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Generalized q-deformed correlation functions as spectral functions of hyperbolic geometry

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Abstract We analyze the role of vertex operator algebra and 2d amplitudes from the point of view of the representation theory of infinite-dimensional Lie algebras, MacMahon and Ruelle functions. By definition p-dimensional MacMahon function, with $p \leq 3$, is the generating function of p-dimensional partitions of integers. These functions can be represented as amplitudes of a two-dimensional $c = 1$ CFT, and, as such, they can be generalized to $p > 3$. With some abuse of language we call the latter amplitudes generalized MacMahon functions. In this paper we show that generalized p-dimensional MacMahon functions can be rewritten in terms of Ruelle spectral functions, whose spectrum is encoded in the Patterson–Selberg function of three-dimensional hyperbolic geometry.

1 Introduction

This paper, whose main focus is the relation among CFT correlators, MacMahon and Ruelle functions, is motivated by the steady, if not growing, interest in the application of symmetric functions, in particular of the two-dimensional MacMahon function and its higher-dimensional generalizations to physical systems. This occurs in many areas of statistical physics [1–3] and topological string theory [4–6], BPS black holes, models of branes wrapping collapsed cycles in Calabi–Yau orbifolds, and quiver gauge theories [7–9]. Generalized MacMahon functions are used, in particular, in the computation of amplitudes of the A-model topological string [10–13], more specifically as regards the so-called topological vertex.

A p-dimensional MacMahon function for $p \leq 3$ is the generating function of a p-dimensional partition of integers, which is the number of different ways in which we can split an integer using distinct p-dimensional arrays of other non-negative integers. As we shall see these functions can be represented as amplitudes of a two-dimensional CFT. We extend this representation as a correlator to generic p and with some abuse of language we call the resulting objects *generalized MacMahon functions*. For $p \geq 4$ these functions do not coincide precisely with the generating functions of p-dimensional partition of integers, but they have all the same remarkable properties. In particular we will show in this paper that generalized p-dimensional MacMahon functions can be rewritten in terms of Ruelle spectral functions, whose spectrum is encoded in the Patterson–Selberg function of three-dimensional hyperbolic geometry. This may lead to an interpretation of the above results in terms of the ADS/CFT correspondence, an attractive possibility which we leave for future investigation.

There is also another side of our work we would like to recall. We have remarked elsewhere the important connection between quantum generating functions in physics and formal power series associated with dimensions of chains and homologies of suitable infinite-dimensional Lie algebras. MacMahon and symmetric functions play an important role in the homological aspects of this connection; its application to partition functions of minimal three-dimensional gravities in the space-time asymptotic to AdS_3, which also describe the three-dimensional Euclidean black holes, pure supergravity, elliptic genera, and associated q-series were studied in [14,15]. On the other hand special applications of symmetric functions appear in the representation theory of infinite-dimensional Lie algebras [14–17]. The usefulness of symmetric function techniques can be demonstrated in providing concrete realizations of the (quantum) affine algebra,

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for instance in calculating the trace of products of currents of this algebra. These functions are, respectively, the appropriate character $\mathrm{ch}_{\mathbb{R}}$ of the basic representations of the $\mathfrak{sl}(\infty)$ and the affine algebra $\widehat{\mathfrak{sl}(\infty)}$ at large central charge c [18]. Note that all simple (twisted and untwisted) Kac–Moody algebras can be embedded in the $\mathfrak{sl}(\infty)$ algebra of infinite matrices with a finite number of non-zero entries, which has a realization in terms of the generators of a Clifford algebra. It has been observed that in the limit $c \to \infty$ the basic representation of $\widehat{\mathfrak{sl}(\infty)}$ is related to the partition function of a three-dimensional field theory [7,19].

Needless to say, although these links are suggestive, the general panorama looks still inarticulate, and more models and examples are needed to accommodate them into a precise scheme. One of the purposes of the present paper is to better understand the role of vertex algebra and 2d amplitudes from the point of view of the representation theory of infinite-dimensional Lie algebras, generalized MacMahon and Ruelle functions. In this regard particularly important is the correspondence between Ruelle spectral functions and the Poincaré q-series associated with conformal structure in two dimensions.

The organization of the paper is as follows. In Sect. 2 we introduce the algebra of q-deformed vertex operators (of the $c = 1$ 2d conformal model) and consider their generalizations and the properties essential for the next sections. In Sect. 3 we reformulate the generalized MacMahon functions in terms of the Ruelle spectral functions of hyperbolic geometry. We also broach and briefly discuss the topic of higher-dimensional partitions, as such, originally introduced by MacMahon. We analyze correlation functions of vertex operators, the MacMahon's conjecture (see (3.15)) and their possible interpretation as p-dimensional partition functions.

In Sect. 4 we consider multipartite (vector valued) generating functions and utilize well-known formulas for Bell polynomials. We derive the infinite hierarchy of q-deformed vertex operators and factorized partition functions and represent them by means of spectral functions.

Finally in Sect. 5 we conclude with a summary of the main results accompanied by discussions and suggestions.

In the appendix we give a few formulas involving Ruelle and Patterson–Selberg spectral functions of hyperbolic three-geometry.

