Eur. Phys. J. C (2014) 74:3101
DOI 10.1140/epjc/s10052-014-3101-2

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

Lagrangian for Frenkel electron and position's non-commutativity due to spin

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Received: 13 June 2014 / Accepted: 12 September 2014 / Published online: 16 October 2014

Abstract We construct a relativistic spinning-particle Lagrangian where spin is considered as a composite quantity constructed on the base of a non-Grassmann vector-like variable. The variational problem guarantees both a fixed value of the spin and the Frenkel condition on the spin-tensor. The Frenkel condition inevitably leads to relativistic corrections of the Poisson algebra of the position variables: their classical brackets became noncommutative. We construct the relativistic quantum mechanics in the canonical formalism (in the physical-time parametrization) and in the covariant formalism (in an arbitrary parametrization). We show how state vectors and operators of the covariant formulation can be used to compute the mean values of physical operators in the canonical formalism, thus proving its relativistic covariance. We establish relations between the Frenkel electron and positive-energy sector of the Dirac equation. Various candidates for the position and spin operators of an electron acquire clear meaning and interpretation in the Lagrangian model of the Frenkel electron. Our results argue in favor of Pryce's (d)-type operators as the spin and position operators of Dirac theory. This implies that the effects of non-commutativity could be expected already at the Compton wavelength. We also present the manifestly covariant form of the spin and position operators of the Dirac equation.

1 Introduction and outlook

A quantum description of spin is based on the Dirac equation, whereas the most popular classical equations of the electron have been formulated by Frenkel [1,2] and Bargmann, Michel and Telegdi (F-BMT) [3]. They almost exactly reproduce the spin dynamics of polarized beams in uniform fields, and this agrees with the calculations based on Dirac theory. Hence we expect that these models might be a proper classical analog for the Dirac theory. The variational formulation for the F-BMT equations represents a rather non-trivial problem [4–14] (note that one needs a Hamiltonian to study, for instance, Zeeman effect). In this work we continue the systematic analysis of these equations, started in [14]. We develop their Lagrangian formulation considering spin as a composite quantity (inner angular momentum) constructed from a non-Grassmann vector-like variable and its conjugated momentum [10–17].

Nonrelativistic spinning particles with reasonable properties can be constructed [15,18] starting from the singular Lagrangian which implies the following Dirac constraints:

$$\boldsymbol{\pi}^2 - a_3 = 0, \quad \boldsymbol{\omega}^2 - a_4 = 0, \quad \boldsymbol{\omega\pi} = 0, \tag{1}$$

where $a_3 = \frac{3\hbar^2}{4a_4}$, while the relativistic form of these constraints reads

$$T_3 = \pi^2 - a_3 = 0, \quad T_4 = \omega^2 - a_4 = 0, \quad T_5 = \omega\pi = 0, \tag{2}$$

$$T_6 = p\omega = 0, \qquad T_7 = p\pi = 0. \tag{3}$$

Besides, we have the standard mass-shell constraint in the position sector, $T_1 = p^2 + (mc)^2 = 0$. We denote the basic variables of spin by $\omega^\mu = (\omega^0, \boldsymbol{\omega})$, $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$; then $\omega\pi = -\omega^0\pi^0 + \boldsymbol{\omega\pi}$ and so on. π^μ and p^μ are the conjugate momenta for ω^μ and the position x^μ.

Since the constraints are written for the phase-space variables, it is easy to construct the corresponding action functional in a the Hamiltonian formulation. We simply take $L_H = p\dot{x} + \pi\dot{\omega} - H$, with the Hamiltonian in the form of a linear combination, the constraints T_i multiplied by auxiliary variables g_i, $i = 1, 3, 4, 5, 6, 7$. The Hamiltonian action with six auxiliary variables admits an interaction with an arbitrary electromagnetic field and gives a unified variational formulation of both Frenkel and BMT equations; see [14]. In Sect. 2 we develop a Lagrangian formulation of these equations.

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Excluding the conjugate momenta from L_H, we obtain the Lagrangian action. Further, excluding the auxiliary variables, one after another, we obtain various equivalent formulations of the model. We briefly discuss all them, as they will be useful when we switch on the interaction with external fields [19,20]. At the end, we get the "minimal" formulation without auxiliary variables. This reads

$$S = \int d\tau \left(\sqrt{a_3 \dot{\omega} N \dot{\omega}} - mc\sqrt{-\dot{x} N \dot{x}} - \frac{g_4}{2}(\omega^2 - a_4) \right), \quad (4)$$

where $N^{\mu\nu} \equiv \eta^{\mu\nu} - \frac{\omega^\mu \omega^\nu}{\omega^2}$ is the projector on the plane transverse to the direction of ω^μ. The last term in (4) represents a velocity-independent constraint which is well known from classical mechanics. So, we might follow the classical-mechanics prescription to exclude the g_4 as well. But this would lead to the loss of the manifest relativistic invariance of the formalism. The action is written in a parametrization τ which obeys

$$\frac{dt}{d\tau} > 0, \quad \text{this implies} \quad g_1(\tau) > 0, \quad p^0 > 0. \quad (5)$$

To explain this restriction, we note that in the absence of spin we expect an action of a spinless particle. Switching off the spin variables ω^μ from Eq. (4), we obtain $L = -mc\sqrt{-\dot{x}^2}$. Let us compare this with a spinless particle interacting with electromagnetic field. In terms of the physical variables $\mathbf{x}(t)$ this reads $L = -mc\sqrt{c^2 - \dot{\mathbf{x}}^2} + eA_0 + \frac{e}{c}\mathbf{A}\dot{\mathbf{x}}$. If we restrict ourselves to the class of increasing parameterizations of the world-line, this reads $L = -mc\sqrt{-\dot{x}^2} + \frac{e}{c}A\dot{x}$, in correspondence with the spinless limit of (4).

Assuming $\frac{dt}{d\tau} < 0$ we arrive at another Lagrangian, $L = mc\sqrt{-\dot{x}^2} + \frac{e}{c}A_\mu \dot{x}^\mu$. So a variational formulation with both positive and negative parameterizations would describe simultaneously two classical theories. In quantum theory they correspond to positive- and negative-energy solutions of the Klein–Gordon equation [21].

In [18] we discussed the geometry behind the constraints (1)–(3). The phase-space surface (1) can be identified with group manifold $SO(3)$. It has the natural structure of a fiber bundle with the base being a two-dimensional sphere, thus providing a connection with the approach of Souriau [22,23]. The components of non-relativistic spin-vector are defined by $S_i = \epsilon_{ijk}\omega_j \pi_k$. At the end, they turn out to be functions of coordinates which parameterize the base. The set (2), (3) is just a Lorentz-covariant form of the constraints (1). In the covariant formulation, S_i is included into the antisymmetric spin-tensor $J^{\mu\nu} = 2\omega^{[\mu}\pi^{\nu]}$ according to the Frenkel rule, $J^{ij} = 2\epsilon^{ijk}S^k$.

In the dynamical theory, these constraints can be interpreted as follows. First, the spin-sector constraints (2) fix the value of the spin, $J^{\mu\nu}J_{\mu\nu} = 6\hbar^2$. As in the rest frame we have $\mathbf{S}^2 = \frac{1}{8}J^{\mu\nu}J_{\mu\nu} = \frac{3\hbar^2}{4}$, and this implies the right value of the three-dimensional spin, as well as the right number of spin degrees of freedom.

Second, the first-class constraint $\pi^2 - a_3 = 0$ provides an additional local symmetry (spin-plane symmetry) of variational problem. The spin-plane symmetry has a clear geometric interpretation as transformations of the structure group of the fiber bundle acting independently at each instance of time. They rotate the pair ω^μ, π^μ in the plane formed by these vectors. In contrast, $J^{\mu\nu}$ turns out to be invariant under the symmetry. Hence the spin-plane symmetry determines the physical sector of the spinning particle: the basic variable ω^μ is gauge non-invariant, so it does not represent an observable quantity, while $J^{\mu\nu}$ does.

Reparametrization symmetry is well known to be crucial for the Lorentz-covariant description of a spinless particle. The spin-plane symmetry, as it determines the physical sector, turns out to be crucial for the description of a spinning particle. We point out that this appears already in the non-relativistic model [$\boldsymbol{\pi}^2 - a_3 = 0$ represents the first-class constraint in the set (1)]. The local symmetry group of the minimal action will be discussed in some detail in Sect. 2.3. A curious property here is that the standard reparametrization symmetry turns out to be a combination of two independent local symmetries.

Equation (3) guarantee the Frenkel-type condition $J^{\mu\nu}p_\nu = 0$. They form a pair of second-class constraints which involve both spin-sector and position-sector variables. This leads to new properties as compared with the nonrelativistic formulation. The second-class constraints must be taken into account by a transition from Poisson to Dirac bracket. As the constraints involve conjugate momenta p^μ for x^μ, this leads to nonvanishing Dirac brackets for the position variables,

$$\{x^\mu, x^\nu\}_D = -\frac{J^{\mu\nu}}{2p^2}. \quad (6)$$

We can pass from the parametric $x^\mu(\tau)$ to the physical variables $x^i(t)$. They also obey a noncommutative algebra; see Eq. (49) below. We remind the reader that in a theory with second-class constraints one can find special coordinates on the constraints surface with canonical (that is, Poisson) brackets; see (58). Functions of special coordinates are candidates for observable quantities. The Dirac bracket (more exactly, its nondegenerate part) is just the canonical bracket rewritten in terms of initial coordinates [24]. For the present case, namely the initial coordinates [they are $x^i(t)$], they are of physical interest,[1] as they represent the position of a particle. So, while there are special coordinates with canonical symplectic structure, the physically interesting coordinates obey the non-commutative algebra.

[1] In the interacting theory namely the initial coordinates obey the F-BMT equations.

In the result, the position space is endowed, in a natural way, with a noncommutative structure by accounting for the spin degrees of freedom. The relations between spin and non-commutativity appeared already in the work of Matthisson [25,26]. It is well known that dynamical systems with second-class constraints allow one to incorporate noncommutative geometry into the framework of classical and quantum theory [5,27–33]. Our model represents an example of a situation when a physically interesting noncommutative particle (6) emerges in this way. For this case, the "parameter of non-commutativity" is proportional to the spin-tensor (spin non-commutativity imposed by hand in quantum theory is considered in [30,31]).

We point out that the nonrelativistic model (1) implies the canonical algebra of the position operators; see [15,18]. So the deformation (6) arises as a relativistic correction induced by spin of the particle.

While the emergence of a noncommutative structure in the classical theory is nothing more than a mathematical game, this became crucial in quantum theory. Quantization of a theory with second-class constraints on the base of Poisson brackets is not consistent, and we are forced to look for a quantum realization of the Dirac brackets. Instead of the standard quantization rule of the position, $x \to \hat{x} = x$, we need to set $x \to \hat{x} = x + \hat{\delta}$ with some operator $\hat{\delta}$ which provides the desired algebra (6). This leads to interesting consequences concerning the relation between classical and quantum theories, which we start to discuss in this work.

A natural way to construct quantum observables is based on the correspondence principle between classical and quantum descriptions. However, this straightforward approach is mostly restricted to simple models like non-relativistic point particle. Elementary particles with spin were initially studied from the quantum perspective, because systematically constructed classical models of a spinning particle were not known. The construction of quantum observables for an electron involves the analysis of the Dirac equation and the representation theory of Lorentz group. Newton and Wigner found possible position operator, $\hat{\mathbf{x}}_{\text{NW}}$, by the analysis of localized states in relativistic theory [34]. Foldy and Wouthuysen invented a convenient representation for the Dirac equation [35]. In this representation the Newton–Wigner position operator simply becomes the multiplication operator, $\hat{\mathbf{x}}_{\text{NW}} = \mathbf{x}$. Pryce noticed that the notion of a center-of-mass in relativistic theory is not unique [36] and suggested the list of possible operators. The Pryce center-of-mass (*e*) has commuting components and coincides with the Newton–Wigner position operator, while the Pryce center-of-mass (*d*) is defined as a covariant object though it has non-commutative components.

Notion of position observables in the theory of the Dirac equation [37–39] is in close relation with the notion of relativistic spin. The current interest in covariant spin operators is related with a broad range of physical problems concerning consistent definition of the relativistic spin operator and the Lorentz-covariant spin density matrix in quantum information theory [40–47]. Consideration of *Zitterbewegung* [48] and spin currents [49] in condensed matter studies involves Heisenberg equations for position and spin observables. Precession of spin in gravitational fields gives a useful tools to test general relativity [50]. Surprisingly, coupling of spin to gravitational fields may be important already in the acceleration experiments due to so-called spin-rotation coupling [51]. In these applications a better understanding of the spinning particle at the classical level may be very useful.

