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QUADRATIC ALGEBRAS RELATED TO ELLIPTIC CURVES

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We construct quadratic finite-dimensional Poisson algebras corresponding to a rank-N degree-one vector bundle over an elliptic curve with n marked points and also construct the quantum version of the algebras. The algebras are parameterized by the moduli of curves. For $N = 2$ and $n = 1$, they coincide with Sklyanin algebras. We prove that the Poisson structure is compatible with the Lie–Poisson structure defined on the direct sum of n copies of $sl(N)$. The origin of the algebras is related to the Poisson reduction of canonical brackets on an affine space over the bundle cotangent to automorphism groups of vector bundles.

Keywords: Poisson structure, integrable system

1. Introduction

We construct quadratic Poisson algebras (classical Sklyanin–Feigin–Odessky algebras) based on the commutation relations involving the elliptic $sl(N,\mathbb{C})$ Belavin–Drinfeld r-matrix [1] and also construct the quantum version of the algebras corresponding to vertex elliptic matrices [2]. These algebras are parameterized by moduli spaces of complex structures of elliptic curves with n marked points and in the quantum case by the Planck constant corresponding to the curve. For $N = 2$ and $n = 1$, we recover the Sklyanin algebra [3]. The constructed algebras are a special case of the general construction [4] but are finitely generated in contrast to the general case. We explicitly describe the Poisson brackets between the generators and describe the corresponding quadratic relations in the quantum case in terms of quasiperiodic functions on the moduli space. On one hand, in the classical case, Poisson algebras have the form of quadratic algebras on the direct product of n copies of $GL(N,\mathbb{C})$ with a nontrivial mixing of the components. On the other hand, there exists the standard linear Lie–Poisson structure on the direct sum of n copies of $GL(N,\mathbb{C})$-valued Lie groups. We prove that these two structures are compatible. Classical algebras underlie the symmetries of an elliptic generalization of the Schlesinger and Garnier systems [5], [6].

In Sec. 2, we use the Poisson reduction to derive a classical vertex r-matrix and an $GL(N,\mathbb{C})$-valued Lax matrix with n simple poles from the canonical brackets on a certain generalization of the cotangent bundle to the two-loop group of $GL(N,\mathbb{C})$. In Sec. 3, we obtain the exact form of the brackets and prove that they are compatible with the Lie–Poisson bracket. Section 4 is devoted to the quantum case.

2. Classical commutation relations of the two-loop group $GL(N,\mathbb{C})$

2.1. Degree-one bundles over elliptic curves. Let $\Sigma_\tau = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ be an elliptic curve with the modular parameter τ, $\operatorname{Im}\tau > 0$. We consider a rank-$N$ vector bundle E_N over Σ_τ. It is described by

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its sections $s = (s_1(z,\bar{z}),\ldots,s_N(z,\bar{z}))$ with the monodromies

$$s^{\mathrm{T}}(z+1,\bar{z}+1) = Qs^{\mathrm{T}}(z,\bar{z}), \qquad s^{\mathrm{T}}(z+\tau,\bar{z}+\bar{\tau}) = \widetilde{\Lambda}s^{\mathrm{T}}(z,\bar{z}),$$

where

$$Q = \mathrm{diag}(1,\mathbf{e}_N,\ldots,\mathbf{e}_N^{N-1}), \qquad \mathbf{e}_N = e^{2\pi i/N}, \qquad \widetilde{\Lambda} = \mathbf{e}_N^{-(z+\tau/2)}\Lambda, \qquad \Lambda = (E_{j,j+1}),$$

and $E_{j,j+1}$ is the matrix with the unit as its $(j,j{+}1)$th entry. Because $\det Q = \pm 1$ and $\det\widetilde{\Lambda} = \pm\mathbf{e}_1^{-(z+\tau/2)}$, the determinants of the shift matrices have the same quasiperiods as the Jacobi theta functions. The theta functions have simple poles in the fundamental period parallelogram of Σ_τ. The vector bundle E_N therefore has degree one.