2 Algebra of vertex operators

In this section we introduce the notation and quote some earlier results we need. To this end we follow mostly [12] and, in particular, the subsequent elaboration by [20]. Let us consider the hierarchy of generalized local q-deformed vertex operators $\Gamma^{(p)}_{\pm}(z,q)\ \ (p > 1)$

$$\Gamma^{(p)}_{\pm}(z,q) = \exp\left(\sum_{n=1}^{\infty} \frac{\mp i}{n}\,\frac{z^{\mp n}}{(1-q^n)^{p-1}} J_{\pm n}\right). \tag{2.1}$$

J_n are the modes of a standard holomorphic $U(1)$ Kac–Moody $J(z)$ with Laurent expansion is $J(z) = \sum_{n\in\mathbb{Z}} z^{-n-1} J_n,\ J_n = \oint (z^n/2\pi i) J(z)\mathrm{d}z$. The Heisenberg algebra is $[J_n, J_m] = n\delta_{n+m,0},\ J_n^{\dagger} = J_{-n}$ and $J_n|0\rangle = 0$ for $n \geq 1$. The commuting mode J_0 is disregarded.

One can use the identities $z^{\pm n}/(1-q^n)^{p-1} = \sum_{j=1}^{p-1} z^n q^{\pm n}/(1-q^n)^j + z^{\pm n}$ to obtain the recursive relations

$$\Gamma^{(p)}_{-}(z,q) = \Gamma^{(1)}_{-}(z) \prod_{j=2}^{p-1} \Gamma^{(j)}_{-}(qz,q),$$
$$\Gamma^{(p)}_{+}(z,q) = \Gamma^{(1)}_{+}(z) \prod_{j=2}^{p-1} \Gamma^{(j)}_{+}(z/q,q). \tag{2.2}$$

The local operators $\Gamma^{(1)}_{\pm}(z) := \Gamma_{\pm}(z)$ act on the Hilbert space states of the $c = 1$ 2d conformal field theory and exhibit properties inherited from the algebra of the $J_{\pm n}$. They obey the following algebra:

$$\Gamma_{\pm}(x)\Gamma_{\pm}(y) = \Gamma_{\pm}(y)\Gamma_{\pm}(x), \qquad x, y \in \mathbb{C},$$
$$\Gamma_{+}(x)\Gamma_{-}(y) = (1-y/x)^{-1}\Gamma_{-}(y)\Gamma_{+}(x). \tag{2.3}$$

It is interesting to note that introducing $L_0 = \sum_{n>0}^{\infty} J_{-n}J_n$, we get

$$[L_0, \Gamma^{(p)}_{\pm}(z,q)] = z\frac{\mathrm{d}}{\mathrm{d}z}\Gamma^{(p)}_{\pm}(z,q). \tag{2.4}$$

Thus $\Gamma^{(p)}_{\pm}(z,q)$ are 'weight 0 primaries'. It follows, in particular, that the operator q^{L_0} acts on $\Gamma_{\pm}(z,q)$ as follows: $q^{L_0}\Gamma_{\pm}(z,q)q^{-L_0} = \Gamma_{\pm}(qz,q)$.

Due to the properties $J_n|0\rangle = 0$, $\langle 0|J_{-n} = 0$ for $n > 0$, the operators $\Gamma_{\pm}(z)$ act on the vacuum as the identity operator: $\Gamma_{+}(z)|0\rangle = |0\rangle$, $\langle 0|\Gamma_{-}(z) = \langle 0|$ it follows that

$$\langle 0|\Gamma_{+}(z)\Gamma_{-}(w)|0\rangle = \frac{1}{1-\frac{w}{z}}. \tag{2.5}$$

As noted in [20], $\Gamma_{-}(z)$ contains all the monomials in $J_{-n_j},\ J^{\lambda}_{-\mathbf{n}} \equiv \prod_{j\geq 1}\left(J_{-n_j}\right)^{\lambda_j}$, where $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a 2d partitions; the state $\Gamma_{-}(z)|0\rangle$ is reducible and is given by a sum over all possible 2d partitions λ. In the case $z = 1$, $\Gamma_{-}(1)|0\rangle = \sum_{(\text{2d partitions}\lambda)} |\lambda\rangle$. A similar relation is valid for $\langle 0|\Gamma_{+}(1)$.

With the mathematical tools just introduced we can proceed to higher-dimensional generalizations. For example, one can rewrite $\Gamma^{(2)}_{\pm}(z,q)$ as follows:

$$\Gamma_-^{(2)}(z,q) = \prod_{t=-\infty}^{-1} \Gamma_-(1)(z)\, q^{L_0} = \prod_{k=0}^{\infty} \Gamma_-(1)\left(q^k z\right)$$
$$= \exp\left(\sum_{n\geq 1} \frac{i}{n}\frac{z^n}{(1-q^n)} J_{-n}\right), \qquad (2.6)$$
$$\Gamma_+^{(2)}(z,q) = \prod_{t=0}^{\infty} q^{L_0}\Gamma_+(1)(z) = \prod_{k=0}^{\infty} \Gamma_+(1)\left(q^{-k} z\right)$$
$$= \exp\left(-\sum_{n\geq 1} \frac{i}{n}\frac{z^{-n}}{(1-q^n)} J_{n}\right). \qquad (2.7)$$

The products $\prod_{t=-\infty}^{-1}(\text{—})$ in these equations are taken over diagonal slices of the 3d partitions. They are reminiscent of the transfer matrix method, where a 3d partition is thought of as an amplitude between the slice at $t=-\infty$ (*in-state*) and the slice at $t=\infty$ (*out-state*). The algebra of these vertex operators takes the following form:

$$q^{L_0}\Gamma_\pm^{(2)}(z,q)q^{-L_0} = \Gamma_\pm^{(2)}(qz,q),$$
$$\Gamma_\pm^{(2)}(z,q)\Gamma_\pm^{(2)}(w,q) = \Gamma_\pm^{(2)}(w,q)\Gamma_\pm^{(2)}(z,q). \qquad (2.8)$$

