Kangtunyakarn *Fixed Point Theory and Applications* 2013, **2013**:157
http://www.fixedpointtheoryandapplications.com/content/2013/1/157

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

A new mapping for finding a common element of the sets of fixed points of two finite families of nonexpansive and strictly pseudo-contractive mappings and two sets of variational inequalities in uniformly convex and 2-smooth Banach spaces

Atid Kangtunyakarn*

*Correspondence:
beawrock@hotmail.com
Department of Mathematics, Faculty of Science, King Mongkut's Institute of Technology Ladkrabang, Bangkok 10520, Thailand

Abstract

In this paper we introduce a new mapping in a uniformly convex and 2-smooth Banach space to prove a strong convergence theorem for finding a common element of the set of fixed points of a finite family of nonexpansive mappings and the set of fixed points of a finite family of strictly pseudo-contractive mappings and two sets of solutions of variational inequality problems. Moreover, we also obtain a strong convergence theorem for a finite family of the set of solutions of variational inequality problems and the set of fixed points of a finite family of strictly pseudo-contractive mappings by using our main result.

Keywords: nonexpansive mapping; strictly pseudo-contractive mapping; variational inequality problem

1 Introduction

Throughout this paper, we use E and E^* to denote a real Banach space and a dual space of E, respectively. For any pair $x \in E$ and $f \in E^*$, $\langle x, f\rangle$ instead of $f(x)$. The duality mapping $J : E \to 2^{E^*}$ is defined by $J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2, \|x\| = \|x^*\|\}$ for all $x \in E$. It is well known that if E is a Hilbert space, then $J = I$, where I is the identity mapping. Recall the following definitions.

Definition 1.1 A Banach space E is said to be uniformly convex iff for any ϵ, $0 < \epsilon \leq 2$, the inequalities $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \epsilon$ imply there exists a $\delta > 0$ such that $\|\frac{x+y}{2}\| \leq 1 - \delta$.

Definition 1.2 A Banach space E is said to be smooth if for each $x \in S_E = \{x \in E : \|x\| = 1\}$, there exists a unique functional $j_x \in E^*$ such that $\langle x, j_x\rangle = \|x\|$ and $\|j_x\| = 1$.

It is obvious that if E is smooth, then J is single-valued which is denoted by j.

Definition 1.3 Let E be a Banach space. Then a function $\rho_E : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be the modulus of smoothness of E if

$$\rho_E(t) = \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = t\right\}.$$

A Banach space E is said to be uniformly smooth if

$$\lim_{t\to 0}\frac{\rho_E(t)}{t} = 0.$$

It is well known that every uniformly smooth Banach space is smooth.

Let $q > 1$. A Banach space E is said to be q-uniformly smooth if there exists a fixed constant $c > 0$ such that $\rho_E(t) \leq ct^q$. It is easy to see that if E is q-uniformly smooth, then $q \leq 2$ and E is uniformly smooth.

A mapping $T : C \to C$ is called a nonexpansive mapping if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$.

T is called an η-strictly pseudo-contractive mapping if there exists a constant $\eta \in (0, 1)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \eta\big\|(I-T)x - (I-T)y\big\|^2 \tag{1.1}$$

for every $x, y \in C$ and for some $j(x - y) \in J(x - y)$. It is clear that (1.1) is equivalent to the following:

$$\langle (I-T)x - (I-T)y, j(x-y)\rangle \geq \eta\big\|(I-T)x - (I-T)y\big\|^2 \tag{1.2}$$

for every $x, y \in C$ and for some $j(x - y) \in J(x - y)$. We give some examples for a strictly pseudo-contractive mapping as follows.

Example 1.1 Let $\mathbb{R}$ be a real line endowed with the Euclidean norm and let $C = (0, \infty)$. Define the mapping $T : C \to C$ by

$$Tx = \frac{2x^2}{3+2x}, \quad \forall x \in C.$$

Then T is a $\frac{1}{9}$-strictly pseudo-contractive mapping.

Example 1.2 (See [1]) Let $\mathbb{R}$ be a real line endowed with the Euclidean norm. Let $C = [-1, 1]$ and let $T : C \to C$ be defined by

$$Tx = \begin{cases} x & \text{if } x \in [-1, 0]; \\ x - x^2 & \text{if } x \in (0, 1]. \end{cases}$$

Then T is a λ-strictly pseudo-contractive mapping where $\lambda \leq \min\{\lambda_1, \lambda_2\}$ and $\lambda_1 \leq \frac{1}{2}$, $\lambda_2 < 1$.

Let C and D be nonempty subsets of a Banach space E such that C is nonempty closed convex and $D \subset C$, then a mapping $P : C \to D$ is sunny [2] provided $P(x + t(x - P(x))) = P(x)$ for all $x \in C$ and $t \geq 0$, whenever $x + t(x - P(x)) \in C$. A mapping $P : C \to D$ is called a retraction if $Px = x$ for all $x \in D$. Furthermore, P is a sunny nonexpansive retraction from C onto D if P is a retraction from C onto D which is also sunny and nonexpansive.

Subset D of C is called a sunny nonexpansive retraction of C if there exists a sunny nonexpansive retraction from C onto D.

An operator A of C into E is said to be *accretive* if there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq 0, \quad \forall x, y \in C.$$

A mapping $A : C \to E$ is said to be α-*inverse strongly accretive* if there exist $j(x - y) \in J(x - y)$ and $\alpha > 0$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Remark 1.3 From (1.1) and (1.2), if T is an η-strictly pseudo-contractive mapping, then $I - T$ is η-inverse strongly accretive.

The variational inequality problem in a Banach space is to find a point $x^* \in C$ such that for some $j(x - x^*) \in J(x - x^*)$,

$$\langle Ax^*, j(x - x^*)\rangle \geq 0, \quad \forall x \in C. \tag{1.3}$$

This problem was considered by Aoyama *et al.* [3]. The set of solutions of the variational inequality in a Banach space is denoted by $S(C, A)$, that is,

$$S(C, A) = \{u \in C : \langle Au, J(v - u)\rangle \geq 0,\ \forall v \in C\}. \tag{1.4}$$

Several problems in pure and applied science, numerous problems in physics and economics reduce to finding an element in (1.4); see, for instance, [4–6].

Recall that normal Mann's iterative process was introduced by Mann [7] in 1953. The normal Mann's iterative process generates a sequence $\{x_n\}$ in the following manner:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad \forall n \geq 1, \end{cases} \tag{1.5}$$

where the sequence $\{\alpha_n\} \subset (0, 1)$. If T is a nonexpansive mapping with a fixed point and the control sequence $\{\alpha_n\}$ is chosen so that $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by normal Mann's iterative process (1.5) converges weakly to a fixed point of T.

In 1967, Halpern has introduced the iteration method guaranteeing the strong convergence as follows:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1 - \alpha_n)x_1 + \alpha_n Tx_n, \quad \forall n \geq 1, \end{cases} \tag{1.6}$$

where $\{\alpha_n\} \subset (0,1)$. Such an iteration is called *Halpern iteration* if T is a nonexpansive mapping with a fixed point. He also pointed out that the conditions $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$ are necessary for the strong convergence of $\{x_n\}$ to a fixed point of T.

Many authors have modified the iteration (1.6) for a strong convergence theorem; see, for instance, [8–10].

In 2008, Zhou [11] proved a strong convergence theorem for the modification of normal Mann's iteration algorithm generated by a strict pseudo-contraction in a real 2-uniformly smooth Banach space as follows.

Theorem 1.4 *Let C be a closed convex subset of a real 2-uniformly smooth Banach space E and let $T : C \to C$ be a λ-strict pseudo-contraction such that $F(T) \neq \emptyset$. Given $u, x_0 \in C$ and sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ in $(0,1)$, the following control conditions are satisfied*:

(i) $a \leq \alpha_n \leq \frac{\lambda}{K^2}$ *for some $a > 0$ and for all $n \geq 0$*,

(ii) $\beta_n + \gamma_n + \delta_n = 1$ *for all $n \geq 0$*,

(iii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$,

(iv) $\alpha_{n+1} - \alpha_n \to 0$, *as $n \to \infty$*,

(v) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$.

Let a sequence $\{x_n\}$ be generated by

$$\begin{cases} y_n = \alpha_n Tx_n + (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n u + \gamma_n x_n + \delta_n y_n, \quad n \geq 0. \end{cases}$$

Then $\{x_n\}$ converges strongly to $x^* \in F(T)$, where $x^* = Q_{F(T)}(u)$ and $Q_{F(T)} : C \to F(T)$ is the unique sunny nonexpansive retraction from C onto $F(T)$.

In 2006, Aoyama *et al.* introduced a Halpern-type iterative sequence and proved that such a sequence converges strongly to a common fixed point of nonexpansive mappings as follows.

Theorem 1.5 *Let E be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and let C be a nonempty closed convex subset of E. Let $\{T_n\}$ be a sequence of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{N} F(T_i)$ is nonempty and let $\{\alpha_n\}$ be a sequence of $[0,1]$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let $\{x_n\}$ be a sequence of C defined as follows: $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x + (1-\alpha_n) T_n x_n$$

for every $n \in \mathbb{N}$. Suppose that $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty$ for any bounded subset B of C. Let T be a mapping of C into itself defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$ and *suppose that* $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. *If either*

(i) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or*

(ii) $\alpha_n \in (0,1]$ *for every* $n \in \mathbb{N}$ *and* $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}}$,

then $\{x_n\}$ *converges strongly to* Qx, *where* Q *is the sunny nonexpansive retraction of* E *onto* $F(T) = \bigcap_{i=1}^{\infty} F(T_n)$.

In 2005, Aoyama *et al.* [3] proved a weak convergence theorem for finding a solution of problem (1.3) as follows.