We choose a holomorphic section, $\bar{\partial}s = 0$, as

$$s(z) = \left(\theta\begin{bmatrix}\frac{1}{N}\\ 0\end{bmatrix}(z;N\tau),\ldots,\theta\begin{bmatrix}1\\ 0\end{bmatrix}(z;N\tau)\right).$$

We define transformations $s^{\mathrm{T}} \to f(z,\bar{z})s^{\mathrm{T}}$ by the smooth maps

$$f\colon \Sigma_\tau \to GL(N,\mathbb{C}), \qquad f \in \Omega^{(0,0)}_{C^\infty}(\Sigma_\tau, GL(N,\mathbb{C})),$$

with the monodromies

$$f(z+1,\bar{z}+1) = Q^{-1}f(z,\bar{z})Q, \qquad f(z+\tau,\bar{z}+\bar{\tau}) = \widetilde{\Lambda}^{-1}f(z,\bar{z})\widetilde{\Lambda}.$$

These transformations preserve the degree of E_N and therefore generate the gauge group $\mathcal{G} = \{f(z,\bar{z})\}$ of the bundle E_N.

In the general case, the operators

$$d_{\bar{A}} = \bar{\partial} + \bar{A}\colon \Omega^{(0,0)}(\Sigma_\tau, E_N) \to \Omega^{(0,1)}(\Sigma_\tau, E_N)$$

define a complex structure on E_N. A section is holomorphic if $d_{\bar{A}}(s^{\mathrm{T}}) = 0$. We here assume that the operator $\bar{A}$ has the same monodromies as the sections of E_N:

$$\bar{A}(z+1,\bar{z}+1) = Q^{-1}\bar{A}(z,\bar{z})Q, \qquad \bar{A}(z+\tau,\bar{z}+\bar{\tau}) = \widetilde{\Lambda}^{-1}\bar{A}(z,\bar{z})\widetilde{\Lambda}. \tag{2.1}$$

Two complex structures defined by connections $\bar{A}$ and $\bar{A}^f$ are said to be equivalent if they are related by a gauge transformation,

$$\bar{A}^f = f^{-1}\bar{A}f + f^{-1}\bar{\partial}f, \quad f \in \mathcal{G}. \tag{2.2}$$

The quotient space of the generated connections $\mathcal{A} = \{\bar{A}\}$ with respect to the action of $\mathcal{G}$ is the moduli space $\mathrm{Bun}(E_N) = \mathcal{A}/\mathcal{G}$ of holomorphic bundles.

We consider the two-loop group $LL(GL(N,\mathbb{C}))$ represented by the space of sections $\{g(z,\bar{z})\} = \Omega^{(0,0)}_{C^\infty}(\Sigma_\tau, GL(N,\mathbb{C}))$ with the monodromies

$$g(z+1,\bar{z}+1) = Q^{-1}g(z,\bar{z})Q, \qquad g(z+\tau,\bar{z}+\bar{\tau}) = \widetilde{\Lambda}^{-1}g(z,\bar{z})\widetilde{\Lambda}. \tag{2.3}$$

The two-loop group $LL(GL(N,\mathbb{C}))$ with these quasiperiodicity conditions is the automorphism group $\mathrm{Aut}\,E_N$ of the degree-one vector bundle E_N.

2.2. Poisson structure on $\mathcal{R}$. On the space

$$\mathcal{R} = \mathcal{A} \times \Omega^{(0,0)}_{C^\infty}(\Sigma_\tau, GL(N,\mathbb{C})) = \{(\bar\partial + \bar A, g)\},$$

there is the symplectic form

$$\omega = \int_{\Sigma_\tau} K\langle d(\bar A g^{-1}) \wedge dg\rangle + \frac{1}{2}\int_{\Sigma_\tau} K\langle g^{-1}dg \wedge \bar\partial(g^{-1}dg)\rangle, \tag{2.4}$$

where $\langle\cdot\rangle$ is the trace taken in the vector representation, K is a section of the canonical bundle over Σ_τ, and $K \in \Omega^{(1,0)}(\Sigma_\tau)$; we choose $K = dz$. The space $\mathcal{R}$ is an affine space over the cotangent bundle $T^*(LL(GL(N,\mathbb{C})))$ to the two-loop group.