We can replicate recursively this construction. This leads to the following hierarchy of composite vertex operators:

$$\Gamma_-^{(n+1)}(z,q) = \prod_{t_n=1}^{\infty}\cdots\prod_{t_2=1}^{\infty}\prod_{t_1=1}^{\infty}\left(\Gamma_-(z)\, q^{L_0}\right)\cdot q^{L_0}\cdots q^{L_0}. \qquad (2.9)$$

A similar expression can be written down for $\Gamma_+^{(n+1)}(z,q)$. For $n=0$, we have just $\Gamma_-(z)$. It is not difficult to check that the explicit expression of the vertex operators $\Gamma_-^{(p)}(z,q)$ ($p\geq 1$) acting on the vacuum is given by

$$\Gamma_-^{(p)}(z,q)\,|0\rangle = \exp\left(\sum_{n=1}^{\infty}\frac{i z^n J_{-n}}{n(1-q^n)^{p-1}}\right)|0\rangle$$
$$= \Gamma_-(z)\prod_{k=2}^{p-1}\Gamma_-^{(k)}(qz,q)\,|0\rangle.$$

3 MacMahon, partitions, and Ruelle spectral functions

In this section, using the formulas in the appendix, we transcribe the generalized MacMahon partition functions in terms of spectral functions of hyperbolic geometry.

The 1d and 2d MacMahon function can be interpreted as the two-point correlation of the vertex operators $\Gamma_+(1)$ and $\Gamma_-(q)$

$$Z_{1d} = \langle 0|\Gamma_+(1)\,\Gamma_-(q)\,|0\rangle = \left\langle 0|\,\Gamma_+(1)\,q^{L_0}\Gamma_-(1)\,|0\right\rangle = (1-q)^{-1}, \qquad (3.1)$$
$$Z_{2d} = \langle 0|\Gamma_+(1)\,q^{L_0}\prod_{k\geq 1}\Gamma_-(1)q^{L_0}|0\rangle = \langle 0|\Gamma_+(1)\prod_{k\geq 1}\Gamma_-(q^k)|0\rangle$$
$$= \prod_{k\geq 1}\left(1-q^k\right)^{-1} \overset{\text{by Eq. (6.4)}}{=\!=\!=} [\mathcal{R}(s=1-i\varrho(\tau))]^{-1}. \qquad (3.2)$$

In the previous section we have introduced a hierarchy of level p vertex operators $\Gamma_\pm^{(p)}$. In perfect analogy with Z_{1d} and Z_{2d} we can introduce and compute Z_{3d}:

$$Z_{3d} = \left\langle 0|\left(\prod_{t=0}^{\infty} q^{L_0}\Gamma_+(1)\right) q^{L_0}\left(\prod_{t=-\infty}^{-1}\Gamma_-(1)q^{L_0}\right)|0\right\rangle$$
$$= \left\langle 0|\,\prod_{t=0}^{\infty}\Gamma_+^{(1)}\left(q^{-t-\frac12}\right)\prod_{\ell=1}^{\infty}\Gamma_-^{(1)}\left(q^{\ell-\frac12}\right)\,|0\right\rangle$$
$$= \prod_{\ell=0}^{\infty}\prod_{j=1}^{\infty}\left[1-q^{j+\ell}\right]^{-1} \overset{k:=j+\ell}{=\!=\!=} \prod_{k=1}^{\infty}\prod_{j=1}^{k}\left[1-q^k\right]^{-1}$$
$$= \prod_{k=1}^{\infty}\left[1-q^k\right]^{-k}. \qquad (3.3)$$

When obtaining the second line in (3.3) we split q^{L_0} as $q^{L_0/2}q^{L_0/2}$ and commute each of the operators $q^{L_0/2}$ to the left and the other to the right. The last product in (3.3) is precisely the usual form of the 3d MacMahon function, which, again, can be rewritten in terms of the spectral Ruelle functions:

$$\prod_{k=1}^{\infty}\left[1-q^k\right]^{-k} \overset{\text{by Eq. (6.8)}}{=\!=\!=} \prod_{n=1}^{\infty}[\mathcal{R}(s=n(1-i\varrho(\tau)))]^{-1}. \qquad (3.4)$$

***p*-dimensional partition functions.** The structure of the p-dimensional partition function Z_{pd} can be analyzed in terms of the vertex operators $\Gamma_\pm^{(p)}(z,q)$ introduced before. As we have seen above, (2.9), the latter can be interpreted as the level p generalization of $\Gamma_-(z)$, and they obey the relations $\Gamma_-^{(p)}(z,q) = q^{L_0}\Gamma_-^{(p)}(1,q)\,q^{-L_0}$, $p\geq 0$. The p-dimensional partition functions Z_{pd} can be defined as [20]

$$Z_{pd} = \left\langle 0|\Gamma_+(1)\,\Gamma_-^{(p)}(z,q)|0\right\rangle$$
$$\equiv \left\langle 0|\Gamma_+(1)\,q^{L_0}\Gamma_-^{(p)}(1,q)\,|0\right\rangle, \quad p\geq 0. \qquad (3.5)$$

We can rewrite these partition functions in terms of Ruelle spectral function. Indeed, commuting $\Gamma_-^{(p)}(z,q)$ to the left of $\Gamma_+(1,q)$ for $p\geq 2$, one gets by induction (see for details [20])

$$Z_{pd} = \prod_{k=1}^{\infty}\left[1-q^k\right]^{-C(k,p)}$$
$$\underset{=\!=\!=}{\text{by Eq.(6.6)}} \prod_{n=1}^{\infty}\left[\frac{\mathcal{R}(s=n(1-i\varrho(\tau)))}{\mathcal{R}(s=(n+1)(1-i\varrho(\tau)))}\right]^{-C(n,p)}, \quad (3.6)$$
$$C(n,p) = \frac{(n+p-3)!}{(n-1)!(p-2)!}. \quad (3.7)$$

