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Research Article

Generating Functions for the Mean Value of a Function on a Sphere and Its Associated Ball in $\mathbb{R}^n$

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We define two functions which determine the properties and the representation of the mean value of a function on a ball and on its associated sphere. Using these two functions, we obtain Pizzetti's formula in $\mathbf{R}^n$ as well as a similar formula for the mean value of a function on the ball associated to the sphere. We also give the expressions of the remainders in these two formulas, using the surface integral on a sphere.

1. Introduction

The mean function over values of a sphere and over its associated ball are very important in the study of some mathematical-physics problems as well as in the theory of a potential and in partial differential equations.

A representation for the mean values of a function over a sphere in $\mathbf{R}^n$ was given by Pizzetti [1] using polyharmonic operators.

In [2, volume II chapter IV, Section 3, page 258], using the second Green's formula, it is given the proof for Pizzetti's formula and then its generalization in $\mathbf{R}^n$.

In this paper, we define two functions which determine the properties and the representations of the mean values of a function over a sphere and over its associated ball.

These functions are called generating functions of these two averages and using these functions we obtain Pizzetti's formula in $\mathbf{R}^n$ as well as a new formula for the mean values of a function over a ball.

There are given the expression of the remainders using an integral over a sphere.

The properties of the generating functions are established using the formality concerning the calculus for higher order derivatives for functions of several variables.

For this purpose we use two formulas of N. Ya. Sonin and of Dirichlet [3, page 365], [4, page 671], respectively.

There are defined the corresponding quantities of some scalar quantities using the differential operators ∇, Δ, $\Delta^{(h)}$.

This fact allows us to prove two new formulas which determine the properties of the generating functions.

It is very important to mention that in this paper the way of deducting Pizzetti's formula is totally different from the way used in [3, page 73] as well as in [5, page 104].

2. General results

Let $\Omega \subset \mathbf{R}^n$ be a bounded set and $f : \Omega \to \mathbf{R}$, $f \in C^{(2m+1)}(\Omega)$.

We denote by $S_r(a) = \{x,\ x \in \mathbf{R}^n,\ |x-a| = r\}$ the sphere of radius r, centered in $a = (a_1, a_2, \dots, a_n)$, $B_r(a) = \{x,\ x \in \mathbf{R}^n,\ |x-a| \le r\}$ the ball of radius r and centered in a which is associated to $S_r(a)$.

We will also denote by R, $R = \max\{r = |x-a|^n\}$ for that $B_r(a) \subset \Omega$. We define the functions

$$\phi : [-R, R] \longrightarrow \mathbf{R}, \quad \phi(r) = \int_{B_1(a)} f(a + r(x-a))\,dx, \tag{2.1}$$

$$\varphi : [-R, R] \longrightarrow \mathbf{R}, \quad \varphi(r) = \int_{S_1(a)} f(a + r(x-a))\,dS_1, \tag{2.2}$$

where dx, dS_1 represent the volume element and area element for the unit ball $B_1(a)$ and for the unit sphere, respectively.

The mean values $M_r^s(f)$ and $M_r^b(f)$ for $f : \Omega \to \mathbf{R}$ over $S_r(a) \subset \Omega$ and over $B_r(a) \subset \Omega$, respectively, are given by the following expressions:

$$M_r^s(f) = \frac{1}{|S_r(a)|}\int_{S_r(a)} f(x)\,dS, \quad 0 \le r \le R, \tag{2.3}$$

$$M_r^b(f) = \frac{1}{|B_r(a)|}\int_{B_r(a)} f(x)\,dx, \quad 0 \le r \le R, \tag{2.4}$$

where (see [6, page 22]) $|S_r(a)| = (2\pi^{n/2}/\Gamma(n/2))r^{n-1}$ represents the area of the sphere $S_r(a)$ and $|B_r(a)| = (2\pi^{n/2}/\Gamma(n/2))(r^n/n)$ represents the volume of the ball $B_r(a)$, Γ being beta-function.

Between the cartesian coordinates $x = (x_1, x_2, \dots, x_n)$ and spherical coordinates $y = (\rho, \theta_1, \dots, \theta_{n-1})$ centered in $a = (a_1, a_2, \dots, a_n)$ there are the relations

$$\begin{aligned}
x_1 &= a_1 + \rho \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-2} \sin\theta_{n-1} = a_1 + \rho h_1, \\
x_2 &= a_2 + \rho \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-2} \cos\theta_{n-1} = a_1 + \rho h_2, \\
x_3 &= a_3 + \rho \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-3} \cos\theta_{n-2} = a_3 + \rho h_3, \\
&\vdots \\
x_{n-2} &= a_{n-2} + \rho \sin\theta_1 \sin\theta_2 \cos\theta_3 = a_{n-2} + \rho h_{n-2}, \\
x_{n-1} &= a_{n-1} + \rho \sin\theta_1 \cos\theta_2 = a_{n-1} + \rho h_{n-1}, \\
x_n &= a_n + \rho \cos\theta_1 = a_n + \rho h_n,
\end{aligned} \tag{2.5}$$

where

$$\rho \geq 0, \quad \theta_i \in [0,\pi], \quad i = \overline{1, n-2}, \quad \theta_{n-1} \in [0, 2\pi]. \tag{2.6}$$