Transformations (2.2) and the transformations

$$g \to f^{-1} g f \tag{2.5}$$

are canonical with respect to symplectic form (2.4). The Hamiltonian vector fields V_ϵ, $\epsilon \in \mathrm{Lie}(\mathcal{G})$, on $\mathcal{R}$ ($V_\epsilon \omega = d\mu^*$) are generated by the Hamiltonian

$$\mu^*(\epsilon; \bar A, g) = \int_{\Sigma_\tau} K\langle \epsilon(g\bar A g^{-1} - \bar\partial g g^{-1} - \bar A)\rangle. \tag{2.6}$$

Remark 1. Let $\Phi \in \Omega^{(1,0)}_{C^\infty}(\Sigma_\tau, \operatorname{End} E_N)$ be a Higgs field and $g = e^{\hbar K^{-1}\Phi}$, where $\hbar \in \mathbb{C}$. In the limit as $\hbar \to 0$,

$$g \sim K^{-1}(\mathrm{Id} + \hbar\Phi + \dots). \tag{2.7}$$

Form (2.4) in the first order becomes the canonical form on the Higgs bundle $\{(d_{\bar A}, \Phi)\}$. Symmetries determine the Hamiltonian

$$\mu^*(\epsilon; \bar A, \Phi) = \int_{\Sigma_\tau} \langle \epsilon(\bar\partial\Phi + [\bar A, \Phi])\rangle.$$

Hence, $\mathcal{R}$ is a deformation of the Higgs bundle.

Inversion of the form in (2.4) defines a Poisson structure on $\mathcal{R}$. In terms of coordinates in basis (B.1), $\bar A = \sum_{\alpha \in \widetilde{\mathbb{Z}}^{(2)}_N} \bar A_\alpha T_\alpha$, $g = \sum_{a \in \mathbb{Z}^{(2)}_N} g_a T_a$, and the inversion of (2.4) becomes

$$K\{\bar A_\alpha(z,\bar z), \bar A_\beta(w,\overline{w})\} = C_{\alpha+\beta}\bar A_{\alpha+\beta}\delta(z-w, \bar z - \overline{w}) + \bar\partial\delta(z-w, \bar z - \overline{w})\delta_{\alpha,-\beta}, \tag{2.8}$$

$$K\{g_a(z,\bar z), \bar A_\beta(w,\overline{w})\} = \mathbf{e}_N^{(a\times\beta)} g_{a+\beta}(z,\bar z)\delta(z-w, \bar z - \overline{w}), \tag{2.9}$$

$$\{g_a(z,\bar z), g_b(w,\overline{w})\} = 0. \tag{2.10}$$

The brackets define a Poisson algebra $\mathcal{O}(\mathcal{R})$ with the symmetry group $\mathcal{G}$.

We define a Poisson subalgebra $\mathbf{P}_{\Sigma_\tau}$ of the Poisson algebra $\mathcal{O}(\mathcal{R})$. The subalgebra satisfies the following conditions:

1. The connection $\bar A$ takes values in the $sl(N,\mathbb{C})$ subalgebra, and g takes values in $GL(N,\mathbb{C})$.
2. The subalgebra $\mathbf{P}_{\Sigma_\tau}$ is generated by holomorphic functionals over $\mathcal{R}$ with test functions vanishing at $z = 0$.

The subalgebra $\mathbf{P}_{\Sigma_\tau}$ has the center $\mathcal{Z}$ generated by $\det g(z,\bar z)$. The symmetry group $\mathcal{G}^{\mathrm{s}} \subset \mathcal{G}$ of $\mathbf{P}_{\Sigma_\tau}$ is generated by smooth maps $f\colon \Sigma_\tau \to SL(N,\mathbb{C})$.

2.3. Poisson reduction. Our aim is to evaluate the Poisson structure reduced with respect to the action of $\mathcal{G}^{\rm s}$. The standard Poisson reduction $\mathbf{P}^{\rm red}_{\Sigma_\tau}$ of the subalgebra $\mathbf{P}_{\Sigma_\tau}$ is described as follows. Let $\mathbf{P}^{\mathcal{G}^{\rm s}}_{\Sigma_\tau}$ be the invariant Poisson subalgebra and $I^{\mathcal{G}^{\rm s}} = \{\mu^*(\epsilon)F(\bar{A}, g) \mid F(\bar{A}, g) \in \mathbf{P}_{\Sigma_\tau}\}$ be the ideal in $\mathbf{P}^{\mathcal{G}^{\rm s}}_{\Sigma_\tau}$ generated by the functional $\mu^*(\epsilon)$ in (2.6) with $\epsilon \in \mathrm{Lie}(\mathcal{G}^{\rm s})$. The reduced Poisson algebra $\mathbf{P}^{\rm red}_{\Sigma_\tau}$ is the quotient

$$\mathbf{P}^{\rm red}_{\Sigma_\tau} = \mathbf{P}^{\mathcal{G}^{\rm s}}_{\Sigma_\tau}/I^{\mathcal{G}^{\rm s}} := \mathbf{P}_{\Sigma_\tau}//\mathcal{G}^{\rm s}. \tag{2.11}$$