Theorem 1.6 *Let* E *be a uniformly convex and* 2*-uniformly smooth Banach space and let* C *be a nonempty closed convex subset of* E. *Let* Q_C *be a sunny nonexpansive retraction from* E *onto* C, *let* $\alpha > 0$ *and let* A *be an* α*-inverse strongly accretive operator of* C *into* E *with* $S(C,A) \neq \emptyset$. *Suppose that* $x_1 = x \in C$ *and* $\{x_n\}$ *is given by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) Q_C(x_n - \lambda_n A x_n)$$

for every $n = 1, 2, \dots$, *where* $\{\lambda_n\}$ *is a sequence of positive real numbers and* $\{\alpha_n\}$ *is a sequence in* $[0,1]$. *If* $\{\lambda_n\}$ *and* $\{\alpha_n\}$ *are chosen so that* $\lambda_n \in [a, \frac{\alpha}{K^2}]$ *for some* $a > 0$ *and* $\alpha_n \in [b,c]$ *for some* b, c *with* $0 < b < c < 1$, *then* $\{x_n\}$ *converges weakly to some element* z *of* $S(C,A)$, *where* K *is the* 2*-uniformly smoothness constant of* E.

In 2009, Kangtunykarn and Suantai [12] introduced the S-mapping generated by a finite family of mappings and real numbers as follows.

Definition 1.4 Let C be a nonempty convex subset of a real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of mappings of C into itself. For each $j = 1, 2, \dots, N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I \in [0,1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Define the mapping $S : C \to C$ as follows:

$$\begin{aligned}
U_0 &= I, \\
U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I, \\
U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I, \\
U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I, \\
&\vdots \\
U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I, \\
S = U_N &= \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I.
\end{aligned} \tag{1.7}$$

This mapping is called the *S-mapping* generated by $T_1, T_2, \dots, T_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$.

For every $i = 1, 2, \dots, N$, put $\alpha_3^j = 0$ in (1.7), then the S-mapping generated by $T_1, T_2, \dots, T_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$ reduces to the K-mapping generated by $T_1, T_2, \dots, T_N$ and $\alpha_1^1, \alpha_1^2, \dots, \alpha_1^N$, which is defined by Kangtunyakarn and Suantai [13].

Recently, Kangtunyakarn [14] introduced an iterative scheme by the modification of Mann's iteration process for finding a common element of the set of solutions of a finite family of variational inequality problems and the set of fixed points of an η-strictly pseudo-contractive mapping and a nonexpansive mapping as follows.

Theorem 1.7 *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E. Let Q_C be the sunny nonexpansive retraction from E onto C. For every $i = 1,2,\dots,N$, let $A_i : C \to E$ be an α_i-inverse strongly accretive mapping. Define a mapping $G_i : C \to C$ by $Q_C(I - \lambda_i A_i)x = G_i x$ for all $x \in C$ and $i = 1,2,\dots,N$, where $\lambda_i \in (0, \frac{\alpha_i}{K^2})$, K is the 2-uniformly smooth constant of E. Let $B : C \to C$ be the K-mapping generated by $G_1, G_2, \dots, G_N$ and $\rho_1, \rho_2, \dots, \rho_N$, where $\rho_i \in (0,1)$, $\forall i = 1,2,\dots,N-1$ and $\rho_N \in (0,1]$. Let $T : C \to C$ be a nonexpansive mapping and $S : C \to C$ be an η-strictly pseudo-contractive mapping with $\mathcal{F} = F(S) \cap F(T) \cap \bigcap_{i=1}^N S(C, A_i) \neq \emptyset$. Define a mapping $B_A : C \to C$ by $T((1-\alpha)I + \alpha S)x = B_A x$, $\forall x \in C$ and $\alpha \in (0, \frac{\eta}{K^2})$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n B x_n + \delta_n B_A x_n, \quad \forall n \geq 1, \tag{1.8}$$

where $f : C \to C$ is a contractive mapping and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subseteq [0,1]$, $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ and satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\{\gamma_n\}, \{\delta_n\} \subseteq [c,d] \subset (0,1)$ *for some* $c, d > 0$ *and* $\forall n \geq 1$,

(iii) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n|$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n|$, $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$,

(iv) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ converses strongly to $q \in \mathcal{F}$, which solves the following variational inequality:

$$\langle q - f(q), j(q-p) \rangle \leq 0, \quad \forall p \in \mathcal{F}.$$

Question How can we prove a strong convergence theorem for the set of fixed points of a finite family of nonexpansive mappings and the set of fixed points of a finite family of strictly pseudo-contractive mappings and the set of solutions of variational inequality problems in a uniformly convex and 2-uniformly smooth Banach space?

Motivated by the S-mapping, we define a new mapping in the next section to answer the above question, and from Theorems 1.4, 1.5, 1.6 and 1.7 we modify the Halpern iteration for finding a common element of two sets of solutions of (1.3) and the set of fixed points of a finite family of nonexpansive mappings and the set of fixed points of a finite family of strictly pseudo-contractive mappings in a uniformly convex and 2-uniformly smooth Banach space. Moreover, by using our main result, we also obtain a strong convergence theorem for a finite family of the set of solutions of (1.3) and the set of fixed points of a finite family of strictly pseudo-contractive mappings.

2 Preliminaries

In this section we collect and prove the following lemmas to use in our main result.

Lemma 2.1 (See [15]) *Let E be a real 2-uniformly smooth Banach space with the best smooth constant K. Then the following inequality holds*:

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, J(x)\rangle + 2\|Ky\|^2$$

for any $x, y \in E$.

Lemma 2.2 (See [16]) *Let X be a uniformly convex Banach space and $B_r = \{x \in X : \|x\| \le r\}$, $r > 0$. Then there exists a continuous, strictly increasing and convex function $g : [0,\infty] \to [0,\infty]$, $g(0) = 0$ such that*

$$\|\alpha x + \beta y + \gamma z\|^2 \le \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta g\big(\|x-y\|\big)$$

for all $x, y, z \in B_r$ and all $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$.

Lemma 2.3 (See [3]) *Let C be a nonempty closed convex subset of a smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C and let A be an accretive operator of C into E. Then, for all $\lambda > 0$,*

$$S(C,A) = F\big(Q_C(I - \lambda A)\big).$$

Lemma 2.4 (See [15]) *Let $r > 0$. If E is uniformly convex, then there exists a continuous, strictly increasing and convex function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$ such that for all $x, y \in B_r(0) = \{x \in E : \|x\| \le r\}$ and for any $\alpha \in [0,1]$, we have $\|\alpha x + (1-\alpha)y\|^2 \le \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)g(\|x-y\|)$.*

Lemma 2.5 (See [17]) *Let C be a closed and convex subset of a real uniformly smooth Banach space E and let $T : C \to C$ be a nonexpansive mapping with a nonempty fixed point $F(T)$. If $\{x_n\} \subset C$ is a bounded sequence such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Then there exists a unique sunny nonexpansive retraction $Q_{F(T)} : C \to F(T)$ such that*

$$\limsup_{n\to\infty}\langle u - Q_{F(T)}u, J(x_n - Q_{F(T)}u)\rangle \le 0$$

for any given $u \in C$.

Lemma 2.6 (See [18]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} = (1-\alpha_n)s_n + \delta_n, \quad \forall n \ge 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\displaystyle\sum_{n=1}^{\infty}\alpha_n = \infty$,

$$(2)\quad \limsup_{n\to\infty}\frac{\delta_n}{\alpha_n}\le 0 \quad or \quad \sum_{n=1}^{\infty}|\delta_n|<\infty.$$

Then $\lim_{n\to\infty} s_n = 0$.

From the S-mapping, we define the mapping generated by two sets of finite families of the mappings and real numbers as follows.

Definition 2.1 Let C be a nonempty convex subset of a Banach space. Let $\{S_i\}_{i=1}^N$ and $\{T_i\}_{i=1}^N$ be two finite families of mappings of C into itself. For each $j = 1,2,\dots,N$, let $\alpha_j = (\alpha_1^j,\alpha_2^j,\alpha_3^j) \in I\times I\times I$, where $I \in [0,1]$ and $\alpha_1^j+\alpha_2^j+\alpha_3^j = 1$. We define the mapping $S^A : C \to C$ as follows:

$$\begin{aligned}
&U_0 = T_1 = I,\\
&U_1 = T_1\bigl(\alpha_1^1 S_1U_0+\alpha_2^1U_0+\alpha_3^1 I\bigr),\\
&U_2 = T_2\bigl(\alpha_1^2 S_2U_1+\alpha_2^2U_1+\alpha_3^2 I\bigr),\\
&U_3 = T_3\bigl(\alpha_1^3 S_3U_2+\alpha_2^3U_2+\alpha_3^3 I\bigr),\\
&\vdots\\
&U_{N-1} = T_{N-1}\bigl(\alpha_1^{N-1} S_{N-1}U_{N-2}+\alpha_2^{N-1}U_{N-2}+\alpha_3^{N-1} I\bigr),\\
&S^A = U_N = T_N\bigl(\alpha_1^N S_NU_{N-1}+\alpha_2^NU_{N-1}+\alpha_3^N I\bigr).
\end{aligned}\tag{2.1}$$

This mapping is called the S^A-mapping generated by $S_1,S_2,\dots,S_N$, $T_1,T_2,\dots,T_N$ and $\alpha_1,\alpha_2,\dots,\alpha_N$.