Plane partitions. So far we have called Z_{pd} a p-dimensional partition function without any comment. Here we would like to motivate this term at least for the cases $p \leq 3$. It comes from the fact that correlation functions of the corresponding vertex operators admit a presentation that can be associated with higher-dimensional partitions. Recall that a higher-dimensional partition of n is an array of numbers whose sum is n:

$$n = \sum_{j_1,\ldots,j_r\geq 0} n_{j_1 j_2\ldots j_r}, \quad \text{where} \quad n_{j_1 j_2 \ldots j_r} \geq n_{k_1 k_2\ldots k_r} \quad (3.8)$$

whenever $j_1 \geq k_1, j_2 \geq k_2, \ldots, j_r \geq k_r$, and all $n_{j_1 j_2\ldots j_r}$ nonnegative integers. Let us introduce also plane partitions: these are two-dimensional arrays of nonnegative integers subject to a nonincreasing condition along rows and columns. It is worth recalling that Young tableaux with strict decrease along columns are essentially equivalent to plane partitions. They were originally used by Alfred Young in his work on invariant theory. Young tableaux have played an important role in the representation theory of the symmetric group; they also occur in algebraic geometry and in many combinatorial problems.

Let us denote $\pi_r(n_1, n_2, \ldots, n_k; q)$ the generating function for plane partitions with at most r columns, at most k rows, and with n_i the first entry in the ith row. The functions $\pi_r(n_1, n_2, \ldots, n_k; q)$ are completely determined by the following recurrence and initial condition:

$$\pi_{r+1}(n_1, n_2, \ldots, n_k; q) = q^{\sum_{j=1}^{k}(n_j)}$$
$$\times \sum_{m_k=0}^{n_k}\ \sum_{m_{k-1}=m_k}^{n_k-1} \cdots \times \sum_{m_1=m_2}^{n_1} \pi_r(m_1,\ldots,m_k;q), \quad (3.9)$$
$$\pi_1(n_1, n_2, \ldots, n_k; q) = q^{\sum_{j=1}^{k}(n_j)}. \quad (3.10)$$

$\pi_{r+1}(n_1, n_2, \ldots, n_k; q)$ can be represented as a determinant (see for details [21]):

$$\pi_r(n_1, n_2, \ldots, n_k; q) = q^{\sum_{j=1}^{k}(n_j)}$$
$$\times \det\left[q^{(i-j)(i-j-1)/2}\binom{n_j+r-1}{r-i+j-1}\right]_{1\leq i,\, j\leq k}. \quad (3.11)$$

Define the number of plane partitions of m, $p_{k,r}(m,n)$, with at most r columns, at most k rows, and with each entry $\leq n$, and let $\pi_{k,r}(n;q) := \sum_{m=0}^{\infty} p_{k,r}(m,n)q^m$. Then one can observe that

$$\pi_{k,r}(n;q) \underset{=\!=\!=}{\text{by Eq.(3.9)}} \sum_{n_k\leq\cdots\leq n_1\leq n} \pi_r(n_1,n_2,\ldots,n_k;q)$$
$$= q^{-kn}\pi_{r+1}(n_1,n_2,\ldots,n_k;q)$$
$$\underset{=\!=\!=}{\text{by Eq.(3.11)}} \det\left[q^{(i-j)(i-j-1)/2}\binom{n+r}{r-i+j}\right]_{1\leq i,\, j\leq k}. \quad (3.12)$$

As a result MacMahon's formulas for the generating function of k-rowed plane partitions $\pi_{k,\infty}(\infty;q)$ follow [21]:

$$\sum_{m=0}^{\infty} p_{k,\infty}(m,\infty)q^m = \prod_{n=1}^{\infty}(1-q^n)^{-\min(k,n)}, \quad (3.13)$$
$$\sum_{m=0}^{\infty} p_{\infty,\infty}(m,\infty)q^m = \prod_{n=1}^{\infty}(1-q^n)^{-n} = Z_{3d}. \quad (3.14)$$

Let $\mu_k(j)$ be the number of k-dimensional partitions of j, then due to the MacMahon's conjecture

$$\sum_{j=0}^{\infty}\mu_k(j)q^j = \prod_{n=1}^{\infty}(1-q^n)^{-C(n,k)},$$
$$C(n,k) = \frac{(n+k-2)!}{(n-1)!}. \quad (3.15)$$

MacMahon eventually came to doubt the truth of (3.15) in general; in fact, its falsehood in general was established in the late 1960s [21]. However, this conjecture is certainly true for $k=1$ and 2. It is remarkable that

$$\sum_{j=0}^{\infty}\mu_1(j)q^j = \prod_{n=1}^{\infty}(1-q^n)^{-1} = Z_{2d}\,, \quad (3.16)$$
$$\sum_{j=0}^{\infty}\mu_2(j)q^j = \prod_{n=1}^{\infty}(1-q^n)^{-n} = Z_{3d}\,. \quad (3.17)$$

Comparing (3.15) and (3.6), (3.7) we get the relations $C(n,k=1) = C(n,p=2)$, $C(n,k=2) = C(n,p=3)$. The MacMahon's conjecture for the case $k>2$ can be corrected by using the comparison between $C(n,k)$ and the power $C(n,p)$ in the q-expansion of p-dimensional partition function Z_{pd}.