In our construction, we use another ideal $\mathbf{P}^{\mathcal{G}^{\rm s}}_{\Sigma_\tau}$ defined below.

We first evaluate the bracket in the invariant subalgebra $\mathbf{P}^{\mathcal{G}^{\rm s}}_{\Sigma_\tau}$. By the monodromy condition in (2.1), the generated field $\bar{A}$ is gauge equivalent to the trivial field $f^{-1}\bar{A}f + f^{-1}\bar{\partial}f = 0$, and therefore

$$\bar{A} = -\bar{\partial}f[\bar{A}]f^{-1}[\bar{A}]. \tag{2.12}$$

We note that the monodromies of gauge matrices (2.3) serve as a "protection" against nontrivial residual symmetries. Let $f[\bar{A}](z, \bar{z})$ be a solution of Eq. (2.12). We consider the transformation of g under the action of a solution of (2.12):

$$L[\bar{A}, g](z, \bar{z}) = f[\bar{A}](z, \bar{z})g(z, \bar{z})f^{-1}[\bar{A}](z, \bar{z}). \tag{2.13}$$

The gauge invariant subalgebra $\mathbf{P}^{\mathcal{G}^{\rm s}}_{\Sigma_\tau}$ is generated by the matrices L:

$$\mathbf{P}^{\mathcal{G}^{\rm s}}_{\Sigma_\tau} = \{\Psi(\bar{A}, g) = \Psi(0, L)\}.$$

Proposition 1. *The brackets on* $\mathbf{P}^{\mathcal{G}^{\rm s}}_{\Sigma_\tau}$ *take the form of the classical commutation relation*

$$\{L_1(z, \bar{z}), L_2(w, \overline{w})\} = [r(z - w), L_1(z, \bar{z}) \otimes L_2(w, \overline{w})], \tag{2.14}$$

where $L_1(z, \bar{z}) = L(z, \bar{z}) \otimes \mathrm{Id}$, $L_2(w, \overline{w}) = \mathrm{Id} \otimes L(w, \overline{w})$, *and* $r(z, w)$ *is the classical elliptic Belavin–Drinfeld* r*-matrix* [1].

Proof. Calculating the brackets in $\mathbf{P}^{\mathcal{G}^{\rm s}}_{\Sigma_\tau}$ amounts to calculating Poisson brackets between matrix elements (2.13) on the level surface $\bar{A} = 0$, $f = \mathrm{Id}$ using (2.8)–(2.10). For this, we need only the expressions

$$r_{\alpha,\beta}(z, \bar{z}; z', \bar{z}') = \frac{\delta f_\alpha(z, \bar{z})}{\delta \bar{A}_\beta(z', \bar{z}')}\bigg|_{\bar{A}=0}. \tag{2.15}$$

Direct calculation of the $\{L, L\}$ bracket given in [7] yields the sought r-matrix form (2.14).

We find the r-matrix. By Eq. (2.12), r is the Green's function of the operator $\bar{\partial}$,

$$\bar{\partial}r_{\alpha,\beta}(z, \bar{z}; z', \bar{z}') = \delta_{\alpha+\beta,0}\delta(z - z', \bar{z} - \bar{z}'), \tag{2.16}$$

with the quasiperiods

$$r(z + 1, \bar{z} + 1) = (Q^{-1} \otimes \mathrm{Id})r(z, \bar{z})(Q \otimes \mathrm{Id}),$$

$$r(z + \tau, \bar{z} + \bar{\tau}) = (\Lambda^{-1} \otimes \mathrm{Id})r(z, \bar{z})(\Lambda \otimes \mathrm{Id}).$$

It follows from (2.16) that $r_{\alpha,\beta}$ is a meromorphic and singular function on the diagonal,

$$\lim_{z' \to z} r_{\alpha,\beta}(z, z') = \frac{1}{z - z'}T_\alpha \otimes T_\beta \delta_{\alpha+\beta,0}. \tag{2.17}$$

Taking (A.5), (B.5), and (B.7) into account, we obtain

$$r(z, w) = r(z - w) = \sum_\alpha \varphi_\alpha(z - w)T_\alpha \otimes T_{-\alpha}. \tag{2.18}$$

This is indeed the classical Belavin–Drinfeld r-matrix [1]. It satisfies the classical Yang–Baxter equation, which ensures that the Jacobi identity is satisfied for bracket (2.14).