There are a lot of operators proposed for the position and spin of relativistic electron; see [4,34–36,45,52]. Which one is the conventional position (spin) operator? Widely assumed as the best candidate is the pair of Foldy–Wouthuysen [$\sim$ Newton–Wigner $\sim$ Pryce (*e*)] mean position and spin operators. The components of the mean-position operator commute with each other, spin obeys the $so(3)$ algebra. However, they do not represent Lorentz-covariant quantities.

To clarify these long-standing questions, in Sects. 3–5 we construct relativistic quantum mechanics of the F-BMT electron. In Sect. 3, quantizing our Lagrangian in a physical-time parametrization, we obtain the operators corresponding to the classical position and spin of our model. Our results argue in favor of covariant Pryce (*d*) position and spin operators.[2] This implies that the effects of non-commutativity could be present at the Compton wavelength, in contrast to conventional expectations [53] of non-commutativity at the Planck length.

In Sect. 4, we construct Hamiltonian formulation in the covariant form (in an arbitrary parametrization). The constraints $p^2 + (mc)^2 = 0$ and $S^2 = \frac{3\hbar^2}{4}$ appeared in classical model can be identified with Casimir operators of Poincaré group. That is, the spin one-half representation of Poincaré group represents a natural quantum realization of our model. According to Wigner [54–56], this is given by the Hilbert space of solutions to the two-component Klein–Gordon (KG) equation. The two-component KG field has been considered by Feynman and Gell-Mann [57] to describe the weak interaction of spin one-half particle in quantum field theory, and by Brown [58] as a starting point for QED. In contrast to KG equation for a scalar field, the two-component KG equation admits the covariant positively defined conserved current

$$I^\mu = \frac{1}{(mc)^2}(\bar{\sigma}\hat{p}\psi)^\dagger\sigma^\mu(\bar{\sigma}\hat{p}\psi) - \psi^\dagger\bar{\sigma}^\mu\psi, \tag{7}$$

which can be used to construct a relativistic quantum mechanics of this equation. This is done in Sect. 5.1; then in Sect. 5.2 we show its equivalence with the quantum mechanics

[2] Pryce (*e*)-operators corresponds to the special variables mentioned above; see Sect. 3.2.

of the Dirac equation. Taking into account the condition (5), we conclude that the F-BMT electron corresponds to the positive-energy sector of the KG quantum mechanics; see Sect. 5.3. In Sect. 5.4, we establish the correspondence between canonical and covariant formulations of F-BMT electron, thus proving relativistic invariance of the physical-time formalism of Sect. 3.2. In particular, we find the manifestly covariant operators

$$\hat{x}^{\mu}_{rp} = x^{\mu} + \frac{1}{2\hat{p}^2}(\sigma\hat{p})^{\mu}, \tag{8}$$

$$\hat{j}^{\mu\nu} = \sigma^{\mu\nu} + \frac{\hat{p}^{\mu}(\sigma\hat{p})^{\nu} - \hat{p}^{\nu}(\sigma\hat{p})^{\mu}}{\hat{p}^2}, \tag{9}$$

and show how they can be used to compute the mean values of the physical [that is, Pryce (d)] operators of position and spin. In other words, they represent a manifestly covariant form of Pryce (d)-operators.

Using the equivalence between KG and Dirac quantum mechanics, we then found the form of these operators on the space of Dirac spinors. They also can be used to compute position and spin of the Frenkel electron; see Sect. 5.5.

2 Search for Lagrangian

2.1 Variational problem with auxiliary variables

To start with, we take the Hamiltonian action [14]

$$S_H = \int d\tau \left(p_{\mu}\dot{x}^{\mu} + \pi_{\mu}\dot{\omega}^{\mu} + \pi_{gi}\dot{g}_i - H\right), \tag{10}$$

$$H = \frac{g_1}{2}(p^2 + m^2c^2) + \frac{g_3}{2}(\pi^2 - a_3) + \frac{g_4}{2}(\omega^2 - a_4) + g_5(\omega\pi) + g_6(p\omega) + g_7(p\pi) + \lambda_{gi}\pi_{gi}. \tag{11}$$

Here π_{gi} are the conjugate momenta for the auxiliary variables g_i. We have denoted by λ_{gi} the Lagrangian multipliers for the primary constraints $\pi_{gi} = 0$. Variation of the action with respect to λ_{gi} gives the equations $\pi_{gi} = 0$, and this implies $\dot{\pi}_{gi} = 0$. Using this in the equations $\frac{\delta S_H}{\delta g_i} = 0$ we obtain[3] the desired constraints (2) and (3). Our model is manifestly Poincaré invariant. The auxiliary variables g_i, are scalars under the Poincaré transformations. The remaining variables transform according to the rule

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu} + a^{\mu}, \quad p'^{\mu} = \Lambda^{\mu}{}_{\nu}p^{\nu}, \tag{12}$$

$$\omega'^{\mu} = \Lambda^{\mu}{}_{\nu}\omega^{\nu}, \quad \pi'^{\mu} = \Lambda^{\mu}{}_{\nu}\pi^{\nu}. \tag{13}$$

[3] ω^{μ} obeys the Hamiltonian equation $\dot{\omega}^{\mu} = g_3\pi^{\mu}$. Together with $\pi^2 > 0$, this implies $\dot{\omega}^2 > 0$.

Local symmetries form a two-parametric group of transformations. It is composed of the standard reparameterizations

$$\delta x^{\mu} = \alpha\dot{x}^{\mu}, \quad \delta p^{\mu} = \alpha\dot{p}^{\mu}, \tag{14}$$

$$\delta\omega^{\mu} = \alpha\dot{\omega}^{\mu}, \quad \delta\pi^{\mu} = \alpha\dot{\pi}^{\mu}, \tag{15}$$

$$\delta g_i = (\alpha g_i)^{\cdot}, \quad \delta\lambda_{g_i} = (\delta g_i)^{\cdot}, \tag{16}$$

as well as by spin-plane transformations with the parameter $\beta(\tau)$:

$$\delta\omega^{\mu} = s\beta\pi^{\mu}, \quad \delta\pi^{\mu} = -\frac{1}{s}\beta\omega^{\mu}, \tag{17}$$

$$\delta g_3 = s\dot{\beta} - 2sg_5\beta, \quad \delta g_4 = \frac{1}{s}\dot{\beta} + 2\frac{1}{s}g_5\beta,$$

$$\delta g_6 = \frac{1}{s}\beta g_7, \quad \delta g_7 = -s\beta g_6,$$

$$\delta g_5 = \frac{1}{s}\beta g_3 - s\beta g_4, \quad \delta\lambda_{gi} = (\delta g_i)^{\cdot}. \tag{18}$$

We have denoted $s \equiv \sqrt{\frac{a_4}{a_3}}$. Equation (17) represents the infinitesimal form of the structure-group transformations of the spin-fiber bundle [18].

The coordinates x^{μ}, Frenkel spin-tensor $J^{\mu\nu}$ and BMT vector s^{μ}_{BMT}

$$J^{\mu\nu}(\tau) = 2(\omega^{\mu}\pi^{\nu} - \omega^{\nu}\pi^{\mu}), \tag{19}$$

$$s^{\mu}_{\text{BMT}}(\tau) \equiv \frac{1}{4\sqrt{-p^2}}\epsilon^{\mu\nu\alpha\beta}p_{\nu}J_{\alpha\beta}, \tag{20}$$

are β-invariant quantities. For their properties see Appendix 1. Note that the spatial components, s^i_{BMT}, coincide with the Frenkel spin,

$$S^i = \frac{1}{4}\epsilon^{ijk}J_{jk}, \tag{21}$$

only in the rest frame. Both transform as a vector under spatial rotations, but they have different transformation laws under a Lorentz boost. In an arbitrary frame they are related by

$$S^i = \frac{p^0}{\sqrt{-p^2}}\left(\delta_{ij} - \frac{p_i p_j}{(p^0)^2}\right)s^j_{\text{BMT}}. \tag{22}$$

where this does not lead to misunderstanding, we denote s^{μ}_{BMT} as s^{μ}.

The Lagrangian of a given Hamiltonian theory with constraints can be restored with the well-known procedure [24,29]. For the present case, it is sufficiently to solve the Hamiltonian equations of motion for x^{μ} and ω^{μ} with respect to p^{μ} and π^{μ}, and substitute them into the Hamiltonian action (10). Let us do this for a more general Hamiltonian action, obtaining a closed formula which will be repeatedly used below.

Consider mechanics with the configuration-space variables $Q^a(\tau)$, $g_i(\tau)$, and with the Lagrangian action

$$S = \frac{1}{2}\int d\tau \left(G_{ab}DQ^aDQ^b - K_{ab}Q^aQ^b - M\right). \tag{23}$$

We have denoted $DQ^a \equiv \dot{Q}^a - H^a{}_b Q^b$, and $G(g,Q)$, $K(g,Q)$, $H(g,Q)$, and $M(g)$ are some functions of the indicated variables. Let us construct the Hamiltonian action functional of this theory. Denoting the conjugate momenta as P_a, π_{gi}, the equations for P_a can be solved,

$$P_a = \frac{\partial L}{\partial \dot{Q}^a} = G_{ab} DQ^b, \quad \Rightarrow \quad \dot{Q}^a = \tilde{G}^{ab} P_b + H^a{}_b Q^b, \tag{24}$$

where $\tilde{G}^{ab}$ is the inverse of the matrix G_{ab}. The equations for the remaining momenta turn out to be the primary constraints, $\pi_{gi} = 0$. Then the Hamiltonian action reads

$$S_H = \int d\tau \left(P_a \dot{Q}^a + \pi_{gi} \dot{g}_i - H \right), \tag{25}$$

$$H = \frac{1}{2}\tilde{G}^{ab} P_a P_b + P_a H^a{}_b Q^b + \frac{1}{2} K_{ab} Q^a Q^b + \frac{1}{2} M + \lambda_{gi} \pi_{gi}. \tag{26}$$

Thus the Hamiltonian (25) and the Lagrangian (23) variational problems are equivalent. We point out that choosing an appropriate set of auxiliary variables g_i, the action (23) can be used to produce any desired quadratic constraints of the variables Q, P.

Let us return to our problem (11). Comparing the Hamiltonian of our interest (11) with the expression (26), we define the "doublets" $Q^a = (x^\mu, \omega^\nu)$, $P_a = (p_\mu, \pi_\nu)$, as well as the matrices

$$\tilde{G}^{ab} = \begin{pmatrix} g_1 & g_7 \\ g_7 & g_3 \end{pmatrix}, \; H^a{}_b = \begin{pmatrix} 0 & g_6 \\ 0 & g_5 \end{pmatrix}, \; K_{ab} = \begin{pmatrix} 0 & 0 \\ 0 & g_4 \end{pmatrix},$$

where $g_1 = g_1 \eta^{\mu\nu}$ and so on. Besides, we take the "mass" term in the form $M = g_1 m^2 c^2 - a_3 g_3 - a_4 g_4$. With this choice, Eq. (26) turns into our Hamiltonian (11). So the corresponding Lagrangian action reads from (23) as follows:

$$S = \int d\tau \frac{1}{2 \det \tilde{G}} \left[g_3 (Dx)^2 - 2 g_7 (Dx D\omega) + g_1 (D\omega)^2 \right] - \frac{1}{2} g_1 m^2 c^2 + \frac{1}{2} g_3 a_3 - \frac{1}{2} g_4 (\omega^2 - a_4). \tag{27}$$

We have denoted

$$Dx^\mu = \dot{x}^\mu - g_6 \omega^\mu, \qquad D\omega^\mu = \dot{\omega}^\mu - g_5 \omega^\mu.$$

Using the inverse matrix

$$G_{ab} = \frac{1}{\det \tilde{G}} \begin{pmatrix} g_3 & -g_7 \\ -g_7 & g_1 \end{pmatrix},$$

the action can be written in the form

$$S = \int d\tau \frac{1}{2} G_{ab} DQ^a DQ^b + \frac{a_3 g_{11} - m^2 c^2 g_{22}}{2 \det G} - \frac{1}{2} g_4 (\omega^2 - a_4), \tag{28}$$

where $DQ^a = (Dx, D\omega)$.

2.2 Variational problem without auxiliary variables

Eliminating the auxiliary variables one by one, we get various equivalent formulations of the model (27). At the end, we arrive at the Lagrangian action without auxiliary variables g_i.

