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Research Article

Continuous Dependence for the Pseudoparabolic Equation

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We determine the continuous dependence of solution on the parameters in a Dirichlet-type initial-boundary value problem for the pseudoparabolic partial differential equation.

1. Introduction

We consider the following initial-boundary value problem:

$$u_t - \alpha\Delta u_t - \beta\Delta u = f(u), \quad x \in \Omega,\ t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x),\ x \in \Omega, \tag{1.2}$$

$$u(x,t) = 0,\ x \in \partial\Omega,\ t > 0, \tag{1.3}$$

where α and β are positive constants, $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$, and $f(u)$ is a given nonlinear function which satisfies

$$0 \geq F(u) \geq f(u) \cdot u, \tag{1.4}$$

$$\left|f(u)\right| \leq c_1\left(1 + |u|^p\right), \tag{1.5}$$

where $F(u) = \int_0^u f(s)ds$, c_1 is a positive constant, and $p \leq n/(n-2)$.

Equation (1.1) is an example of a general class of equations of Sobolev type, sometimes referred to as Sobolev-Galpern type.

A mixed-boundary value problem for the one-dimensional case of (1.1) appears in the study of nonsteady flow of second-order fluids [1] where u represents the velocity of the fluid.

Equation (1.1) can be assumed as a model for the heat conduction involving a thermodynamic temperature $\theta = u - \alpha \Delta u$ and a conductive temperature u; see [2].

Equations of the form (1.1) have been called pseudoparabolic by Showalter and Ting [3], because well posed initial-boundary value problems for parabolic equations are also well-posed for (1.1). Moreover, in certain cases, the solution of a parabolic initial-boundary value problem can be obtained as a limit of solutions to the corresponding problem for (1.1) when α goes to zero; see [4].

In [5], Karch proved well-posedness for a Cauchy problem for the pseudoparabolic (1.1).

2. A Priori Estimates

In this section, we obtain a priori estimates for the problem (1.1)–(1.3).

Lemma 2.1. *Let $u_0 \in H_0^1(\Omega)$. Under assumption* (1.4), *if u is a solution of the problem* (1.1)–(1.3) *then one has the following estimate:*

$$\|\nabla u\|^2 \leq D_1, \tag{A}$$

$$\int_0^t \|\nabla u_s\|^2 ds \leq D_2, \tag{B}$$

where $D_1 > 0$ and $D_2 > 0$ depend on the initial data and parameters of (1.1).

Proof. We multiply (1.1) by u_t and integrate over Ω. We get

$$\frac{d}{dt}[E(t)] + \alpha \|\nabla u_t\|^2 + \|u_t\|^2 = 0, \tag{2.1}$$

where $E(t) = (\beta/2)\|\nabla u\|^2 - \int_\Omega F(u)dx$. We integrate (2.1) on the interval $(0, t)$, and from (1.4) we get

$$\frac{\beta}{2}\|\nabla u\|^2 + \alpha \int_0^t \|\nabla u_s\|^2 ds \leq E(0). \tag{2.2}$$

Hence (A) and (B) follow from (2.2). $\square$

3. Continuous Dependence on the Coefficient α

In this section we prove that the solution of the problem (1.1)–(1.3) depends continuously on the coefficient α in $H^1(\Omega)$ norm.

We now assume that u and v are the solutions of the following problems, respectively:

$$
\begin{gathered}
u_t - \alpha_1 \Delta u_t - \beta \Delta u = f(u), \quad x \in \Omega,\ t > 0, \\
u(x,0) = u_0(x), \quad x \in \Omega, \\
u(x,t) = 0, \quad x \in \partial\Omega,\ t > 0, \\
v_t - \alpha_2 \Delta v_t - \beta \Delta v = f(v), \quad x \in \Omega,\ t > 0, \\
v(x,0) = u_0(x), \quad x \in \Omega, \\
v(x,t) = 0, \quad x \in \partial\Omega,\ t > 0.
\end{gathered}
\tag{3.1}
$$

Let $w = u - v$, $\alpha = \alpha_1 - \alpha_2$. Then w is a solution of the problem

$$w_t - \alpha_1 \Delta w_t - \alpha \Delta v_t - \beta \Delta w = f(u) - f(v), \quad x \in \Omega,\ t > 0, \tag{3.2}$$

$$w(x,0) = 0, \quad x \in \Omega, \tag{3.3}$$

$$w(x,t) = 0, \quad x \in \partial\Omega,\ t > 0. \tag{3.4}$$

The following theorem establishes continuous dependence of the solution of (1.1)–(1.3) on the coefficient α in $H^1(\Omega)$ norm.

