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Existence of positive solutions for third-order boundary value problems with integral boundary conditions on time scales

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Abstract

In this paper, four functionals fixed point theorem is used to verify the existence of at least one positive solution for third-order boundary value problems with integral boundary conditions for an increasing homeomorphism and homomorphism on time scales. We also provide an example to demonstrate our results.

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Keywords: Green's function; time scales; fixed point theorem; increasing homeomorphism and positive homomorphism; integral boundary conditions; positive solution

1 Introduction

The theory of time scales was introduced by Hilger [1] in his PhD thesis in 1988. Theoretically, this new theory has not only unified continuous and discrete equations, but has also exhibited much more complicated dynamics on time scales. Moreover, the study of dynamic equations on time scales has led to several important applications, for example, insect population models, biology, neural networks, heat transfer, and epidemic models, see [2–12].

Recently, scientists have noticed that the boundary conditions in many areas of applied mathematics and physics come down to integral boundary conditions. For instance, the models on chemical engineering, heat conduction, thermo-elasticity, plasma physics, and underground water flow can be reduced to the nonlocal problems with integral boundary conditions. For more information about this subject, we refer the readers to the excellent survey by Corduneanu [13], and Agarwal and O'Regan [14]. In addition, such kind of boundary value problem in a Banach space has been studied by some researchers, we refer the readers to [15–18] and the references therein. However, to the best of our knowledge, little work has been done on the existence of positive solutions for third-order boundary value problem with integral boundary conditions on time scales. This paper attempts to fill this gap in literature.

In [19], Ma considered the existence and multiplicity of positive solutions for the m-boundary value problems

$$\begin{cases} (p(t)u')' - q(t)u + f(t,u) = 0, \quad 0 < t < 1, \\ au(0) - bp(0)u'(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \\ cu(1) + dp(1)u'(1) = \sum_{i=1}^{m-2} \beta_i u(\xi_i). \end{cases}$$

The main tool is Guo-Krasnoselskii fixed point theorem.

In [15], Boucherif considered the second-order boundary value problem with integral boundary conditions

$$\begin{cases} y'' = f(t, y(t)), \quad 0 < t < 1, \\ y(0) - ay'(0) = \int_0^1 g_0(s)y(s)\,ds, \\ y(1) - by'(1) = \int_0^1 g_1(s)y(s)\,ds. \end{cases}$$

By using Krasnoselskii's fixed point theorem, he obtained the existence criteria of at least one positive solution.

In [17], Li and Zhang were concerned with the second-order p-Laplacian dynamic equations with integral boundary conditions on time scales

$$\begin{cases} (\phi_p(x^{\triangle}(t)))^{\nabla} + \lambda f(t, x(t), x^{\triangle}(t)) = 0, \quad t \in (0, T)_{\mathbb{T}}, \\ x^{\triangle}(0) = 0, \qquad \alpha x(T) - \beta x(0) = \int_0^T g(s)x(s)\nabla s. \end{cases}$$

By using Legget-Williams fixed point theorem, they obtained the existence criteria of at least three positive solutions.

In [18], Zhao *et al.* considered the third-order differential equation

$$x'''(t) + f\bigl(t, x(t)\bigr) = 0, \quad t \in J,$$

subject to one of the following integral boundary conditions

$$x(0) = 0, \qquad x''(0) = 0, \qquad x(1) = \int_0^1 g(t)x(t)\,dt,$$
$$x(0) = \int_0^1 g(t)x(t)\,dt, \qquad x''(0) = 0, \qquad x(1) = 0.$$

They investigated the existence, nonexistence, and multiplicity of positive solutions for a class of nonlinear boundary value problems of third-order differential equations with integral boundary conditions in ordered Banach spaces by means of fixed-point principle in cone and the fixed-point index theory for strict set contraction operator.

In [16], Fu and Ding considered the third-order boundary value problems with integral boundary conditions

$$\begin{cases} (\varphi(-x''(t)))' = f(t, x(t)), \quad t \in J, \\ x(0) = 0, \qquad x''(0) = 0, \qquad x(1) = \int_0^1 g(t)x(t)\,dt. \end{cases}$$

The arguments were based upon the fixed-point principle in cone for strict set contraction operators.

In [7], Han and Kang were concerned with the existence of multiple positive solutions of the third-order p-Laplacian dynamic equation on time scales

$$\begin{cases} (\phi_p(u^{\triangle\triangle}(t)))^{\nabla} + f(t, u(t)) = 0, \quad t \in [a, b], \\ \alpha u(\rho(a)) - \beta u^{\triangle}(\rho(a)) = 0, \qquad \gamma u(b) + \delta u^{\triangle}(b) = 0, \qquad u^{\triangle\triangle}(\rho(a)) = 0, \end{cases}$$

where $\phi_p(s)$ = is p-Laplacian operator, *i.e.*, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_p^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$. By using fixed point theorems in cones, they obtained the existence of multiple positive solutions for singular nonlinear boundary value problem.