Lemma 2.7 *Let C be a nonempty closed convex subset of a 2-uniformly smooth and uniformly convex Banach space. Let $\{S_i\}_{i=1}^N$ be a finite family of κ_i-strict pseudo-contractions of C into itself and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(S_i)\cap\bigcap_{i=1}^N F(T_i)\neq\emptyset$ and $\kappa=\min\{\kappa_i : i=1,2,\dots,N\}$ with $K^2\le\kappa$, where K is the 2-uniformly smooth constant of E. Let $\alpha_j=(\alpha_1^j,\alpha_2^j,\alpha_3^j)\in I\times I\times I$, where $I=[0,1]$, $\alpha_1^j+\alpha_2^j+\alpha_3^j=1$, $\alpha_1^j\in(0,1]$, $\alpha_2^j\in[0,1]$ and $\alpha_3^j\in(0,1)$ for all $j=1,2,\dots,N$. Let S^A be the S^A-mapping generated by $S_1,S_2,\dots,S_N$, $T_1,T_2,\dots,T_N$ and $\alpha_1,\alpha_2,\dots,\alpha_N$. Then $F(S^A)=\bigcap_{i=1}^N F(S_i)\cap\bigcap_{i=1}^N F(T_i)$ and S^A is a nonexpansive mapping.*

Proof Let $x_0\in F(S^A)$ and $x^*\in\bigcap_{i=1}^N F(S_i)\cap\bigcap_{i=1}^N F(T_i)$, we have

$$\begin{aligned}
\|x_0-x^*\|^2 &= \bigl\|T_N\bigl(\alpha_1^NS_NU_{N-1}+\alpha_2^NU_{N-1}+\alpha_3^NI\bigr)x_0-x^*\bigr\|^2\\
&\le \bigl\|\alpha_1^N\bigl(S_NU_{N-1}x_0-x^*\bigr)+\alpha_2^N\bigl(U_{N-1}x_0-x^*\bigr)+\alpha_3^N\bigl(x_0-x^*\bigr)\bigr\|^2\\
&= \biggl\|\bigl(1-\alpha_3^N\bigr)\biggl(\frac{\alpha_1^N}{1-\alpha_3^N}\bigl(S_NU_{N-1}x_0-x^*\bigr)+\frac{\alpha_2^N}{1-\alpha_3^N}\bigl(U_{N-1}x_0-x^*\bigr)\biggr)\\
&\quad+\alpha_3^N\bigl(x_0-x^*\bigr)\biggr\|^2\\
&\le \bigl(1-\alpha_3^N\bigr)\biggl\|\frac{\alpha_1^N}{1-\alpha_3^N}\bigl(S_NU_{N-1}x_0-x^*\bigr)+\frac{\alpha_2^N}{1-\alpha_3^N}\bigl(U_{N-1}x_0-x^*\bigr)\biggr\|^2
\end{aligned}$$

$$
\begin{aligned}
&\quad + \alpha_3^N \|x_0 - x^*\|^2 \\
&= (1-\alpha_3^N)\left\| \frac{\alpha_1^N}{1-\alpha_3^N}(S_N U_{N-1}x_0 - x^*) + \left(1 - \frac{\alpha_1^N}{1-\alpha_3^N}\right)(U_{N-1}x_0 - x^*)\right\|^2 \\
&\quad + \alpha_3^N \|x_0 - x^*\|^2 \\
&= (1-\alpha_3^N)\left\| \frac{\alpha_1^N}{1-\alpha_3^N}(S_N U_{N-1}x_0 - U_{N-1}x_0) + U_{N-1}x_0 - x^*\right\|^2 + \alpha_3^N \|x_0 - x^*\|^2 \\
&\leq (1-\alpha_3^N)\biggl(\|U_{N-1}x_0 - x^*\|^2 \\
&\quad + 2\frac{\alpha_1^N}{1-\alpha_3^N}\langle S_N U_{N-1}x_0 - U_{N-1}x_0, j(U_{N-1}x_0 - x^*)\rangle \\
&\quad + 2K^2\left(\frac{\alpha_1^N}{1-\alpha_3^N}\right)^2 \|S_N U_{N-1}x_0 - U_{N-1}x_0\|^2\biggr) + \alpha_3^N \|x_0 - x^*\|^2 \\
&= (1-\alpha_3^N)\biggl(\|U_{N-1}x_0 - x^*\|^2 + 2\frac{\alpha_1^N}{1-\alpha_3^N}\langle S_N U_{N-1}x_0 - x^*, j(U_{N-1}x_0 - x^*)\rangle \\
&\quad + 2\frac{\alpha_1^N}{1-\alpha_3^N}\langle x^* - U_{N-1}x_0, j(U_{N-1}x_0 - x^*)\rangle \\
&\quad + 2K^2\left(\frac{\alpha_1^N}{1-\alpha_3^N}\right)^2 \|S_N U_{N-1}x_0 - U_{N-1}x_0\|^2\biggr) + \alpha_3^N \|x_0 - x^*\|^2 \\
&\leq (1-\alpha_3^N)\biggl(\|U_{N-1}x_0 - x^*\|^2 \\
&\quad + 2\frac{\alpha_1^N}{1-\alpha_3^N}\bigl(\|U_{N-1}x_0 - x^*\|^2 - \kappa\|(I-S_N)U_{N-1}x_0\|^2\bigr) \\
&\quad - 2\frac{\alpha_1^N}{1-\alpha_3^N}\|x^* - U_{N-1}x_0\|^2 + 2K^2\left(\frac{\alpha_1^N}{1-\alpha_3^N}\right)^2 \|S_N U_{N-1}x_0 - U_{N-1}x_0\|^2\biggr) \\
&\quad + \alpha_3^N \|x_0 - x^*\|^2 \\
&= (1-\alpha_3^N)\biggl(\|U_{N-1}x_0 - x^*\|^2 - 2\frac{\alpha_1^N}{1-\alpha_3^N}\kappa\|(I-S_N)U_{N-1}x_0\|^2 \\
&\quad + 2K^2\left(\frac{\alpha_1^N}{1-\alpha_3^N}\right)^2 \|S_N U_{N-1}x_0 - U_{N-1}x_0\|^2\biggr) + \alpha_3^N \|x_0 - x^*\|^2 \\
&= (1-\alpha_3^N)\biggl(\|U_{N-1}x_0 - x^*\|^2 \\
&\quad - 2\frac{\alpha_1^N}{1-\alpha_3^N}\left(\kappa - K^2\left(\frac{\alpha_1^N}{1-\alpha_3^N}\right)\right)\|(I-S_N)U_{N-1}x_0\|^2\biggr) + \alpha_3^N \|x_0 - x^*\|^2 \\
&\leq (1-\alpha_3^N)\|U_{N-1}x_0 - x^*\|^2 + \alpha_3^N \|x_0 - x^*\|^2 \\
&\leq (1-\alpha_3^N)\bigl((1-\alpha_3^{N-1})\|U_{N-2}x_0 - x^*\|^2 + \alpha_3^{N-1}\|x_0 - x^*\|^2\bigr) + \alpha_3^N \|x_0 - x^*\|^2 \\
&= \prod_{j=N-1}^{N}(1-\alpha_3^j)\|U_{N-2}x_0 - x^*\|^2 + \left(1 - \prod_{j=N-1}^{N}(1-\alpha_3^j)\right)\|x_0 - x^*\|^2 \\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
&\le \prod_{j=3}^{N}\bigl(1-\alpha_3^j\bigr)\bigl\|U_2x_0-x^*\bigr\|^2+\left(1-\prod_{j=3}^{N}\bigl(1-\alpha_3^j\bigr)\right)\bigl\|x_0-x^*\bigr\|^2\\
&= \prod_{j=3}^{N}\bigl(1-\alpha_3^j\bigr)\bigl\|T_2\bigl(\alpha_1^2S_2U_1+\alpha_2^2U_1+\alpha_3^2I\bigr)x_0-x^*\bigr\|^2\\
&\quad+\left(1-\prod_{j=3}^{N}\bigl(1-\alpha_3^j\bigr)\right)\bigl\|x_0-x^*\bigr\|^2\\
&\le \prod_{j=3}^{N}\bigl(1-\alpha_3^j\bigr)\bigl\|\alpha_1^2\bigl(S_2U_1x_0-x^*\bigr)+\alpha_2^2\bigl(U_1x_0-x^*\bigr)+\alpha_3^2\bigl(x_0-x^*\bigr)\bigr\|^2\\
&\quad+\left(1-\prod_{j=3}^{N}\bigl(1-\alpha_3^j\bigr)\right)\bigl\|x_0-x^*\bigr\|^2\\
&= \prod_{j=3}^{N}\bigl(1-\alpha_3^j\bigr)\Bigl\|\bigl(1-\alpha_3^2\bigr)\left(\frac{\alpha_1^2}{1-\alpha_3^2}\bigl(S_2U_1x_0-x^*\bigr)+\frac{\alpha_2^2}{1-\alpha_3^2}\bigl(U_1x_0-x^*\bigr)\right)\\
&\quad+\alpha_3^2\bigl(x_0-x^*\bigr)\Bigr\|^2+\left(1-\prod_{j=3}^{N}\bigl(1-\alpha_3^j\bigr)\right)\bigl\|x_0-x^*\bigr\|^2\\
&\le \prod_{j=3}^{N}\bigl(1-\alpha_3^j\bigr)\left(\bigl(1-\alpha_3^2\bigr)\left\|\frac{\alpha_1^2}{1-\alpha_3^2}\bigl(S_2U_1x_0-x^*\bigr)\right.\right.\\
&\quad\left.\left.+\frac{\alpha_2^2}{1-\alpha_3^2}\bigl(U_1x_0-x^*\bigr)\right\|^2+\alpha_3^2\bigl\|x_0-x^*\bigr\|^2\right)+\left(1-\prod_{j=3}^{N}\bigl(1-\alpha_3^j\bigr)\right)\bigl\|x_0-x^*\bigr\|^2\\
&= \prod_{j=3}^{N}\bigl(1-\alpha_3^j\bigr)\left(\bigl(1-\alpha_3^2\bigr)\left\|\frac{\alpha_1^2}{1-\alpha_3^2}\bigl(S_2U_1x_0-x^*\bigr)+\left(1-\frac{\alpha_1^2}{1-\alpha_3^2}\right)\bigl(U_1x_0-x^*\bigr)\right\|^2\right.\\
&\quad\left.+\alpha_3^2\bigl\|x_0-x^*\bigr\|^2\right)+\left(1-\prod_{j=3}^{N}\bigl(1-\alpha_3^j\bigr)\right)\bigl\|x_0-x^*\bigr\|^2\\
&= \prod_{j=3}^{N}\bigl(1-\alpha_3^j\bigr)\left(\bigl(1-\alpha_3^2\bigr)\left\|\frac{\alpha_1^2}{1-\alpha_3^2}(S_2U_1x_0-U_1x_0)+U_1x_0-x^*\right\|^2\right.\\
&\quad\left.+\alpha_3^2\bigl\|x_0-x^*\bigr\|^2\right)+\left(1-\prod_{j=3}^{N}\bigl(1-\alpha_3^j\bigr)\right)\bigl\|x_0-x^*\bigr\|^2\\
&\le \prod_{j=3}^{N}\bigl(1-\alpha_3^j\bigr)\left(\bigl(1-\alpha_3^2\bigr)\left(\bigl\|U_1x_0-x^*\bigr\|^2\right.\right.\\
&\quad+2\frac{\alpha_1^2}{1-\alpha_3^2}\bigl\langle S_2U_1x_0-U_1x_0,j\bigl(U_1x_0-x^*\bigr)\bigr\rangle\\
&\quad\left.\left.+2K^2\left(\frac{\alpha_1^2}{1-\alpha_3^2}\right)\bigl\|S_2U_1x_0-U_1x_0\bigr\|^2\right)+\alpha_3^2\bigl\|x_0-x^*\bigr\|^2\right)\\
&\quad+\left(1-\prod_{j=3}^{N}\bigl(1-\alpha_3^j\bigr)\right)\bigl\|x_0-x^*\bigr\|^2
\end{aligned}
$$