Theorem 3.1. *Assume that*

$$\left| f(u) - f(v) \right| \le K\left(1 + |u|^{p-1} + |v|^{p-1}\right)|u - v|, \tag{3.5}$$

where $1 < p \le n/(n-2)$ *if* $n > 2$, $p \in [1,\infty)$ *if* $n = 2$. *Let* w *be the solution of the problem* (3.2)–(3.4). *Then* w *satisfies the estimate*

$$\|w\|^2 + \alpha_1 \|\nabla w\|^2 \le D(\alpha_1 - \alpha_2)^2 e^{M_1 t}. \tag{3.6}$$

Here K, D, *and* M_1 *are positive constants.*

Proof. We multiply (3.2) by w and integrate over Ω. We get

$$\frac{1}{2}\frac{d}{dt}\left[\|w\|^2 + \alpha_1 \|\nabla w\|^2\right] + \alpha \int_\Omega \nabla v_t \nabla w\, dx + \beta \|\nabla w\|^2 = \int_\Omega (f(u) - f(v)) w\, dx. \tag{3.7}$$

Using the Cauchy-Schwarz inequality and (3.5), we get

$$\begin{aligned}\frac{d}{dt}\left[\frac{1}{2}\|w\|^2+\frac{\alpha_1}{2}\|\nabla w\|^2\right]+\beta\|\nabla w\|^2 \\ \leq |\alpha|\|\nabla v_t\|\|\nabla w\|+K\int_\Omega\left(1+|u|^{p-1}+|v|^{p-1}\right)|w|^2dx.\end{aligned}\tag{3.8}$$

Making use of Holder's inequality, we estimate the second term at the right-hand side of (3.8) as follows

$$K\int_\Omega\left(1+|u|^{p-1}+|v|^{p-1}\right)|w|^2dx\leq K\|w\|^2+K_1\left(\|u\|^{p-1}_{(p-1)n}+\|v\|^{p-1}_{(p-1)n}\right)\|w\|_{2n/(n-2)}\|w\|.\tag{3.9}$$

Inequality $\|w\|_{2n/(n-2)}\leq K_2\|\nabla w\|$ is valid for all $w\in H^1_0(\Omega)$. Using the Sobolev inequality and (A), we obtain the estimate

$$\|u\|^{p-1}_{(p-1)n}+\|v\|^{p-1}_{(p-1)n}\leq d_1\left(\|\nabla u\|^{p-1}+\|\nabla v\|^{p-1}\right)\leq K_3.\tag{3.10}$$

Therefore using Poincare's inequality from (3.9) and (3.10), we get

$$K\int_\Omega\left(1+|u|^{p-1}+|v|^{p-1}\right)|w|w\,dx\leq K_4\|\nabla w\|^2,\tag{3.11}$$

where d_1 and K_i $(i=1,2,3,4)$ are positive constants. By using (3.11) in (3.8) we get

$$\frac{d}{dt}\left[\frac{1}{2}\|w\|^2+\frac{\alpha_1}{2}\|\nabla w\|^2\right]+\beta\|\nabla w\|^2\leq|\alpha|\|\nabla v_t\|\|\nabla w\|+K_4\|\nabla w\|^2.\tag{3.12}$$

Using arithmetic-geometric mean inequality, we have

$$\frac{d}{dt}E_1(t)\leq\frac{|\alpha|^2}{2}\|\nabla v_t\|^2+M_1E_1(t),\tag{3.13}$$

where $E_1(t)=(1/2)\|w\|^2+(\alpha_1/2)\|\nabla w\|^2$ and $M_1=\max\{(2/\alpha_1)\ (K_4+1/2),1\}$. Solving the first-order differential inequality (3.13) and from (B), we obtain

$$E_1(t)\leq\frac{D_2}{2}|\alpha|^2e^{M_1t}.\tag{3.14}$$

The last estimate implies the desired inequality. □

4. Continuous Dependence on the Coefficient β

In this section we prove that the solution of the problem (1.1)–(1.3) depends continuously on the coefficient β in $H^1(\Omega)$ norm.