Concluding this section, the term MacMahon partition function for Z_{pd} is fully justified for $p \leq 3$. Following [20] we call Z_{pd} for $p \geq 4$ *generalized MacMahon functions* due to the straightforward way they are obtained by generalizing the definition for $p \leq 3$. The relation with the original MacMahon's definition is still an intriguing open problem: its solution may shed light also into the corresponding 2d CFT mentioned above.

4 Multipartite generating functions and infinite hierarchy of q-deformed vertex operators

Multipartite generating functions. Let consider, for any ordered ℓ-tuple of nonnegative integers not all zeros, $(k_1, k_2, \ldots, k_\ell) = \mathbf{k}$ (referred to as "ℓ-partite" or *multipartite* numbers), the (multi)partitions, i.e. distinct representations of $(k_1, k_2, \ldots, k_\ell)$ as sums of multipartite numbers. Let us call $\mathcal{C}_-^{(u,\ell)}(\mathbf{k}) = \mathcal{C}_-^{(\ell)}(u; k_1, k_2, \ldots, k_\ell)$ the number of such multipartitions, and introduce in addition the symbol $\mathcal{C}_+^{(u,\ell)}(\mathbf{k}) = \mathcal{C}_+^{(\ell)}(u; k_1, k_2, \ldots, k_\ell)$. Their generating functions are defined by

$$\mathcal{F}(u) := \prod_{\mathbf{k}\geq 0}\left(1 - ux_1^{k_1}x_2^{k_2}\cdots x_\ell^{k_\ell}\right)^{-1} = \sum_{\mathbf{k}\geq 0}\mathcal{C}_-^{(u,\ell)}(\mathbf{k})x_1^{k_1}x_2^{k_2}\cdots x_\ell^{k_\ell}\,, \tag{4.1}$$

$$\mathcal{G}(u) := \prod_{\mathbf{k}\geq 0}\left(1 + ux_1^{k_1}x_2^{k_2}\cdots x_\ell^{k_\ell}\right) = \sum_{\mathbf{k}\geq 0}\mathcal{C}_+^{(u,\ell)}(\mathbf{k})x_1^{k_1}x_2^{k_2}\cdots x_\ell^{n_\ell}\,. \tag{4.2}$$

Therefore,

$$\begin{aligned}\log\mathcal{F}(u) &= -\sum_{\mathbf{k}\geq 0}\log\left(1 - ux_1^{k_1}x_2^{k_2}\cdots x_\ell^{k_\ell}\right)\\ &= \sum_{\mathbf{k}\geq 0}\sum_{m=1}^{\infty}\frac{u^m}{m}x_1^{mk_1}x_2^{mk_2}\cdots x_\ell^{mk_\ell}\\ &= \sum_{m=1}^{\infty}\frac{u^m}{m}(1-x_1^m)^{-1}(1-x_2^m)^{-1}\cdots(1-x_\ell^m)^{-1}\\ &= \sum_{m=1}^{\infty}\frac{u^m}{m}\prod_{j=1}^{r}(1-x_j^m)^{-1},\end{aligned} \tag{4.3}$$

$$\log\mathcal{G}(-u) = \log\mathcal{F}(u)\,. \tag{4.4}$$

Finally,

$$\mathcal{F}(u) = \sum_{\mathbf{k}\geq 0}\mathcal{C}_-^{(u,\ell)}(\mathbf{k})x_1^{k_1}x_2^{k_2}\cdots x_\ell^{k_\ell} = \exp\left(\sum_{m=1}^{\infty}\frac{u^m}{m}\prod_{j=1}^{r}(1-x_j^m)^{-1}\right), \tag{4.5}$$

$$\mathcal{G}(u) = \sum_{\mathbf{k}\geq 0}\mathcal{C}_+^{(u,\ell)}(\mathbf{k})x_1^{k_1}x_2^{k_2}\cdots x_\ell^{n_\ell} = \exp\left(\sum_{m=1}^{\infty}\frac{(-u)^m}{m}\prod_{j=1}^{r}(1-x_j^m)^{-1}\right). \tag{4.6}$$

It is known that the Bell polynomials are very useful in many problems in combinatorics. We would like to note their application in multipartite partition problem [21]. The Bell polynomials technique can be used for the calculation $\mathcal{C}_-^{(\ell)}(\mathbf{k})$ and $\mathcal{C}_+^{(\ell)}(\mathbf{k})$. Let

$$\mathcal{F}(u) := 1 + \sum_{j=1}^{\infty}\mathcal{P}_j(x_1, x_2, \ldots, x_\ell)u^j, \qquad \mathcal{P}_j = 1 + \sum_{\mathbf{k}>0}P(\mathbf{k}; j)x_1^{n_1}\cdots x_\ell^{n_\ell}, \tag{4.7}$$

$$\mathcal{G}(u) := 1 + \sum_{j=1}^{\infty}\mathcal{Q}_j(x_1, x_2, \ldots, x_\ell)u^j, \qquad \mathcal{Q}_j = 1 + \sum_{\mathbf{k}>0}Q(\mathbf{k}; j)x_1^{n_1}\cdots x_\ell^{n_\ell}. \tag{4.8}$$

Useful expressions for the recurrence relation of the Bell polynomial $Y_n(g_1, g_2, \ldots, g_n)$ and generating function $\mathcal{B}(u)$ have the forms [21]:

$$Y_{n+1}(g_1, g_2, \ldots, g_{n+1}) = \sum_{k=0}^{n}\binom{n}{k} \times Y_{n-k}(g_1, g_2, \ldots, g_{n-k})g_{k+1}, \tag{4.9}$$

$$\mathcal{B}(u) = \sum_{n=0}^{\infty}\frac{Y_n u^n}{n!} \Longrightarrow \log\mathcal{B}(u) = \sum_{n=1}^{\infty}\frac{g_n u^n}{n!}\,. \tag{4.10}$$