Remark 2. If (2.7) holds, then the only nontrivial brackets in (2.9) are given by

$$\{\Phi_\alpha(z,\bar z), \bar A_\beta(w,\overline{w})\} = \delta_{\alpha,-\beta}\delta(z-w,\bar z-\overline{w}),$$

and (2.14) is replaced with the linear brackets

$$\{L_1(z,\bar z), L_2(w,\overline{w})\} = [r(z-w), L(z,\bar z)\otimes \mathrm{Id} + \mathrm{Id}\otimes \mathrm{Id}\, L_2(w,\overline{w})].$$

We fix a divisor D_n of noncoincident points on Σ_τ,

$$D_n = (x_1,\dots,x_n), \quad x_j \neq x_k, \quad x_j \in \Sigma_\tau,$$

and define the subalgebra $\mathrm{Lie}(D_n)(\mathcal{G}^{\mathrm{s}}) \subset \mathrm{Lie}(\mathcal{G}^{\mathrm{s}})$,

$$\mathrm{Lie}(D_n)(\mathcal{G}^{\mathrm{s}}) = \{\varepsilon \in \mathrm{Lie}(\mathcal{G}^{\mathrm{s}}) \mid \varepsilon(x_j,\bar x_j) = 0,\ x_j \in D_n\}.$$

We consider the ideal $I(D_n)$ generated by the functional

$$\mu^*_{D_n}(\varepsilon;\bar A, g) = \mu^*_{D_n}(\varepsilon; L) = \int_{\Sigma_\tau} \langle \varepsilon\bar\partial L(z,\bar z)\rangle, \tag{2.19}$$

where ε belongs to the subalgebra $\mathrm{Lie}(\mathcal{G}^{\mathrm{s}}(D_n))$. Because the functional $\mu^*_{D_n}$ depends only on L, it follows that $I(D_n) \subset \mathbf{P}^{\mathcal{G}^{\mathrm{s}}}_{\Sigma_\tau}$.

We consider the quotient Poisson algebra $\mathbf{P}^{\mathcal{G}^{\mathrm{s}}}_{\Sigma_\tau}/I(D_n)$.

Proposition 2. *The reduced Poisson algebra*

$$\mathbf{P}^{\mathrm{red}}_{\Sigma_\tau,D_n} = \mathbf{P}^{\mathcal{G}^{\mathrm{s}}}_{\Sigma_\tau}/I(D_n) \tag{2.20}$$

is finitely generated, $\dim \mathbf{P}^{\mathrm{red}}_{\Sigma_\tau,D_n} = nN^2$. *The matrix* $L(z)$ (*the Lax matrix*) *involved in the classical commutation relations takes the form*

$$L = S_0T_0 + \sum_{j=1}^n (S_0^j E_1(z-x_j)T_0 + \widetilde L_j), \qquad \widetilde L_j = \sum_\alpha S^j_\alpha \varphi_\alpha(z-x_j)T_\alpha, \tag{2.21}$$

where

$$\sum_{j=1}^n S_0^j = 0, \tag{2.22}$$

the functions $\varphi_\alpha(z-x_j)$ *are defined in* (B.5), *and* $E_1(z-x_j)$ *is the first function in Eisenstein basis* (A.1).