We now assume that u and v are the solutions of the following problems, respectively:

$$\begin{gathered}
u_t - \alpha\Delta u_t - \beta_1\Delta u = f(u), \quad x \in \Omega,\ t > 0, \\
u(x,0) = u_0(x), \quad x \in \Omega, \\
u(x,t) = 0, \quad x \in \partial\Omega,\ t > 0, \\
v_t - \alpha\Delta v_t - \beta_2\Delta v = f(v), \quad x \in \Omega,\ t > 0, \\
v(x,0) = u_0(x),\ x \in \Omega, \\
v(x,t) = 0, \quad x \in \partial\Omega,\ t > 0.
\end{gathered} \tag{4.1}$$

Let $w = u - v$, $\beta = \beta_1 - \beta_2$. Then w is a solution of the problem

$$w_t - \alpha\Delta w_t - \beta_1\Delta w - \beta\Delta v = f(u) - f(v), \quad x \in \Omega,\ t > 0, \tag{4.2}$$

$$w(x,0) = 0, \quad x \in \Omega, \tag{4.3}$$

$$w(x,t) = 0, \quad x \in \partial\Omega,\ t > 0. \tag{4.4}$$

The main result of this section is the following theorem.

Theorem 4.1. *Assume that* (3.5) *holds. Let w be the solution of the problem* (4.2)–(4.4). *Then w satisfies the estimate*

$$\|w\|^2 + \alpha\|\nabla w\|^2 \le D_1(\beta_1 - \beta_2)^2 e^{M_2 t}, \tag{4.5}$$

where M_2 is constant.

Proof. We multiply (4.2) by w and integrate over Ω. We get

$$\frac{1}{2}\frac{d}{dt}\left[\|w\|^2 + \alpha\|\nabla w\|^2\right] + \beta\int_\Omega \nabla v\nabla w\,dx + \beta_1\|\nabla w\|^2 = \int_\Omega (f(u) - f(v))w\,dx. \tag{4.6}$$

By using Cauchy-Schwarz inequality and (3.5) in (4.6) we get

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}\|w\|^2 + \frac{\alpha}{2}\|\nabla w\|^2\right] + \beta_1\|\nabla w\|^2 \\
&\quad \le |\beta|\|\nabla v\|\|\nabla w\| + K\int_\Omega \left(1 + |u|^{p-1} + |v|^{p-1}\right)|w|^2 dx,
\end{aligned} \tag{4.7}$$

and by using (3.11) in (4.7) we obtain

$$\frac{d}{dt}\left[\frac{1}{2}\|w\|^2+\frac{\alpha}{2}\|\nabla w\|^2\right]+\beta_1\|\nabla w\|^2\leq\left|\beta\right|\|\nabla v\|\|\nabla w\|+K_4\|\nabla w\|^2. \tag{4.8}$$

Using arithmetic-geometric mean inequality, we have

$$\frac{d}{dt}E_2(t)\leq\frac{\left|\beta\right|^2}{2}\|\nabla v\|^2+M_2E_2(t), \tag{4.9}$$

where $E_2(t) = (1/2)\|w\|^2 + (\alpha/2)\|\nabla w\|^2$ and $M_2 = \max\{(2/\alpha)(K_4 + 1/2), 1\}$. Solving the first-order differential inequality (4.9) and from (A), we obtain

$$E_2(t)\leq\frac{D_1}{2}\left|\beta\right|^2e^{M_2t}. \tag{4.10}$$

Hence the proof is completed. □

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