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Superradiant instability of charged scalar field in stringy black hole mirror system

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Abstract It has been shown that the mass of a charged scalar field in the background of a charged stringy black hole is never able to generate a potential well outside the event horizon to trap the superradiant modes. This is to say that the charged stringy black hole is stable against massive charged scalar perturbations. In this paper we will study the superradiant instability of the massless scalar field in the background of charged stringy black hole due to a mirror-like boundary condition. The analytical expression of the frequencies of unstable superradiant modes is derived by using the asymptotic matching method. It is also pointed out that the black hole mirror system becomes extremely unstable for a large charge q of the scalar field and a small mirror radius r_m.

1 Introduction

Long ago, there was proposal of building a black hole bomb [1] by using the classical superradiance phenomenon [2–7]. It seems that the mechanism of a black hole bomb is very simple. When an impinging bosonic wave with the frequency satisfying the superradiant condition is scattered by the event horizon of the rotating black hole, the amplitude of this bosonic wave will be enlarged. If one places a mirror outside of the hole, the enlarged wave will be reflected into the hole once again. Then this wave will be bounced back and forth between the event horizon and the mirror. Meanwhile, the energy of this wave can become sufficiently big in this black hole mirror system until the mirror is destroyed.

The black hole bomb mechanism firstly proposed by Press and Teukolsky [1] was studied by Cardoso et al. in [8] recently. It is found that there exists a minimum mirror's radius to make the black hole mirror system unstable. See also Refs. [9–14] for recent studies on this topic. The black hole bomb mechanism can be generalized to other cases. The first case is to study the massive bosonic field in rotating black holes, for example in [15–26], where the mass term can play the role of the reflecting mirror. In this case, the wave will be trapped in the potential well outside of the hole and the amplitude will grow exponentially, which triggers the instability of the system. The second case is to study the bosonic field perturbation in a black hole background with the Dirichlet boundary condition at asymptotic infinity. These background spacetimes include black holes in AdS spacetime [27–33], black holes in a Gödel universe [34,35], and black holes in a linear dilaton background [36,37]. In all these spacetimes, the Dirichlet boundary condition provides the reflecting mirror, which results in the instabilities of the systems.

For a charged scalar wave in the background of the spherical symmetric charged black hole, if the frequency of this impinging wave satisfies the superradiant condition, the wave will also undergo the superradiant process when scattered by the horizon [38]. But it is pointed out in [18] that there is no unstable mode of a scalar field in a Reissner–Nordström (RN) black hole. More recently, it was proved by Hod in [39,40] that, for the Reissner–Nordström (RN) black holes, the existence of a trapping potential well outside the black hole and superradiant amplification of the trapped modes cannot be satisfied simultaneously. This means that the RN black holes are stable under the perturbations of massive charged scalar fields. Soon after, Degollado et al. [41,42] found that the same system can be made unstable by adding a mirror-like boundary condition like the case of the Kerr black hole. However, whether all of the charged black holes have similar properties to the RN black hole is still an interesting question that deserves further studies.

In [43], we have shown that the mass term of the scalar field in the charged stringy black hole is never able to generate a potential well outside the event horizon to trap the superradiant modes. This is to say that the charged stringy black hole is stable against massive charged scalar perturbations. In this paper, we will further study the superradiant

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instability of the massless scalar field in the background of the charged stringy black hole due to a mirror-like boundary condition.

This black hole is the static spherical symmetric charged black hole in the low energy effective theory of heterotic string theory in four dimensions, which was first found by Gibbons and Maeda in [44] and independently found by Garfinkle et al. in [45] a few years later. The metric is given by

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 \\ +r\left(r-\frac{Q^2}{M}\right)(d\theta^2+\sin^2\theta d\phi^2), \quad (1)$$

and the electric field and the dilaton field are

$$A_t = -\frac{Q}{r}, \\ e^{2\Phi} = 1-\frac{Q^2}{Mr}. \quad (2)$$

The parameters M and Q are the mass and electric charge of the black hole, respectively. The event horizon of the black hole is located at $r = 2M$. The area of the sphere of the charged stringy black hole approaches zero when $r = Q^2/M$. Therefore, the sphere surface of the radius $r = Q^2/M$ is singular. When $Q^2 \leq 2M^2$, this singular surface is surrounded by the event horizon. We will consider the black hole with the parameters satisfying the condition $Q^2 \leq 2M^2$ in this paper. When $Q^2 = 2M^2$, the singular surface coincides with the event horizon. This is the case of the extremal black hole.