Proof. We analyze the solutions of the equation

$$\mu^*_{D_n}(\varepsilon; L) = \int_{\Sigma_\tau} \langle \varepsilon\bar\partial L(z,\bar z)\rangle = 0. \tag{2.23}$$

The solutions are meromorphic quasiperiodic maps with simple poles at the marked points. Let $L(z) = \sum_a L_a(z)T_a$ be a decomposition of L with respect to the basis T_a of the group $GL(N,\mathbb{C})$. It follows from (2.3) that

$$L_a(z+1) = \mathbf{e}_N^{a_2} L_a(z), \qquad L_a(z+\tau) = \mathbf{e}_N^{-a_1} L_a(z), \qquad a = (a_1,a_2).$$

The functions $\varphi_a(z-x_j)$ in (B.5) have these monodromies (B.7) and simple poles at the x_j. They define an n-dimensional basis in the space of quasiperiodic functions (B.13) with poles at the x_j. If $a=(0,0)$, then $L_0(z)$ is a doubly periodic function with simple poles at the x_j. A basis in this space is given by unity and the Eisenstein functions $E_1(z-x_j)$, the sum of their residues being equal to zero. Hence, the space has the dimension n, and relations (2.21) and (2.22) are satisfied.

As we noted above, the $\det g$ generate Casimir functionals in $\mathbf{P}^{\rm red}_{\Sigma_\tau}$. As a result, the brackets on $\mathbf{P}^{\rm red}_{\Sigma_\tau,D_n}$ are degenerate. The functions $\det L(z)$ are generating functions for the Casimir elements $C^\mu(j)$. Because $\det L(z)$ is a doubly periodic function, it can be decomposed with respect to the basis of elliptic functions (A.3):

$$\det L(z) = C^0 + \sum_j^n C^1(j)E_1(z-x_j) + C^2(j)E_2(z-x_j) + \cdots + C^N(j)E_N(z-x_j). \tag{2.24}$$

In view of the condition

$$\sum_{j=1}^n C^1(j) = 0, \tag{2.25}$$

the number of independent Casimir elements is Nn. The symplectic leaf in the general position

$$\mathcal{R}^2_{n,N} = \mathbf{P}^{\rm red}_{\Sigma_\tau,D_n}/\{(C^\mu(j) = C(j)^\mu_{(0)}),\ \mu = 1,\ldots,N,\ j = 1,\ldots,N\}$$

has the dimension

$$\dim(\mathcal{R}^2_{n,N}) = nN(N-1). \tag{2.26}$$

We note that this quantity coincides with the sum of dimensions of n coadjoint orbits of $GL(N,\mathbb{C})$ in the general position.

3. The structure of the reduced Poisson space

3.1. Exact form of the quadratic brackets. Proposition 2 gives the reduced Poisson algebra $\mathbf{P}^{\rm red}_{\Sigma_\tau,D_n}$ with the generators

$$\left\{S_0,\ (S^j_0, \mathbf{S}^j = \{S^j_\alpha\}\,, j = 1,\ldots,n)\ \middle|\ \sum_{j=1}^n S^j_0 = 0\right\}. \tag{3.1}$$

The brackets between the generators were evaluated in [6].

Proposition 3. *In terms of generators* (3.1), *the Poisson brackets on the space* $\mathbb{C}^{nN^2}$ *take the form*[1]

$$\{S_0, S^j_0\}_2 = \{S^j_0, S^k_0\}_2 = \{S^j_\alpha, S^k_\alpha\}_2 = 0, \tag{3.2}$$

$$\{S_0, S^k_\alpha\}_2 = \sum_{\gamma\neq\alpha} \mathbf{C}(\alpha,\gamma)\left(S^k_{\alpha-\gamma}S^k_\gamma E_2(\breve\gamma) - \sum_{j\neq k} S^j_{-\gamma}S^k_{\alpha+\gamma}f_\gamma(x_k-x_j)\right), \tag{3.3}$$

$$\begin{aligned}\{S^k_\alpha, S^k_\beta\}_2 = {} & \mathbf{C}(\alpha,\beta)S_0S^k_{\alpha+\beta} + \sum_{\gamma\neq\alpha,-\beta}\mathbf{C}(\gamma,\alpha-\beta)S^k_{\alpha-\gamma}S^k_{\beta+\gamma}\mathbf{f}_{\alpha,\beta,\gamma} + {} \\ & + \mathbf{C}(\alpha,\beta)S^k_0S^k_{\alpha+\beta}(E_1(\breve\alpha+\breve\beta) - E_1(\breve\alpha) - E_1(\breve\beta)) - {} \\ & - \mathbf{C}(\alpha,\beta)\sum_{j\neq k}[S^k_0S^j_{\alpha+\beta}\varphi_{\alpha+\beta}(x_k-x_j) - S^j_0S^k_{\alpha+\beta}E_1(x_k-x_j)] - {} \\ & - 2\sum_{j\neq k}\mathbf{C}(\gamma,\alpha-\beta)S^k_{\alpha-\gamma}S^k_{\beta+\gamma}\varphi_{\beta+\gamma}(x_k-x_j),\end{aligned} \tag{3.4}$$