Motivated by the results above, in this study, we consider the following third-order boundary value problem (BVP) on time scales:

$$\begin{cases} (\phi(-u^{\triangle\triangle}(t)))^{\triangle} + q(t)f(t,u(t),u^{\triangle}(t)) = 0, & t \in [0,1]_{\mathbb{T}}, \\ au(0) - bu^{\triangle}(0) = \int_0^1 g_1(s)u(s)\triangle s, \\ cu(1) + du^{\triangle}(1) = \int_0^1 g_2(s)u(s)\triangle s, \\ u^{\triangle\triangle}(1) = 0, \end{cases} \tag{1.1}$$

where $\mathbb{T}$ is a time scale, $0, 1 \in \mathbb{T}$, $[0,1]_{\mathbb{T}} = [0,1] \cap \mathbb{T}$, $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism and positive homomorphism with $\phi(0) = 0$. A projection $\phi : \mathbb{R} \to \mathbb{R}$ is called an increasing homeomorphism and positive homomorphism if the following conditions are satisfied:

(i) If $x \le y$, then $\phi(x) \le \phi(y)$ for all $x, y \in \mathbb{R}$;
(ii) ϕ is a continuous bijection, and its inverse mapping is also continuous;
(iii) $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in \mathbb{R}$.

Throughout this paper, we assume that the following conditions hold:

(C1) $a, b, c, d \in [0,\infty)$ with $ac + ad + bc > 0$,
(C2) $f \in \mathcal{C}([0,1]_{\mathbb{T}} \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$,
(C3) q, g_1 and $g_2 \in \mathcal{C}([0,1]_{\mathbb{T}}, \mathbb{R}^+)$.

By using the four functionals fixed point theorem [20], we get the existence of at least one positive solution for BVP (1.1). In fact, our result is also new when $\mathbb{T} = \mathbb{R}$ (the differential case) and $\mathbb{T} = \mathbb{Z}$ (the discrete case). Therefore, the result can be considered as a contribution to this field.

This paper is organized as follows. In Section 2, we provide some definitions and preliminary lemmas, which are the key tools for our main result. We give and prove our main result in Section 3. Finally, in Section 4, we give an example to demonstrate our result.

2 Preliminaries

In this section, to state the main results of this paper, we need the following lemmas.

We define $\mathbb{B} = \mathcal{C}^{\triangle}[0,1]$, which is a Banach space with the norm

$$\|u\| = \max\Bigl\{\max_{t\in[0,1]_{\mathbb{T}}} \bigl|u(t)\bigr|, \max_{t\in[0,1]_{\mathbb{T}}} \bigl|u^{\triangle}(t)\bigr|\Bigr\}.$$

Define the cone $\mathcal{P} \subset \mathbb{B}$ by

$$\mathcal{P} = \Bigl\{u \in \mathbb{B} : u(t) \text{ is nonnegative, nondecreasing on } [0,1]_{\mathbb{T}}, u^{\triangle}(t) \text{ is nonincreasing on } [0,1]_{\mathbb{T}}, \text{ and } au(0) - bu^{\triangle}(0) = \int_0^1 g_1(s)u(s)\triangle s\Bigr\}.$$

Denote by θ and φ, the solutions of the corresponding homogeneous equation

$$\bigl(\phi\bigl(-u^{\triangle\triangle}(t)\bigr)\bigr)^{\triangle} = 0, \quad t \in [0,1]_{\mathbb{T}}, \tag{2.1}$$

under the initial conditions,

$$\begin{cases} \theta(0) = b, & \theta^{\triangle}(0) = a, \\ \varphi(1) = d, & \varphi^{\triangle}(1) = -c. \end{cases} \tag{2.2}$$

Using the initial conditions (2.2), we can deduce from equation (2.1) for θ and φ the following equations:

$$\theta(t) = b + at, \qquad \varphi(t) = d + c(1-t). \tag{2.3}$$

Set

$$\Delta := \begin{vmatrix} -\int_0^1 g_1(s)(b+as)\triangle s & \rho - \int_0^1 g_1(s)(d+c(1-s))\triangle s \\ \rho - \int_0^1 g_2(s)(b+as)\triangle s & -\int_0^1 g_2(s)(d+c(1-s))\triangle s \end{vmatrix} \tag{2.4}$$

and

$$\rho := ad + ac + bc. \tag{2.5}$$

Lemma 2.1 *Let* (C1)-(C3) *hold*. *Assume that*

(C4) $\Delta \neq 0$.