We start with analyzing the scalar field perturbation in the background of the charged stringy black hole. The dynamics of the charged massless scalar field perturbation is governed by the Klein–Gordon equation,

$$(\nabla_\nu - iqA_\nu)(\nabla^\nu - iqA^\nu)\Psi = 0, \quad (3)$$

where q denotes the charge of the scalar field. By taking the ansatz of the scalar field $\Psi = e^{-i\omega t}R(r)Y_{lm}(\theta,\phi)$, where ω is the conserved energy of the mode, l is the spherical harmonic index, and m is the azimuthal harmonic index with $-l \leq k \leq l$, one can deduce the radial wave equation in the form of

$$\Delta\frac{d}{dr}\left(\Delta\frac{dR}{dr}\right) + UR = 0, \quad (4)$$

where we have introduced a new function $\Delta = (r-r_+)(r-r_-)$ with $r_+ = 2M$ and $r_- = Q^2/M$, and the potential function is given by

$$U = \left(r-\frac{Q^2}{M}\right)^2(\omega r - qQ)^2 - \Delta l(l+1). \quad (5)$$

The classical superradiance phenomenon for the scalar field perturbation is present in a charged stringy black hole [46,47]. In particular, by studying the asymptotic solutions of the radial wave equation near the horizon and at spatial infinity with the appropriate boundary conditions, one can obtain the superradiant condition of the charged scalar field [43]:

$$\omega < q\Phi_H, \quad (6)$$

with $\Phi_H = \frac{Q}{2M}$ being the electric potential at the horizon.

It has been shown by analyzing the behavior of the effective potential that for both the nonextremal black holes and the extremal black holes there is no potential well which is separated from the horizon by a potential barrier. Thus, the superradiant modes of the charged scalar field cannot be trapped and lead to the instabilities of the black holes. This indicates that the extremal and the nonextremal charged black holes in string theory are stable against charged scalar field perturbations [43].

In this paper, we will make the black hole unstable by placing a reflecting mirror outside of the hole. More precisely, we will impose the mirror's boundary condition that the scalar field vanishes at the mirror's location r_m, i.e.

$$\Psi(r = r_m) = 0. \quad (7)$$

The complex frequencies satisfying the purely ingoing boundary at the black hole horizon and the mirror's boundary condition are called boxed quasinormal (BQN) frequencies [8]. The scalar modes in the superradiant regime will bounce back and forth between event horizon and mirror. Meanwhile, the energy extracted from the black hole by means of the superradiance process will grow exponentially. This will cause the instability of the black hole mirror system. In the following, we will present an analytical calculation of the BQN frequencies in a certain limit and show the instability in the superradiant regime caused by the mirror's boundary condition.

Now we will employ the matched asymptotic expansion method [48,49] to compute the unstable modes of a charged scalar field in this black hole mirror system. We shall assume that the Compton wavelength of the scalar particles is much larger than the typical size of the black hole, i.e. $1/\omega \gg M$. With this assumption, we can divide the space outside the event horizon into two regions, namely, a near-region, $r - r_+ \ll 1/\omega$, and a far-region, $r-r_+ \gg M$. The approximated solution can be obtained by matching the near-region solution and the far-region solution in the overlapping region $M \ll r-r_+ \ll 1/\omega$. Finally, we can impose the mirror's boundary condition to obtain the analytical expression of the unstable modes in this system.

Firstly, let us focus on the near-region in the vicinity of the event horizon, $\omega(r-r_+) \ll 1$. The radial wave function can be reduced to the form

$$\Delta\partial_r(\Delta\partial_r R(r))+\Big[(r_+-r_-)^2\varpi^2-l(l+1)\Delta\Big]R(r)=0, \quad (8)$$

with the parameter ϖ given by

$$\varpi=r_+(\omega-q\Phi_H). \quad (9)$$

Introducing the new coordinate variable

$$z=\frac{r-r_+}{r-r_-}, \quad (10)$$

the near-region radial wave equation can be rewritten in the form of

$$z\partial_z(z\partial_z R(z))+\left[\varpi^2-l(l+1)\frac{z}{(1-z)^2}\right]R(z)=0. \quad (11)$$