The Jacobian of this punctual transform is

$$J(\rho, \theta_1, \ldots, \theta_{n-1}) = \frac{\partial(x)}{\partial(y)} = \rho^{n-1} \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \sin^{n-4}\theta_3 \cdots \sin\theta_{n-2}. \tag{2.7}$$

The volume element dy written in spherical coordinates has the expression

$$dy = J(\rho, \theta_1, \ldots, \theta_{n-1}) d\rho\, d\theta_1 \cdots d\theta_{n-1} \tag{2.8}$$

and the area element for $S_\rho(a)$ is

$$dS_\rho = \rho^{n-1} y^*(\theta_1, \ldots, \theta_{n-2}) d\theta_1 d\theta_2 \cdots d\theta_{n-1}, \tag{2.9}$$

where

$$J^*(\theta_1, \ldots, \theta_{n-2}) = \frac{J(\rho, \theta_1, \ldots, \theta_{n-1})}{\rho^{n-1}} = \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \cdots \sin^2\theta_{n-3} \sin\theta_{n-2}. \tag{2.10}$$

From (2.8) and (2.9) we have

$$dy = d\rho\, dS_\rho = \rho^{n-1} d\rho\, dS_1(\theta_1, \ldots, \theta_{n-1}), \tag{2.11}$$

where $dS_1(\theta_1, \ldots, \theta_{n-1})$ represents the area element of the unit sphere $S_1(a)$ in spherical coordinates.

Proposition 2.1. *Between the functions $\phi, \psi : [-R, R] \to \mathbf{R}$ defined in* (2.1) *and* (2.2)*, there is the relation*

$$\phi(r) = \int_0^1 u^{n-1} \psi(ru) du, \quad |r| \leq R. \tag{2.12}$$

Proof. Using the spherical coordinates and the relations (2.9) and (2.11) we obtain the following:

$$\begin{aligned} \psi(r) &= \int_\Delta f(\ldots, a_i + rh_i, \ldots) J^*(\theta_1, \ldots, \theta_{n-2}) d\theta_1 \cdots d\theta_{n-1} \\ &= \int_\Delta f(\ldots, a_i + rh_i, \ldots) dS_1(\theta_1, \ldots, \theta_{n-1}), \end{aligned} \tag{2.13}$$

$$\begin{aligned} \phi(r) &= \int_0^1 \int_\Delta f(\ldots, a_i + ruh_i, \ldots) u^{n-1} du\; dS_1(\theta_1, \ldots, \theta_{n-1}) \\ &= \int_0^1 u^{n-1} \left[\int_\Delta f(\ldots, a_i + ruh_i, \ldots) dS_1(\theta_1, \ldots, \theta_{n-1}) \right] du, \end{aligned} \tag{2.14}$$

where $\Delta = \underbrace{[0,\pi] \times \cdots \times [0,\pi] \times [0, 2\pi]}_{n-2 \text{ times}}$. From (2.13) we note that

$$\psi(ru) = \int_\Delta f(\ldots, a_i + ruh_i, \ldots) dS_1(\theta_1, \ldots, \theta_{n-1}) \tag{2.15}$$

and taking into account (2.14), the relation (2.12) is proved. □

Concerning the dependence between the functions ϕ, ψ and the mean values $M_r^s(f)$, $M_r^b(f)$ of a function f over a sphere and over the associated ball, respectively, we can state the following.

Proposition 2.2. *Between the functions ϕ, ψ defined by* (2.1) *and* (2.2)*, respectively, and the mean values $M_r^f(f)$, $M_r^b(f)$ defined by* (2.3) *and* (2.4)*, there are the following relations:*

$$M_r^b(f) = \frac{r^n}{|B_r(a)|}\phi(r) = \frac{n\Gamma(n/2)}{2\pi^{n/2}}\phi(r), \quad 0 \le r \le R, \tag{2.16}$$

$$M_r^s(f) = \frac{r^{n-1}}{|S_r(a)|}\psi(r) = \frac{n\Gamma(n/2)}{2\pi^{n/2}}\psi(r), \quad 0 \le r \le R. \tag{2.17}$$

Proof. For $0 \le r \le R$, from (2.14) makingthe substitution $ur = \rho$ and taking into account (2.4) we obtain

$$\begin{aligned}\phi(r) &= \frac{1}{r^n}\int_0^r \rho^{n-1}\left[\int_\Delta f(\dots,a_i+\rho h_i,\dots)dS_1(\theta_1,\dots,\theta_{n-1})\right]d\rho \\ &= \frac{1}{r^n}\int_{B_r(a)} f(x)dx = \frac{2\pi^{n/2}}{n\Gamma(n/2)}M_r^b(f)\end{aligned} \tag{2.18}$$

and so, (2.16) is proved.

Using the spherical coordinates, we have

$$\begin{aligned}M_r^s(f) &= \frac{1}{|S_r(a)|}\int_{S_r(x_0)} f(x)dx \\ &= \frac{1}{|S_r(a)|}\int_\Delta f(\dots,a_i+rh_i,\dots)dS_r(\theta_1,\dots,\theta_{n-1}) \\ &= \frac{r^{n-1}}{|S_r(a)|}\int_\Delta f(\dots,a_i+rh_i,\dots)dS_1(\theta_1,\dots,\theta_{n-1}).\end{aligned} \tag{2.19}$$

Taking into account (2.13) we have

$$M_r^s(f) = \frac{r^{n-1}}{|S_r(a)|}\psi(r) = \frac{\Gamma(n/2)}{2\pi^{n/2}}\psi(r) \tag{2.20}$$

which is (2.17).