First, we write the equations for g_5 and g_6 following from (27). They imply $(\omega D\omega) = 0$ and $(\omega Dx) = 0$, and then

$$g_5 = \frac{(\dot{\omega}\omega)}{\omega^2}, \quad g_6 = \frac{(\dot{x}\omega)}{\omega^2}.$$

We substitute the solution[4] into the action (27), this reads

$$S = \int d\tau \frac{1}{2 \det \tilde{G}} \left[g_3 (\dot{x} N \dot{x}) - 2 g_7 (\dot{x} N \dot{\omega}) + g_1 (\dot{\omega} N \dot{\omega}) \right] - \frac{1}{2} g_1 m^2 c^2 + \frac{1}{2} g_3 a_3 - \frac{1}{2} g_4 (\omega^2 - a_4). \tag{29}$$

It has been denoted

$$N^{\mu\nu} \equiv \eta^{\mu\nu} - \frac{\omega^\mu \omega^\nu}{\omega^2}, \quad \text{then} \quad N^{\mu\nu}\omega_\nu = 0. \tag{30}$$

Together with $\tilde{N}^{\mu\nu} \equiv \frac{\omega^\mu \omega^\nu}{\omega^2}$, this forms a pair of projectors $N + \tilde{N} = 1$, $N^2 = N$, $\tilde{N}^2 = \tilde{N}$, $N\tilde{N} = 0$. Any vector V^μ can be decomposed on the transverse and longitudinal parts with respect to ω^μ, $V^\mu = V^\mu_\perp + V^\mu_\parallel$, where $V^\mu_\perp = N^\mu{}_\nu V^\nu$, then $V^\mu_\perp \omega_\mu = 0$; and $V^\mu_\parallel = \tilde{N}^\mu{}_\nu V^\nu = \frac{(\omega V)}{\omega^2}\omega^\mu \sim \omega^\mu$. Further, in the action (29) we put $g_7 = 0$,

$$S = \int d\tau \frac{1}{2 g_1} (\dot{x} N \dot{x}) - \frac{1}{2} g_1 m^2 c^2 + \frac{1}{2 g_3} (\dot{\omega} N \dot{\omega}) + \frac{1}{2} a_3 g_3 - \frac{1}{2} g_4 (\omega^2 - a_4). \tag{31}$$

This does not alter the dynamical equations, whereas the constraint $\omega\pi = 0$ appears as the third-stage constraint.

The first two terms in Eq. (31) (as well as the third and the fourth terms) have a structure similar to that of a spinless particle, $\frac{1}{2e}\dot{x}^2 - \frac{em^2c^2}{2}$. It is well known that for this case we can substitute the equations of motion for e back into the Lagrangian; this leads to an equivalent variational problem. So, we solve the equation for g_3, $g_3 = \sqrt{\frac{\dot{\omega} N \dot{\omega}}{a_3}}$, and substitute this back into (31); this gives

$$S = \int d\tau \frac{1}{2 g_1} (\dot{x} N \dot{x}) - \frac{1}{2} g_1 m^2 c^2 + \sqrt{a_3} \sqrt{\dot{\omega} N \dot{\omega}} - \frac{1}{2} g_4 (\omega^2 - a_4). \tag{32}$$

[4] There is no guarantee that this gives an equivalent variational problem, the equivalence must be verified by direct computations. Fortunately, for our case the trick works well.

Analogously, we solve the equation for g_1, $g_1 = \frac{\sqrt{-\dot{x}N\dot{x}}}{mc}$ and substitute this into (32), this gives the "minimal" action

$$S = \int d\tau\left[\sqrt{a_3\dot{\omega}N\dot{\omega}} - mc\sqrt{-\dot{x}N\dot{x}} - \frac{g_4}{2}(\omega^2 - a_4)\right]. \quad (33)$$

This depends only on the transverse parts of the velocities $\dot{x}^\mu$ and $\dot{\omega}^\mu$. The second term from (33) appeared as a Lagrangian of the particle [59,60] inspired by the bag model [61] in hadron physics.

2.3 Local symmetries of the minimal action

Our model is invariant under two local symmetries. For the initial formulation (10) they have been written in Eqs. (12) and (14). Let us see how they look for the minimal action. This is invariant under reparametrization of the lines $x^\mu(\tau)$ and $\omega^\mu(\tau)$ supplemented by a proper transformation of the auxiliary variable $g_4(\tau)$. We use the projectors N and $\tilde{N}$ to decompose an infinitesimal reparametrization as follows:

$$\delta x^\mu = \alpha\dot{x}^\mu = \alpha\tilde{N}\dot{x}^\mu + \alpha N\dot{x}^\mu,$$
$$\delta\omega^\mu = \alpha\dot{\omega}^\mu = \alpha N\dot{\omega}^\mu + \alpha\tilde{N}\dot{\omega}^\mu,$$
$$\delta g_4 = (\alpha g_4)^{\cdot} = \left(\frac{\alpha\sqrt{a_3\dot{\omega}N\dot{\omega}}}{\omega^2}\right)^{\cdot} + \left(\alpha g_4 - \alpha\frac{\sqrt{a_3\dot{\omega}N\dot{\omega}}}{\omega^2}\right)^{\cdot}. \quad (34)$$

Our observation is that each projection

$$\delta_\beta x^\mu = \beta\tilde{N}\dot{x}^\mu, \qquad \delta_\beta\omega^\mu = \beta N\dot{\omega}^\mu,$$
$$\delta_\beta g_4 = \left(\beta\frac{\sqrt{a_3}\sqrt{\dot{\omega}N\dot{\omega}}}{\omega^2}\right)^{\cdot}. \quad (35)$$
$$\delta_\gamma x^\mu = \gamma N\dot{x}^\mu, \qquad \delta_\gamma\omega^\mu = \gamma\tilde{N}\dot{\omega}^\mu,$$
$$\delta_\gamma g_4 = \left(\gamma g_4 - \gamma\frac{\sqrt{a_3}\sqrt{\dot{\omega}N\dot{\omega}}}{\omega^2}\right)^{\cdot} \quad (36)$$

separately turns out to be a symmetry of the minimal action. It can be verified using the intermediate expressions

$$\delta_\beta\omega^2 = 0, \quad \delta_\beta\sqrt{-\dot{x}N\dot{x}} = 0,$$
$$\delta_\beta N^{\mu\nu} = -\frac{\beta}{\omega^2}((N\dot{\omega})^\mu\omega^\nu + (\mu \leftrightarrow \nu)),$$
$$\delta_\beta\sqrt{\dot{\omega}N\dot{\omega}} = \left(\beta\sqrt{\dot{\omega}N\dot{\omega}}\right)^{\cdot} - (\omega^2)^{\cdot}\frac{\beta\sqrt{\dot{\omega}N\dot{\omega}}}{2\omega^2},$$
$$\delta_\gamma\omega^2 = \gamma(\omega^2)^{\cdot}, \quad \delta_\gamma N^{\mu\nu} = 0,$$
$$\delta_\gamma\sqrt{-\dot{x}N\dot{x}} = (\gamma\sqrt{-\dot{x}N\dot{x}})^{\cdot},$$
$$\delta_\gamma\sqrt{\dot{\omega}N\dot{\omega}} = (\omega^2)^{\cdot}\frac{\gamma\sqrt{\dot{\omega}N\dot{\omega}}}{2\omega^2}.$$

Any pair among the transformations (34)–(36) can be taken as independent symmetries of the minimal action.

Let the functions $x(\tau)$, $\omega(\tau)$, $g_4(\tau)$ represent a solution to equations of motion. Then they obey $(\omega\dot{\omega}) = (\omega\dot{x}) = 0$ and $g_4 = \frac{\sqrt{a_3}\sqrt{\dot{\omega}N\dot{\omega}}}{\omega^2}$. Using this expressions, the transformations (35) and (36) acquire the form

$$\delta_\beta x^\mu = 0, \quad \delta_\beta\omega^\mu = \beta\dot{\omega}^\mu, \quad \delta_\beta g_4 = (\beta g_4)^{\cdot}. \quad (37)$$
$$\delta_\gamma x^\mu = \gamma\dot{x}^\mu, \quad \delta_\gamma\omega^\mu = 0, \quad \delta_\gamma g_4 = 0. \quad (38)$$

Hence on the true trajectories the symmetries have a simple meaning. The γ-transformations (38) represent reparametrizations of the configuration-space trajectory x^μ, whereas the β-transformations (37) represent reparametrizations of the inner-space trajectory ω^μ. Their sum gives the standard reparametrization transformation of the theory, Eq. (34).

3 Minimal action in the physical-time parametrization

3.1 Position's non-commutativity due to spin

Using reparametrization invariance of the Lagrangian (33), we take the physical time as the evolution parameter, $\tau = t$. Now we work with the physical dynamical variables $x^\mu = (ct, \mathbf{x}(t))$ and $\omega^\mu = (\omega^0(t), \boldsymbol{\omega}(t))$ in the expression (33). In this section the dot means a derivative with respect to t, $\dot{x}^\mu = (c, \frac{d\mathbf{x}}{dt})$ and so on. Let us construct the Hamiltonian formulation of the model (33).

Computing the conjugate momenta, we obtain the primary constraint $\pi_{g4} = 0$, and the expressions

$$p_i = mc\frac{N\dot{x}^i}{\sqrt{-\dot{x}N\dot{x}}}, \quad (39)$$
$$\pi^\mu = \sqrt{a_3}\frac{N\dot{\omega}^\mu}{\sqrt{\dot{\omega}N\dot{\omega}}}. \quad (40)$$

Comparing expressions for $\mathbf{p}^2$ and $\mathbf{p}\boldsymbol{\omega}$, after tedious computations we obtain the equality which does not involve the time derivative, $\mathbf{p}^2 + (mc)^2 = (\frac{\mathbf{p}\boldsymbol{\omega}}{\omega^0})^2$. Hence Eq. (39) implies the constraint

$$-\sqrt{\mathbf{p}^2 + (mc)^2}\omega^0 + \mathbf{p}\boldsymbol{\omega} = 0.$$

This is the analog of the covariant constraint $p^\mu\omega_\mu = 0$. Equation (40) together with Eq. (30) implies more primary constraints, $\omega\pi = 0$, $\pi^2 - a_3 = 0$. Computing the Hamiltonian, $P\dot{Q} - L + \lambda_a\Phi_a$, we obtain

$$H = c\sqrt{\mathbf{p}^2 + (mc)^2} + \lambda_3(\pi^2 - a_3) + \frac{1}{2}g_4(\omega^2 - a_4)$$
$$+ \lambda_5(\omega\pi) + \lambda_6(-\sqrt{\mathbf{p}^2 + (mc)^2}\omega^0 + \mathbf{p}\boldsymbol{\omega}) + \lambda_4\pi_{g4}. \quad (41)$$

Preservation in time of the primary constraints implies the following chains of algebraic consequences:

$$\pi_{g4} = 0, \Rightarrow \omega^2 - a_4 = 0, \Rightarrow \lambda_5 = 0.$$
$$(\omega\pi) = 0, \Rightarrow \lambda_3 = \frac{a_4}{2a_3}g_4.$$

$$-\sqrt{\mathbf{p}^2+(mc)^2}\omega^0+\mathbf{p}\boldsymbol{\omega}=0,\ \Rightarrow$$
$$-\sqrt{\mathbf{p}^2+(mc)^2}\pi^0+\mathbf{p}\boldsymbol{\pi}=0,\ \Rightarrow\ \lambda_6=0.$$

Three Lagrangian multipliers have been determined in the process, $\lambda_5=\lambda_6=0$ and $\lambda_3=\frac{a_4}{2a_3}g_4$, whereas λ_1 and λ_4 remain arbitrary functions. For the use of the latter, let us denote

$$p^0\equiv\sqrt{(mc)^2+\mathbf{p}^2},\qquad g^{\mu\nu}\equiv\eta^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}. \tag{42}$$

Besides the constraints, the action implies the Hamiltonian equations

$$\frac{\mathrm{d}x^i}{\mathrm{d}t}=c\frac{p^i}{p^0},\quad \frac{\mathrm{d}p^i}{\mathrm{d}t}=0, \tag{43}$$
$$\dot g_4=\lambda_4,\quad \dot\pi_{g4}=0; \tag{44}$$
$$\dot\omega^\mu=\frac{a_4}{a_3}g_4\pi^\mu,\quad \dot\pi^\mu=-g_4\omega^\mu. \tag{45}$$

Equations (43) describe a free-moving particle with a speed less than the speed of light,

$$x^i=x^i_0+v^it,\quad v^i=c\frac{p^i}{\sqrt{(mc)^2+\mathbf{p}^2}},\quad p^i=\text{const}. \tag{46}$$

The spin-sector variables have ambiguous evolution, because a general solution to (45) depends on the arbitrary function g_4. So they do not represent the observable quantities. As candidates for the physical variables of spin-sector, we can take either the Frenkel spin-tensor,

$$\frac{\mathrm{d}J^{\mu\nu}}{\mathrm{d}t}=0,\quad J^{\mu\nu}p_\nu=0,\quad J^2=6\hbar^2, \tag{47}$$

or, equivalently, the BMT vector

$$\frac{\mathrm{d}s^\mu}{\mathrm{d}t}=0,\quad s^\mu p_\nu=0,\quad s^2=\frac{3\hbar^2}{4}. \tag{48}$$