$$\begin{aligned}
&\leq \prod_{j=3}^{N}\left(1-\alpha_3^j\right)\left(\left(1-\alpha_3^2\right)\left(\left\|U_1x_0-x^*\right\|^2\right.\right.\\
&\quad\left.\left. -2\frac{\alpha_1^2}{1-\alpha_3^2}\left(\kappa-K^2\left(\frac{\alpha_1^2}{1-\alpha_3^2}\right)\right)\left\|(I-S_2)U_1x_0\right\|^2\right)\right.\\
&\quad\left.+\alpha_3^2\left\|x_0-x^*\right\|^2\right)+\left(1-\prod_{j=3}^{N}\left(1-\alpha_3^j\right)\right)\left\|x_0-x^*\right\|^2\\
&\leq \prod_{j=3}^{N}\left(1-\alpha_3^j\right)\left(1-\alpha_3^2\right)\left(\left\|U_1x_0-x^*\right\|^2+\alpha_3^2\left\|x_0-x^*\right\|^2\right)\\
&\quad+\left(1-\prod_{j=3}^{N}\left(1-\alpha_3^j\right)\right)\left\|x_0-x^*\right\|^2\\
&= \prod_{j=2}^{N}\left(1-\alpha_3^j\right)\left\|U_1x_0-x^*\right\|^2+\left(1-\prod_{j=2}^{N}\left(1-\alpha_3^j\right)\right)\left\|x_0-x^*\right\|^2\\
&= \prod_{j=2}^{N}\left(1-\alpha_3^j\right)\left\|\alpha_1^1\left(S_1U_0x_0-x^*\right)+\alpha_2^1\left(U_0x_0-x^*\right)+\alpha_3^1\left(x_0-x^*\right)\right\|^2\\
&\quad+\left(1-\prod_{j=2}^{N}\left(1-\alpha_3^j\right)\right)\left\|x_0-x^*\right\|^2\\
&= \prod_{j=2}^{N}\left(1-\alpha_3^j\right)\left\|\alpha_1^1\left(S_1x_0-x^*\right)+\left(1-\alpha_1^1\right)\left(x_0-x^*\right)\right\|^2\\
&\quad+\left(1-\prod_{j=2}^{N}\left(1-\alpha_3^j\right)\right)\left\|x_0-x^*\right\|^2\\
&= \prod_{j=2}^{N}\left(1-\alpha_3^j\right)\left\|\alpha_1^1(S_1x_0-x_0)+x_0-x^*\right\|^2+\left(1-\prod_{j=2}^{N}\left(1-\alpha_3^j\right)\right)\left\|x_0-x^*\right\|^2\\
&\leq \prod_{j=2}^{N}\left(1-\alpha_3^j\right)\left(\left\|x_0-x^*\right\|^2+2\alpha_1^1\left\langle S_1x_0-x_0,j\left(x_0-x^*\right)\right\rangle\right.\\
&\quad\left.+2K^2\left(\alpha_1^1\right)^2\|S_1x_0-x_0\|^2\right)+\left(1-\prod_{j=2}^{N}\left(1-\alpha_3^j\right)\right)\left\|x_0-x^*\right\|^2\\
&= \prod_{j=2}^{N}\left(1-\alpha_3^j\right)\left(\left\|x_0-x^*\right\|^2+2\alpha_1^1\left\langle S_1x_0-x^*,j\left(x_0-x^*\right)\right\rangle\right.\\
&\quad+2\alpha_1^1\left\langle x^*-x_0,j\left(x_0-x^*\right)\right\rangle\\
&\quad\left.+2K^2\left(\alpha_1^1\right)^2\|S_1x_0-x_0\|^2\right)+\left(1-\prod_{j=2}^{N}\left(1-\alpha_3^j\right)\right)\left\|x_0-x^*\right\|^2\\
&\leq \prod_{j=2}^{N}\left(1-\alpha_3^j\right)\left(\left\|x_0-x^*\right\|^2+2\alpha_1^1\left(\left\|x_0-x^*\right\|-\kappa\|S_1x_0-x_0\|^2\right)\right.\\
&\quad-2\alpha_1^1\left\|x^*-x_0\right\|^2
\end{aligned}$$

$$
\begin{aligned}
&\quad + 2K^2(\alpha_1^1)^2\|S_1x_0 - x_0\|^2) + \left(1 - \prod_{j=2}^{N}(1-\alpha_3^j)\right)\|x_0 - x^*\|^2 \\
&= \prod_{j=2}^{N}(1-\alpha_3^j)\left(\|x_0 - x^*\|^2 - 2\alpha_1^1(\kappa - K^2\alpha_1^1)\|S_1x_0 - x_0\|^2\right) \\
&\quad + \left(1 - \prod_{j=2}^{N}(1-\alpha_3^j)\right)\|x_0 - x^*\|^2 \\
&= \|x_0 - x^*\|^2 - \prod_{j=2}^{N}(1-\alpha_3^j)2\alpha_1^1(\kappa - K^2\alpha_1^1)\|S_1x_0 - x_0\|^2 \\
&\le \|x_0 - x^*\|^2. \qquad (2.2)
\end{aligned}
$$

For every $j = 1, 2, \dots, N$ and (2.2), we have

$$
\|U_jx_0 - x^*\|^2 \le \|x_0 - x^*\|^2. \qquad (2.3)
$$

For every $k = 1, 2, \dots, N-1$ and (2.2) we have

$$
\begin{aligned}
\|x_0 - x^*\|^2 &\le \prod_{j=k+1}^{N}(1-\alpha_3^j)\|U_kx_0 - x^*\|^2 + \left(1 - \prod_{j=k+1}^{N}(1-\alpha_3^j)\right)\|x_0 - x^*\|^2 \\
&= \prod_{j=k+1}^{N}(1-\alpha_3^j)\|T_k(\alpha_1^kS_kU_{k-1} + \alpha_2^kU_{k-1} + \alpha_3^kI)x_0 - x^*\|^2 \\
&\quad + \left(1 - \prod_{j=k+1}^{N}(1-\alpha_3^j)\right)\|x_0 - x^*\|^2 \\
&\le \prod_{j=k+1}^{N}(1-\alpha_3^j)\|\alpha_1^k(S_kU_{k-1}x_0 - x^*) + \alpha_2^k(U_{k-1}x_0 - x^*) + \alpha_3^k(x_0 - x^*)\|^2 \\
&\quad + \left(1 - \prod_{j=k+1}^{N}(1-\alpha_3^j)\right)\|x_0 - x^*\|^2 \\
&= \prod_{j=k+1}^{N}(1-\alpha_3^j)\left\|(1-\alpha_3^k)\left(\frac{\alpha_1^k}{1-\alpha_3^k}(S_kU_{k-1}x_0 - x^*) + \frac{\alpha_2^k}{1-\alpha_3^k}(U_{k-1}x_0 - x^*)\right)\right. \\
&\quad \left. + \alpha_3^k(x_0 - x^*)\right\|^2 + \left(1 - \prod_{j=k+1}^{N}(1-\alpha_3^j)\right)\|x_0 - x^*\|^2 \\
&\le \prod_{j=k+1}^{N}(1-\alpha_3^j)\left((1-\alpha_3^k)\left\|\frac{\alpha_1^k}{1-\alpha_3^k}(S_kU_{k-1}x_0 - x^*) + \frac{\alpha_2^k}{1-\alpha_3^k}(U_{k-1}x_0 - x^*)\right\|^2\right. \\
&\quad \left. + \alpha_3^k\|x_0 - x^*\|^2\right) + \left(1 - \prod_{j=k+1}^{N}(1-\alpha_3^j)\right)\|x_0 - x^*\|^2 \\
&= \prod_{j=k+1}^{N}(1-\alpha_3^j)\left((1-\alpha_3^k)\left\|\frac{\alpha_1^k}{1-\alpha_3^k}(S_kU_{k-1}x_0 - x^*)\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \left(1 - \frac{\alpha_1^k}{1-\alpha_3^k}\right)\left(U_{k-1}x_0 - x^*\right)\Bigg\|^2 \\
&\quad + \alpha_3^k \|x_0 - x^*\|^2\Bigg) + \left(1 - \prod_{j=k+1}^{N}\left(1-\alpha_3^j\right)\right)\|x_0 - x^*\|^2 \\
&= \prod_{j=k+1}^{N}\left(1-\alpha_3^j\right)\left(\left(1-\alpha_3^k\right)\left\|\frac{\alpha_1^k}{1-\alpha_3^k}(S_kU_{k-1}x_0 - U_{k-1}x_0) + U_{k-1}x_0 - x^*\right\|^2\right. \\
&\quad \left. + \alpha_3^k \|x_0 - x^*\|^2\right) + \left(1 - \prod_{j=k+1}^{N}\left(1-\alpha_3^j\right)\right)\|x_0 - x^*\|^2 \\
&\le \prod_{j=k+1}^{N}\left(1-\alpha_3^j\right)\Bigg(\left(1-\alpha_3^k\right)\Bigg(\|U_{k-1}x_0 - x^*\|^2 \\
&\quad + 2\frac{\alpha_1^k}{1-\alpha_3^k}\left\langle S_kU_{k-1}x_0 - U_{k-1}x_0, j\left(U_{k-1}x_0 - x^*\right)\right\rangle \\
&\quad + 2K^2\left(\frac{\alpha_1^k}{1-\alpha_3^k}\right)^2 \|S_kU_{k-1}x_0 - U_{k-1}x_0\|^2\Bigg) + \alpha_3^k\|x_0 - x^*\|^2\Bigg) \\
&\quad + \left(1 - \prod_{j=k+1}^{N}\left(1-\alpha_3^j\right)\right)\|x_0 - x^*\|^2 \\
&= \prod_{j=k+1}^{N}\left(1-\alpha_3^j\right)\Bigg(\left(1-\alpha_3^k\right)\Bigg(\|U_{k-1}x_0 - x^*\|^2 \\
&\quad + 2\frac{\alpha_1^k}{1-\alpha_3^k}\left\langle S_kU_{k-1}x_0 - x^*, j\left(U_{k-1}x_0 - x^*\right)\right\rangle \\
&\quad + 2\frac{\alpha_1^k}{1-\alpha_3^k}\left\langle x^* - U_{k-1}x_0, j\left(U_{k-1}x_0 - x^*\right)\right\rangle \\
&\quad + 2K^2\left(\frac{\alpha_1^k}{1-\alpha_3^k}\right)^2 \|S_kU_{k-1}x_0 - U_{k-1}x_0\|^2\Bigg) + \alpha_3^k\|x_0 - x^*\|^2\Bigg) \\
&\quad + \left(1 - \prod_{j=k+1}^{N}\left(1-\alpha_3^j\right)\right)\|x_0 - x^*\|^2 \\
&\le \prod_{j=k+1}^{N}\left(1-\alpha_3^j\right)\Bigg(\left(1-\alpha_3^k\right)\Bigg(\|U_{k-1}x_0 - x^*\|^2 \\
&\quad + 2\frac{\alpha_1^k}{1-\alpha_3^k}\left(\|U_{k-1}x_0 - x^*\|^2 - \kappa\left\|(I-S_k)U_{k-1}x_0\right\|\right) \\
&\quad - 2\frac{\alpha_1^k}{1-\alpha_3^k}\|x^* - U_{k-1}x_0\|^2 \\
&\quad + 2K^2\left(\frac{\alpha_1^k}{1-\alpha_3^k}\right)^2 \|S_kU_{k-1}x_0 - U_{k-1}x_0\|^2\Bigg) + \alpha_3^k\|x_0 - x^*\|^2\Bigg) \\
&\quad + \left(1 - \prod_{j=k+1}^{N}\left(1-\alpha_3^j\right)\right)\|x_0 - x^*\|^2
\end{aligned}
$$