The relations (2.16) and (2.17) show that for $0 \le r \le R$, the functions $\phi, \psi : [-R, R] \to \mathbf{R}$, determine the properties of the mean values $M_r^b(f)$ and $M_r^s(f)$ and permit the calculus of these two quantities. □

These relations justify the introduction of the following.

Definition 2.3. The functions $\phi, \psi : [-R, R] \to \mathbf{R}$ defined by (2.1) and (2.2) are called generating functions for the mean values $M_r^b(f)$ and $M_r^s(f)$ of a function $f : \Omega \to \mathbf{R}$ over the ball $B_r(a) \subset \Omega$ and over the sphere $S_r(a) \subset \Omega$, respectively. Particularly, if $f \ge 0$ on $\Omega \subset \mathbf{R}^n$, then this function can be considered as the mass density on $S_r(a)$ and on $B_r(a)$, respectively.

Consequently, taking into account (2.3) and (2.17), the total mass m_r^s of the sphere $S_r(a)$ is given by the expression

$$m_r^s = \int_{S_r(a)} f(x)dS = |S_r(a)|M_r^s(f) = r^{n-1}\psi(r), \quad 0 \le r \le R. \tag{2.21}$$

In this case, $M_r^s(f)$ represents the mean value of the mass density on $S_r(a)$. Similarly, from (2.4) and (2.16) we have the following expression for the total mass on $B_r(a)$:

$$m_r^b = \int_{B_r(a)} f(x)dx = |B_r(a)|M_r^b(f) = r^n\phi(r), \quad 0 \le r \le R. \tag{2.22}$$

The expressions (2.21) and (2.22) prove that the generating functions ϕ and ψ, when $f \ge 0$ and $0 \le r \le R$, have a mechanical meaning, allowing the calculus of the mass for $S_r(a)$ and for $B_r(a)$ with the density $\rho(x) = f(x)$, $x \in \Omega \subset \mathbf{R}^n$.

Next, for the study of the properties of the generating functions ψ and ϕ, we will use M. Ya. Sonin Formula.

Let $m_i \in R$, $i = \overline{1,n}$ be real numbers and $k \in \mathbf{N}_0$. Then (see [3, page 365], [4, page 671]) we have Sonin formula

$$\begin{aligned} S &= \int_{B_1(0)} (m_1x_1 + \cdots + m_nx_n)^{2k} dx_1 \cdots dx_n \\ &= \pi^{(n-1)/2} \frac{\Gamma(k+1/2)}{\Gamma(n/2+k+1)} (m_1^2 + \cdots + m_n^r)^k. \end{aligned} \tag{2.23}$$

Denoting $m = (m_1, m_2, \dots, m_n) \in \mathbf{R}^n$ and "$\langle\ \rangle$" the scalar product, (2.23) becomes

$$S = \int_{B_1(0)} \langle m, x\rangle^{2k} = \pi^{(n-1)/2} \frac{\Gamma(k+1/2)}{\Gamma(n/2+k+1)} \langle m, m\rangle^k. \tag{2.24}$$

We mention that this result can be justified (see [3, page 365]) on the basis of Dirichlet formula

$$D = \int_{B_1(0)} x_1^{2u_1} \cdots x_n^{2u_n} dx_1 \cdots dx_n = \frac{\Gamma(\mu_1 + 1/2) \cdots \Gamma(\mu_n + 1/2)}{\Gamma(n/2+k+1)}, \tag{2.25}$$

where $\mu_i \in \mathbf{N}_0$, $i = \overline{1,n}$ and $k = \mu_1 + \cdots + \mu_n$.

Using $\mu = (\mu_1 + \cdots + \mu_n) \in \mathbf{N}_0^n$, where $|\mu| = \mu_1 + \cdots + \mu_n = k$, Dirichlet formula becomes

$$D = \int_{B_1(0)} x^{2\mu} dx = \frac{\Gamma(\mu_1 + 1/2) \cdots \Gamma(\mu_n + 1/2)}{\Gamma(n/2+k+1)}. \tag{2.26}$$

Making the substitutions $u_i = x_i - a_i$, $i = \overline{1,n}$ and using $J(u_1, \dots, u_m) = \partial(x)/\partial(u) = 1$ the

Jacobian of the transform, we have

$$S = \int_{B_1(a)} \langle m, x-a\rangle^{2k} dx = \int_{B_1(0)} \langle m, x\rangle^{2k} dx = \pi^{(n-1)/2} \frac{\Gamma(k+1/2)}{\Gamma(n/2+k+1)} \langle m, m\rangle^{k}, \tag{2.27}$$

$$D = \int_{B_1(a)} (x-a)^{2\mu} dx = \int_{B_1(0)} x^{2\mu} dx = \frac{\Gamma(\mu_1+1/2)\cdots\Gamma(\mu_n+1/2)}{\Gamma(n/2+k+1)}, \tag{2.28}$$

where $B_1(a)$ represents the unit ball centered in $a \in \mathbf{R}^n$.