If $u \in C^{\triangle}[0,1]$ *is a solution of the equation*

$$\begin{aligned} u(t) = {} & \int_0^1 G(t,s)\phi^{-1}\left(\int_s^1 q(\tau)f\big(\tau, u(\tau), u^{\triangle}(\tau)\big)\triangle\tau\right)\triangle s \\ & + A(f)(b+at) + B(f)\big(d + c(1-t)\big), \end{aligned} \tag{2.6}$$

where

$$G(t,s) = \frac{1}{\rho}\begin{cases} (b + a\sigma(s))(d + c(1-t)), & \sigma(s) \le t, \\ (b+at)(d + c(1-\sigma(s))), & t \le s, \end{cases} \tag{2.7}$$

$$A(f) := \frac{1}{\Delta}\begin{vmatrix} \int_0^1 g_1(s)H_f(s)\triangle s & \rho - \int_0^1 g_1(s)(d+c(1-s))\triangle s \\ \int_0^1 g_2(s)H_f(s)\triangle s & -\int_0^1 g_2(s)(d+c(1-s))\triangle s \end{vmatrix}, \tag{2.8}$$

$$B(f) := \frac{1}{\Delta}\begin{vmatrix} -\int_0^1 g_1(s)(b+as)\triangle s & \int_0^1 g_1(s)H_f(s)\triangle s \\ \rho - \int_0^1 g_2(s)(b+as)\triangle s & \int_0^1 g_2(s)H_f(s)\triangle s \end{vmatrix} \tag{2.9}$$

and

$$H_f(s) := \int_0^1 G(s,r)\phi^{-1}\left(\int_r^1 q(\tau)f\big(\tau, u(\tau), u^{\triangle}(\tau)\big)\triangle\tau\right)\triangle r, \tag{2.10}$$

then u *is a solution of the boundary value problem* (1.1).

Proof Let u satisfy the integral equation (2.6), then u is a solution of problem (1.1). Then we have

$$\begin{aligned} u(t) = {} & \int_0^1 G(t,s)\phi^{-1}\left(\int_s^1 q(\tau)f\big(\tau, u(\tau), u^{\triangle}(\tau)\big)\triangle\tau\right)\triangle s \\ & + A(f)(b+at) + B(f)\big(d + c(1-t)\big), \end{aligned}$$

i.e.,

$$\begin{aligned}
u(t) &= \int_0^t \frac{1}{\rho}\big(b+a\big(\sigma(s)\big)\big)\big(d+c(1-t)\big)\phi^{-1}\left(\int_s^1 q(\tau)f\big(\tau,u(\tau),u^{\triangle}(\tau)\big)\triangle\tau\right)\triangle s \\
&\quad + \int_t^1 \frac{1}{\rho}(b+at)\big(d+c\big(1-\sigma(s)\big)\big)\phi^{-1}\left(\int_s^1 q(\tau)f\big(\tau,u(\tau),u^{\triangle}(\tau)\big)\triangle\tau\right)\triangle s \\
&\quad + A(f)(b+at) + B(f)\big(d+c(1-t)\big), \\
u^{\triangle}(t) &= -\int_0^t \frac{c}{\rho}\big(b+a\big(\sigma(s)\big)\big)\phi^{-1}\left(\int_s^1 q(\tau)f\big(\tau,u(\tau),u^{\triangle}(\tau)\big)\triangle\tau\right)\triangle s \\
&\quad + \int_t^1 \frac{a}{\rho}\big(d+c\big(1-\sigma(s)\big)\big)\phi^{-1}\left(\int_s^1 q(\tau)f\big(\tau,u(\tau),u^{\triangle}(\tau)\big)\triangle\tau\right)\triangle s \\
&\quad + A(f)a - B(f)c.
\end{aligned}$$

Therefore,

$$\begin{aligned}
u^{\triangle\triangle}(t) &= \frac{1}{\rho}\big(-c\big(b+a\big(\sigma(t)\big)\big) - a\big(d+c\big(1-\sigma(t)\big)\big)\big)\phi^{-1}\left(\int_t^1 q(\tau)f\big(\tau,u(\tau),u^{\triangle}(\tau)\big)\triangle\tau\right) \\
&= \frac{1}{\rho}\big(-(ad+ac+bc)\big)\phi^{-1}\left(\int_t^1 q(\tau)f\big(\tau,u(\tau),u^{\triangle}(\tau)\big)\triangle\tau\right) \\
&= -\phi^{-1}\left(\int_t^1 q(\tau)f\big(\tau,u(\tau),u^{\triangle}(\tau)\big)\triangle\tau\right), \\
\phi\big(-u^{\triangle\triangle}(t)\big) &= \int_t^1 q(\tau)f\big(\tau,u(\tau),u^{\triangle}(\tau)\big)\triangle\tau, \\
\big(\phi\big(-u^{\triangle\triangle}(t)\big)\big)^{\triangle} &= -q(t)f\big(t,u(t),u^{\triangle}(t)\big).
\end{aligned}$$