[1] The subscript 2 on the braces in $\{\,\cdot\,,\cdot\,\}_2$ indicates the quadratic bracket.

where the quantities $\mathbf{f}_{\alpha,\beta,\gamma}$, $E_2(\breve{\alpha})$, *and* $E_1(\breve{\alpha})$ *are defined in* (B.12) *and* (B.4). *For* $j \neq k$,

$$\{S^j_\alpha, S^k_\beta\}_2 = \sum_{\gamma\neq\alpha,-\beta} \mathbf{C}(\gamma, \alpha-\beta) S^j_{\alpha-\gamma} S^k_{\beta+\gamma} \varphi_\gamma(x_j - x_k) - \\ - \mathbf{C}(\alpha,\beta)(S^j_0 S^k_{\alpha+\beta}\varphi_\alpha(x_j - x_k) - S^k_0 S^j_{\alpha+\beta}\varphi_{-\beta}(x_k - x_j)), \tag{3.5}$$

$$\{S^j_0, S^k_\beta\}_2 = \begin{cases} 2\sum_\gamma \mathbf{C}(\gamma,-\beta) S^j_{-\gamma} S^k_{\beta+\gamma}\varphi_\gamma(x_k - x_j), & j \neq k, \\ -2\sum_{m\neq k}\sum_\gamma \mathbf{C}(\gamma,-\beta) S^k_{-\gamma} S^m_{\beta+\gamma}\varphi_{\beta+\gamma}(x_k - x_m), & j = k. \end{cases} \tag{3.6}$$

This algebra is an exact special form of the general construction of quadratic Poisson algebras [4]. This algebra was derived in [8] for $n = 1$ and is the classical Sklyanin algebra for $n = 1$ and $N = 2$ [3].

3.2. Twisted bundles. We need an equivalent form of the Poisson algebra on $\mathbb{C}^{nN^2}$. We consider the twisted bundle $E'_N = \mathrm{Aut}(E_N) \otimes \mathcal{L}$, where $\mathcal{L}$ is the trivial linear bundle over Σ_τ. Sections of E'_N are sections of E_N times $\vartheta(z+\eta)/\vartheta(z)$, $\eta \in \Sigma_\tau$. Therefore, the shift functions of E'_N have the form

$$\mathrm{ad}(Q)\colon z \to z+1, \qquad e^{-2\pi i\eta}\,\mathrm{ad}(\widetilde{\Lambda})\colon z \to z+\tau.$$

It follows from (B.6) and (B.8) that the solution of Eq. (2.23) with these monodromies and with simple poles at the divisor $\widetilde{D}_n = (\tilde{x}_1, \ldots, \tilde{x}_n)$ is given by

$$L^\eta_{\widetilde{D}_n} = \sum_{j=1}^n \biggl[\widetilde{S}^j_0 \varphi_\eta(z - \tilde{x}_j) T_0 + \sum_\alpha \widetilde{S}^j_\alpha \varphi_{\alpha,\eta}(z - \tilde{x}_j) T_\alpha\biggr], \tag{3.7}$$

where $\varphi_\eta = \varphi_{0,\eta}$. The corresponding algebra $\widetilde{\mathbf{P}}^{\mathrm{red}}_{\Sigma_\tau,\widetilde{D}_n}$ is defined as above by the classical commutation relations

$$\{L^\eta_{1,\widetilde{D}_n}(z), L^\eta_{2,\widetilde{D}_n}(w)\} = [r(z-w), L^\eta_{1,\widetilde{D}_n}(z) \otimes L^\eta_{2,\widetilde{D}_n}(w)] \tag{3.8}$$

with the set of nN^2 generators

$$\widetilde{\mathbf{S}}^j = \{\widetilde{S}^j_a\}, \quad a = (a_1, a_2) \in \mathbb{Z}^{(2)}_N, \quad j = 1, \ldots, n.$$