Let us consider the integrals

$$S^* = \int_{S_1(a)} \langle m, x-a\rangle^{2k} dS_1, \qquad D^* = \int_{S_1(a)} (x-a)^{2\mu} dS_1, \tag{2.29}$$

where $S_1(a)$ is the unit sphere, centered in $a \in \mathbf{R}^n$.

Proposition 2.4. *Between the pairs of integrals* (S^*, S) *and* (D^*, D)*, there are the following relations:*

$$S^* = (2k+n)S = 2\pi^{(n-1)/2} \frac{\Gamma(k+1/2)}{\Gamma(k+n/2)} \langle m, m\rangle^{k}, \tag{2.30}$$

$$D^* = (2k+n)D = 2\frac{\Gamma(\mu_1+1/2)\cdots\Gamma(\mu_n+1/2)}{\Gamma(k+n/2)}. \tag{2.31}$$

Proof. Using the spherical coordinates (2.5), we have

$$x - a = h = (h_1, h_2, \ldots, h_n). \tag{2.32}$$

Taking into account (2.11), Sonin's integral (2.27) becomes

$$\begin{aligned} S &= \int_{B_1(a)} \langle m, x-a\rangle^{2k} dx \\ &= \int_0^1 \int_{\Delta} \langle m, \rho h\rangle^{2k} \rho^{n-1} d\rho \, dS_1(\theta_1, \ldots, \theta_{n-1}) \\ &= \frac{1}{2k+n} \int_{\Delta} \langle m, h\rangle^{2k} dS_1(\theta_1, \ldots, \theta_{n-1}) \\ &= \frac{1}{2k+n} \int_{S_1(a)} \langle m, x-a\rangle^{2k} dS_1, \end{aligned} \tag{2.33}$$

where $\Delta = \underbrace{[0,\pi] \times \cdots \times [0,\pi]}_{n-2} \times [0, 2\pi]$. We obtain

$$\begin{aligned} S^* &= \int_{S_1(a)} \langle m, x-a\rangle^{2k} dS_1 \\ &= (2k+n)S \\ &= (2k+n) \frac{\Gamma(k+1/2)}{\Gamma(n/2+k+1)} \pi^{(n-1)/2} \langle m, m\rangle^{k} \\ &= 2\frac{\Gamma(k+1/2)}{\Gamma(n/2+k)} \pi^{(n-1)/2} \langle m, m\rangle^{k}. \end{aligned} \tag{2.34}$$

Using the same procedure, Dirichlet's integral (2.28) becomes

$$\begin{aligned} D &= \int_{B_1(a)} (x-a)^{2\mu} dx \\ &= \int_0^1 \int_{\Delta} \langle m, \rho h\rangle^{2\mu} \rho^{n-1} d\rho \, dS_1(\theta_1, \dots, \theta_{n-1}) \\ &= \frac{1}{2k+n} \int_{\Delta} h^{2\mu} dS_1(\theta_1, \dots, \theta_{n-1}) \\ &= \frac{1}{2k+n} \int_{S_1(a)} (x-a)^{2\mu} dS_1. \end{aligned} \tag{2.35}$$

Hence

$$\begin{aligned} D^* &= \int_{S_1(a)} (x-a)^{2\mu} dS_1 \\ &= (2k+n)D \\ &= (2k+n)\frac{\Gamma(\mu_1+1/2)\cdots\Gamma(\mu_n+1/2)}{\Gamma(k+n/2+1)} \\ &= 2\frac{\Gamma(\mu_1+1/2)\cdots\Gamma(\mu_n+1/2)}{\Gamma(n/2+k)}. \end{aligned} \tag{2.36}$$

So the proposition is proved. □

Next, using the formulas concerning the calculus of the higher order differential for functions of several variables, we will define the correspondings for some scalar quantities which appear in the expressions of S and S^*, respectively (2.27) and (2.29).

The corresponding scalar quantities are defined using some differential operators, this fact leads us to new expressions similar to (2.24) and (2.30).

On the basis of these formulas, we will establish the properties of the generating functions ψ and ϕ and of the mean values $M_r^s(f)$ and $M_r^b(f)$.

Let $f \in C^p(\Omega)$ with $B_1(a) \subset \Omega$ and $2k \leq p$. We will define the following correspondings:

(1)

$$m = (m_1, \dots, m_n) \longrightarrow \nabla f(a) = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) f(a), \tag{2.37}$$

where $\nabla = (\partial/\partial x_1, \dots, \partial/\partial x_n)$ represents the operator "nabla";

(2)

$$|m|^2 = \langle m, m\rangle \longrightarrow \langle \nabla, \nabla\rangle f(a) = \Delta f(a), \tag{2.38}$$

where $|\cdot|$ represents the norm of a vector and $\Delta = \langle \nabla, \nabla\rangle = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$ the Laplace operator in $\mathbf{R}^n$;

(3)

$$|m|^{2k} = \langle m, m\rangle^k \longrightarrow \langle \nabla, \nabla\rangle^k f(a) = \Delta^{(k)} f(a), \tag{2.39}$$

where $\Delta^{(k)} = \underbrace{\Delta \cdot \Delta \cdots \Delta}_{k} = (\partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2)^{(k)}$ represents the polyharmonic operator of order k;

(4)

$$\langle m, x-a\rangle \longrightarrow \langle \nabla, x-a\rangle f(a) = \left(\frac{\partial}{\partial x_1}(x_1 - a_1) + \cdots + \frac{\partial}{\partial x_n}(x_n - a_n)\right) f(a) = df(a), \tag{2.40}$$

where $d = (\partial/\partial x_1)(x_1 - a_1) + \cdots + (\partial/\partial x_n)(x_n - a_n)$ represents the differential operator;

(5)

$$\langle m, x-a\rangle^{2k} \longrightarrow \langle \nabla, x-a\rangle^{2k} f(a) = d^{2n} f(a), \tag{2.41}$$

where $d^{2k} = ((\partial/\partial x_1)(x_1 - a_1) + \cdots + (\partial/\partial x_n)(x_n - a_n))^{2k}$ represents the differential operator of order $2k$.