$$\begin{aligned}
&= \prod_{j=k+1}^{N} \left(1-\alpha_3^j\right)\left(\left(1-\alpha_3^k\right)\left(\left\|U_{k-1}x_0 - x^*\right\|^2\right.\right. \\
&\quad \left.\left. - 2\frac{\alpha_1^k}{1-\alpha_3^k}\left(\kappa - K^2\left(\frac{\alpha_1^k}{1-\alpha_3^k}\right)\right)\left\|(I-S_k)U_{k-1}x_0\right\|^2\right) + \alpha_3^k\left\|x_0 - x^*\right\|^2\right) \\
&\quad + \left(1 - \prod_{j=k+1}^{N}\left(1-\alpha_3^j\right)\right)\left\|x_0 - x^*\right\|^2 \\
&\le \prod_{j=k+1}^{N} \left(1-\alpha_3^j\right)\left(\left(1-\alpha_3^k\right)\left(\left\|x_0 - x^*\right\|^2\right.\right. \\
&\quad \left.\left. - 2\frac{\alpha_1^k}{1-\alpha_3^k}\left(\kappa - K^2\left(\frac{\alpha_1^k}{1-\alpha_3^k}\right)\right)\left\|(I-S_k)U_{k-1}x_0\right\|^2\right) + \alpha_3^k\left\|x_0 - x^*\right\|^2\right) \\
&\quad + \left(1 - \prod_{j=k+1}^{N}\left(1-\alpha_3^j\right)\right)\left\|x_0 - x^*\right\|^2,
\end{aligned}$$

which implies that

$$U_{k-1}x_0 = S_k U_{k-1}x_0 \tag{2.4}$$

for every $k = 1, 2, \ldots, N-1$.

From (2.2), it implies that $x_0 = S_1 x_0$, that is, $x_0 \in F(S)$. From the definition of S^A, we have

$$U_1 x_0 = T_1\left(\alpha_1^1 S_1 U_0 x_0 + \alpha_2^1 U_0 x_0 + \alpha_3^1 x_0\right) = T_1 x_0 = x_0. \tag{2.5}$$

From (2.2) and $U_1 x_0 = x_0$, we have

$$\begin{aligned}
\left\|x_0 - x^*\right\|^2 &\le \prod_{j=3}^{N}\left(1-\alpha_3^j\right)\left(\left(1-\alpha_3^2\right)\left(\left\|U_1 x_0 - x^*\right\|^2\right.\right. \\
&\quad \left. - 2\frac{\alpha_1^2}{1-\alpha_3^2}\left(\kappa - K^2\left(\frac{\alpha_1^2}{1-\alpha_3^2}\right)\right)\left\|(I-S_2)U_1 x_0\right\|^2\right) \\
&\quad \left. + \alpha_3^2\left\|x_0 - x^*\right\|^2\right) + \left(1 - \prod_{j=3}^{N}\left(1-\alpha_3^j\right)\right)\left\|x_0 - x^*\right\|^2 \\
&= \prod_{j=3}^{N}\left(1-\alpha_3^j\right)\left(\left(1-\alpha_3^2\right)\left(\left\|x_0 - x^*\right\|^2\right.\right. \\
&\quad \left. - 2\frac{\alpha_1^2}{1-\alpha_3^2}\left(\kappa - K^2\left(\frac{\alpha_1^2}{1-\alpha_3^2}\right)\right)\left\|(I-S_2)x_0\right\|^2\right) \\
&\quad \left. + \alpha_3^2\left\|x_0 - x^*\right\|^2\right) + \left(1 - \prod_{j=3}^{N}\left(1-\alpha_3^j\right)\right)\left\|x_0 - x^*\right\|^2 \\
&= \prod_{j=3}^{N}\left(1-\alpha_3^j\right)\left(1-\alpha_3^2\right)\left(\left\|x_0 - x^*\right\|^2\right. \\
&\quad \left. - 2\frac{\alpha_1^2}{1-\alpha_3^2}\left(\kappa - K^2\left(\frac{\alpha_1^2}{1-\alpha_3^2}\right)\right)\left\|(I-S_2)x_0\right\|^2\right)
\end{aligned}$$

$$
\begin{aligned}
&+\prod_{j=3}^{N}(1-\alpha_3^j)\alpha_3^2\|x_0-x^*\|^2+\left(1-\prod_{j=3}^{N}(1-\alpha_3^j)\right)\|x_0-x^*\|^2\\
&=\prod_{j=2}^{N}(1-\alpha_3^j)\left(\|x_0-x^*\|^2-2\frac{\alpha_1^2}{1-\alpha_3^2}\left(\kappa-K^2\left(\frac{\alpha_1^2}{1-\alpha_3^2}\right)\right)\|(I-S_2)x_0\|^2\right)\\
&\quad+\left(1-\prod_{j=2}^{N}(1-\alpha_3^j)\right)\|x_0-x^*\|^2.
\end{aligned}
$$

It implies that $x_0 = S_2x_0$.

From the definition of S^A and $x_0 = S_2x_0$, we have

$$U_2x_0 = T_2(\alpha_1^2S_2U_1+\alpha_2^2U_1+\alpha_3^2I)x_0 = T_2x_0. \tag{2.6}$$

From the definition of U_3 and (2.4), we have

$$U_3x_0 = T_3(\alpha_1^3S_3U_2+\alpha_2^3U_2+\alpha_3^3I)x_0 = T_3((1-\alpha_3^3)U_2x_0+\alpha_3^3x_0). \tag{2.7}$$

From (2.2), (2.6), (2.7) and E is uniformly convex, we have

$$
\begin{aligned}
\|x_0-x^*\|^2 &\le \prod_{j=4}^{N}(1-\alpha_3^j)\|U_3x_0-x^*\|^2+\left(1-\prod_{j=4}^{N}(1-\alpha_3^j)\right)\|x_0-x^*\|^2\\
&=\prod_{j=4}^{N}(1-\alpha_3^j)\|T_3((1-\alpha_3^3)U_2x_0+\alpha_3^3x_0)-x^*\|^2\\
&\quad+\left(1-\prod_{j=4}^{N}(1-\alpha_3^j)\right)\|x_0-x^*\|^2\\
&\le\prod_{j=4}^{N}(1-\alpha_3^j)\|(1-\alpha_3^3)(U_2x_0-x^*)+\alpha_3^3(x_0-x^*)\|^2\\
&\quad+\left(1-\prod_{j=4}^{N}(1-\alpha_3^j)\right)\|x_0-x^*\|^2\\
&=\prod_{j=4}^{N}(1-\alpha_3^j)\|(1-\alpha_3^3)(T_2x_0-x^*)+\alpha_3^3(x_0-x^*)\|^2\\
&\quad+\left(1-\prod_{j=4}^{N}(1-\alpha_3^j)\right)\|x_0-x^*\|^2\\
&\le\prod_{j=4}^{N}(1-\alpha_3^j)\Big((1-\alpha_3^3)\|T_2x_0-x^*\|^2+\alpha_3^3\|x_0-x^*\|^2\\
&\quad-\alpha_3^3(1-\alpha_3^3)g_2(\|T_2x_0-x_0\|)\Big)\\
&\quad+\left(1-\prod_{j=4}^{N}(1-\alpha_3^j)\right)\|x_0-x^*\|^2
\end{aligned}
$$

$$\begin{aligned}&\le \prod_{j=4}^{N}(1-\alpha_3^j)\big(\|x_0-x^*\|^2-\alpha_3^3(1-\alpha_3^3)g_2(\|T_2x_0-x_0\|)\big)\\&\quad+\left(1-\prod_{j=4}^{N}(1-\alpha_3^j)\right)\|x_0-x^*\|^2.\end{aligned}$$

It implies that

$$g_2(\|T_2x_0-x_0\|)=0. \tag{2.8}$$

Assume that $T_2x_0\neq x_0$, then we have $\|T_2x_0-x_0\|>0$. From the properties of g_2, we have

$$0=g(0)<g(\|T_2x_0-x_0\|)=0. \tag{2.9}$$

This is a contradiction. Then we have $T_2x_0=x_0$. From (2.6), we have $x_0=T_2x_0=U_2x_0$.