The constraints $\pi^2-a_3=0$ and $\pi_{g4}=0$ belong to first-class, the other form the second-class set. To take the latter into account, we construct the corresponding Dirac bracket. The nonvanishing Dirac brackets are

$$\{x^i,x^j\}_D=\frac{\epsilon^{ijk}s_k}{mcp^0},\quad \{x^i,p^j\}_D=\delta^{ij}, \tag{49}$$
$$\{p^i,p^j\}_D=0, \tag{50}$$
$$\{J^{\mu\nu},J^{\alpha\beta}\}_D=2\left(g^{\alpha[\mu}J^{\nu]\beta}-g^{\beta[\mu}J^{\nu]\alpha}\right), \tag{51}$$
$$\{x^\mu,J^{\alpha\beta}\}_D=\frac{1}{(mc)^2}\left(J^{\mu[\beta}p^{\alpha]}-\frac{p^\mu}{p^0}J^{0[\beta}p^{\alpha]}\right), \tag{52}$$
$$\{s^i,s^j\}_D=\frac{p^0}{mc}\epsilon^{ijk}\left(s_k-\frac{(\mathbf{s}\,\mathbf{p})p_k}{p_0^2}\right), \tag{53}$$
$$\{x^i,s^j\}_D=\left(s^i-\frac{(\mathbf{s}\,\mathbf{p})p^i}{p_0^2}\right)\frac{p^j}{(mc)^2}, \tag{54}$$

where p^0 and $g^{\mu\nu}$ have been specified in (42). After the transition to the Dirac brackets the second-class constraints can be used as strong equalities. In particular, we can present s^0 in terms of independent variables,

$$s^0=\frac{(\mathbf{s}\,\mathbf{p})}{\sqrt{\mathbf{p}^2+(mc)^2}},$$

and in the expression for Hamiltonian (41) only the first and second terms survive. Besides, we omit the second term, as it does not contribute to the equations for the spin-plane invariant variables. In the result, we obtain the physical Hamiltonian

$$H_{ph}=c\sqrt{\mathbf{p}^2+(mc)^2}. \tag{55}$$

As it should be, Eqs. (43), (47) and (48) follow from the physical Hamiltonian with use of the Dirac bracket, $\dot Q=\{Q,H_{ph}\}_D$.

3.2 Operators of physical observables: F-BMT electron chooses Pryce's (d)-type spin and position

Both operators (except $\hat p_i$) and the abstract state vectors of the physical-time formalism we denote by capital letters, $\hat Q$, $\Psi(t,\mathbf{x})$. In order to quantize the model, the classical Dirac-bracket algebra should be realized by operators, $[\hat Q_1,\hat Q_2]=i\hbar\,\{Q_1,Q_2\}_D|_{Q_i\to\hat Q_i}$. To start with, we look for classical variables which have canonical Dirac brackets, thus simplifying the quantization procedure. Consider the spin variables $\tilde s_j$ defined by the following transformation:

$$\tilde s_j=\left(\delta_{jk}-\frac{p_jp_k}{p^0(p^0+mc)}\right)s^k,$$
$$s_j=\left(\delta_{jk}+\frac{p_jp_k}{mc(p^0+mc)}\right)\tilde s^k.$$

The vector $\tilde{\mathbf{s}}$ is nothing but the spin in the rest frame. Its components have the following Dirac brackets:

$$\{\tilde s^i,\tilde s^j\}_D=\epsilon^{ijk}\tilde s_k,$$
$$\{x^i,\tilde s^j\}_D=\frac{1}{mc(p^0+mc)}\left(\tilde s^ip^j-\delta^{ij}(\mathbf{p}\,\tilde{\mathbf{s}})\right). \tag{56}$$

The last equation together with the following Dirac bracket: $\{\epsilon^{ikm}\tilde s_kp_m,\tilde s^j\}=\tilde s^ip^j-\delta^{ij}(\mathbf{p}\,\tilde{\mathbf{s}})$, suggests one to consider the variables

$$\tilde x^j=x^j-\frac{1}{mc(p^0+mc)}\epsilon^{jkm}\tilde s_kp_m. \tag{57}$$

The canonical variables $\tilde x^j$, p_i, and $\tilde S^j$ have a simple algebra

$$\{\tilde x^j,\tilde x^i\}_D=0,\qquad \{\tilde x^i,p^j\}_D=\delta^{ij}, \tag{58}$$
$$\{\tilde x^j,\tilde s^i\}_D=0,\qquad \{\tilde s^i,\tilde s^j\}_D=\epsilon^{ijk}\tilde s_k. \tag{59}$$

Besides, the constraints (48) on s^μ imply $\tilde{\mathbf{s}}^2=\frac{3}{4}\hbar^2$. So the corresponding operators $\hat{\tilde S}^j$ should realize an irreducible

Table 1 Position/spin operators for the relativistic electron [36]

$\beta=\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \alpha^i=\begin{pmatrix}0&\sigma^i\\\sigma^i&0\end{pmatrix}, \Sigma^i=\begin{pmatrix}\sigma^i&0\\0&\sigma^i\end{pmatrix}$

	Dirac representation, $i\hbar\partial_t\Psi_D=c(\alpha^i p_i+mc\beta)\Psi_D$	F–W representation, $i\hbar\partial_t\Psi=c\beta\hat{p}^0\Psi$	Classical model
$\hat{X}^j_{P(d)}$	$x^j+\frac{i\hbar}{2mc}\beta\left(\alpha^j-\frac{\alpha^k\hat{p}_k\hat{p}^j}{(\hat{p}^0)^2}\right)$	$x^j-\frac{\hbar\epsilon^{jkm}\hat{p}_k\Sigma_m}{2mc(\hat{p}^0+mc)}$	Position x^j
$\hat{S}^j_{P(d)}$	$\frac{1}{2m^2c^2}\left(m^2c^2\Sigma^j-imc\beta\epsilon^{jkl}\alpha_k\hat{p}_l\right)$	$\frac{\hbar}{2mc}\beta\left(\hat{p}^0\Sigma^j-\frac{\hat{p}^k\Sigma_k\hat{p}^j}{(\hat{p}^0+mc)}\right)$	Frenkel spin S^j
$\hat{X}^j_{P(e)}=\hat{x}^j_{\rm FW}$	$x^j+\frac{\hbar}{2\hat{p}^0}\left(i\beta\alpha^j+\frac{\epsilon^{jkm}\hat{p}_k\Sigma_m}{\hat{p}^0+mc}-\frac{i\beta\alpha^k\hat{p}_k\hat{p}^j}{\hat{p}^0(\hat{p}^0+mc)}\right)$	x^j	$\tilde{x}^j$
$\hat{S}^j_{P(e)}=\hat{S}^j_{\rm FW}$	$\frac{\hbar}{2\hat{p}^0}\left(mc\Sigma^j-im\beta\epsilon^{jkl}\alpha_k\hat{p}_l+\frac{\Sigma^k\hat{p}_k\hat{p}^j}{\hat{p}^0+mc}\right)$	$\frac{\hbar}{2}\Sigma^j$	$\tilde{s}^j$
$\hat{X}^j_{P(c)}$	$x^j+\frac{\hbar}{2(\hat{p}^0)^2}\left(\epsilon^{jkm}\hat{p}_k\Sigma_m+imc\beta\alpha^j\right)$	$x^j+\frac{\hbar\epsilon^{jkm}\hat{p}_k\Sigma_m}{2\hat{p}^0(\hat{p}^0+mc)}$	
$\hat{S}^j_{P(c)}$	$\frac{\hbar}{2(\hat{p}^0)^2}\left(m^2c^2\Sigma^j-imc\beta\epsilon^{jkl}\alpha_k\hat{p}_l+\Sigma^k\hat{p}_k\hat{p}^j\right)$	$\frac{\hbar}{2\hat{p}^0}\beta\left(mc\Sigma^j+\frac{\hat{p}^k\Sigma_k\hat{p}^j}{(\hat{p}^0+mc)}\right)$	$s^j_{\rm BMT}$

representation of $SO(3)$ with spin $s=1/2$. The quantization in terms of these variables becomes straightforward. The Hilbert space consists of two-component functions $\Psi_a(t,\mathbf{x})$, $a=1,2$. A realization of the Dirac-brackets algebra by operators has the standard form

$$p_j\to\hat{p}_j=-i\hbar\partial_j,$$
$$\tilde{x}^j\to\hat{\tilde{X}}^j=x^j,$$
$$\tilde{s}^j_{\rm BMT}\to\hat{\tilde{S}}^j_{\rm BMT}=\frac{\hbar}{2}\sigma^j.$$

The conversion formulas between canonical and initial variables have no ordering ambiguities, so we immediately obtain the operators corresponding to the physical position and spin of the classical theory,

$$x^i\to\hat{X}^i=x^i-\frac{\hbar}{2mc(\hat{p}^0+mc)}\epsilon^{ijk}\hat{p}_j\sigma_k, \tag{60}$$

$$\hat{J}^{0i}=-\frac{\hbar}{mc}\epsilon^{ijk}\hat{p}_j\sigma_k, \tag{61}$$

$$\hat{J}^{ij}=\frac{\hbar}{mc}\epsilon^{ijk}\left(\hat{p}^0\sigma_k-\frac{1}{(\hat{p}^0+mc)}(\hat{\mathbf{p}}\boldsymbol{\sigma})\hat{p}_k\right), \tag{62}$$

$$\hat{S}^i=\frac{1}{4}\epsilon^{ijk}\hat{J}_{jk}=\frac{\hbar}{2mc}\left(-\hat{p}^0\sigma^i-\frac{1}{(\hat{p}^0+mc)}(\hat{\mathbf{p}}\boldsymbol{\sigma})\hat{p}^i\right). \tag{63}$$

The BMT operator reads

$$\hat{S}^0_{\rm BMT}=-\frac{\hbar}{2mc}(\hat{\mathbf{p}}\boldsymbol{\sigma}), \tag{64}$$

$$\hat{S}^j_{\rm BMT}=\frac{\hbar}{2}\left(\sigma^j+\frac{1}{mc(\hat{p}^0+mc)}(\hat{\mathbf{p}}\boldsymbol{\sigma})\hat{p}^j\right). \tag{65}$$

The energy operator (55) determines the evolution of a state vector by the Schrödinger equation

$$i\hbar\frac{{\rm d}\Psi}{{\rm d}t}=c\sqrt{\hat{\mathbf{p}}^2+(mc)^2}\Psi, \tag{66}$$

as well the evolution of operators by Heisenberg equations. The scalar product can be defined as follows:

$$\langle\Psi,\Phi\rangle=\int d^3x\Psi^\dagger\Phi. \tag{67}$$

By construction, the abstract vector $\Psi(t,\mathbf{x})$ of the Hilbert space can be identified with the amplitude of the probability density of the canonical coordinate $\tilde{x}^i$. Since our position operators $\hat{x}^i$ are noncommutative, the issue of the wave function requires special discussion, which we postpone for the future.

To compare our operators with those known in the literature, we remind the reader that Pryce [36] wrote his operators acting on the space of Dirac spinor Ψ_D; see the first column in Table 1. Foldy and Wouthuysen [35] found unitary transformation which maps the Dirac equation $i\hbar\partial_t\Psi_D=c(\alpha^i p_i+mc\beta)\Psi_D$ into the pair of square-root equations $i\hbar\partial_t\Psi=c\beta\hat{p}^0\Psi$. Applying the FW transformation, the Pryce operators acquire a block-diagonal form on the space Ψ; see the second column. Our operators act on the space of solutions of square-root equation (66), so we compare them with positive-energy parts (upper-left blocks) of Pryce operators of the second column.

Our operators of canonical variables $\hat{\tilde{X}}^j=x^j$ and $\hat{\tilde{S}}^j$ correspond to the Pryce (e) (∼ Foldy–Wouthuysen ∼ Newton–Wigner) position and spin operators.

However, operators of position x^j and spin S^j of our model are $\hat{X}^j$ and $\hat{S}^j$. They correspond to the Pryce (d)-operators.

Operator of BMT-vector $\hat{S}^j_{\rm BMT}$ is the Pryce (c) spin.

While we have started from relativistic theory (33), working with the physical variables we have lost, from the beginning, the manifest relativistic covariance. Is the quantum mechanics thus obtained a relativistic theory? Below we present a manifestly covariant formalism and confirm that scalar products, mean values, and transition probabilities can be computed in a covariant form.