So, we get

$$\big(\phi\big(-u^{\triangle\triangle}(t)\big)\big)^{\triangle} + q(t)f\big(t,u(t),u^{\triangle}(t)\big) = 0.$$

Since

$$\begin{aligned}
u(0) &= \int_0^1 \frac{b}{\rho}\big(d+c\big(1-\sigma(s)\big)\big)\phi^{-1}\left(\int_s^1 q(\tau)f\big(\tau,u(\tau),u^{\triangle}(\tau)\big)\triangle\tau\right)\triangle s \\
&\quad + A(f)b + B(f)(d+c), \\
u^{\triangle}(0) &= \int_0^1 \frac{a}{\rho}\big(d+c\big(1-\sigma(s)\big)\big)\phi^{-1}\left(\int_s^1 q(\tau)f\big(\tau,u(\tau),u^{\triangle}(\tau)\big)\triangle\tau\right)\triangle s \\
&\quad + A(f)a - B(f)c,
\end{aligned}$$

we have

$$\begin{aligned}
au(0) - bu^{\triangle}(0) = B(f)\rho \\
&= \int_0^1 g_1(s)\left(\int_0^1 G(s,r)\phi^{-1}\left(\int_r^1 q(\tau)f\big(\tau,u(\tau),u^{\triangle}(\tau)\big)\triangle\tau\right)\triangle r\right. \\
&\quad \left. + A(f)(b+as) + B(f)\big(d+c(1-s)\big)\right)\triangle s. \qquad (2.11)
\end{aligned}$$

Since

$$\begin{aligned}u(1) &= \int_0^1 \frac{d}{\rho}\big(b + a(\sigma(s))\big)\phi^{-1}\left(\int_s^1 q(\tau)f\big(\tau, u(\tau), u^{\triangle}(\tau)\big)\triangle\tau\right)\triangle s \\ &\quad + A(f)(b+a) + B(f)d, \\ u^{\triangle}(1) &= -\int_0^1 \frac{c}{\rho}\big(b + a(\sigma(s))\big)\phi^{-1}\left(\int_s^1 q(\tau)f\big(\tau, u(\tau), u^{\triangle}(\tau)\big)\triangle\tau\right)\triangle s \\ &\quad + A(f)a - B(f)c,\end{aligned}$$

we have

$$\begin{aligned}cu(1) + du^{\triangle}(1) &= A(f)\rho \\ &= \int_0^1 g_2(s)\left(\int_0^1 G(s,r)\phi^{-1}\left(\int_r^1 q(\tau)f\big(\tau, u(\tau), u^{\triangle}(\tau)\big)\triangle\tau\right)\triangle r\right. \\ &\quad \left. + A(f)(b+as) + B(f)\big(d + c(1-s)\big)\right)\triangle s. \end{aligned} \tag{2.12}$$

From (2.11) and (2.12), we get

$$\begin{cases} [-\int_0^1 g_1(s)(b+as)\triangle s]A(f) + [\rho - \int_0^1 g_1(s)(d + c(1-s))\triangle s]B(f) \\ \quad = \int_0^1 g_1(s)H_f(s)\triangle s, \\ [\rho - \int_0^1 g_2(s)(b+as)\triangle s]A(f) + [-\int_0^1 g_2(s)(d + c(1-s))\triangle s]B(f) \\ \quad = \int_0^1 g_2(s)H_f(s)\triangle s, \end{cases}$$

which implies that $A(f)$ and $B(f)$ satisfy (2.8) and (2.9), respectively. □

Lemma 2.2 *Let* (C1)-(C3) *hold. Assume that*
(C5) $\Delta < 0$, $\rho - \int_0^1 g_2(s)(b+as)\triangle s > 0$, $a - \int_0^1 g_1(s)\triangle s > 0$.
Then for $u \in \mathcal{C}^{\triangle}[0,1]$, *the solution* u *of problem* (1.1) *satisfies*

$$u(t) \ge 0 \quad \textit{for } t \in [0,1]_{\mathbb{T}}.$$

Proof It is an immediate subsequence of the facts that $G \ge 0$ on $[0,1]_{\mathbb{T}} \times [0,1]_{\mathbb{T}}$ and $A(f) \ge 0$, $B(f) \ge 0$. □

Lemma 2.3 *Let* (C1)-(C3) *and* (C5) *hold. Assume that*
(C6) $c - \int_0^1 g_2(s)\triangle s < 0$.
Then the solution $u \in \mathcal{C}^{\triangle}[0,1]$ *of problem* (1.1) *satisfies* $u^{\triangle}(t) \ge 0$ *for* $t \in [0,1]_{\mathbb{T}}$.