To verify the second formula in (4.10) we need to differentiate with respect to u and observe that a comparison of the coefficients of u^n in the resulting equation produces an identity equivalent to (4.9). From (4.9) one can obtain the following explicit formula for the Bell polynomials (known as Faa di Bruno's formula):

$$Y_n(g_1, g_2, \ldots, g_n) = \sum_{\mathbf{k}\vdash n}\frac{n!}{k_1!\cdots k_n!}\prod_{j=1}^{n}\left(\frac{g_j}{j!}\right)^{k_j}. \tag{4.11}$$

Let $\beta_r(m) := \prod_{j=1}^{r}(1-x_j^m)^{-1}$; the following result holds (see for details [21]):

$$\mathcal{P}_j = \frac{1}{j!}Y_j\left(0!\beta_r(1), 1!\beta_r(2)\,, \ldots,\, (j-1)!\beta_r(j)\right), \tag{4.12}$$

$$\mathcal{Q}_j = \frac{1}{(-1)^j j!}Y_j\left(-0!\beta_r(1), -1!\beta_r(2)\,, \ldots,\, -(j-1)!\beta_r(j)\right). \tag{4.13}$$

As an example, let us calculate $\mathcal{P}_2$ coefficient. Using the recurrence relation (4.9) we obtain $\mathcal{P}_2 = (1/2)Y_2(\beta_r(1), \beta_r(2)) = (1/2)Y_2(\beta_r(1)^2, \beta_r(2)) = (1/2)\left(\prod_{j=1}^{r}(1-x_j^2)^{-1} + \prod_{j=1}^{r}(1-x_j^2)\right)$.

The infinite hierarchy. Let us consider again the hierarchy $\Gamma_-^{(p)}(z,q)$ of q-deformed vertex operators. We have $\Gamma_+(1)\,\Gamma_-^{(p)}(z,q) = M_p(q)\,\Gamma_-^{(p)}(z,q)\,\Gamma_+(1)$, where $M_p(q)$ is precisely the generalized p-dimensional MacMahon function. The general relation is the following:

$$\langle 0|\,\Gamma_+(z_1)\,\Gamma_-^{(\ell+1)}(z_\ell,q)\,|0\rangle = \prod_{k_\ell=0}^{\infty}\cdots\prod_{k_1=0}^{\infty}\prod_{k_0=0}^{\infty}\left[1-q^{k_1+\ldots+k_\ell}\frac{z_\ell}{z_1}\right]^{-1}, \qquad (4.14)$$

with $z_\ell/z_1 = q^{k_0}$. In the case $x_1 = x_2 = \cdots = x_\ell = q$ we get

$$\mathcal{F}(u) = \prod_{\mathbf{k}\geq 0}\left(1-uq^{k_1+k_2+\cdots+k_\ell}\right)^{-1} = \exp\left(-\sum_{m=1}^{\infty}\frac{u^m}{m}(1-q^m)^{-\ell}\right), \qquad (4.15)$$

$$\mathcal{G}(u) = \prod_{\mathbf{k}\geq 0}\left(1+uq^{k_1+k_2+\cdots+k_\ell}\right) = \exp\left(-\sum_{m=1}^{\infty}\frac{(-u)^m}{m}(1-q^m)^{-\ell}\right). \qquad (4.16)$$

These formulas could be interpreted as the ℓ copies of free CFT_2 representations. Indeed, setting $uq^{k_1+\cdots+k_\ell} = Q_{\mathbf{k}}q^{k_0}$ with $Q_{\mathbf{k}} = q^{k_1+\cdots+k_l}$ ($\mathbf{k} = (k_1,\ldots,k_\ell)$) we get

$$Z_2(Q_{\mathbf{k}},q) = \prod_{k_0=0}^{\infty}\left[1-Q_{\mathbf{k}}q^{k_0}\right]^{-1} = [(1-Q_{\mathbf{k}})\mathcal{R}(s=(k_1+\cdots+k_\ell)(1-i\varrho(\tau)))]^{-1}. \qquad (4.17)$$

Therefore the right hand side of (4.14) can be factorized as $\prod_{\mathbf{k}\geq\mathbf{0}} Z_2(Q_{\mathbf{k}},q)$. We can treat this factorization as a product of infinite copies, each of them is $Z_2(Q_{\mathbf{k}},q)$ and corresponds to a free CFT_2.

5 Conclusions

We have shown above that all p-dimensional partition functions that have been considered in this paper can be written in terms of Ruelle functions, a spectral function related to hyperbolic geometry in three dimensions (see the appendix). Thus they cannot only be interpreted as correlators in a 2d CFT, but they suggest a possible interpretation in terms of three-dimensional physics. This relation is, however, still to be unveiled. In this last section we would like nevertheless to recall that in some specific cases the interpretation in terms three-dimensional physics has been possible.

