}(\alpha)/\xi_{1,j}^2)} = C\mathrm{e}^{2s\min_{1\le j\le n}\xi_{1,j}(\rho)\min_{1\le j\le n}(\xi_{1,j}(\alpha)/\xi_{1,j}^2)}. \tag{5.8}$$

The expectation $\mathbf{E}_{s\rho}\exp(-4D_2/A_3)$ is similar. Again, in the same way as in the proof of [12, Lemma 6.2] we show that $\mathbf{E}_{s\rho}\exp(-4D_2/A_3)$ is bounded by $CM_1(s\rho)^{\min_j(\xi_{1,j}(\alpha)/\xi_{1,j}^2,\xi_{2,j}(\alpha)/\xi_{2,j}^2,\xi_3(\alpha)/\xi_3^2)}$, where

$$M_1(s\rho) = \max_{1\le j\le n}\left(\mathrm{e}^{2\xi_{1,j}(s\rho)}, \mathrm{e}^{2\xi_{2,j}(s\rho)}, \mathrm{e}^{2\xi_3(s\rho)}\right) = \mathrm{e}^{2s\min_{1\le j\le n}(\xi_{1,j}(\rho),\xi_{2,j}(\rho))}.$$

Hence,

$$\begin{aligned}&\mathbf{E}_{s\rho}\exp(-4D_2/A_3)\\ &\quad\le C\mathrm{e}^{2s\min_{1\le j\le n}(\xi_{1,j}(\rho),\xi_{2,j}(\rho))\min_j(\xi_{1,j}(\alpha)/\xi_{1,j}^2,\xi_{2,j}(\alpha)/\xi_{2,j}^2,\xi_3(\alpha)/\xi_3^2)}. \end{aligned} \tag{5.9}$$

Now we estimate the second expectation on the right in (5.4) by

$$\begin{aligned}&\sum_{j=1}^n \mathbf{E}_{s\rho}A_0^{-1}A_{1,j}^{1/2}\exp\left(-\frac{D}{A_1}-\frac{D}{A_3}\right)\\ &= \sum_{j=1}^n \mathbf{E}_{s\rho}A_{2,\Pi}^{-1/2}A_3^{-1/2}\prod_{k\ne j}A_{1,k}^{-1/2}\exp\left(-\frac{D}{A_1}-\frac{D}{A_3}\right)\\ &= \sum_{j=1}^n \mathrm{e}^{-\sum_k\xi_{2,k}(s\rho)}\mathrm{e}^{-\xi_3(s\rho)}\mathrm{e}^{-\sum_{k\ne j}\xi_{2,i}(s\rho)}\\ &\quad\times\mathbf{E}_0 A_{2,\Pi}^{-1/2}A_3^{-1/2}\prod_{k\ne j}A_{1,k}^{-1/2}\exp\left(-\frac{D}{A_1}-\frac{D}{A_3}\right). \end{aligned}$$

Since $s < 0$, $\mathrm{e}^{-\sum_k \xi_{2,k}(s\rho)}\mathrm{e}^{-\xi_3(s\rho)}\mathrm{e}^{-\sum_{k\neq j}\xi_{2,i}(s\rho)} \le \mathrm{e}^{-\rho_0(s\rho)}$. To estimate the last one expectation above we proceed as in (5.5) and (5.6) and get the same estimate. Hence, the estimate (5.1) holds on K_1.

Now we have to consider the set $K_2 = K \cap \{(m, v) : \|v\|_\infty \ge \varepsilon\}$. On this set (5.5) simplifies and, using Lemma 2.3, (5.6), (5.7) and (5.8) as above, we get

$$\mathbf{E}_{s\rho} A_0^{-1} \exp\left(-\frac{D_1}{A_1}\right) \le \left(\mathbf{E}_{s\rho}(A_0^{-1})^2\right)^{1/2} \left(\mathbf{E}_{s\rho} \exp\left(-\frac{2D_1}{A_1}\right)\right)^{1/2}$$
$$\le \mathrm{e}^{-\rho_0(s\rho)}\mathrm{e}^{s \min_{1\le j\le n}\xi_{1,j}(\rho)\min_{1\le j\le n}(\xi_{1,j}(\alpha)/\xi_{1,j}^2)}.$$

As in the previous case the second expectation in (5.4) has the same estimate. Hence, the estimate (5.2) holds on K_2. Finally, we consider the set $K_3 = K \cap \{(m, v) : \phi(m) \ge \varepsilon\}$. Then,

$$\mathbf{E}_{s\rho} A_0^{-1} \exp\left(-\frac{D_2}{A_3}\right) \le \left(\mathbf{E}_{s\rho}(A_0^{-1})^2\right)^{1/2} \left(\mathbf{E}_{s\rho} \exp\left(-\frac{2D_2}{A_3}\right)\right)^{1/2}$$
$$\le \mathrm{e}^{-\rho_0(s\rho)}\mathrm{e}^{s \min_{1\le j\le n}(\xi_{1,j}(\rho),\xi_{2,j}(\rho))\min_j(\xi_{1,j}(\alpha)/\xi_{1,j}^2,\xi_{2,j}(\alpha)/\xi_{2,j}^2,\xi_3/\xi_3^2)}.$$

Again, the second expectation in (5.4) has the same estimate. Thus, (5.3) is proved. □

5.1. Proof of Theorem 1.2

Proof of Theorem 1.2. Recall that we have $h = (x, y, z) = (m, v)$ with $v = x$ and $(y, z) = m$. It is clear that for h with the norm $|h|_\rho \le 1$ we have $\nu(h) \le C_\rho$. Let $\delta_t^\rho = \mathrm{Ad}((\log t)\rho)$. Then, $|\delta_t^\rho h|_\rho = t|h|_\rho$. Let $h = \delta_{\exp(-s)}^\rho h_0$ with $|h_0|_\rho = 1$ and $s < 0$. Then $|h|_\rho = \mathrm{e}^{-s} > 1$. Let $K = \{h_0 : |h_0|_\rho = 1\}$. By definition (2.7), $\nu(h) = \nu(\delta_{\exp(-s)}^\rho h_0) = \nu((s\rho)^{-1}h_0(s\rho)) = \mathrm{e}^{\rho_0(s\rho)}\nu^{s\rho}(h_0)$, where $\rho_0 = \sum_j(\xi_{1,j}+\xi_{2,j})+\xi_3$, and the result follows from Theorem 5.1. □

5.2. Example

Here we compare the above upper bound given by Theorem 1.2 with the result from [12]. Consider $k = 2$ and $\xi_{1,j} = (1, 0)$, $\xi_{2,j} = (0, 1)$. Theorem 1.1 gives $\nu(x, y, z) \le C(1 + |(x, y, z)|_\rho)^{-\frac{C_1\rho_0(\rho)\gamma(\alpha)}{4}}$, where $\gamma(\alpha) = 2\min(\alpha_1, \alpha_2)$ for some constant C_1 which depends on ρ and can be computed. Take $\rho = (1, 2)$. To compute C_1 we proceed similarly as in [12, Example 1] getting that [12] gives

$$\nu(x, y, z) \le C(1 + |(x, y, z)|_\rho)^{-\frac{\min(\alpha_1,\alpha_2)}{2}},$$

whereas Theorem 1.2 gives, for example for $\phi(y, z) > 1$ and $\|y\|_\infty > 1$,

$$\nu(x, y, z) \le C(1 + |(x, y, z)|_\rho)^{-\frac{\alpha_1}{2}-\frac{\min(\alpha_1,\alpha_2)}{2}}.$$

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