Proof Assume that the inequality $u^{\triangle}(t) < 0$ holds. Since $u^{\triangle}(t)$ is nonincreasing on $[0,1]_{\mathbb{T}}$, one can verify that

$$u^{\triangle}(1) \le u^{\triangle}(t), \quad t \in [0,1]_{\mathbb{T}}.$$

From the boundary conditions of problem (1.1), we have

$$-\frac{c}{d}u(1) + \frac{1}{d}\int_0^1 g_2(s)u(s)\triangle s \le u^{\triangle}(t) < 0.$$

The last inequality yields

$$-cu(1) + \int_0^1 g_2(s)u(s)\triangle s < 0.$$

Therefore, we obtain that

$$\int_0^1 g_2(s)u(1)\triangle s < \int_0^1 g_2(s)u(s)\triangle s < cu(1),$$

i.e.,

$$\left(c - \int_0^1 g_2(s)\triangle s\right)u(1) > 0.$$

According to Lemma 2.2, we have that $u(1) \geq 0$. So, $c - \int_0^1 g_2(s)\triangle s > 0$. However, this contradicts to condition (C6). Consequently, $u^{\triangle}(t) \geq 0$ for $t \in [0,1]_{\mathbb{T}}$. □

Lemma 2.4 *If* (C1)-(C6) *hold, then* $\max_{t\in[0,1]_{\mathbb{T}}} u(t) \leq M \max_{t\in[0,1]_{\mathbb{T}}} u^{\triangle}(t)$ *for* $u \in \mathcal{P}$, *where*

$$M = 1 + \frac{b + \int_0^1 sg_1(s)\triangle s}{a - \int_0^1 g_1(s)\triangle s}. \tag{2.13}$$

Proof For $u \in \mathcal{P}$, since $u^{\triangle}(t)$ is nonincreasing on $[0,1]_{\mathbb{T}}$, one arrives at

$$\frac{u(t) - u(0)}{t} \leq u^{\triangle}(0),$$

i.e., $u(t) - u(0) \leq tu^{\triangle}(0)$. Hence,

$$\int_0^1 g_1(s)u(s)\triangle s - \int_0^1 g_1(s)\triangle su(0) \leq \int_0^1 sg_1(s)\triangle su^{\triangle}(0).$$

By $au(0) - bu^{\triangle}(0) = \int_0^1 g_1(s)u(s)\triangle s$, we get

$$u(0) \leq \frac{b + \int_0^1 sg_1(s)\triangle s}{a - \int_0^1 g_1(s)\triangle s}u^{\triangle}(0).$$

Hence,

$$\begin{aligned} u(t) &= \int_0^t u^{\triangle}(s)\triangle s + u(0) \leq tu^{\triangle}(0) + u(0) \\ &\leq tu^{\triangle}(0) + \frac{b + \int_0^1 sg_1(s)\triangle s}{a - \int_0^1 g_1(s)\triangle s}u^{\triangle}(0) \leq \left(1 + \frac{b + \int_0^1 sg_1(s)\triangle s}{a - \int_0^1 g_1(s)\triangle s}\right)u^{\triangle}(0) \\ &= Mu^{\triangle}(0), \end{aligned}$$

i.e.,

$$\max_{t\in[0,1]_{\mathbb{T}}} u(t) \leq M \max_{t\in[0,1]_{\mathbb{T}}} u^{\triangle}(t).$$

The proof is finalized. □

From Lemma 2.4, we obtain

$$\begin{aligned}\|u\| &= \max\Big\{\max_{t\in[0,1]_{\mathbb{T}}}\big|u(t)\big|, \max_{t\in[0,1]_{\mathbb{T}}}\big|u^{\triangle}(t)\big|\Big\}\\ &\le \max\Big\{M\max_{t\in[0,1]_{\mathbb{T}}}u^{\triangle}(t), \max_{t\in[0,1]_{\mathbb{T}}}u^{\triangle}(t)\Big\}\\ &= M\max_{t\in[0,1]_{\mathbb{T}}}u^{\triangle}(t).\end{aligned}$$

Now, define an operator $T:\mathcal{P}\to\mathbb{B}$ by

$$\begin{aligned}Tu(t) = {}&\int_0^1 G(t,s)\phi^{-1}\left(\int_0^s q(\tau)f\big(\tau,u(\tau),u^{\triangle}(\tau)\big)\triangle\tau\right)\triangle s\\ &+ A(f)(b+at) + B(f)\big(d+c(1-t)\big),\end{aligned}\tag{2.14}$$

where G, $A(f)$ and $B(f)$ are defined as in (2.7), (2.8) and (2.9), respectively.