Using these correspondences and taking into account (2.23), (2.27), (2.29), and (2.30), we obtain

$$\int_{B_1(a)} d^{2k} f(a)dx = \pi^{(n-1)/2} \frac{\Gamma(k+1/2)}{\Gamma(n/2+k+1)} \Delta^{(k)} f(a), \tag{2.42}$$

$$\int_{S_1(a)} d^{2k} f(a)dS_1 = 2\pi^{(n-1)/2} \frac{\Gamma(k+1/2)}{\Gamma(n/2+k)} \Delta^{(k)} f(a). \tag{2.43}$$

We note that the expressions (2.42) and (2.43) represent the correspondings for (2.27) and (2.30). We will prove the availability of these relations.

Proposition 2.5. *For $f \in C^p(\Omega)$ with $B_1(a) \subset \Omega$ and $2k \le p$ the relations* (2.42) *and* (2.43) *held.*

Proof. We have

$$\begin{aligned}\int_{B_1(a)} d^{2k} f(a)dx &= \int_{B_1(a)} \left(\frac{\partial}{\partial x_1}(x_1 - a_1) + \cdots + \frac{\partial}{\partial x_n}(x_n - a_n)\right)^{2k} f(a)dx_1 \cdots dx_n \\ &= \sum_{2\mu_1+\cdots+2\mu_n=2k} \frac{(2k)!}{(2\mu_1)!\cdots(2\mu_n)!} \left(\frac{\partial^2}{\partial x_1^2}\right)^{\mu_1} \cdots \left(\frac{\partial^2}{\partial x_n^2}\right)^{\mu_n} f(a) \cdot \int_{B_1(a)} (x-a)^{2\mu} dx.\end{aligned} \tag{2.44}$$

On the basis of $(2k)! = 2^k k!(2k-1)!! = 2^{2k} k!(\Gamma(k+1/2)/\sqrt{\pi})$, we obtain

$$\frac{(2k)!}{(2\mu_1)!\cdots(2\mu_n)!} = \frac{k!}{\mu_1!\cdots\mu_n!} \frac{\pi^{(n-1)/2}\Gamma(k+1/2)}{\Gamma(\mu_1+1/2)\cdots\Gamma(\mu_n+1/2)}. \tag{2.45}$$

Taking into account (2.45) and Dirichlet Formulas (2.43), the expression (2.44) becomes

$$\int_{B_1(a)} d^{2k} f(a)dx = \pi^{(n-1)/2} \frac{\Gamma(k+1/2)}{\Gamma(n/2+k+1)} \sum_{\mu_1+\mu_2+\cdots+\mu_n=k} \frac{k!}{\mu_1!\cdots\mu_n!} \left(\frac{\partial^2}{\partial x_1^2}\right)^{\mu_1} \cdots \left(\frac{\partial^2}{\partial x_n^2}\right)^{\mu_n} f(a), \tag{2.46}$$

so that

$$\begin{aligned}\int_{B_1(a)} d^{2k} f(a)dx &= \pi^{(n-1)/2} \frac{\Gamma(k+1/2)}{\Gamma(n/2+k+1)} \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)^{(k)} f(a) \\ &= \pi^{(n-1)/2} \frac{\Gamma(k+1/2)}{\Gamma(n/2+k+1)} \Delta^{(k)} f(a).\end{aligned} \tag{2.47}$$

Using the same method, we obtain

$$\begin{aligned}&\int_{S_1(a)} d^{2k} f(a)dx \\ &\quad= \int_{S_1(a)} \left(\frac{\partial}{\partial x_1}(x_1 - a_1) + \cdots + \frac{\partial}{\partial x_n}(x_n - a_n)\right)^{2k} f(a)dS_1 \\ &\quad= \sum_{2\mu_1+\cdots+2\mu_n=2k} \frac{(2k)!}{(2\mu_1)!\cdots(2\mu_n)!} \left(\frac{\partial^2}{\partial x_1^2}\right)^{\mu_1} \cdots \left(\frac{\partial^2}{\partial x_n^2}\right)^{\mu_n} f(a) \cdot \int_{S_1(a)} (x-a)^{2\mu} dS_1.\end{aligned} \tag{2.48}$$

On the basis of (2.45) and (2.31), we have

$$\begin{aligned}&\int_{S_1(a)} d^{2k} f(a)dS_1 \\ &\quad= 2\pi^{(n-1)/2} \frac{\Gamma(k+1/2)}{\Gamma(n/2+k)} \sum_{\mu_1+\cdots+\mu_n=k} \frac{k!}{\mu_1!\cdots\mu_n!} \left(\frac{\partial^2}{\partial x_1^2}\right)^{\mu_1} \cdots \left(\frac{\partial^2}{\partial x_n^2}\right)^{\mu_n} f(a) \\ &\quad= 2\pi^{(n-1)/2} \frac{\Gamma(k+1/2)}{\Gamma(n/2+k)} \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)^{(k)} f(a) \\ &\quad= 2\pi^{(n-1)/2} \frac{\Gamma(k+1/2)}{\Gamma(n/2+k)} \Delta^{(k)} f(a)\end{aligned} \tag{2.49}$$

and so the proposition is proved. □

For the generating functions ϕ and ψ we can state the following.