Through defining

$$R=z^{i\varpi}(1-z)^{l+1}F(z), \quad (12)$$

the near-region radial wave equation becomes the standard hypergeometric equation

$$z(1-z)\partial_z^2F(z)+[c-(1+a+b)]\partial_z F(z)-abF(z)=0, \quad (13)$$

with the parameters

$$a=l+1+2i\varpi,$$
$$b=l+1,$$
$$c=1+2i\varpi. \quad (14)$$

In the neighborhood of $z=0$, the general solution of the radial wave equation is then given in terms of the hypergeometric function [50]

$$R=Az^{-i\varpi}(1-z)^{l+1}F(l+1,l+1-2i\varpi,1-2i\varpi,z)$$
$$+Bz^{i\varpi}(1-z)^{l+1}F(l+1,l+1+2i\varpi,1+2i\varpi,z). \quad (15)$$

It is obvious that the first term represents the ingoing wave at the horizon, while the second term represents the outgoing wave at the horizon. This can be observed by calculating the group velocity of the wave [24]. At the horizon, the first term behaves as $\Psi(t,r)|_{r\to r_+}\sim e^{-i\omega t}e^{-i\varpi\ln(r-r_+)}$. The normalized group velocity, v_{gr}, at the horizon is $v_{gr}=\frac{1}{r_+}\frac{d(-\varpi)}{d\omega}=-1$, which is independent of the value of ω. This implies the first term is the ingoing wave solution. The second term is obviously the outgoing wave solution, which is reflected by the fact that the corresponding group velocity is positive.

Because we are considering the classical superradiance process, the ingoing boundary condition at the horizon should be employed. Then we have to set $B=0$. The physical solution of the radial wave equation corresponding to the ingoing wave at the horizon is then given by

$$R=Az^{-i\varpi}(1-z)^{l+1}F(l+1,l+1-2i\varpi,1-2i\varpi,z). \quad (16)$$

In the far-region, $r-r_+\gg M$, the effects induced by the black hole can be neglected. The metric is reduced to the Minkowski metric in spherical coordinates. Then the radial wave equation reduces to the wave equation of a scalar field in the flat background,

$$\partial_r^2(rR(r))+\left[\omega^2-\frac{l(l+1)}{r^2}\right](rR(r))=0. \quad (17)$$

This equation can be solved by the Bessel function, which is given by [50]

$$R=r^{-1/2}\left[\alpha J_{l+1/2}(\omega r)+\beta J_{-l-1/2}(\omega r)\right]. \quad (18)$$

In order to match the far-region solution with the near-region solution, we should study the large r behavior of the near-region solution and the small r behavior of the far-region solution. For the sake of this purpose, we can us the $z\to 1-z$ transformation law for the hypergeometric function [50],

$$F(a,b,c;z)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b,a+b-c+1;1-z)$$
$$+(1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}$$
$$\times F(c-a,c-b,c-a-b+1;1-z). \quad (19)$$

By employing this formula and using the properties of the hypergeometric function $F(a,b,c,0)=1$, we can get the large r behavior of the near-region solution as

$$R\sim A\Gamma(1-2i\varpi)\left[\frac{(r_+-r_-)^{-l}\Gamma(2l+1)}{\Gamma(l+1)\Gamma(l+1-2i\varpi)}r^l\right.$$
$$\left.+\frac{(r_+-r_-)^{l+1}\Gamma(-2l-1)}{\Gamma(-l)\Gamma(-l-2i\varpi)}r^{-l-1}\right]. \quad (20)$$

On the other hand, using the asymptotic form of the Bessel function [50], $J_\nu(z)=(z/2)^\nu/\Gamma(\nu+1)$ $(z\ll 1)$, one finds the small r behavior of the far-region solution as

$$R\sim\alpha\frac{(\omega/2)^{l+1/2}}{\Gamma(l+3/2)}r^l+\beta\frac{(\omega/2)^{-l-1/2}}{\Gamma(-l+1/2)}r^{-l-1}. \quad (21)$$

By comparing the large r behavior of the near-region solution with the small r behavior of the far-region solution, one can conclude that there exists an overlapping region $M\ll r-r_+\ll 1/\omega$ where the two solutions should match. This matching yields the relation

$$\frac{\beta}{\alpha}=\frac{\Gamma(-l+1/2)}{\Gamma(l+3/2)}\frac{\Gamma(l+1)}{\Gamma(2l+1)}\frac{\Gamma(-2l-1)}{\Gamma(-l)}\frac{\Gamma(l+1-2i\varpi)}{\Gamma(-l-2i\varpi)}$$
$$\times\left(\frac{\omega}{2}\right)^{2l+1}(r_+-r_-)^{2l+1}. \quad (22)$$