Lemma 2.5 *Let* (C1)-(C6) *hold. Then* $T:\mathcal{P}\to\mathcal{P}$ *is completely continuous.*

Proof By Arzela-Ascoli theorem, we can easily prove that operator T is completely continuous. □

3 Main results

Let α and Ψ be nonnegative continuous concave functionals on $\mathcal{P}$, and let β and Φ be nonnegative continuous convex functionals on $\mathcal{P}$, then for positive numbers r, j, l and R, we define the sets

$$\begin{aligned}&Q(\alpha,\beta,r,R) = \big\{u\in\mathcal{P}: r\le\alpha(u), \beta(u)\le R\big\},\\ &U(\Psi,j) = \big\{u\in Q(\alpha,\beta,r,R): j\le\Psi(u)\big\},\\ &V(\Phi,l) = \big\{u\in Q(\alpha,\beta,r,R): \Phi(u)\le l\big\}.\end{aligned}\tag{3.1}$$

Lemma 3.1 [20] *If* $\mathcal{P}$ *is a cone in a real Banach space* $\mathbb{B}$, α *and* Ψ *are nonnegative continuous concave functionals on* $\mathcal{P}$, β *and* Φ *are nonnegative continuous convex functionals on* $\mathcal{P}$, *and there exist positive numbers* r, j, l *and* R *such that*

$$T: Q(\alpha,\beta,r,R)\to\mathcal{P}$$

is a completely continuous operator, and $Q(\alpha,\beta,r,R)$ *is a bounded set. If*

(i) $\{u\in U(\Psi,j):\beta(u)<R\}\cap\{u\in V(\Phi,l): r<\alpha(u)\}\neq\emptyset$;
(ii) $\alpha(Tu)\ge r$ *for all* $u\in Q(\alpha,\beta,r,R)$ *with* $\alpha(u)=r$ *and* $l<\Phi(Tu)$;
(iii) $\alpha(Tu)\ge r$ *for all* $u\in V(\Phi,l)$ *with* $\alpha(u)=r$;
(iv) $\beta(Tu)\le R$ *for all* $u\in Q(\alpha,\beta,r,R)$ *with* $\beta(u)=R$ *and* $\Psi(Tu)<j$;
(v) $\beta(Tu)\le R$ *for all* $u\in U(\Psi,j)$ *with* $\beta(u)=R$.

Then T *has a fixed point* u *in* $Q(\alpha,\beta,r,R)$.

Suppose that $\omega, z\in\mathbb{T}$ with $0<\omega<z<1$. For the convenience, we take the notations

$$A := \frac{1}{\Delta}\begin{vmatrix}\int_0^1 g_1(s)H(s)\triangle s & \rho-\int_0^1 g_1(s)(d+c(1-s))\triangle s\\ \int_0^1 g_2(s)H(s)\triangle s & -\int_0^1 g_2(s)(d+c(1-s))\triangle s\end{vmatrix},$$

$$H(s) := \int_0^1 G(s,r)\phi^{-1}\left(\int_r^1 q(\tau)\triangle\tau\right)\triangle r,$$

$$\Omega = \int_\omega^z G(\omega,s)\phi^{-1}\left(\int_s^z q(\tau)\triangle\tau\right)\triangle s,$$

$$\Lambda = \int_0^1 \frac{1}{\rho}a\big(d + c\big(1 - \sigma(s)\big)\big)\phi^{-1}\left(\int_0^1 q(\tau)\triangle\tau\right)\triangle s + Aa,$$

$$N = \frac{a - \int_0^1 g_1(s)\triangle s}{b + \int_0^1 sg_1(s)\triangle s},$$

and define the maps

$$\alpha(u) = \min_{t\in[\omega,z]_{\mathbb{T}}} u(t), \qquad \Phi(u) = \max_{t\in[0,1]_{\mathbb{T}}} u(t), \qquad \beta(u) = \Psi(u) = \max_{t\in[0,1]_{\mathbb{T}}} u^{\triangle}(t). \tag{3.2}$$

Let $Q(\alpha,\beta,r,R)$, $U(\Psi,j)$ and $V(\Phi,l)$ be defined by (3.1).