4 Minimal action in covariant formalism: covariant form of a noncommutative algebra of positions

Obtaining the minimal action (4) we have made various tricks. So, let us confirm that the action indeed leads to the desired constraints (2) and (3). Computing the conjugate momenta we obtain the primary constraint $\pi_{g4}=0$ and the expressions

$$p^\mu = mc\frac{N\dot{x}^\mu}{\sqrt{-\dot{x}N\dot{x}}}, \quad \pi^\mu = \sqrt{a_3}\frac{N\dot{\omega}^\mu}{\sqrt{\dot{\omega}N\dot{\omega}}}.$$

Due to Eq. (30), they imply more primary constraints, $p\omega=0$, $p^2+(mc)^2=0$, $\omega\pi=0$, and $\pi^2-a_3=0$. Computing the Hamiltonian, $P\dot{Q}-L+\lambda_a\Phi_a$, we obtain

$$H=\frac{1}{2}\lambda_1(p^2+m^2c^2)+\lambda_3(\pi^2-a_3)+\frac{1}{2}g_4(\omega^2-a_4) + \lambda_5(\omega\pi)+\lambda_6(p\omega)+\lambda_4\pi_{g4}. \qquad (68)$$

The preservation in time of the primary constraints implies the following chain of algebraic consequences:

$$\pi_{g4}=0, \Rightarrow \omega^2-a_4=0, \Rightarrow \lambda_5=0.$$
$$(\omega\pi)=0, \Rightarrow \lambda_3=\frac{a_4}{2a_3}g_4.$$
$$(p\omega)=0, \Rightarrow (p\pi)=0, \Rightarrow \lambda_6=0.$$

As the result, the minimal action generates all the desired constraints (2) and (3). Three Lagrangian multipliers have been determined in the process, $\lambda_5=\lambda_6=0$ and $\lambda_3=\frac{a_4}{2a_3}g_4$, whereas λ_1 and λ_4 remain as arbitrary functions.

Besides the constraints, the action implies the Hamiltonian equations $\dot{g}_4=\lambda_4$, $\dot{\pi}_{g4}=0$, $\dot{x}^\mu=\lambda_1 p^\mu$, $\dot{p}^\mu=0$, $\dot{\omega}^\mu=\frac{a_4}{a_3}g_4\pi^\mu$, $\dot{\pi}^\mu=-g_4\omega^\mu$. The general solution to these equations in arbitrary and proper-time parameterizations is presented in Appendix 2.

To take into account the second-class constraints T_4, T_5, T_6, and T_7, we pass from Poisson to Dirac bracket. We write them for the spin-plane invariant variables, they are x^μ, p^μ, and either the Frenkel spin-tensor or BMT four-vector (19). The non-vanishing Dirac brackets are as follows.

Spatial sector:

$$\{x^\mu, x^\nu\}=-\frac{1}{2p^2}J^{\mu\nu}, \quad \{x^\mu, p^\nu\}=\eta^{\mu\nu}, \quad \{p^\mu, p^\nu\}=0. \qquad (69)$$

Frenkel sector:

$$\{J^{\mu\nu}, J^{\alpha\beta}\}=2(g^{\mu\alpha}J^{\nu\beta}-g^{\mu\beta}J^{\nu\alpha}-g^{\nu\alpha}J^{\mu\beta}+g^{\nu\beta}J^{\mu\alpha}), \qquad (70)$$

$$\{x^\mu, J^{\alpha\beta}\}=\frac{1}{p^2}J^{\mu[\alpha}p^{\beta]}, \qquad (71)$$

BMT-sector:

$$\{s^\mu, s^\nu\}=-\frac{1}{\sqrt{-p^2}}\epsilon^{\mu\nu\alpha\beta}p_\alpha s_\beta=\frac{1}{2}J^{\mu\nu}, \qquad (72)$$

$$\{x^\mu, s^\nu\}=-\frac{s^\mu p^\nu}{p^2}=-\frac{1}{4\sqrt{-p^2}}\epsilon^{\mu\nu\alpha\beta}J_{\alpha\beta}-\frac{p^\mu s^\nu}{p^2}. \qquad (73)$$

In Eq. (70) we have written $g^\mu{}_\nu\equiv\delta^\mu{}_\nu-\frac{p^\mu p_\nu}{p^2}$. Together with $\tilde{g}^\mu{}_\nu\equiv\frac{p^\mu p_\nu}{p^2}$, this forms a pair of projectors $g+\tilde{g}=1$, $g^2=g$, $\tilde{g}^2=\tilde{g}$, $g\tilde{g}=0$. The transition to spin-plane invariant variables does not spoil manifest covariance. So, we write the equations of motion in terms of these variables:

$$\dot{x}^\mu=\lambda_1 p^\mu, \qquad \dot{p}^\mu>0, \qquad (74)$$
$$\dot{J}^{\mu\nu}=0, \quad J^{\mu\nu}p_\nu=0, \quad J^2=6\hbar^2. \qquad (75)$$
$$\dot{S}^\mu=0, \quad S^\mu p_\nu=0, \quad S^2=\frac{3\hbar^2}{4}. \qquad (76)$$

Besides, we have the first-class constraint

$$p^2+(mc)^2=0, \quad \text{where} \quad p^0>0. \qquad (77)$$

Let us compare these results with non-manifestly covariant formalism of previous section. The evolution of the physical variables can be obtained from Eqs. (74)–(77) assuming that the functions $Q^\mu(\tau)$ represent the physical variables $Q^i(t)$ in the parametric form. Using the formula $\frac{dF}{dt}=c\frac{\dot{F}(\tau)}{\dot{x}^0(\tau)}$, this gives Eqs. (43), (47), and (48). The brackets (49)–(54) of the physical variables appear, if we impose the physical-time gauge $x^0-\tau=0$ for the constraint (77), and pass from (69)–(73) to the Dirac bracket, which takes into account this second-class pair. The physical Hamiltonian (55) can be obtained from (68) considering the physical-time gauge as a canonical transformation [24].

Summarizing, in classical mechanics all basic relations for the physical variables can be obtained from the covariant formalism. In the next section we discuss how far we can proceed toward a formulation of quantum mechanics in a manifestly covariant form.

5 Manifestly covariant form of quantum mechanics of the Frenkel electron

According to Wigner [54–56], with an elementary particle in QFT we associate the Hilbert space of the representation of the Poincaré group. The space can be described in a manifestly covariant form as a space of solutions to the Klein–Gordon (KG) equation for a properly chosen multicomponent field $\psi_i(x^\mu)$. The one-component field corresponds to a spin-zero particle. A two-component field has been considered by Feynman and Gell-Mann [57] to describe the weak interaction of a spin one-half particle, and by Brown as a starting point for QED [58]. It is well known, that the one-component KG field has no quantum-mechanical interpretation. In contrast, the two-component KG equation does admit the probabilistic interpretation: the four-vector (80) represents a positively defined conserved current of this equation. On this base, we consider below the relativistic quantum mechanics of the two-component KG equation and show its equivalence with the quantum mechanics of the Dirac equation. Then we show that the covariantly quantized F-BMT electron corresponds to the positive-energy sector of this quantum mechanics. Finally, we establish the correspondence between canonical and covariant formulations, thus proving relativistic invariance of the physical-time formalism of Sect. 3.2.

5.1 Relativistic quantum mechanics of two-component Klein–Gordon equation

We denote states and operators of the covariant formalism by lower case letters, to distinguish them from the quantities of canonical formalism. Consider the space of abstract state vectors composed by two-component Weyl spinors $\psi_a(x^\mu)$, $a = 1, 2$. The generators of Poincaré transformations in this space read

$$\hat{m}^{\mu\nu} = x^\mu \hat{p}^\nu - x^\nu \hat{p}^\mu + \frac{1}{2}\sigma^{\mu\nu}, \quad \hat{p}_\mu = -i\hbar\partial_\mu, \tag{78}$$

where the Lorentz generators

$$\sigma^{\mu\nu} = -\frac{i\hbar}{2}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu),$$

are built from standard Pauli matrices σ^i combined into the sets

$$\sigma^\mu = (\mathbf{1}, \sigma^i), \qquad \bar{\sigma}^\mu = (-\mathbf{1}, \sigma^i).$$

They are Hermitian and obey $\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu = 2\eta^{\mu\nu}, \bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu = 2\eta^{\mu\nu}$. Further, on the Poincaré-invariant subspace selected by two-component KG equation,

$$(\hat{p}^2 + m^2c^2)\psi = 0, \tag{79}$$

we define an invariant and positive-defined scalar product as follows. The four-vector[5]

$$I^\mu[\psi, \phi] = \frac{1}{m^2c^2}(\bar{\sigma}\hat{p}\psi)^\dagger\sigma^\mu\bar{\sigma}\hat{p}\phi - \psi^\dagger\bar{\sigma}^\mu\phi \tag{80}$$

represents a conserved current of Eq. (79), that is, $\partial_\mu I^\mu = 0$, when ψ and ϕ satisfy Eq. (79). Then the integral

$$(\psi, \phi) = \int_\Omega d\Omega_\mu I^\mu, \qquad d\Omega_\mu = \frac{d^4x}{dx_\mu} \tag{81}$$

does not depend on the choice of a space-like three-dimensional hyperplane Ω (an inertial coordinate system). As a consequence, this does not depend on time. So we can restrict ourselves to the hyperplane Ω defined by the equation $x^0 = \text{const}$, and then

$$(\psi, \phi) = \int d^3x I^0. \tag{82}$$

Besides, this scalar product is positive-defined,[6] since

$$I^0[\psi, \psi] = \frac{1}{m^2c^2}(\bar{\sigma}\hat{p}\psi)^\dagger\bar{\sigma}\hat{p}\psi + \psi^\dagger\psi > 0. \tag{83}$$

So, this can be considered as a probability density of the operator $\hat{\mathbf{x}} = \mathbf{x}$. We point out that the transformation properties of the column ψ are in agreement with this scalar product: if ψ transforms as a (right) Weyl spinor, then I^μ represents a four-vector.

Now we can confirm relativistic invariance of the scalar product (67) of the canonical formalism. The operator $\hat{p}^0$ is Hermitian on the subspace of positive-energy solutions ψ, so we can write

$$\begin{aligned}(\psi, \phi) &= \int d^3x \frac{1}{m^2c^2}(\bar{\sigma}\hat{p}\psi)^\dagger\bar{\sigma}\hat{p}\phi + \psi^\dagger\bar{\phi} \\ &= \int d^3x \left[\left(\frac{1}{mc}\bar{\sigma}\hat{p} + i\right)\psi\right]^\dagger\left(\frac{1}{mc}\bar{\sigma}\hat{p} + i\right)\phi.\end{aligned} \tag{84}$$

This suggests the map $W : \{\psi\} \to \{\Psi\}, \Psi = W\psi$,

$$W = \frac{\bar{\sigma}\hat{p}}{mc} + i, \quad W^{-1} = \frac{1}{2\hat{p}^0}\left(i\sigma\hat{p} - mc\right), \tag{85}$$

which respects the scalar products (67) and (82), and thus proves relativistic invariance of the scalar product $\langle\Psi, \Phi\rangle$,

$$\langle\Psi, \Phi\rangle = (\psi, \phi). \tag{86}$$

We note that the map W is determined up to an isometry, and we can multiply W from the left by an arbitrary unitary operator U, $W \to W' = UW$, $U^\dagger U = 1$. Here † denotes Hermitian conjugation with respect to the scalar product $\langle, \rangle$. The ambiguity in the definition of W can be removed by the polar decomposition of the operator [63]. A bounded operator between Hilbert spaces admits the following factorization: $W = PV$, where $V = (W^\dagger W)^{1/2}$, $P = WV^{-1}$. The positively defined operator $W^\dagger W > 0$ has a unique square root, $(W^\dagger W)^{1/2}$. Moreover, $W^\dagger W = W'^\dagger W'$, therefore V defines a map from $\{\psi\}$ to $\{\Psi\}$ without ambiguity. We present the explicit form of V in Sect. 5.4.

[5] † denotes the usual Hermitian conjugation, $\hat{a}^\dagger = (\hat{a}^*)^T$, $(\hat{a}\hat{b})^\dagger = \hat{b}^\dagger\hat{a}^\dagger$, then $(\hat{p}^\mu f)^\dagger = -\hat{p}^\mu(f)^\dagger$.

[6] See also a detailed discussion of positively defined scalar products for the Klein–Gordon-type equations [62].