By using the property of Gamma function, $\Gamma(x+1) = x\Gamma(x)$, one can easily derive the relations

$$\frac{\Gamma(-l+1/2)}{\Gamma(l+3/2)} = \frac{(-1)^l 2^{2l+1}}{(2l-1)!!(2l+1)!!},$$
$$\frac{\Gamma(-2l-1)}{\Gamma(-l)} = \frac{(-1)^{l+1} l!}{(2l+1)!},$$
$$\frac{\Gamma(l+1-2i\varpi)}{\Gamma(-l-2i\varpi)} = (-1)^{l+1} 2i\varpi \prod_{k=1}^{l} (k^2+4\varpi^2). \tag{23}$$

Applying these formulas to the matching condition, one can derive

$$\frac{\beta}{\alpha} = 2i\varpi \frac{(-1)^l}{(2l+1)} \left(\frac{l!}{(2l-1)!!}\right)^2 \frac{(r_+-r_-)^{2l+1}}{(2l)!(2l+1)!} \times \prod_{k=1}^{l}(k^2+4\varpi^2)\omega^{2l+1}. \tag{24}$$

Now we want to impose the mirror's boundary condition to study the unstable modes. We assume that the mirror is placed near infinity at a radius $r = r_m$. The far-region radial solution should vanish when reflected by the mirror. This yields the extra condition between the amplitudes α and β of the far-region radial solution, which is given by

$$\frac{\beta}{\alpha} = -\frac{J_{l+1/2}(\omega r_m)}{J_{-l-1/2}(\omega r_m)}. \tag{25}$$

This mirror condition together with the matching condition gives us the following equation, which determines the BQN frequencies of the scalar field in this black hole mirror system:

$$\frac{J_{l+1/2}(\omega r_m)}{J_{-l-1/2}(\omega r_m)} = 2i\varpi \frac{(-1)^{l+1}}{(2l+1)} \left(\frac{l!}{(2l-1)!!}\right)^2 \frac{(r_+-r_-)^{2l+1}}{(2l)!(2l+1)!} \times \prod_{k=1}^{l}(k^2+4\varpi^2)\omega^{2l+1}. \tag{26}$$

For very small ω, the analytical solution of the BQN frequencies can be found from the above relation. In this case, the right hand side of the above relation is very small and then can be set to be zero. This means that

$$J_{l+1/2}(\omega r_m) = 0. \tag{27}$$

The real zeros of the Bessel functions were well studied. We shall label the nth positive zero of the Bessel function $J_{l+1/2}$ as $j_{l+1/2,n}$. Then we get

$$\omega r_m = j_{l+1/2,n}. \tag{28}$$

In the first approximation for BQN frequencies, the solution of (26) has a small imaginary part, which can be written as

$$\omega_{BQN} = \frac{j_{l+1/2,n}}{r_m} + i\delta, \tag{29}$$

where the introduced imaginary part δ is small enough comparing the real part of BQN frequency. It can be considered as a correction to (28). For the small δ, we can use the Taylor expansion of Bessel function $J_{l+1/2}(\omega r_m) = i\delta r_m J'_{l+1/2}(j_{l+1/2,n})$. Then (26) can be reduced to

$$i\delta r_m \frac{J'_{l+1/2}(j_{l+1/2,n})}{J_{-l-1/2}(j_{l+1/2,n})} = 2i\varpi \frac{(-1)^{l+1}}{(2l+1)} \left(\frac{l!}{(2l-1)!!}\right)^2 \times \frac{(r_+-r_-)^{2l+1}}{(2l)!(2l+1)!} \prod_{k=1}^{l}(k^2+4\varpi^2) \left(\frac{j_{l+1/2,n}}{r_m}\right)^{2l+1}. \tag{30}$$