Theorem 3.1 *Assume that* (C1)-(C6) *hold. If there exist constants* r, j, l, R *with* $\max\{\frac{r}{\omega}, R\} \le l$, $\max\{\frac{N+1}{N}j, \frac{N+1}{N\omega+1}r\} < R$, *and suppose that* f *satisfies the following conditions*:

(C7) $f(t,u,u^{\triangle}) \ge \phi(\frac{r}{\Omega})$ *for* $(t,u,u^{\triangle}) \in [\omega,z]_{\mathbb{T}} \times [r,l] \times [0,R]$;

(C8) $f(t,u,u^{\triangle}) \le \phi(\frac{R}{\Lambda})$ *for* $(t,u,u^{\triangle}) \in [0,1]_{\mathbb{T}} \times [0,MR] \times [0,R]$.

Then BVP (1.1) *has at least one positive solution* $u \in \mathcal{P}$ *such that*

$$\min_{t\in[\omega,z]_{\mathbb{T}}} u(t) \ge r, \qquad \max_{t\in[0,1]_{\mathbb{T}}} u(t) \le R.$$

Proof The boundary value problem (1.1) has a solution $u = u(t)$ if and only if u solves the operator equation $u = Tu$. Thus, we set out to verify that the operator T satisfies four functionals fixed point theorem, which will prove the existence of a fixed point of T.

We first show that $Q(\alpha,\beta,r,R)$ is bounded, and $T : Q(\alpha,\beta,r,R) \to \mathcal{P}$ is completely continuous. For all $u \in Q(\alpha,\beta,r,R)$ with Lemma 2.4, we have

$$\|u\| \le M \max_{t\in[0,1]_{\mathbb{T}}} u^{\triangle}(t) = M\beta(u) \le MR,$$

which means that $Q(\alpha,\beta,r,R)$ is a bounded set. According to Lemma 2.5, it is clear that $T : Q(\alpha,\beta,r,R) \to \mathcal{P}$ is completely continuous.

Let

$$u_0 = \frac{R}{N+1}(Nt+1).$$

Clearly, $u_0 \in \mathcal{P}$. By direct calculation,

$$\alpha(u_0) = u_0(\omega) = \frac{R}{N+1}(N\omega+1) > r,$$

$$\beta(u_0) = \frac{R}{N+1}N < R,$$

$$\Psi(u_0) = \beta(u_0) = \frac{R}{N+1}N \ge j,$$

$$\Phi(u_0) = u_0(1) = \frac{R}{N+1}(N+1) = R \le l.$$

So, $u_0 \in \{u \in U(\Psi, j) : \beta(u) < R\} \cap \{u \in V(\Phi, l) : r < \alpha(u)\}$, which means that (i) in Lemma 3.1 is satisfied.

For all $u \in Q(\alpha, \beta, r, R)$ with $\alpha(u) = r$ and $l < \Phi(Tu)$, since $u^{\triangle}$ is nonincreasing on $[0,1]_{\mathbb{T}}$, we have

$$\alpha(Tu) = Tu(\omega) \geq \omega Tu(1) = \omega\Phi(Tu) > \omega l \geq r.$$

So, $\alpha(Tu) > r$. Hence, (ii) in Lemma 3.1 is fulfilled.

For all $u \in V(\Phi, l)$ with $\alpha(u) = r$,

$$\begin{aligned}
\alpha(Tu) &= \min_{t\in[\omega,z]_{\mathbb{T}}} Tu(t) = (Tu)(\omega) \\
&= \int_0^1 fG(\omega,s)\phi^{-1}\left(\int_s^1 q(\tau)f\big(\tau,u(\tau),u^{\triangle}(\tau)\big)\triangle\tau\right)\triangle s \\
&\quad + A(f)(b+a\omega) + B(f)\big(d+c(1-\omega)\big) \\
&\geq \int_0^1 G(\omega,s)\phi^{-1}\left(\int_s^1 q(\tau)f\big(\tau,u(\tau),u^{\triangle}(\tau)\big)\triangle\tau\right)\triangle s \\
&\geq \int_\omega^z G(\omega,s)\phi^{-1}\left(\int_s^z q(\tau)f\big(\tau,u(\tau),u^{\triangle}(\tau)\big)\triangle\tau\right)\triangle s \\
&\geq \frac{r}{\Omega}\int_\omega^z G(\omega,s)\phi^{-1}\left(\int_s^z q(\tau)\triangle\tau\right)\triangle s = r,
\end{aligned}$$

and for all $u \in U(\Psi, j)$ with $\beta(u) = R$,

$$\begin{aligned}
\beta(Tu) &= \max_{t\in[0,1]_{\mathbb{T}}} (Tu)^{\triangle}(t) = (Tu)^{\triangle}(0) \\
&\leq \int_0^1 \frac{1}{\rho}a\big(d+c\big(1-\sigma(s)\big)\big)\phi^{-1}\left(\int_0^1 q(\tau)f\big(\tau,u(\tau),u^{\triangle}(\tau)\big)\triangle\tau\right)\triangle s \\
&\quad + A(f)a \\
&\leq \frac{R}{\Lambda}\left(\int_0^1 \frac{1}{\rho}a\big(d+c\big(1-\sigma(s)\big)\big)\phi^{-1}\left(\int_0^1 q(\tau)\triangle\tau\right)\triangle s + Aa\right) \\
&= R.
\end{aligned}$$

Thus, (iii) and (v) in Lemma 3.1 hold. We finally prove that (iv) in Lemma 3.1 holds.