From the definition of U_3, we have

$$U_3x_0=T_3\big((1-\alpha_3^3)U_2x_0+\alpha_3^3x_0\big)=T_3x_0.$$

By using the same method as above, we have

$$x_0=U_3x_0=T_3x_0.$$

Continuing on this way, we can conclude that

$$x_0=U_ix_0=T_ix_0 \tag{2.10}$$

for every $i=1,2,\dots,N-1$. From (2.2) and (2.10), we have

$$\begin{aligned}\|x_0-x^*\|^2&\le(1-\alpha_3^N)\bigg(\|U_{N-1}x_0-x^*\|^2\\&\quad-2\frac{\alpha_1^N}{1-\alpha_3^N}\bigg(\kappa-K^2\bigg(\frac{\alpha_1^N}{1-\alpha_3^N}\bigg)\bigg)\|(I-S_N)U_{N-1}x_0\|^2\bigg)+\alpha_3^N\|x_0-x^*\|^2\\&=(1-\alpha_3^N)\bigg(\|x_0-x^*\|^2-2\frac{\alpha_1^N}{1-\alpha_3^N}\bigg(\kappa-K^2\bigg(\frac{\alpha_1^N}{1-\alpha_3^N}\bigg)\bigg)\|(I-S_N)x_0\|^2\bigg)\\&\quad+\alpha_3^N\|x_0-x^*\|^2.\end{aligned}$$

It implies that

$$x_0=S_Nx_0. \tag{2.11}$$

From the definition of S^A and (2.10), we have

$$x_0=S^Ax_0=U_Nx_0=T_N\big(\alpha_1^NS_NU_{N-1}+\alpha_2^NU_{N-1}+\alpha_3^NI\big)x_0=T_Nx_0.$$

Then we have

$$x_0 \in \bigcap_{i=1}^{N} F(T_i) \quad \text{and} \quad x_0 \in \bigcap_{i=1}^{N} F(U_i). \tag{2.12}$$

Since $S_k U_{k-1} x_0 = U_{k-1} x_0$ for every $k = 1, 2, \ldots, N-1$ and $x_0 \in \bigcap_{i=1}^{N} F(U_i)$, then we have

$$S_k x_0 = x_0$$

for every $k = 1, 2, \ldots, N-1$. From (2.11), it implies that

$$x_0 \in \bigcap_{i=1}^{N} F(S_i). \tag{2.13}$$

From (2.12) and (2.13), we have

$$x_0 \in \bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} F(S_i). \tag{2.14}$$

Hence, $F(S^A) \subseteq \bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} F(S_i)$. It is easy to see that $\bigcap_{i=1}^{N} F(T_i) \cap \bigcap_{i=1}^{N} F(S_i) \subseteq F(S^A)$.

Applying (2.2), we have that the mapping S^A is nonexpansive. □

Lemma 2.8 [19] *Let C be a closed convex subset of a strictly convex Banach space E. Let T_1 and T_2 be two nonexpansive mappings from C into itself with $F(T_1) \cap F(T_2) \neq \emptyset$. Define a mapping S by*

$$Sx = \lambda T_1 x + (1-\lambda) T_2 x, \quad \forall x \in C,$$

where λ is a constant in $(0,1)$. Then S is nonexpansive and $F(S) = F(T_1) \cap F(T_2)$.

Applying Lemma 2.8, we have the following lemma.

Lemma 2.9 *Let C be a closed convex subset of a strictly convex Banach space E. Let T_1, T_2 and T_3 be three nonexpansive mappings from C into itself with $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$. Define a mapping S by*

$$Sx = \alpha T_1 x + \beta T_2 x + \gamma T_3 x, \quad \forall x \in C,$$

where α, β, γ is a constant in $(0,1)$ and $\alpha + \beta + \gamma = 1$. Then S is nonexpansive and $F(S) = F(T_1) \cap F(T_2) \cap F(T_3)$.

Proof For every $x \in C$ and the definition of the mapping S, we have

$$\begin{aligned} Sx &= \alpha T_1 x + \beta T_2 x + \gamma T_3 x \\ &= \alpha T_1 x + (1-\alpha)\left(\frac{\beta}{1-\alpha} T_2 x + \frac{\gamma}{1-\alpha} T_3 x\right) \end{aligned}$$

$$= \alpha T_1x + (1-\alpha)\left(\frac{\beta}{1-\alpha}T_2x + \left(1-\frac{\beta}{1-\alpha}\right)T_3x\right)$$
$$= \alpha T_1x + (1-\alpha)S_1x, \tag{2.15}$$

where $S_1 = \frac{\beta}{1-\alpha}T_2 + (1-\frac{\beta}{1-\alpha})T_3$. From Lemma 2.8, we have $F(S_1) = F(T_2) \cap F(T_3)$ and S_1 is a nonexpansive mapping. From Lemma 2.8 and (2.15), we have $F(S) = F(T_1) \cap F(S_1)$ and S is a nonexpansive mapping. Hence we have $F(S) = F(T_1) \cap F(T_2) \cap F(T_3)$. □

3 Main results

Theorem 3.1 *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C and let A, B be α- and β-inverse strongly accretive mappings of C into E, respectively. Let $\{S_i\}_{i=1}^N$ be a finite family of κ_i-strict pseudo-contractions of C into itself and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\mathcal{F} = \bigcap_{i=1}^N F(S_i) \cap \bigcap_{i=1}^N F(T_i) \cap S(C,A) \cap S(C,B) \neq \emptyset$ and $\kappa = \min\{\kappa_i : i = 1,2,\ldots,N\}$ with $K^2 \leq \kappa$, where K is the 2-uniformly smooth constant of E. Let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0,1]$, $\alpha_2^j \in [0,1]$ and $\alpha_3^j \in (0,1)$ for all $j = 1,2,\ldots,N$. Let S^A be the S^A-mapping generated by $S_1, S_2, \ldots, S_N$, $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. Let $\{x_n\}$ be the sequence generated by $x_1, u \in C$ and*

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Q_C(I-aA)x_n + \delta_n Q_C(I-bB)x_n + \eta_n S^A x_n, \quad \forall n \geq 1, \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{\eta_n\} \in [0,1]$ and $\alpha_n + \beta_n + \gamma_n + \delta_n + \eta_n = 1$ and satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$

(ii) $\{\gamma_n\}, \{\delta_n\}, \{\eta_n\} \subseteq [c,d] \subset (0,1),$ *for some* $c, d > 0$, $\forall n \geq 1$,

(iii) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n|, \quad \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n|, \quad \sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n|,$
$\sum_{n=1}^{\infty} |\eta_{n+1} - \eta_n|, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$

(iv) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1,$

(v) $a \in \left(0, \frac{\alpha}{K^2}\right)$ *and* $b \in \left(0, \frac{\beta}{K^2}\right)$.

Then $\{x_n\}$ converges strongly to $z_0 = Q_{\mathcal{F}}u$, where $Q_{\mathcal{F}}$ is the sunny nonexpansive retraction of C onto $\mathcal{F}$.

Proof First we show that $Q_C(I-aA)$ and $Q_C(I-bB)$ are nonexpansive mappings. Let $x, y \in C$, we have

$$\|Q_C(I-aA)x - Q_C(I-aA)y\|^2 \leq \|x-y-a(Ax-Ay)\|^2$$
$$\leq \|x-y\|^2 - 2a\langle Ax-Ay, j(x-y)\rangle + 2K^2a^2\|Ax-Ay\|^2$$

$$\begin{aligned}
&\leq \|x-y\|^2 - 2a\alpha\|Ax-Ay\|^2 + 2K^2a^2\|Ax-Ay\|^2 \\
&= \|x-y\|^2 - 2a(\alpha - K^2a)\|Ax-Ay\|^2 \\
&\leq \|x-y\|^2. \qquad (3.2)
\end{aligned}$$

Then we have $Q_C(I-aA)$ is a nonexpansive mapping. By using the same methods as (3.2), we have $Q_C(I-bB)$ is a nonexpansive mapping.

Let $x^* \in \mathcal{F}$. From Lemma 2.3, we have $x^* \in F(Q_C(I-aA))$ and $x^* \in F(Q_C(I-bB))$. By the definition of x_n, we have

$$\begin{aligned}
\|x_{n+1}-x^*\| &\leq \alpha_n\|u-x^*\| + \beta_n\|x_n-x^*\| + \gamma_n\|Q_C(I-aA)x_n-x^*\| \\
&\quad + \delta_n\|Q_C(I-bB)x_n-x^*\| + \eta_n\|S^Ax_n-x^*\| \\
&\leq \alpha_n\|u-x^*\| + (1-\alpha_n)\|x_n-x^*\| \\
&\leq \max\{\|u-x^*\|, \|x_1-x^*\|\}.
\end{aligned}$$

By induction, we have $\|x_n-x^*\| \leq \max\{\|u-x^*\|, \|x_1-x^*\|\}$. We can imply that the sequence $\{x_n\}$ is bounded and so are $\{S^Ax_n\}$, $\{Q_C(I-aA)x_n\}$ and $\{Q_C(I-bB)x_n\}$.