5.2 Relation with Dirac equation

Here we demonstrate the equivalence of the quantum mechanics of the KG and the Dirac equations. To this aim, let us replace two equations of second order, (79), by an equivalent system of four equations of the first order. To achieve this, with the aid of the identity $\hat{p}^\mu \hat{p}_\mu = \sigma^\mu \hat{p}_\mu \bar{\sigma}^\nu \hat{p}_\nu$, we represent (79) in the form

$$\sigma^\mu \hat{p}_\mu \bar{\sigma}^\nu \hat{p}_\nu \psi + m^2c^2\psi = 0. \tag{87}$$

Consider an auxiliary two-component function $\bar{\xi}$ (Weyl spinor of opposite chirality), and define the evolution of ψ and $\bar{\xi}$ according to the equations[7]

$$\sigma^\mu \hat{p}_\mu (\bar{\sigma}^\nu \hat{p}_\nu)\psi + m^2c^2\psi = 0, \tag{88}$$
$$(\bar{\sigma}^\nu \hat{p}_\nu)\psi - mc\bar{\xi} = 0. \tag{89}$$

That is, the dynamics of ψ is determined by (87), while $\bar{\xi}$ accompanies ψ: $\bar{\xi}$ is determined from the well-known ψ taking its derivative, $\bar{\xi} = \frac{1}{mc}(\bar{\sigma}\hat{p})\psi$. Evidently, the systems (79) and (88), (89) are equivalent. Rewriting the system (88), (89) in a more symmetric form, we recognize the Dirac equation

$$\begin{pmatrix} 0 & \sigma^\mu \hat{p}_\mu \\ -\bar{\sigma}^\nu \hat{p}_\nu & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \bar{\xi} \end{pmatrix} + mc \begin{pmatrix} \psi \\ \bar{\xi} \end{pmatrix} = 0,$$
$$(\gamma^\mu_W \hat{p}_\mu + mc)\Psi = 0, \tag{90}$$

for the Dirac spinor $\Psi = (\psi, \bar{\xi})$ in the Weyl representation of γ-matrices

$$\gamma^0_W = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \qquad \gamma^i_W = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

This gives one-to-one correspondence among two spaces. With each solution ψ to the KG equation we associate the solution

$$\Psi[\psi] = \begin{pmatrix} \psi \\ \frac{1}{mc}(\bar{\sigma}\hat{p})\psi \end{pmatrix}$$

to the Dirac equation. Below we also use the Dirac representation of the γ-matrices

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{91}$$

In this representation, the Dirac spinor corresponding to ψ reads

$$\Psi_D[\psi] = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ -\mathbf{1} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \psi \\ \frac{1}{mc}(\bar{\sigma}\hat{p})\psi \end{pmatrix} = \frac{1}{\sqrt{2}mc} \begin{pmatrix} [(\bar{\sigma}\hat{p}) + mc]\psi \\ [(\bar{\sigma}\hat{p}) - mc]\psi \end{pmatrix}. \tag{92}$$

The conserved current (80) of the KG equation (79), after being rewritten in terms of the Dirac spinor, coincides with the Dirac current

$$I^\mu[\psi_1, \psi_2] = \bar{\Psi}[\psi_1]\gamma^\mu \Psi[\psi_2]. \tag{93}$$

Therefore, the scalar product (81) coincides with that of Dirac.

5.3 Covariant operators of F-BMT electron

In a covariant scheme, we need to construct operators $\hat{x}^\mu$, $\hat{p}^\mu$, $\hat{j}^{\mu\nu}$, $\hat{s}^\mu_{\text{BMT}}$ whose commutators,

$$[\hat{q}_1, \hat{q}_2] = i\hbar\, \{q_1, q_2\}_D|_{q_i \to \hat{q}_i}\,, \tag{94}$$

are defined by the Dirac brackets (69)–(73). Inspection of the classical equations $S^2 = \frac{3\hbar^2}{4}$ and $p^2 + (mc)^2 = 0$ suggests that we can look for a realization of the operators in the Hilbert space constructed in Sect. 5.1.

With the spin-sector variables we associate the operators

$$s^\mu_{\text{BMT}} \to \hat{s}^\mu_{\text{BMT}} = \frac{1}{4\sqrt{-\hat{p}^2}} \epsilon^{\mu\nu\alpha\beta} \hat{p}_\nu \sigma_{\alpha\beta}, \tag{95}$$

$$J^{\mu\nu} \to \hat{j}^{\mu\nu} \equiv -\frac{2}{\sqrt{-\hat{p}^2}} \epsilon^{\mu\nu\alpha\beta} \hat{p}_\alpha \hat{s}_{\text{BMT}\beta} = \sigma^{\mu\nu} + \frac{\hat{p}^\mu(\sigma\hat{p})^\nu - \hat{p}^\nu(\sigma\hat{p})^\mu}{\hat{p}^2}. \tag{96}$$

They obey the desired commutators (94), (72), (70). To find the position operator, we separate the inner angular momentum $\hat{j}^{\mu\nu}$ in the expression (78) of the Poincaré generator

$$\hat{m}^{\mu\nu} = \left[x^\mu + \frac{(\sigma\hat{p})^\mu}{2\hat{p}^2}\right]\hat{p}^\nu - \left[x^\nu + \frac{(\sigma\hat{p})^\nu}{2p^2}\right]\hat{p}^\mu + \frac{1}{2}\hat{j}^{\mu\nu}. \tag{97}$$

This suggests the operator of "relativistic position"[8]

$$x^\mu \to \hat{x}^\mu_{rp} = \hat{x}^\mu + \frac{1}{2\hat{p}^2}(\sigma\hat{p})^\mu, \tag{98}$$

where $\hat{x}^\mu\psi = x^\mu\psi$. The operators $\hat{p}_\mu = -i\hbar\partial_\mu$, (95), (96), and (98) obey the algebra (94), (69)–(73).

Equation (76) in this realization states that the square of the second Casimir operator of the Poincaré group has the fixed value $\frac{3\hbar^2}{4}$, and in the representation chosen is satisfied identically. Equations (77) just state that we work in the positive-energy subspace of the Hilbert space of KG equation (79).

We thus completed our covariant quantization procedure by matching the classical variables of the reparametrization-invariant formulation to operators acting on the Hilbert space

[7] Note that $\bar{\xi}$ can be considered as the conjugated momentum for ψ, then the passage from (87) to (90) is just the passage from the Lagrangian to the Hamiltonian formulation. A similar interpretation can be developed for the Schrödinger equation; see [64].

[8] The classical analog of this operator also appeared as a gauge-invariant variable in a mechanical model of the Dirac equation; see [12].

of two-component spinors with the scalar product (81). The construction presented is manifestly Poincaré covariant. In the next Sect. we discuss the connection between canonical and manifestly covariant formulations of the F-BMT electron.

5.4 Relativistic invariance of canonical formalism

The relativistic invariance of the scalar product (67) has already been shown in Sect. 5.1. Here we show how the covariant formalism can be used to compute the mean values and probability rates of the canonical formulation, thus proving its relativistic covariance. Namely, we confirm the following.

Proposition *The Hilbert space of the canonical formulation is*

$$H^+_{can} = \left\{ \Psi(t,\mathbf{x})\,;\, i\hbar\frac{d\Psi}{dt} = \sqrt{\hat{\mathbf{p}}^2+(mc)^2}\Psi, \right.$$
$$\left. \times\ \langle\Psi,\Phi\rangle = \int d^3x\Psi^\dagger\Phi \right\} \tag{99}$$

and

$$H_{\rm cov} = \left\{ \psi(x^\mu)\,;\,(\hat{p}^2+m^2c^2)\psi=0, \right.$$
$$\left. \times\ (\psi,\phi) = \int_\Omega d\Omega_\mu I^\mu[\psi,\phi] \right\} \tag{100}$$

is the Hilbert space of the two-component KG equation.

With a state vector Ψ we associate ψ as follows:

$$\psi = V^{-1}\Psi, \quad V^{-1} = \frac{1}{2\sqrt{\hat{p}^0(\hat{p}^0+mc)}}[mc-\sigma\hat{p}]. \tag{101}$$

Then $\langle\Psi,\Phi\rangle = (\psi,\phi)$. Besides, the mean values of the physical position and spin operators (60)–(63) can be computed as follows:

$$\langle\Psi,\hat{X}^i\Phi\rangle = {\rm Re}(\psi,\hat{x}^i_{rp}\phi), \qquad \langle\Psi,\hat{J}^{ij}\Phi\rangle = (\psi,\hat{j}^{ij}\phi),$$
$$\langle\Psi,\hat{S}^i\Phi\rangle = \frac{1}{4}\epsilon^{ijk}(\psi,\hat{j}^{jk}\phi),$$

where $\hat{x}^i_{rp}$ and $\hat{j}^{ij}$ are the spatial components of the manifestly covariant operators

$$\hat{x}^\mu_{rp} = \hat{x}^\mu + \frac{(\sigma\hat{p})^\mu}{2\hat{p}^2}, \quad \hat{j}^{\mu\nu} = \sigma^{\mu\nu} + \frac{\hat{p}^\mu(\sigma\hat{p})^\nu - \hat{p}^\nu(\sigma\hat{p})^\mu}{\hat{p}^2}.$$

We also show that the map V can be identified with the Foldy–Wouthuysen transformation applied to the Dirac spinor (92).

It will be convenient to work in the momentum representation, $\psi(x^\mu) = \int d^4p\psi(p^\mu)e^{\frac{i}{\hbar}px}$. The transition to the momentum representation implies the substitution

$$\hat{p}_\mu \to p_\mu, \qquad \hat{x}_\mu \to i\hbar\frac{\partial}{\partial p^\mu},$$

in the expressions of covariant operators (95), (96), (98), and so on.

An arbitrary solution to the KG equation reads

$$\psi(t,\mathbf{x}) = \int d^3p\left(\psi(\mathbf{p})e^{\frac{i\omega_p x^0}{\hbar}} + \psi_-(\mathbf{p})e^{\frac{-i\omega_p x^0}{\hbar}}\right)e^{-\frac{i(\mathbf{px})}{\hbar}},$$
$$\omega_p \equiv \sqrt{\mathbf{p}^2+(mc)^2},$$

where $\psi(\mathbf{p})$ and $\psi_-(\mathbf{p})$ are arbitrary functions of three-momentum, they correspond to positive- and negative-energy solutions. The scalar product can then be written as follows:

$$(\psi,\phi) = 2\int\frac{d^3p\,\omega_p}{m^2c^2}\left[\psi^\dagger(\bar{\sigma}p)\phi - \psi^\dagger_-(\sigma p)\phi_-\right],$$

where

$$(\bar{\sigma}p) = \omega_p + (\boldsymbol{\sigma}\mathbf{p}), \quad (\sigma p) = -\omega_p + (\boldsymbol{\sigma}\mathbf{p}).$$

We see that this scalar product separates positive and negative energy parts of the state vectors. Since our classical theory contains only positive energies, we restrict our further considerations to the positive-energy solutions only. In the result, in the momentum representation the scalar product (82) reads in terms of the non-trivial metric ρ

$$(\psi,\phi) = \int d^3p\psi^\dagger\rho\phi, \qquad \rho = \frac{2\omega_p}{m^2c^2}(\bar{\sigma}p). \tag{102}$$

Now our basic space is composed of arbitrary functions $\psi(\mathbf{p})$. The operators $\hat{x}^i$, $\hat{s}^\mu$ and $\hat{j}^{\mu\nu}$ act on this space as before, with the only modification that $\hat{p}^0\psi(\mathbf{p}) = \omega_p\psi(\mathbf{p})$. The operator $\hat{x}^0$ and, as a consequence, the operator $\hat{x}^0_{rp}$, do not act in this space. Fortunately, they are not necessary to prove the proposition formulated above.

Given the operator $\hat{A}$ we denote its Hermitian conjugate in the space $H^+_{\rm can}$ as $\hat{A}^\dagger$. Hermitian operators in the space $H^+_{\rm can}$ have both real eigenvalues and expectation values. Consider an operator $\hat{a}$ in the space $H_{\rm cov}$ with real expectation values $(\psi,\hat{a}\psi) = (\psi,\hat{a}\psi)^*$. It should obey $\hat{a}^\dagger\rho = \rho\hat{a}$. That is, such an operator in $H_{\rm cov}$ should be pseudo-Hermitian. We denote pseudo-Hermitian conjugation in $H_{\rm cov}$ as follows: $\hat{a}_c = \rho^{-1}\hat{a}^\dagger\rho$. Then the pseudo-Hermitian part of an operator $\hat{a}$ is given by $\frac{1}{2}(\hat{a}+\hat{a}_c)$.

Let us check the pseudo-Hermicity properties of the basic operators. From the following identities:

$$(\sigma^{\mu\nu})^\dagger\rho = \rho\left(\sigma^{\mu\nu} + \frac{2i\hbar}{p^2}(\sigma p)(p^\mu\bar{\sigma}^\nu - p^\nu\bar{\sigma}^\mu)\right),$$
$$(\sigma^{\mu\nu}p_\nu)^\dagger\rho = \rho\left(\sigma^{\mu\nu}p_\nu + 2i\hbar[p^\mu - (\sigma p)\bar{\sigma}^\mu]\right),$$
$$(\hat{x}^j_{rp})^\dagger\rho = \rho\left(\hat{x}^j_{rp} + \frac{i\hbar}{m^2c^2\omega_p}\left[\frac{m^2c^2}{\omega_p}p^j - p^j(\boldsymbol{\sigma}\mathbf{p})\right]\right),$$

we see that operators $\sigma^{\mu\nu}$ and $\hat{x}^j_{rp}$ are non-pseudo-Hermitian, while the operators $\hat{p}^\mu$, $\hat{s}^\mu$, $\hat{j}^{\mu\nu}$ and orbital part of $\hat{m}^{ij}$ are pseudo-Hermitian.