Proposition 2.6. *Let* $\Omega \in \mathbf{R}^n$ *be a bounded set and* $f : \Omega \to \mathbf{R}$, $f \in C^{2m+1}(\Omega)$. *Then the functions* $\phi, \psi : [-R, R] \to \mathbf{R}$ *defined by* (2.1) *and* (2.2), *where* $R = \max|x-a|$ *such that* $B_r(a) \subset \Omega$, *have the following properties:*

(1) ϕ, ψ *are even functions and* $\phi, \psi \in C^{2m+1}([-R,R])$,

(2) $\phi^{(k)}(0) = \psi^{(k)}(0) = 0$ *for* k *odd*, $k \leq 2m+1$,

(3) $\psi^{(2k)}(0) = \int_{S_1(a)} d^{2k}(a) dS_1 = 2\pi^{(n-1)/2}(\Gamma(k+1/2)/\Gamma(n/2+k))\Delta^{(k)} f(a)$, $k \leq m$,

(4) $\phi^{(2k)}(0) = (1/(n+2k))\psi^{2k}(0) = \pi^{(n-1)/2}(\Gamma(k+1/2)/\Gamma(n/2+k+1))\Delta^{(k)} f(a)$, $k \leq m$,

(5) $\phi^{(p)}(r) = \int_0^1 u^{n+p-1}\psi^{(p)}(ru)du$, $p \leq 2m+1$,

(6) $\psi^{(p)}(r) = \int_{S_1(a)} d^p f(a + r(x-a)) dS_1$, $p \leq 2m+1$,

(7) $\psi(r) = 2\pi^{n/2}\sum_{k=0}^{m}(1/k!\Gamma(n/2+k))(r/2)^{2k}\Delta^{(k)} f(a) + R_{2m+1}(r)$,

where

$$R_{2m+1}(r) = \frac{\psi^{(2m+1)}(\theta r)}{(2m+1)!} r^{2m+1} = \frac{r^{2m+1}}{(2m+1)!}\int_{S_1(a)} d^{2m+1} f(a + \theta r(x-a)) dS_1, \quad 0 < \theta < 1, \tag{2.50}$$

(8) $\phi(r) = \pi^{n/2}\sum_{k=0}^{m}(1/k!\Gamma(n/2+k+1))(r/2)^{2k}\Delta^{(k)} f(a) + R^*_{2m+1}(r)$,

where

$$R^*_{2m+1}(r) = \frac{\phi^{2m+1}(\theta r)}{(2m+1)!} r^{2m+1} = \frac{r^{2m+1}}{(2m+1)!}\int_0^1\int_{S_1(a)} u^{n+2m} d^{2m+1} f(a + \theta r u(x-a)) du\, dS_1, \quad 0 < \theta < 1. \tag{2.51}$$

Proof. Using the spherical coordinates given by (2.5) and denoting $h = (h_1, \ldots, h_n)$, the expression for ψ given by (2.2) becomes

$$\begin{aligned} \psi(r) &= \int_{S_1(a)} f(a + r(x-a)) dS_1 \\ &= \int_{\Delta} J^*(\theta_1, \ldots, \theta_{n-2}) \left[\int_0^{2\pi} f^*(r, \theta_1, \ldots, \theta_{n-1}) d\theta_{n-1}\right] d\theta_1 \cdots d\theta_{n-2}, \end{aligned} \tag{2.52}$$

where $\Delta = \underbrace{[0,\pi] \times \cdots \times [0,\pi]}_{n-2}$, $f^*(r, \theta_1, \ldots, \theta_{n-1}) = f(a + rh)$ and $J^*(\theta_1, \ldots, \theta_{n-2})$ is given by the formulas (2.10).

Since $f \in C^{2m+1}(\Omega)$, using the differentiation rule for integrals depending on a parameter, it results in $\psi \in C^{2m+1}([-R,R])$.

In order to prove that ψ is an even function we will change the variables $(\theta_1, \ldots, \theta_{n-1}) \to (u_1, \ldots, u_{n-1})$ by the relations

$$\begin{gathered} \theta_i = \pi - u_i, \quad i = \overline{1, n-2}, \quad \theta_{n-1} = \pi + u_{n-1}, \\ u_i \in [0, \pi], \quad i = \overline{1, n-2}, \quad u_{n-1} \in [-\pi, \pi] \end{gathered} \tag{2.53}$$

in the expression (2.52).