Next, we show that $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$. From the definition of x_n, we have

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \|\alpha_nu + \beta_nx_n + \gamma_nQ_C(I-aA)x_n + \delta_nQ_C(I-bB)x_n + \eta_nS^Ax_n \\
&\quad - \alpha_{n-1}u - \beta_{n-1}x_{n-1} - \gamma_{n-1}Q_C(I-aA)x_{n-1} - \delta_{n-1}Q_C(I-bB)x_{n-1} \\
&\quad - \eta_{n-1}S^Ax_{n-1}\| \\
&\leq |\alpha_n-\alpha_{n-1}|\|u\| + \beta_n\|x_n-x_{n-1}\| + |\beta_n-\beta_{n-1}|\|x_{n-1}\| \\
&\quad + \gamma_n\|Q_C(I-aA)x_n - Q_C(I-aA)x_{n-1}\| + |\gamma_n-\gamma_{n-1}|\|Q_C(I-aA)x_{n-1}\| \\
&\quad + \delta_n\|Q_C(I-bB)x_n - Q_C(I-bB)x_{n-1}\| + |\delta_n-\delta_{n-1}|\|Q_C(I-bB)x_{n-1}\| \\
&\quad + \eta_n\|S^Ax_n - S^Ax_{n-1}\| + |\eta_{n-1}-\eta_n|\|S^Ax_n\| \\
&\leq (1-\alpha_n)\|x_n-x_{n-1}\| + |\alpha_n-\alpha_{n-1}|\|u\| + |\beta_n-\beta_{n-1}|\|x_{n-1}\| \\
&\quad + |\gamma_n-\gamma_{n-1}|\|Q_C(I-aA)x_{n-1}\| + |\delta_n-\delta_{n-1}|\|Q_C(I-bB)x_{n-1}\| \\
&\quad + |\eta_{n-1}-\eta_n|\|S^Ax_n\|.
\end{aligned}$$

Applying Lemma 2.6, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \qquad (3.3)$$

Next, we show that

$$\lim_{n\to\infty}\|Q_C(I-aA)x_n-x_n\| = \lim_{n\to\infty}\|Q_C(I-bB)x_n-x_n\| = \lim_{n\to\infty}\|S^Ax_n-x_n\| = 0. \qquad (3.4)$$

From the definition of x_n, we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|\alpha_n(u-x^*) + \beta_n(x_n-x^*) + \gamma_n(Q_C(I-aA)x_n-x^*) \\
&\quad + \delta_n(Q_C(I-bB)x_n-x^*) + \eta_n(S^Ax_n-x^*)\|^2
\end{aligned}$$

$$
\begin{aligned}
&= \Bigg\| \beta_n\big(x_n - x^*\big) + \gamma_n\big(Q_C(I-aA)x_n - x^*\big) + (\alpha_n + \delta_n + \eta_n)\bigg(\frac{\alpha_n(u-x^*)}{\alpha_n + \delta_n + \eta_n} \\
&\quad + \frac{\delta_n(Q_C(I-bB)x_n - x^*)}{\alpha_n + \delta_n + \eta_n} + \frac{\eta_n(S^A x_n - x^*)}{\alpha_n + \delta_n + \eta_n}\bigg)\Bigg\|^2 \\
&= \big\| \beta_n\big(x_n - x^*\big) + \gamma_n\big(Q_C(I-aA)x_n - x^*\big) + c_n z_n\big\|^2,
\end{aligned}
$$

where $c_n = \alpha_n + \delta_n + \eta_n$ and $z_n = \frac{\alpha_n(u-x^*)}{\alpha_n+\delta_n+\eta_n} + \frac{\delta_n(Q_C(I-bB)x_n-x^*)}{\alpha_n+\delta_n+\eta_n} + \frac{\eta_n(S^A x_n-x^*)}{\alpha_n+\delta_n+\eta_n}$.

From Lemma 2.2, we have

$$
\begin{aligned}
\big\|x_{n+1} - x^*\big\|^2 &\le \beta_n \big\|x_n - x^*\big\|^2 + \gamma_n \big\|Q_C(I-aA)x_n - x^*\big\| + c_n\|z_n\|^2 \\
&\quad - \beta_n\gamma_n g_1\big(\big\|x_n - Q_C(I-aA)x_n\big\|\big) \\
&\le (\beta_n + \gamma_n)\big\|x_n - x^*\big\|^2 - \beta_n\gamma_n g_1\big(\big\|x_n - Q_C(I-aA)x_n\big\|\big) \\
&\quad + c_n\bigg(\frac{\alpha_n\|u-x^*\|^2}{\alpha_n+\delta_n+\eta_n} + \frac{\delta_n\|Q_C(I-bB)x_n - x^*\|^2}{\alpha_n+\delta_n+\eta_n} + \frac{\eta_n\|S^A x_n - x^*\|^2}{\alpha_n+\delta_n+\eta_n}\bigg) \\
&\le (\beta_n + \gamma_n)\big\|x_n - x^*\big\|^2 - \beta_n\gamma_n g_1\big(\big\|x_n - Q_C(I-aA)x_n\big\|\big) \\
&\quad + \alpha_n\big\|u - x^*\big\|^2 + (\delta_n + \eta_n)\big\|x_n - x^*\big\|^2 \\
&\le \big\|x_n - x^*\big\|^2 - \beta_n\gamma_n g_1\big(\big\|x_n - Q_C(I-aA)x_n\big\|\big) + \alpha_n\big\|u - x^*\big\|^2,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\beta_n\gamma_n g_1\big(\big\|x_n - Q_C(I-aA)x_n\big\|\big) &\le \big\|x_n - x^*\big\|^2 - \big\|x_{n+1} - x^*\big\|^2 + \alpha_n\big\|u - x^*\big\|^2 \\
&\le \big(\big\|x_n - x^*\big\| + \big\|x_{n+1} - x^*\big\|\big)\big\|x_{n+1} - x_n\big\| \\
&\quad + \alpha_n\big\|u - x^*\big\|^2. \qquad (3.5)
\end{aligned}
$$

From (3.3) and condition (i), we obtain

$$
\lim_{n\to\infty} g_1\big(\big\|x_n - Q_C(I-aA)x_n\big\|\big) = 0. \qquad (3.6)
$$

From the property of g_1, we have

$$
\lim_{n\to\infty} \big\|x_n - Q_C(I-aA)x_n\big\| = 0. \qquad (3.7)
$$

By using the same method as (3.7), we can imply that

$$
\lim_{n\to\infty} \big\|x_n - Q_C(I-bB)x_n\big\| = \lim_{n\to\infty} \big\|x_n - S^A x_n\big\| = 0.
$$

Define $Gx = \alpha S^A x + \beta Q_C(I-aA)x + \gamma Q_C(I-bB)x$ for all $x \in C$ and $\alpha + \beta + \gamma = 1$. From Lemma 2.9, we have $F(G) = F(Q_C(I-aA)) \cap F(Q_C(I-bB)) \cap F(S^A)$. From Lemmas 2.3 and 2.7, we have $\mathcal{F} = F(G) = \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N F(S_i) \cap S(C,A) \cap S(C,B)$. By the definition of G, we obtain

$$
\|Gx_n - x_n\| \le \alpha\big\|S^A x_n - x_n\big\| + \beta\big\|Q_C(I-aA)x_n - x_n\big\| + \gamma\big\|Q_C(I-bB)x_n - x_n\big\|.
$$

From (3.4), we have

$$\lim_{n\to\infty} \|Gx_n - x_n\| = 0. \tag{3.8}$$

From Lemma 2.5 and (3.8), we have

$$\limsup_{n\to\infty} \langle u - z_0, j(x_n - z_0)\rangle \le 0, \tag{3.9}$$

where $z_0 = Q_{\mathcal{F}}u$. Finally, we prove strong convergence of the sequence $\{x_n\}$ to $z_0 = Q_{\mathcal{F}}u$. From the definition of x_n, we have

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &= \big\|\alpha_n(u - z_0) + \beta_n(x_n - z_0) + \gamma_n\big(Q_C(I - aA)x_n - z_0\big) \\
&\quad + \delta_n\big(Q_C(I - bB)x_n - z_0\big) + \eta_n\big(S^A x_n - z_0\big)\big\|^2 \\
&= \bigg\|\alpha_n(u - z_0) + (1 - \alpha_n)\bigg(\frac{\beta_n(x_n - z_0)}{1 - \alpha_n} + \frac{\gamma_n(Q_C(I - aA)x_n - z_0)}{1 - \alpha_n} \\
&\quad + \frac{\delta_n(Q_C(I - bB)x_n - z_0)}{1 - \alpha_n} + \frac{\eta_n(S^A x_n - z_0)}{1 - \alpha_n}\bigg)\bigg\|^2 \\
&\le \bigg\|(1 - \alpha_n)\bigg(\frac{\beta_n(x_n - z_0)}{1 - \alpha_n} + \frac{\gamma_n(Q_C(I - aA)x_n - z_0)}{1 - \alpha_n} \\
&\quad + \frac{\delta_n(Q_C(I - bB)x_n - z_0)}{1 - \alpha_n} + \frac{\eta_n(S^A x_n - z_0)}{1 - \alpha_n}\bigg)\bigg\|^2 + 2\alpha_n\langle u - x_0, j(x_{n+1} - z_0)\rangle \\
&\le (1 - \alpha_n)\|x_n - z_0\|^2 + 2\alpha_n\langle u - x_0, j(x_{n+1} - z_0)\rangle.
\end{aligned}$$

Applying Lemma 2.6 and condition (i), we have $\lim_{n\to\infty} \|x_n - z_0\| = 0$. This completes the proof. □

4 Applications

From our main results, we obtain strong convergence theorems in a Banach space. Before proving these theorem, we need the following lemma which is the result from Lemma 2.7 and Definition 1.4. Therefore, we omit the proof.