To construct the map (101) we look for the square root of the metric, $V=\rho^{1/2}$. The metric ρ is positively defined, therefore the square root is unique [63], and it reads

$$V=\frac{1}{mc}\sqrt{\frac{\omega_p}{\omega_p+mc}}[(\bar{\sigma}p)+mc]. \tag{103}$$

We use this to define the map $H_{\rm cov}\to H^+_{\rm can}, \Psi=V\psi$, which corresponds to the polar decomposition of the map W defined in (85). Then the scalar product (102) can be rewritten as

$$(\psi,\phi)=\int d^3p(V\psi)^\dagger V\phi=\int d^3p\Psi^\dagger\Phi=\langle\Psi,\Phi\rangle.$$

This proves the relativistic invariance of the scalar product $\langle\Psi,\Phi\rangle$ of the canonical formalism.

Our map defined by the operator V turns out to be in a close relation with the Foldy–Wouthuysen transformation. It can be seen applying the Foldy–Wouthuysen unitary transformation

$$U_{\rm FW}=\frac{\omega_p+mc+(\boldsymbol{\gamma}\mathbf{p})}{\sqrt{2(\omega_p+mc)\omega_p}}$$

to the Dirac spinor $\Psi_D[\psi]$,

$$\Psi_{\rm FW}[\psi]=U_{\rm FW}\Psi_D[\psi]=\begin{pmatrix}V\psi\\0\end{pmatrix}=\begin{pmatrix}\Psi\\0\end{pmatrix}.$$

The last equation means that operator V is a restriction of operator $U_{\rm FW}$ to the space of positive-energy right Weyl spinors ψ.

The transformation between the state vectors induces the map of operators

$$\hat{Q}=V\hat{q}V^{-1}, \tag{104}$$

where

$$V^{-1}=\frac{1}{2\sqrt{\omega_p(\omega_p+mc)}}[mc-(\sigma p)].$$

Then

$$\langle\Psi,\hat{Q}\Phi\rangle=(\psi,\hat{q}\phi). \tag{105}$$

Due to the Hermicity of V, $V^\dagger=V$, pseudo-Hermitian operators, $\hat{q}^\dagger V^2=V^2\hat{q}$, transform into Hermitian operators $\hat{Q}^\dagger=\hat{Q}$. For an operator $\hat{q}$ which commutes with the momentum operator, the transformation (104) acquires the following form:

$$\hat{Q}=\frac{1}{2}(\hat{q}+\hat{q}^\dagger)-\frac{1}{2(\omega_p+mc)}(\hat{q}-\hat{q}^\dagger)(\boldsymbol{\sigma}\mathbf{p}).$$

Using this formula, we have checked by direct computation that the covariant operators $\hat{\mathbf{p}}$, $\hat{j}^{\mu\nu}$ and $\hat{s}^\mu_{\rm BMT}$ transform into canonical operators $\hat{\mathbf{p}}$, $\hat{J}^{\mu\nu}$ and $\hat{S}^\mu_{\rm BMT}$, so the spatial part of $\hat{J}^{\mu\nu}$, $\hat{S}^i=\frac{1}{4}\epsilon^{ijk}\hat{J}_{jk}$ represents the classical spin S^i. This observation together with Eq. (105) implies that the mean values of the operators of the canonical formalism are relativistic-covariant quantities.

Concerning the position operator, we first apply the inverse to Eq. (104) to our canonical coordinate $\hat{\hat{X}}^i=i\hbar\frac{\partial}{\partial p^i}$ in the momentum representation

$$\begin{aligned}\hat{\hat{x}}^i_V&=V^{-1}\hat{\hat{X}}^iV=\hat{\hat{X}}^i+[V^{-1},\hat{\hat{X}}^i]V\\&=i\hbar\frac{\partial}{\partial p^i}-\frac{i\hbar p^i(\boldsymbol{\sigma}\mathbf{p})}{2mc\omega_p(\omega_p+mc)}+\frac{i\hbar p^i}{2\omega_p}\\&\quad+\frac{i\hbar\sigma^i}{2mc}+\frac{\hbar\epsilon^{ijk}\sigma_jp_k}{2mc(\omega_p+mc)}.\end{aligned}$$

Our position operator then can be mapped as follows:

$$\begin{aligned}\hat{x}^i_V&=V^{-1}\left(i\hbar\frac{\partial}{\partial p^i}+\frac{\epsilon^{ijk}\hat{S}_jP_k}{mc(\omega_p+mc)}\right)V\\&=i\hbar\frac{\partial}{\partial p^i}+\frac{i\hbar p^i(\boldsymbol{\sigma}\mathbf{p})}{2p^2\omega_p}+\frac{i\hbar p^i}{2\omega_p}-\frac{i\hbar\omega_p\sigma^i}{2p^2}+\frac{\hbar\epsilon^{ijk}p_j\sigma_k}{2p^2}.\end{aligned} \tag{106}$$

We note that the pseudo-Hermitian part of the operator $\hat{x}^i_{rp}$ coincides with the image $\hat{x}^i_V$,

$$\hat{x}^i_V=\frac{1}{2}\left(\hat{x}^i_{rp}+\left[\hat{x}^i_{rp}\right]_c\right).$$

Since $\hat{x}^\mu_{rp}$ has an explicitly covariant form, this also proves the covariant character of the position operator $\hat{X}^i$. Indeed, (104) means that the matrix elements of $\hat{X}^i$ are expressed through the real part of the manifestly covariant matrix elements,

$$\langle\Psi,\hat{X}^i\Phi\rangle=(\psi,\hat{x}^i_V\phi)={\rm Re}(\psi,\hat{x}^i_{rp}\phi).$$

In summary, we have proved the proposition formulated above. The operators $\hat{j}^{\mu\nu}$ and $\hat{x}^\mu_{rp}$, which act on the space of thetwo-component KG equation, represent the manifestly covariant form of the Pryce (d)-operators.

Table 2 summarizes the manifest form of the operators of the canonical formalism and their images in the covariant formalism.

5.5 Manifestly covariant operators of spin and position of the Dirac equation

According to Eq. (93), the scalar product (ψ,ϕ) coincides with that of Dirac. This allows us to find the manifestly covariant operators in the Dirac theory which have the same expectation values as $\hat{j}^{\mu\nu}$ and $\hat{x}^\mu_{rp}$. Consider the following analog of $\hat{j}^{\mu\nu}$ on the space of four-component Dirac spinors:

$$\begin{aligned}\hat{j}^{\mu\nu}_D&=\Sigma^{\mu\nu}+\frac{\hat{p}^\mu\Sigma^{\nu\alpha}\hat{p}_\alpha-\hat{p}^\nu\Sigma^{\mu\alpha}\hat{p}_\alpha}{\hat{p}^2}\\&=\Sigma^{\mu\nu}+\frac{i\hbar}{\hat{p}^2}\left(\hat{p}^\mu\gamma^\nu-\hat{p}^\nu\gamma^\mu\right)(\gamma\hat{p}),\end{aligned} \tag{107}$$

where $\Sigma^{\mu\nu}=\frac{i\hbar}{2}(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)$. This definition is independent from a particular representation of γ-matrices. In

Table 2 Operators of canonical and manifestly covariant formulations in momentum representation

	Canonical formalism $\Psi(\mathbf{p})$	Covariant formalism $\psi(\mathbf{p})$
$\hat{p}_j \to \hat{p}_j$	p_j	p_j
$\hat{S}^i \to \hat{s}^i$	$\frac{\hbar}{2mc}\left(\omega_p\sigma^i - \frac{1}{(\omega_p+mc)}(\mathbf{p}\boldsymbol{\sigma})p^i\right)$	$\frac{\hbar\omega_p}{2(mc)^2}\left(\omega_p\sigma^i - (\mathbf{p}\boldsymbol{\sigma})p^i - i\epsilon_{imn}p^m\sigma^n\right)$
$\hat{X}^i \to \hat{x}^i_V$	$i\hbar\frac{\partial}{\partial p^i} - \frac{\hbar}{2mc(\omega_p+mc)}\epsilon^{ijk}p_j\sigma_k$	$i\hbar\frac{\partial}{\partial p^i} + \frac{i\hbar p^i(\boldsymbol{\sigma}\mathbf{p})}{2p^2\omega_p} + \frac{i\hbar p^i}{2\omega_p} - \frac{i\hbar}{2p^2}\omega_p\sigma^i + \frac{\hbar}{2p^2}\epsilon^{ijk}p_j\sigma_k$
$\hat{J}^{ij} \to \hat{j}^{ij}$	$\frac{\hbar}{mc}\epsilon^{ijk}\left(\omega_p\sigma_k - \frac{1}{(\omega_p+mc)}(\mathbf{p}\boldsymbol{\sigma})p_k\right)$	$\frac{\hbar\omega_p}{m^2c^2}\epsilon^{ijk}\left(\omega_p\sigma_k - (\mathbf{p}\boldsymbol{\sigma})p_k - i\epsilon_{kmn}p^m\sigma^n\right)$
$\hat{J}^{0i} \to \hat{j}^{0i}$	$-\frac{\hbar}{mc}\epsilon^{ijk}p_j\sigma_k$	$-\frac{\hbar}{m^2c^2}\epsilon^{ijk}\left(\omega_p\sigma_k - i\epsilon_{kml}p^m\sigma^l\right)p_j$
$\hat{S}^0_{\mathrm{BMT}} \to \hat{s}^0_{\mathrm{BMT}}$	$\frac{\hbar}{2mc}(\mathbf{p}\boldsymbol{\sigma})$	$\frac{\hbar}{2mc}(\mathbf{p}\boldsymbol{\sigma})$
$\hat{S}^i_{\mathrm{BMT}} \to \hat{s}^i_{\mathrm{BMT}}$	$\frac{\hbar}{2}\left(\sigma^i + \frac{1}{mc(\omega_p+mc)}(\mathbf{p}\boldsymbol{\sigma})p^i\right)$	$\frac{\hbar}{2mc}(\omega_p\sigma^i + i\epsilon^{ijk}p_j\sigma_k)$

the representation (91) this reads

$$\Sigma^{\mu\nu} = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & (\sigma^{\mu\nu})^\dagger \end{pmatrix}$$

and can be used to prove the equality of the matrix elements

$$\int d^3x \Psi[\psi]^\dagger \hat{j}^{\mu\nu}_D \Phi[\phi] = (\psi, \hat{j}^{\mu\nu}\phi),$$

for arbitrary solutions ψ, ϕ of the two-component KG equation. The covariant position operator can be defined as follows:

$$\hat{x}^\mu_D = x^\mu + \frac{\Sigma^{\mu\alpha}\hat{p}_\alpha}{2\hat{p}^2} + \frac{i\hbar(\gamma^5-1)\hat{p}^\mu}{2\hat{p}^2}$$
$$= x^\mu + \frac{i\hbar\gamma^\mu}{2\hat{p}^2}(\gamma\hat{p}) + \frac{i\hbar\gamma^5\hat{p}^\mu}{2\hat{p}^2}, \quad (108)$$

where $\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$. Again, one can check that the matrix elements in the two theories coincide,

$$\int d^3x \Psi[\psi]^\dagger \hat{x}^\mu_D \Phi[\phi] = (\psi, \hat{x}^\mu_{rp}\phi).$$

As a result, the manifestly covariant operators $\hat{j}^{\mu\nu}_D$ and $\hat{x}^\mu_D$ of the Dirac equation represent the position **x** and spin **S** (21) of the Frenkel electron (33). Their mean values can be computed as follows:

$$\langle\Psi, \hat{X}^i\Phi\rangle = \frac{1}{2}\mathrm{Re}(\Psi[\psi], [\hat{x}^i_D + \hat{x}^{i\dagger}_D]\Phi[\phi])_D, \quad (109)$$

$$\langle\Psi, \hat{S}^i\Phi\rangle = \frac{1}{4}\epsilon^{ijk}(\Psi[\psi], \hat{j}^{jk}_D\Phi[\phi])_D. \quad (110)$$

6 Conclusions

The content and the main results of this work have been described in the Sect. 1. So, here we finish with some complementary comments.