The Jacobian of this transform is defined by

$$J_1(u_1,\ldots,u_{n-1}) = \frac{\partial(\theta_1,\ldots,\theta_{n-1})}{\partial(u_1,\ldots,u_{n-1})} = (-1)^{n-2} \tag{2.54}$$

Having this change of variables, (2.52) becomes

$$\psi(r) = \int_{\Delta} J^*(u_1,\ldots,u_{n-2}) \left[\int_{-\pi}^{\pi} f^*(-r,u_1,\ldots,u_{n-1})du_{n-1}\right] du_1\cdots du_{n-2}. \tag{2.55}$$

Since $f^*(-r,u_1,\ldots,u_{n-1})$ is periodical, with the period 2π with respect to the variable u_{n-1}, the expression (2.55) becomes

$$\psi(r) = \int_{\Delta} J^*(u_1,\ldots,u_{n-2}) \left[\int_{0}^{2\pi} f^*(-r,u_1,\ldots,u_{n-1})du_{n-1}\right] du_1\cdots du_{n-2}. \tag{2.56}$$

Makingthe comparison between the expression from above and (2.52) we have $\psi(r) = \psi(-r)$ so ψ is an even function.

Consequently, $\psi^{(2k)}$ is an even function and $\psi^{(2k+1)}$ is odd, it means that $\psi^{(2k+1)}(0) = 0$. From (2.12) we have ϕ is even, $\phi \in C^{2m+1}([-R,R])$, $\phi^{(2k)}$ is even and $\phi^{(2k+1)}$ is odd, so $\phi^{(2k+1)}(0) = 0$.

So we proved the properties (1) and (2).

Remark 2.7. In the case of $f : \mathbf{R}^n \to \mathbf{R}$, $f \in \mathfrak{D}(\mathbf{R}^n)$ where $\mathfrak{D}(\mathbf{R}^n)$ is L. Schwartz space, $f \in C(\mathbf{R}^n)$ and having a compact support, then $\psi : \mathbf{R} \to \mathbf{R}$, $\psi(r) = \int_{S_1(a)} f(a+r(x-a))dS_1$, $\psi(r) \in C(\mathbf{R})$ with compact support.

If $f \in \mathfrak{D}(\mathbf{R}^n)$, we can state that the formulas

$$\psi(r) = \int_{S_1(a)} f(a+r(x-a))dS_1, \tag{2.57}$$

$r \in \mathbf{R}$, define an integral transform which associates $f \in \mathfrak{D}(\mathbf{R}^n)$ to the even function $\psi \in \mathfrak{D}(\mathbf{R})$.

According to [7, page 61], this integral transform is denoted by T_p^n named polare transform. Thus we write $\psi = T_p^n(f)$. From (2.2) applying the differentiation rule for composite functions and denoting $y = a = r(x-a)$, we obtain

$$\begin{aligned}
\psi^{(p)}(r) &= \int_{S_1(a)} \frac{\partial p}{\partial r^p} f(a+r(x-a))dS_1 \\
&= \int_{S_1(a)} \left(\frac{\partial}{\partial y_i}\frac{\partial y_i}{\partial r}\right)^{(p)} f(y)dS_1 \\
&= \int_{S_1(a)} \left(\frac{\partial}{\partial y_1}(x_1-a_1)+\cdots+\frac{\partial}{\partial y_n}(x_n-a_n)\right)^{(p)} f(y)dS_1 \\
&= \int_{S_1(a)} d^p f(a+r(x-a))dS_1.
\end{aligned} \tag{2.58}$$

For $p = 2k \le 2m$ and $r = 0$ we have

$$\psi^{(2k)}(0) = \int_{S_1(a)} \left(\frac{\partial}{\partial x_1}(x_1 - a_1) + \cdots + \frac{\partial}{\partial x_n}(x_n - a_n) \right)^{(2k)} f(a) dS_1 = \int_{S_1(a)} d^{(2k)} f(a) dS_1. \tag{2.59}$$

Taking into account (2.43), we have

$$\psi^{(2k)}(0) = \int_{S_1(a)} d^{2k} f(a) dS_1 = 2\pi^{(n-1)/2} \frac{\Gamma(k+1/2)}{\Gamma(n/2+k)} \Delta^k f(a). \tag{2.60}$$

On the other hand, from (2.12) and (2.60) we obtain

$$\begin{aligned} \phi^{(p)}(r) &= \int_0^1 u^{n+p-1} \psi^{(p)}(ru) du, \quad |r| \le R, \ p \le 2m+1, \\ \phi^{(2k)}(0) &= \frac{1}{n+2k} \psi^{(2k)}(0) \\ &= \frac{2\pi^{(n-1)/2}}{(n+2k)} \frac{\Gamma(k+1/2)}{\Gamma(n/2+k)} \Delta^k f(a) \\ &= \pi^{(n-1)/2} \frac{\Gamma(k+1/2)}{\Gamma(n/2+k+1)} \Delta^k f(a). \end{aligned} \tag{2.61}$$

The formulas (2.60) and (2.61) justify the properties (3), (4), (5), (6) from the proposition.