For the benefit of the reader, let us start explaining the connection between highest weight representations of infinite-dimensional Lie algebras and holomorphic factorized quantum corrections for supergravity in three dimensions. Let $M(c,h)$ $(c,h\in\mathbb{C})$ be the Verma module over Virasoro (Vir) algebras. We have $[L_0,L_{-n}] = nL_{-n}$ and L_0 is diagonalizable on $M(c,h)$ with spectrum $h+\mathbb{Z}_+$ and with eigenspace decomposition $M(c,h) = \bigoplus_{j\in\mathbb{Z}_+}M(c,h)_{h+j}$, where $M(c,h)_{h+j}$ is spanned by elements of the basis of $M(c,h)$. The conformal central charge c acts on $M(c,h)$ as $c\,\mathrm{Id}$. It follows that $W_j = \dim M(c,h)_{h+j}$, where W_j is the partition function [16]. The latter can be rewritten in the form

$$\mathrm{Tr}_{M(c,h)}\,q^{L_0} := \sum_{\lambda} q^{\lambda}\dim M(c,h)_{\lambda} = q^h\prod_{j=1}^{\infty}(1-q^j)^{-1}. \qquad (5.1)$$

The series $\mathrm{Tr}_M\,q^{L_0}$ is called the formal character of the Vir module M.

For three-dimensional gravity in a real hyperbolic space the partition function admits a factorization: it is a product of holomorphic and antiholomorphic functions $W_{0,1}(\tau,\overline{\tau}) = W(\tau)_{\mathrm{hol}}\cdot W(\overline{\tau})_{\mathrm{antihol}}$, where

$$W(\tau)_{\mathrm{hol}} = q^{-k}\prod_{n=1}^{\infty}(1-q^{n+1})^{-1},$$
$$W(\overline{\tau})_{\mathrm{antihol}} = \overline{q}^{-k}\prod_{n=1}^{\infty}(1-\overline{q}^{n+1})^{-1}. \qquad (5.2)$$

The holomorphic contribution in (5.2) corresponds to the formal character of the Vir module. On the other hand the modulus of a Riemann surface Σ of genus one (the conformal boundary of AdS_3) is defined up to $\gamma\cdot\tau = (a\tau+b)/(c\tau+d)$ with $\gamma\in SL(2,\mathbb{Z})$. Therefore the generating function as the sum of known contributions of states of left- and right-moving modes in the conformal field theory takes the form $\sum_{c,d}W_{c,d}(\tau,\overline{\tau}) = \sum_{c,d}W_{0,1}((a\tau+b)/(c\tau+d),\overline{\tau})$. The generating function, represented as the sum over geometries, becomes [14]

$$\sum_{c,d}W_{c,d}(\gamma\cdot\tau,\overline{\tau}) = \sum_{c,d}\left|q^{-k}\prod_{n=2}^{\infty}(1-q^n)^{-1}\right|_{\gamma}^2 = \sum_{c,d}\Big\{|q\overline{q}|^{-k}\cdot[\mathcal{R}(s=2-2i\varrho(\tau))]_{\mathrm{hol}}^{-1} \times[\mathcal{R}(s=2+2i\varrho(\tau))]_{\mathrm{antihol}}^{-1}\Big\}_{\gamma}. \qquad (5.3)$$

Here $|\ldots|_{\gamma}$ denotes the transform of an expression $|\ldots|$ by γ. The summand in (5.3) is independent of the choice of a and b in γ. The sum over c and d in (5.3) should be thought of as a sum over the coset $\mathrm{PSL}(2,\mathbb{Z})/\mathbb{Z}\equiv(\mathrm{SL}(2,\mathbb{Z})/\{\pm 1\})/\mathbb{Z}$. This result can be extended to $\mathcal{N}=1$ supergravity [22]. The infinite series of quantum corrections for the Neveu–Schwarz and Ramond sector of supergravity can be reproduced in terms of Ruelle spectral functions in a holomorphically factorized theory [14].

This is an example of the fact that the Ruelle function represents a bridge between two-dimensional CFT and three-dimensional physics. This might be the meaning also of the formulas we have derived in the previous sections.

In this light it is important to recall that a particular example of MacMahon function, Z_{3d}, is directly linked to the topological vertex [10] in topological string theory. This is an open topological amplitude in a Calabi–Yau background. Analysis of this vertex, $\mathcal{C}_{\lambda\mu\nu}$, and open string partition function leads to a relation $Z_{3d} \sim \mathcal{C}_{\lambda\mu\nu}$ [12,13,20,23]. This can be achieved as $\langle \nu^t|\mathcal{O}_+(\lambda)\mathcal{O}_-(\lambda^t)|\mu\rangle$ where the operators $\mathcal{O}_+$ and $\mathcal{O}_-$ play the role of composite local vertex operators of two-dimensional $c = 1$ conformal theory, and λ, μ, ν represent boundary states described by 2d Young diagrams. Applications of the topological vertex [24] suggest a connection with Chern–Simons theory. This may be the clue for our future investigation.

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Appendix: Spectral functions of hyperbolic three-geometry

In this section we recall some results on the Ruelle (Patterson–Selberg type) spectral functions. For details we refer the reader to [25–27] where spectral functions of hyperbolic three-geometry were considered in connection with three-dimensional Euclidean black holes, pure supergravity, and string amplitudes.

Let $\mathfrak{G}^\gamma \in G = SL(2,\mathbb{C})$ be the discrete group defined by

$$\begin{aligned}\mathfrak{G}^\gamma &= \{\mathrm{diag}(e^{2n\pi(\mathrm{Im}\,\tau+i\mathrm{Re}\,\tau)},\ e^{-2n\pi(\mathrm{Im}\,\tau+i\mathrm{Re}\,\tau)}) : n\in\mathbb{Z}\}\\ &= \{\gamma^n : n\in\mathbb{Z}\},\\ \gamma &= \mathrm{diag}(e^{2\pi(\mathrm{Im}\,\tau+i\mathrm{Re}\,\tau)},\ e^{-2\pi(\mathrm{Im}\,\tau+i\mathrm{Re}\,\tau)}).\end{aligned} \tag{6.1}$$