For all $u \in Q(\alpha, \beta, r, R)$ with $\beta(u) = R$ and $\Psi(Tu) < j$, we have

$$\beta(Tu) = \Psi(Tu) < j < \frac{N}{N+1}R < R.$$

Thus, all conditions of Lemma 3.1 are satisfied. T has a fixed point u in $Q(\alpha, \beta, r, R)$. Therefore, BVP (1.1) has at least one positive solution $u \in \mathcal{P}$ such that

$$\min_{t\in[\omega,z]_{\mathbb{T}}} u(t) \geq r, \qquad \max_{t\in[0,1]_{\mathbb{T}}} u(t) \leq R.$$

The proof is completed. □

4 An example

Example 4.1 In BVP (1.1), suppose that $\mathbb{T} = [0,1]$, $q(t) = g_1(t) = g_2(t) = 1$, $a = 4$, $b = 2$, $c = \frac{1}{2}$, and $d = 8$, *i.e.*,

$$\begin{cases} (\phi(-u^{\triangle\triangle}(t)))^{\triangle} + f(t, u(t), u^{\triangle}(t)) = 0, & t \in [0,1], \\ 4u(0) - 2u^{\triangle}(0) = \int_0^1 u(s)\,ds, \\ \frac{1}{2}u(1) + 8u^{\triangle}(1) = \int_0^1 u(s)\,ds, \\ u^{\triangle\triangle}(1) = 0, \end{cases} \tag{4.1}$$

where

$$f(t, u, u^{\triangle}) = \begin{cases} 82, & u \in [0,5], \\ \frac{17}{179}u + \frac{14{,}593}{179}, & u \geq 5 \end{cases}$$

and

$$\phi(u) = \begin{cases} \frac{u}{2}, & u \leq 0, \\ u, & u \geq 0. \end{cases}$$

By simple calculation, we get $\rho = 35$, $\theta(t) = 2 + 4t$, $\varphi(t) = \frac{17}{2} - \frac{1}{2}t$, $\Delta = -\frac{3{,}185}{4}$, $M = \frac{11}{6}$, $N = \frac{6}{5}$, $A = B = \frac{32}{1{,}911}$, and

$$G(t,s) = \frac{1}{35} \begin{cases} (2 + 4s)(\frac{17}{2} - \frac{1}{2}t), & s \leq t, \\ (2 + 4t)(\frac{17}{2} - \frac{1}{2}s), & t \leq s. \end{cases}$$

Set $\omega = \frac{1}{3}$, $z = \frac{2}{3}$, then we get

$$\Omega = \frac{5}{81}, \qquad \Lambda = \frac{9{,}649}{9{,}555}.$$

Choose $r = 5$, $l = 150$, $j = 10$ and $R = 100$, it is easy to check that $\max\{15, 100\} \leq 150$, $\max\{\frac{55}{3}, \frac{55}{12}\} < 100$, and conditions (C1)-(C6) are satisfied. Now, we show that (C7) and (C8) are satisfied:

$$\begin{aligned} &f(t, u(t), u^{\triangle}(t)) \geq 82 > \phi\left(\frac{r}{\Omega}\right) = 81 \\ &\quad \text{for } (t, u(t), u^{\triangle}(t)) \in \left[\frac{1}{3}, \frac{2}{3}\right] \times [5, 150] \times [0, 100]; \\ &f(t, u(t), u^{\triangle}(t)) < 99 \leq \phi\left(\frac{R}{\Lambda}\right) = 99.0258 \\ &\quad \text{for } (t, u(t), u^{\triangle}(t)) \in [0,1] \times \left[0, \frac{550}{3}\right] \times [0, 100]. \end{aligned}$$

So, all conditions of Theorem 3.1 hold. Thus, by Theorem 3.1, BVP (4.1) has at least one positive solution u such that

$$\min_{t \in [\frac{1}{3}, \frac{2}{3}]} u(t) \geq 5, \qquad \max_{t \in [0,1]} u(t) \leq 100.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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