From this we can easily obtain the small imaginary part of the BQN frequencies as

$$\delta = -\gamma \left(\frac{j_{l+1/2,n}}{r_m} - q\Phi_H\right) \frac{(-1)^l J_{-l-1/2}(j_{l+1/2,n})}{J'_{l+1/2}(j_{l+1/2,n})}, \tag{31}$$

with

$$\gamma = \frac{2}{(2l+1)} \left(\frac{l!}{(2l-1)!!}\right)^2 \frac{r_+(r_+-r_-)^{2l+1}}{r_m(2l)!(2l+1)!} \times \left(\prod_{k=1}^{l}(k^2+4\varpi^2)\right) \left(\frac{j_{l+1/2,n}}{r_m}\right)^{2l+1}. \tag{32}$$

Notice that γ is always greater than zero, and $(-1)^l J_{-l-1/2}(j_{l+1/2,n})$ and $J'_{l+1/2}(j_{l+1/2,n})$ always have the same sign. So we have

$$\delta \propto -(\mathrm{Re}[\omega_{BQN}] - q\Phi_H). \tag{33}$$

It is easy to see that, in the superradiance regime, $\mathrm{Re}[\omega_{BQN}] - q\Phi_H < 0$, the imaginary part of the complex BQN frequency $\delta > 0$. The scalar field has the time dependence $e^{-i\omega t} = e^{-i\mathrm{Re}[\omega]t} e^{\delta t}$, which implies the exponential amplification of superradiance modes. This indicates that the BQN frequencies in the superradiant regime is unstable for the charged scalar field in a stringy black hole with a mirror placed outside of the hole.

Here, we shall discuss our analytical result briefly. The instability time scaling that characterizes the composed black hole mirror system is given by

$$\tau = \frac{1}{\delta}. \tag{34}$$

Firstly, the imaginary part of the complex BQN frequency δ decreases when the mirror's radius r_m increases. This means that the instability time scaling becomes larger for the larger mirror radius.

Secondly, from (29), we can observe the that wave frequencies of these unstable superradiant modes are proportional to the inverse of the mirror radius. When the mirror radius decreases, the allowed wave frequencies will increase. The superradiant condition then restricts the position of the mirror; it cannot be placed very near the horizon. There exists a critical radius r_m^{crit} at which this instability disappears.

From the analytical result, one can obtain the critical radius which is given by

$$r_m^{crit} = \frac{j_{l+1/2,n}}{q\Phi_H}. \tag{35}$$

However, from the above equation, one can see that we can still place the mirror at a very small radius as long as the charge q of the scalar field is big enough.

Finally, one should note that, for the RN black hole in a cavity [13,42], the charged scalar field has a rapid growth of superradiant instability. The expression of the imaginary part of BQN frequencies is very similar to the result given in [42]. For the present case, one can also observe that δ grows with the charge q of the scalar field. This implies the instability becomes stronger as q increases. So one can expect that, for large q and small r_m, the instability time scale of this charged spherical symmetric black hole mirror system will become very short. This result is different from the rotating black hole mirror system. For the rotating black hole [8], the superradiant condition is given by $\omega < m\Phi_H$, where m and Φ_H are the azimuthal number and the angular velocity of the horizon, respectively. The value of m cannot be taken arbitrarily large because of the limit condition $m \leq l$ with l being the spherical harmonic index.

In summary, we have studied the instability of the massless charged scalar field in the stringy black hole mirror system. By imposing the mirror boundary condition, we have analytically calculated the expression of the BQN frequencies. Based on this result, we also point out that the black hole mirror system becomes extremely unstable for the large charge q of the scalar field and the small mirror radius r_m. In [13], it is deduced by Hod using the analytical method that, for the RN black hole, the instability time scale can be made arbitrarily short in a special limit. So, the analytical computation and the numerical simulation are still required to verify the conclusion.