There are a lot of candidates for the spin and position operators of the relativistic electron. Different position observables coincide when we consider the standard quasi-classical limit. So, in the absence of a systematically constructed classical model of an electron it is difficult to understand the difference between these operators. Our approach allows us to do this, after realizing them at the classical level. As we have seen, various non-covariant, covariant, and manifestly covariant operators acquire clear meaning in the Lagrangian model of the Frenkel electron developed in this work.

Starting with the variational formulation we described the relativistic Frenkel electron with the aid of a singular Lagrangian. The equations of motion for the classical model are consistent [19,20] with experimentally tested BMT equations. We showed that the classical variables of position are non-commutative quantities. Selecting a physical-time parametrization in our model in the case of the free electron, we have performed the canonical quantization procedure. As it should be, we arrived at quantum mechanics, which can be identified with the positive-energy part of the Dirac theory in the Foldy–Wouthuysen representation. The Foldy–Wouthuysen mean-position and spin operators correspond to the canonical variables $\tilde{x}_j$ and $\tilde{s}_j$ of the model, whereas the classical position **x** and spin **S** are represented by Pryce (d)-operators. Since all variables obey the same equations in the free theory, the question of which of them are the true position and spin is a matter of convention. The situation changes in the interacting theory, where namely **x** and **S** obey the expected F-BMT equations and thus represent the position and spin.

Concerning the position, in his pioneer work [36], Pryce noticed that "except the particles of spin 0, it does not seem to be possible to find a definition which is relativistically covariant and at the same time yields commuting coordinates". Now we know why this happens. At the classical level, an accurate account of spin (that is, of the Frenkel condition) in a Lagrangian theory yields, inevitably, relativistic corrections to the classical brackets of the position variables.

It seems to be very interesting to study $\hat{X}^j_{P(d)}$ as the "true" relativistic position operator in more detail. The first reason is an interesting modification of the quantum interaction between the electron and background electromagnetic fields coming from the non-local interactions $\hat{p}^\mu \to \hat{p}^\mu - \frac{e}{c}A^\mu(\hat{X}_j)$, $F^{\mu\nu}(\hat{X}_j)\hat{J}_{\mu\nu}$. The second reason is its natural non-commutativity, which could be contrasted with a number of theoretical models where non-commutativity is introduced by hand. We return to these issues in the next paper [19].

We also quantized our model in an arbitrary parameterization, keeping the manifest Lorentz-invariance. The covariant quantization gives the positive-energy sector of the two-component Klein–Gordon equation (a quantum field theory of two-component KG has been proposed by Feynman and Gell-Mann [57]). We have found a covariant conserved current for the two-component KG equation, which allows us to define an invariant, positive-definite scalar product with metric ρ in the space of two-component spinors. The resulting relativistic quantum mechanics represents the one-particle sector of the Feynman–Gell-Mann quantum field theory. The classical spin-plane invariant variables p^μ, S^μ, and $J^{\mu\nu}$ produce manifestly covariant operators.

The square root of the metric, $V = \rho^{1/2}$, defines the map from canonical to covariant formulations. This allows us to establish the relativistic covariance of the canonical formalism: scalar product and mean values of operators of the canonical formalism can be computed using the corresponding quantities of the covariant formalism; see the proposition of Sect. 5.4. Going back, the transformation V allows us to interpret the results of covariant quantization in terms of one-particle observables of an electron in the FW representation (see Table 2). The relativistic-position operator $\hat{x}^\mu_{rp}$ is non-Hermitian and does not correspond to a physical observable. However, the pseudo-Hermitian part of $\hat{x}^j_{rp}$ coincides with the image of the physical-position operator $\hat{x}^j_V = V^{-1}\hat{X}^i V$.

Our classical model may provide us with a unification in modern issues of quantum observables in various theoretical and experimental setups [31–46]. Since the model constructed admits an interaction with electromagnetic and gravitational fields, one can try to extend the obtained results beyond the free relativistic electron.

Acknowledgments This work has been supported by the Brazilian foundation CNPq. AMPM thanks CAPES for the financial support (Program PNPD/2011).

Appendix 1: Some identities

We have

$$\eta^{\mu\nu} = (-,+,+,+), \quad \epsilon^{0123} = 1, \quad \epsilon_{0123} = -1,$$

$$\epsilon^{abcd}\epsilon_{ab\mu\nu} = -2(\delta^c{}_\mu\delta^d{}_\nu - \delta^c{}_\nu\delta^d{}_\mu).$$
$$\epsilon^{\mu abc}\epsilon_{\mu ijk} = -[\delta^a{}_i(\delta^b{}_j\delta^c{}_k - \delta^b{}_k\delta^c{}_j) + \text{cycle}(ijk)].$$

Given quantities $J^{\mu\nu} = -J^{\nu\mu}$ and p^μ, we define the vectors

$$s^\mu = \frac{1}{4\sqrt{-p^2}}\epsilon^{\mu\nu\alpha\beta}p_\nu J_{\alpha\beta}, \quad \text{then} \quad s^\mu p_\mu = 0, \qquad (111)$$
$$\Phi^\mu = J^{\mu\nu}p_\nu, \qquad \text{then} \qquad \Phi^\mu p_\mu = 0.$$

Then both $J^{\mu\nu}$ and its dual, $^*J^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu ab}J_{ab}$, can be decomposed in terms of these vectors,

$$J^{\mu\nu} = \frac{\Phi^\mu p^\nu - \Phi^\nu p^\mu}{p^2} - \frac{2}{\sqrt{-p^2}}\epsilon^{\mu\nu ab}p_a s_b,$$
$$\epsilon^{\mu\nu ab}J_{ab} = 4\frac{p^\mu s^\nu - p^\nu s^\mu}{\sqrt{-p^2}} - \frac{2}{p^2}\epsilon^{\mu\nu ab}p_a\Phi_b.$$

We have the identity

$$s^\mu s_\mu = -\frac{1}{4p^2}(J^{\mu\nu}p_\nu)^2 + \frac{1}{8}J^{\mu\nu}J_{\mu\nu}.$$

If, in addition to this, $J^{\mu\nu}$ obeys

$$J^{\mu\nu}p_\nu = 0,$$

then $J^{\mu\nu}$ and S^μ turn out to be equivalent:

$$s^\mu = \frac{1}{4\sqrt{-p^2}}\epsilon^{\mu\nu\alpha\beta}p_\nu J_{\alpha\beta}, \qquad J^{\mu\nu} = -\frac{2}{\sqrt{-p^2}}\epsilon^{\mu\nu\alpha\beta}p_\alpha s_\beta,$$

and they obey the identities

$$\epsilon^{\mu\nu ab}J_{ab} = 4\frac{p^\mu s^\nu - p^\nu s^\mu}{\sqrt{-p^2}},$$
$$s^\mu s_\mu = \frac{1}{8}J^{\mu\nu}J_{\mu\nu}.$$

In the rest system of p^μ, $p^\mu = (p^0, \mathbf{0})$, $\sqrt{-p^2} = |p^0| = mc$ we have

$$s^0 = 0, \qquad s^i = \frac{p^0}{4|p^0|}\epsilon^{ijk}J_{jk}.$$

The last equality explains our normalization for the BMT vector s^μ, Eq. (111).

Appendix 2: General solution to equations of motion

Lagrangian equations The variation of the minimal action (33) implies the equations

$$\frac{\delta S}{\delta g_4}=0: \quad \omega^2=a_4, \quad \Rightarrow \quad (\omega\dot{\omega})=0, \tag{112}$$

$$\frac{\delta S}{\delta x}=0: -mc\left(\frac{N\dot{x}^\mu}{\sqrt{-\dot{x}N\dot{x}}}\right)^{\cdot}=0, \quad \Rightarrow$$
$$-mc\frac{N\dot{x}^\mu}{\sqrt{-\dot{x}N\dot{x}}}=p^\mu=\text{const}, \quad \Rightarrow$$
$$(p\omega)=0, \quad p^2=-(mc)^2, \quad \Rightarrow \quad (p\dot{\omega})=0, \tag{113}$$

$$\frac{\delta S}{\delta \omega}=0:$$
$$\sqrt{a_3}\left(\frac{N\dot{\omega}^\mu}{\sqrt{\dot{\omega}N\dot{\omega}}}\right)^{\cdot}+\frac{\sqrt{a_3}(\omega\dot{\omega})}{\omega^2\sqrt{\dot{\omega}N\dot{\omega}}}N\dot{\omega}^\mu$$
$$+\frac{(\omega\dot{x})}{\omega^2}\frac{mcN\dot{x}^\mu}{\sqrt{-\dot{x}N\dot{x}}}+g_4\omega^\mu=0. \tag{114}$$

Using the consequences pointed out in Eqs. (112) and (113), we simplify Eq. (114)

$$\sqrt{a_3}\left(\frac{\dot{\omega}^\mu}{\sqrt{\dot{\omega}^2}}\right)^{\cdot}-\frac{(\omega\dot{x})}{a_4}p^\mu+g_4\omega^\mu=0.$$

Contraction of this equation with ω^μ gives the expression for g_4,

$$g_4=\frac{\sqrt{a_3}\sqrt{\dot{\omega}^2}}{a_4},$$

whereas contraction with p^μ implies $(\omega\dot{x})=0$. Collecting all this, the initial Lagrangian equations can be presented in the equivalent form

$$\left(\frac{\dot{x}^\mu}{\sqrt{-\dot{x}^2}}\right)^{\cdot}=0, \tag{115}$$

$$\left(\frac{\dot{\omega}^\mu}{\sqrt{\dot{\omega}^2}}\right)^{\cdot}+\frac{\sqrt{\dot{\omega}^2}}{a_4}\omega^\mu=0, \tag{116}$$

$$\omega^2=a_4, \quad (\omega\dot{x})=0. \tag{117}$$

We have the second-order equations (115) and (116). Besides, there are presented two Lagrangian constraints; see (117).

The general solution to Eqs. (115)–(117) reads

$$x^\mu=x_0^\mu+p^\mu\lambda_1(\tau), \tag{118}$$

$$\omega^\mu=\sqrt{\frac{a_4}{a_3}}A^\mu\sin f(\tau)+\sqrt{\frac{a_4}{a_3}}B^\mu\cos f(\tau), \tag{119}$$

where λ_1 and f are arbitrary functions of the evolution parameter. The constants of integration obey the restrictions

$$p^2=-(mc)^2, \quad (pA)=(pB)=0, \quad A^2=a_3,$$
$$B^2=a_3, \quad (AB)=0. \tag{120}$$

Equation (118) determines the straight line (as geometric place of points) in Minkowski space, whereas (119) is an ellipse which lies on the plane of the inner space formed by the vectors A^μ and B^μ. Due to the arbitrary functions $\lambda(\tau)$ and $f(\tau)$, the evolution along the trajectories is not specified, as it should be in a reparametrization-invariant theory.

General solution to Hamiltonian equations The Hamiltonian formulation leads to the same result. The Hamiltonian constraints and equations written in Sect. 4 do not determine the multipliers λ_1 and λ_4. As a consequence, the variable $g_4(\tau)$ cannot be determined neither with the constraints nor with the dynamical equations. This implies the functional ambiguity in the solutions to the equations of motion for the basic variables x^μ, ω^μ and π^μ: besides the integration constants, the solution depends on these arbitrary functions.

Denoting

$$f(\tau)=\sqrt{\frac{a_4}{a_3}}\int d\tau g_4,$$

the general solution to the Hamiltonian equations is given by Eqs. (118)–(120) and

$$\pi^\mu=A^\mu\cos f(\tau)-B^\mu\sin f(\tau).$$

Covariant dynamics in proper-time parametrization The physical variables obey nondegenerate equations, but they are not manifestly covariant. The standard way to work with nondegenerate equations keeping covariance is to fix the parametrization to be the proper time of the particle, $x^\mu=x^\mu(s)$. Here s is the time measured in the instantaneous rest frame. As the proper time coincides with the interval between the particle positions, in the proper-time parametrization we have the relation[9]

$$(\dot{x}^\mu(s))^2=-c^2.$$

This equation together with Eq. (74) fixes $\lambda_1=\frac{1}{m}$, so we arrive at the deterministic equations

$$\dot{x}^\mu(s)=\frac{p^\mu}{m}, \quad \dot{p}^\mu(s)=0; \quad p^2=-(mc)^2,$$

with the solution being

$$x^\mu=x_0^\mu+\frac{p^\mu}{m}s, \quad p^2=-(mc)^2, \quad p^\mu=\text{const}.$$

As before, the physical dynamical variables $x^i(t)$ is obtained from $x^\mu(s)$ excluding the parameter s.

The spin-sector is described either by Eq. (47) or by Eq. (48).

[9] In an arbitrary parametrization we have $(\dot{x}^\mu(\tau))^2=-c^2\dot{s}^2(\tau)$, which is not interesting.

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