In order to obtain the formula (7) we will apply Mac-Laurin formulas. Thus ψ being even and $\psi \in C^{2m+1}([-R, R])$, on the basis of (2.58), we can write

$$\psi(r) = \sum_0^m \frac{\psi^{(2k)}(0)}{(2k)!} r^{2k} + R_{2m+1}(r), \tag{2.62}$$

where

$$R_{2m+1}(r) = \frac{\psi^{(2m+1)}(\theta r)}{(2m+1)!} r^{2m+1} = \frac{r^{2m+1}}{(2m+1)!} \int_{S_1(a)} d^{2m+1} f(a + \theta r(x-a)) dS_1, \quad 0 < \theta < 1. \tag{2.63}$$

Since $(2k)! = 2^{2k} k! \Gamma(k+1/2) \pi^{-1/2}$ and taking into account (2.60), from (2.62) we obtain (7).

Using the same method, we obtain (8) taking into account the results from (4), (5), and (6). So, Proposition 2.6 is proved. □

In particular, if $f \in C([-R, R])$ and f is analytic, then the remainders $R_{2m+1}(t) \to 0$, $R^*_{2m+1}(r) \to 0$ for $m \to \infty$.

We obtain the following Mac-Laurin Series for $\psi, \phi : [-R, R] \to \mathbf{R}$:

$$\psi(r) = 2\pi^{n/2} \sum_{k=0} \frac{1}{k! \Gamma(n/2+k)} \left(\frac{r}{2} \right)^{2k} \Delta^{(k)} f(a), \quad |r| \le R, \tag{2.64}$$

$$\phi(r) = \pi^{n/2} \sum_{k=0} \frac{1}{k! \Gamma(n/2+k+1)} \left(\frac{r}{2} \right)^{2k} \Delta^{(k)} f(a), \quad |r| \le R, \tag{2.65}$$

The results established in Proposition 2.6 permit to give the corresponding representations for the mean values $M_r^s(f)$ and $M_r^b(f)$ defined in (2.3) and (2.4), by using (2.16) and (2.17).

Thus, for $0 \le r \le R$, taking into account (8), (2.16), and (2.64) we can write

$$\begin{aligned}
M_r^b(f) &= \frac{1}{|B_r(a)|}\int_{B_r(a)} f(x)dx \\
&= \frac{n}{2}\Gamma\left(\frac{n}{2}\right)\sum_{k=0}^{m}\frac{1}{k!\Gamma(n/2+k+1)}\left(\frac{r}{2}\right)^{2k}\Delta^{(k)}f(a) \\
&\quad + \frac{n}{2\pi^{n/2}}\Gamma\left(\frac{n}{2}\right)\frac{r^{2m+1}}{(2m+1)!}\int_0^1\int_{S_1(a)} u^{n+2m}d^{2m+1}f(a+\theta ru(x-a))du\,dS_1, \quad 0<\theta<1, \\
M_r^b(f) &= \frac{n}{2}\Gamma\left(\frac{n}{2}\right)\sum_{k=0}\frac{1}{k!\Gamma(n/2+k+1)}\left(\frac{r}{2}\right)^{2k}\Delta^{(k)}f(a).
\end{aligned} \tag{2.66}$$

Similarly, from (7), (2.62), and (2.17) we have

$$\begin{aligned}
M_r^s(f) &= \frac{1}{|S_r(a)|}\int_{S_r(a)} f(x)dS \\
&= \Gamma\left(\frac{n}{2}\right)\sum_{k=0}^{m}\frac{1}{k!\Gamma(n/2+k)}\left(\frac{r}{2}\right)^{2k}\Delta^{(k)}f(a) \\
&\quad + \frac{\Gamma(n/2)}{2\pi^{n/2}}\cdot\frac{r^{2m+1}}{(2m+1)!}\int_{S_1(a)} d^{2m+1}f(a+\theta r(x-a))dS_1, \quad 0<\theta<1,
\end{aligned} \tag{2.67}$$

$$M_r^s(f) = \Gamma\left(\frac{n}{2}\right)\sum_{k=0}\frac{1}{k!\Gamma(n/2+k)}\left(\frac{r}{2}\right)^{2k}\Delta^{(k)}f(a). \tag{2.68}$$

We remark that the expression of the remainder for $M_r^s(f)$ given in (2.67)

$$R^{**}_{2m+1}(r) = \frac{\Gamma(n/2)}{2\pi^{n/2}}\frac{r^{2m+1}}{(2m+1)!}\int_{S_1(a)} d^{2m+1}f(a+\theta r(x-a))dS_1, \quad 0<\theta<1 \tag{2.69}$$

may have other forms as in [2, volume II, chapter IV, Section 3, page 261] and [8].

For $n = 3$, since $\Gamma(3/2+k) = ((2k+1)!/2^{2k+1}k!)\sqrt{\pi}$, from (2.68), we obtain Pizzetti's Formula [1]:

$$M_r^s(f) = \sum_{k=0}\frac{r^{2k}}{(2k+1)!}\Delta^{(k)}f(a). \tag{2.70}$$

Similarly, since

$$\Gamma\left(\frac{3}{2}+k+1\right) = \left(\frac{3}{2}+k\right)\Gamma\left(\frac{3}{2}+k\right) = \frac{(2k+3)(2k+1)!}{2^{2k+2}k!}\sqrt{\pi}, \tag{2.71}$$

we have

$$M_r^b(f) = 3\sum_{k=0}\frac{r^{2k}}{(2k+1)!(2k+3)}\Delta^{(k)}f(a). \tag{2.72}$$

This last formula constitutes the similarity of Pizzetti's formula for the mean value $M_r^b(f)$ over the ball $B_r(a) \subset \mathbf{R}^3$.

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