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References

1. W.H. Press, S.A. Teukolsky, Nature (London) **238**, 211 (1972)
2. Ya. B. Zel'dovich, Pis'ma Zh. Eksp. Teor. Fiz. **14**, 270 (1971) [JETP Lett. **14**, 180 (1971)]
3. Ya. B. Zel'dovich, Zh. Eksp. Teor. Fiz. **62**, 2076 (1972) [Sov. Phys. JETP **35**, 1085 (1972)]
4. J.M. Bardeen, W.H. Press, S.A. Teukolsky, Astrophys. J. **178**, 347 (1972)
5. C.W. Misner, Bull. Am. Phys. Soc. **17**, 472 (1972)
6. A.A. Starobinsky, Zh. Eksp. Teor. Fiz. **64**, 48 (1973) [Sov. Phys. JETP **37**, 28 (1973)]
7. A.A. Starobinsky, S.M. Churilov, Zh. Eksp. Teor. Fiz. **65**, 3 (1973) [Sov. Phys. JETP **38**, 1 (1973)]
8. V. Cardoso, O.J.C. Dias, J.P.S. Lemos, S. Yoshida, Phys. Rev. D **70**, 044039 (2004)
9. J.G. Rosa, JHEP **1006**, 015 (2010)
10. J.-P. Lee, Mod. Phys. Lett. A **27**, 1250038 (2012)
11. J.-P. Lee, JHEP **1201**, 091 (2012)
12. J.G. Rosa, JHEP **02**, 014 (2013)
13. S. Hod, Phys. Rew. D **88**, 064055 (2013)
14. S. Hod, Phys. Rew. D **88**, 124007 (2013)
15. T. Damour, N. Deruelle, R. Ruffini, Lett. Nuovo Cimento Soc. Ital. Fis. **15**, 257 (1976)
16. S. Detweiler, Phys. Rev. D **22**, 2323 (1980)
17. T.M. Zouros, D.M. Eardley, Ann. Phys. (N.Y.) **118**, 139 (1979)
18. H. Furuhashi, Y. Nambu, Prog. Theor. Phys. **112**, 983 (2004)
19. M.J. Strafuss, G. Khanna, Phys. Rev. D **71**, 024034 (2005)
20. S.R. Dolan, Phys. Rev. D **76**, 084001 (2007)
21. S. Hod, O. Hod, Phys. Rev. D **81**, 061502 (2010)
22. S. Hod, Phys. Lett. B **708**, 320 (2012)
23. R.A. Konoplya, Phys. Lett. B **666**, 283 (2008)
24. O.J.C. Dias, Phys. Rev. D **73**, 124035 (2006)
25. S.-J. Zhang, B. Wang, E. Abdalla. arXiv:1306.0932
26. S.R. Dolan, Phys. Rev. D **87**, 124026 (2013)
27. V. Cardoso, O.J.C. Dias, Phys. Rev. D **70**, 084011 (2004)
28. V. Cardoso, O.J.C. Dias, S. Yoshida, Phys. Rev. D **74**, 044008 (2006)
29. H. Kodama, R.A. Konoplya, A. Zhidenko, Phys. Rev. D **79**, 044003 (2009)
30. A.N. Aliev, O. Delice, Phys. Rev. D **79**, 024013 (2009)
31. N. Uchikata, S. Yoshida, Phys. Rev. D **83**, 064020 (2011)
32. R. Li, Phys. Lett. B **714**, 337 (2012)
33. S.-J. Zhang, Q. Pan, B. Wang, E. Abdalla, JHEP **09**, 101 (2013)
34. R.A. Konoplya, A. Zhidenko, Phys. Rev. D **84**, 104022 (2011)
35. R. Li, Eur. Phys. J. C **73**, 2274 (2013)
36. G. Clement, D. Galtsov, C. Leygnac, Phys. Rev. D **67**, 024012 (2003)
37. R. Li, Eur. Phys. J. C **73**, 2296 (2013)
38. J.D. Bekenstein, Phys. Rev. D **7**, 949 (1973)
39. S. Hod, Phys. Lett. B **718**, 1489 (2013)
40. S. Hod, Phy. Lett. B **713**, 505 (2012)
41. J.C. Degollado, C.A.R. Herdeiro, H.F. Runarsson, Phys. Rev. D **88**, 063003 (2013)
42. J.C. Degollado, C.A.R. Herdeiro, arXiv:1312.4579 [gr-qc]
43. R. Li, Phys. Rev. D **88**, 127901 (2013)
44. G.W. Gibbons, K. Maeda, Nucl. Phys. B **298**, 741 (1998)
45. D. Garfinkle, G.T. Horowitz, A. Strominger, Phys. Rev. D **43**, 3140 (1991)
46. K. Shiraishi, Mod. Phys. Lett. A **7**, 3449 (1992)
47. J. Koga, K. Maeda, Phys. Lett. B **340**, 29 (1994)
48. D.N. Page, Phys. Rev. D **13**, 198 (1976)
49. W.G. Unruh, Phys. Rev. D **14**, 3251 (1976)
50. M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Function* (Dover Publications, New York, 1970)