Lemma 4.1 *Let C be a nonempty closed convex subset of a 2-uniformly smooth and uniformly convex Banach space. Let $\{S_i\}_{i=1}^N$ be a finite family of κ_i-strict pseudo-contractions of C into itself with $\bigcap_{i=1}^N F(S_i) \ne \emptyset$ and $\kappa = \min\{\kappa_i : i = 1, 2, \dots, N\}$ with $K^2 \le \kappa$, where K is the 2-uniformly smooth constant of E. Let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0,1]$, $\alpha_2^j \in [0,1]$ and $\alpha_3^j \in (0,1)$ for all $j = 1, 2, \dots, N$. Let S be the S-mapping generated by $S_1, S_2, \dots, S_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$. Then $F(S) = \bigcap_{i=1}^N F(S_i)$ and S is a nonexpansive mapping.*

Theorem 4.2 *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C and let A, B be α- and β-inverse strongly accretive mappings of C into E, respectively. Let $\{S_i\}_{i=1}^N$ be a finite family of κ_i-strict pseudo-contractions of C into itself with $\mathcal{F} = \bigcap_{i=1}^N F(S_i) \cap S(C,A) \cap S(C,B) \ne \emptyset$ and $\kappa = \min\{\kappa_i : i = 1, 2, \dots, N\}$ with $K^2 \le \kappa$, where K is the 2-uniformly smooth constant of E. Let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0,1]$,*

$\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0,1]$, $\alpha_2^j \in [0,1]$ and $\alpha_3^j \in (0,1)$ for all $j = 1,2,\dots,N$. Let S be the S-mapping generated by $S_1, S_2, \dots, S_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$. Let $\{x_n\}$ be the sequence generated by $x_1, u \in C$ and

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Q_C(I - aA)x_n + \delta_n Q_C(I - bB)x_n + \eta_n S x_n, \quad \forall n \geq 1,$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{\eta_n\} \in [0,1]$ and $\alpha_n + \beta_n + \gamma_n + \delta_n + \eta_n = 1$ and satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$

(ii) $\{\gamma_n\}, \{\delta_n\}, \{\eta_n\} \subseteq [c,d] \subset (0,1)$ *for some* $c, d > 0$, $\forall n \geq 1$,

(iii) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n|, \quad \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n|, \quad \sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n|,$
$\sum_{n=1}^{\infty} |\eta_{n+1} - \eta_n|, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$

(iv) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1,$

(v) $a \in \left(0, \frac{\alpha}{K^2}\right)$ *and* $b \in \left(0, \frac{\beta}{K^2}\right).$

Then $\{x_n\}$ converges strongly to $z_0 = Q_{\mathcal{F}} u$, where $Q_{\mathcal{F}}$ is the sunny nonexpansive retraction of C onto $\mathcal{F}$.

Proof Put $I = T_1 = T_2 = \cdots = T_N$ in Theorem 3.1. From Lemma 4.1 and Theorem 3.1 we can conclude the desired result. □

Theorem 4.3 *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C. For every $i = 1,2,\dots,N$, let A_i, A, B be α_i-, α- and β-inverse strongly accretive mappings of C into E, respectively. Define a mapping $G_i : C \to C$ by $Q_C(I - \lambda_i A_i)x = G_i x$, where $\lambda_i \in (0, \frac{\alpha_i}{K^2})$, K is the 2-uniformly smooth constant of E, for all $x \in C$ and $i = 1,2,\dots,N$. Let $\{S_i\}_{i=1}^N$ be a finite family of κ_i-strict pseudo-contractions of C into itself and with $\mathcal{F} = \bigcap_{i=1}^N F(S_i) \cap \bigcap_{i=1}^N S(C, A_i) \cap S(C, A) \cap S(C, B) \neq \emptyset$ and $\kappa = \min\{\kappa_i : i = 1,2,\dots,N\}$ with $K^2 \leq \kappa$. Let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0,1]$, $\alpha_2^j \in [0,1]$ and $\alpha_3^j \in (0,1)$ for all $j = 1,2,\dots,N$. Let S^A be the S^A-mapping generated by $S_1, S_2, \dots, S_N$, $G_1, G_2, \dots, G_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$. Let $\{x_n\}$ be the sequence generated by $x_1, u \in C$ and*

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Q_C(I - aA)x_n + \delta_n Q_C(I - bB)x_n + \eta_n S^A x_n, \quad \forall n \geq 1,$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}, \{\eta_n\} \in [0,1]$ and $\alpha_n + \beta_n + \gamma_n + \delta_n + \eta_n = 1$ and satisfy the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$

(ii) $\{\gamma_n\}, \{\delta_n\}, \{\eta_n\} \subseteq [c,d] \subset (0,1)$ *for some* $c, d > 0, \forall n \geq 1$,

(iii) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n|, \quad \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n|, \quad \sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n|,$
$\sum_{n=1}^{\infty} |\eta_{n+1} - \eta_n|, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$

(iv) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(v) $a \in \left(0, \frac{\alpha}{K^2}\right)$ *and* $b \in \left(0, \frac{\beta}{K^2}\right)$.

Then $\{x_n\}$ *converges strongly to* $z_0 = Q_{\mathcal{F}}u$, *where* $Q_{\mathcal{F}}$ *is the sunny nonexpansive retraction of* C *onto* $\mathcal{F}$.

Proof By using the same method as (3.2), we can conclude that $\{G_i\}_{i=1}^N$ is a nonexpansive mapping. From Lemma 2.3, we have $F(G_i) = S(C, A_i)$ for all $i = 1, 2, \ldots, N$. From Theorem 3.1 we can conclude the desired conclusion. □

Competing interests
The author declares that they have no competing interests.

Acknowledgements
This research was supported by the Research Administration Division of King Mongkut's Institute of Technology Ladkrabang.

Received: 26 January 2013 Accepted: 28 May 2013 Published: 18 June 2013

References
1. Sahua, DR, Petru'lel, A: Strong convergence of iterative methods by strictly pseudocontractive mappings in Banach spaces. Nonlinear Anal. **74**, 6012-6023 (2011)
2. Reich, S: Asymptotic behavior of contractions in Banach spaces. J. Math. Anal. Appl. **44**, 57-70 (1973)
3. Aoyama, K, Iiduka, H, Takahashi, W: Weak convergence of an iterative sequence for accretive operators in Banach spaces. Fixed Point Theory Appl. **2006**, Article ID 35390 (2006). doi:10.1155/FPTA/2006/35390
4. Chang, SS, Joseph Lee, HW, Chan, CK: A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization. Nonlinear Anal. **70**, 3307-3319 (2009)
5. Cai, G, Bu, S: Strong convergence theorems for variational inequality problems and fixed point problems in uniformly smooth and uniformly convex Banach spaces. J. Glob. Optim. (2012). doi:10.1007/s10898-012-9923-2
6. Jaillet, P, Lamberton, D, Lapeyre, B: Variational inequalities and the pricing of American options. Acta Appl. Math. **21**, 263-289 (1990)
7. Mann, WR: Mean value methods in iteration. Proc. Am. Math. Soc. **4**(3), 506-510 (1953)
8. Buong, N, Lang, ND: Hybrid Mann-Halpern iteration methods for nonexpansive mappings and semigroups. Appl. Math. Comput. **218**, 2459-2466 (2011)
9. Song, Y: A new sufficient condition for strong convergence of Halpern type iterations. Appl. Math. Comput. **198**(2), 721-728 (2007)
10. Qin, X, Cho, YJ, Kang, SM, Zhou, H: Convergence of a modified Halpern-type iteration algorithm for quasi-ϕ-nonexpansive mappings. Appl. Math. Lett. **22**, 1051-1055 (2009)
11. Zhou, H: Convergence theorems for λ-strict pseudo-contractions in 2-uniformly smooth Banach spaces. Nonlinear Anal. **69**, 3160-3173 (2008)
12. Kangtunyakarn, A, Suantai, S: Hybrid iterative scheme for generalized equilibrium problems and fixed point problems of finite family of nonexpansive mappings. Nonlinear Anal. Hybrid Syst. **3**, 296-309 (2009)
13. Kangtunyakarn, A, Suantai, S: A new mapping for finding common solutions of equilibrium problems and fixed point problems of finite family of nonexpansive mappings. Nonlinear Anal. **71**, 4448-4460 (2009)
14. Kangtunyakarn, A: Iterative scheme for a nonexpansive mapping, an η-strictly pseudo-contractive mapping and variational inequality problems in a uniformly convex and 2-uniformly smooth Banach space. Fixed Point Theory Appl. **2013**, Article ID 23 (2013)
15. Xu, HK: Inequalities in Banach spaces with applications. Nonlinear Anal. **16**, 1127-1138 (1991)
16. Cho, YJ, Zhou, HY, Guo, G: Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings. Comput. Math. Appl. **47**, 707-717 (2004)
17. Zhou, H: Convergence theorems for λ-strict pseudocontractions in 2-uniformly smooth Banach spaces. Nonlinear Anal. **69**, 3160-3173 (2008)

18. Xu, HK: An iterative approach to quadratic optimization. J. Optim. Theory Appl. **116**, 659-678 (2003)
19. Bruck, RE: Properties of fixed point sets of nonexpansive mappings in Banach spaces. Trans. Am. Math. Soc. **179**, 251-262 (1973)

doi:10.1186/1687-1812-2013-157
Cite this article as: Kangtunyakarn: A new mapping for finding a common element of the sets of fixed points of two finite families of nonexpansive and strictly pseudo-contractive mappings and two sets of variational inequalities in uniformly convex and 2-smooth Banach spaces. *Fixed Point Theory and Applications* 2013 **2013**:157.