One can construct a zeta function of Selberg-type for the group $\mathfrak{G}^\gamma \equiv \mathfrak{G}^\gamma_{(\alpha,\beta)}$ generated by a single hyperbolic element of the form $\gamma_{(\alpha,\beta)} = \mathrm{diag}(e^z, e^{-z})$, where $z = \alpha + i\beta$ for $\alpha, \beta > 0$. Actually $\alpha = 2\pi\,\mathrm{Im}\,\tau$ and $\beta = 2\pi\,\mathrm{Re}\,\tau$. The Patterson–Selberg spectral function $Z_{\mathfrak{G}^\gamma}(s)$ and its logarithm for $\mathrm{Re}\,s > 0$ can be attached to $H^3/\mathfrak{G}^\gamma$ as follows:

$$Z_{\mathfrak{G}^\gamma}(s) := \prod_{k_1,k_2\geq 0}[1-(e^{i\beta})^{k_1}(e^{-i\beta})^{k_2}e^{-(k_1+k_2+s)\alpha}], \tag{6.2}$$

$$\log Z_{\mathfrak{G}^\gamma}(s) = -\frac{1}{4}\sum_{n=1}^{\infty}\frac{e^{-n\alpha(s-1)}}{n[\sinh^2\left(\frac{\alpha n}{2}\right)+\sin^2\left(\frac{\beta n}{2}\right)]}. \tag{6.3}$$

For more information as regards the analytic properties of this spectral function we refer the reader to the papers [26,28]. Let us introduce the Ruelle functions $\mathcal{R}(s)$, as an alternating product of factors, each of which is a Selberg zeta function ($\mathcal{R}(s)$ and $Z_{\mathfrak{G}^\gamma}(s)$ can be continued meromorphically to the entire complex plane $\mathbb{C}$),

$$\begin{aligned}&\prod_{n=\ell}^{\infty}(1-q^{an+\varepsilon})\\ &\quad= \prod_{p=0,1} Z_{\mathfrak{G}^\gamma}(\underbrace{(a\ell+\varepsilon)(1-i\varrho(\tau))+1-a}_{s}\\ &\quad\quad + a(1+i\varrho(\tau)p)^{(-1)^p} = \mathcal{R}(s\\ &\quad= (a\ell+\varepsilon)(1-i\varrho(\tau))+1-a),\end{aligned} \tag{6.4}$$

$$\begin{aligned}&\prod_{n=\ell}^{\infty}(1+q^{an+\varepsilon})\\ &\quad= \prod_{p=0,1} Z_{\mathfrak{G}^\gamma}(\underbrace{(a\ell+\varepsilon)(1-i\varrho(\tau))+1-a+i\sigma(\tau)}_{s}\\ &\quad\quad + a(1+i\varrho(\tau)p)^{(-1)^p}\\ &\quad= \mathcal{R}(s=(a\ell+\varepsilon)(1-i\varrho(\tau))+1-a+i\sigma(\tau)),\end{aligned} \tag{6.5}$$

where $q \equiv e^{2\pi i\tau}$, $\varrho(\tau) = \mathrm{Re}\,\tau/\mathrm{Im}\,\tau = \beta/\alpha$, $\sigma(\tau) = 1/(2\,\mathrm{Im}\,\tau) = \pi/\alpha$, while a is a real number, $\ell\in\mathbb{Z}_+$ and $\varepsilon\in\mathbb{C}$. We can use the Ruelle functions $\mathcal{R}(s)$ to write the results in the most general form,

$$\begin{aligned}&\prod_{n=\ell}^{\infty}(1-q^{an+\varepsilon})^{C(n)}\\ &\quad= \prod_{n=\ell}^{\infty}\left[\frac{\mathcal{R}(s=(an+\epsilon)(1-i\varrho(\tau))+1-a)}{\mathcal{R}(s=(a(n+1)+\epsilon)(1-i\varrho(\tau))+1-a}\right]^{C(n)},\end{aligned} \tag{6.6}$$

$$\begin{aligned}&\prod_{n=\ell}^{\infty}(1+q^{an+\varepsilon})^{C(n)}\\ &\quad= \prod_{n=\ell}^{\infty}\left[\frac{\mathcal{R}(s=(an+\epsilon)(1-i\varrho(\tau))+1-a+i\sigma(\tau))}{\mathcal{R}(s=(a(n+1)+\epsilon)(1-i\varrho(\tau))+1-a+i\sigma(\tau)}\right]^{C(n)},\end{aligned} \tag{6.7}$$

where $C(n)$ are certain coefficients. In the simplest cases $C(n) = bn$, $b\in\mathbb{R}$, and (6.6) and (6.7) becomes

$$\begin{aligned}&\prod_{n=\ell}^{\infty}(1-q^{an+\epsilon})^{bn} = \mathcal{R}(s=(a\ell+\varepsilon)(1-i\varrho(\tau))+1-a)^{b\ell}\\ &\quad\times\prod_{n=\ell+1}^{\infty}\mathcal{R}(s=(an+\varepsilon)(1-i\varrho(\tau))+1-a)^{b},\end{aligned} \tag{6.8}$$

$$\begin{aligned}&\prod_{n=\ell}^{\infty}(1+q^{an+\epsilon})^{bn} = \mathcal{R}(s=(a\ell+\varepsilon)(1-i\varrho(\tau))+1-a+i\sigma(\tau))^{b\ell}\\ &\quad\times\prod_{n=\ell+1}^{\infty}\mathcal{R}(s=(an+\varepsilon)(1-i\varrho(\tau))+1-a+i\sigma(\tau))^{b}.\end{aligned} \tag{6.9}$$

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