The brackets between the generators can be derived from (3.8) as before. We do not need the exact formulas because we immediately prove the equivalence of these two algebras. In Sec. 4, we consider only the brackets containing $\widetilde{S}^k_0$ in the right-hand side (cf. (3.4) and (3.5)),

$$\{\widetilde{S}^k_\alpha, \widetilde{S}^k_\beta\} = \sum_{\gamma\neq\alpha,-\beta} \mathbf{C}(\gamma,\alpha-\beta)\widetilde{S}^k_{\alpha-\gamma}\widetilde{S}^k_{\beta+\gamma}\tilde{\mathbf{f}}^\eta_{\alpha,\beta,\gamma} + \\ + \mathbf{C}(\alpha,\beta)\widetilde{S}^k_0\widetilde{S}^k_{\alpha+\beta}(E_1(\breve{\alpha}+\breve{\beta}+\eta) - E_1(\breve{\alpha}) - E_1(\breve{\beta}) - E_1(\eta)) - \\ - \mathbf{C}(\alpha,\beta)\sum_{j\neq k}[\widetilde{S}^k_0\widetilde{S}^j_{\alpha+\beta}\varphi_{\alpha+\beta,\eta}(x_k - x_j) - \widetilde{S}^j_0\widetilde{S}^k_{\alpha+\beta}\varphi_\eta(x_k - x_j)] - \\ - 2\sum_{j\neq k}\mathbf{C}(\gamma,\alpha-\beta)\widetilde{S}^k_{\alpha-\gamma}\widetilde{S}^k_{\beta+\gamma}\varphi_{\beta+\gamma,\eta}(x_k - x_j), \tag{3.9}$$

where

$$\tilde{\mathbf{f}}^\eta_{\alpha,\beta,\gamma} = E_1(\breve{\gamma}) - E_1(\breve{\alpha}-\breve{\beta}-\breve{\gamma}) + E_1(\breve{\alpha}-\breve{\gamma}+\eta) - E_1(\breve{\beta}+\breve{\gamma}+\eta), \tag{3.10}$$

$$\{\widetilde{S}^j_\alpha, \widetilde{S}^k_\beta\} = \sum_{\gamma\neq\alpha,-\beta}\mathbf{C}(\gamma,\alpha-\beta)\widetilde{S}^j_{\alpha-\gamma}\widetilde{S}^k_{\beta+\gamma}\varphi_{\gamma,\eta}(x_j - x_k) - \\ - \mathbf{C}(\alpha,\beta)(\widetilde{S}^j_0\widetilde{S}^k_{\alpha+\beta}\varphi_{\alpha,\eta}(x_j - x_k) - \widetilde{S}^k_0\widetilde{S}^j_{\alpha+\beta}\varphi_{-\beta,\eta}(x_k - x_j)). \tag{3.11}$$

We prove the equivalence, setting $\tilde{x}_i = 0$ for some i.

Proposition 4. *We fix two indices $1 \leq i, k \leq n$, $i \neq k$ and define $x_k = -\eta$ and $x_j = \tilde{x}_j$ for $j \neq k$. Then the Poisson algebras $\mathbf{P}^{\mathrm{red}}_{\Sigma_\tau, D_n}$ and $\widetilde{\mathbf{P}}^{\mathrm{red}}_{\Sigma_\tau, \widetilde{D}_n}$ are isomorphic. The corresponding canonical transformations are given by*

$$S_0 = \widetilde{S}^i_0 + \sum_{j\neq i}^{n} \frac{1}{\varphi_\eta(\tilde{x}_j)}(E_1(\tilde{x}_j) + E_1(\eta))\widetilde{S}^j_0,$$

$$S^k_0 = -\sum_{j\neq i}^{n} \frac{\widetilde{S}^j_0}{\varphi_\eta(\tilde{x}_j)}, \qquad S^k_\alpha = \frac{\widetilde{S}^i_\alpha}{\varphi_\alpha(\eta)} + \sum_{j\neq i} \frac{\varphi_{-\alpha}(\tilde{x}_j)\widetilde{S}^j_\alpha}{\varphi_\eta(\tilde{x}_j)\varphi_\alpha(\eta)}, \tag{3.12}$$

$$S^{j\neq k}_0 = \frac{\widetilde{S}^j_0}{\varphi_\eta(\tilde{x}_j)}, \qquad S^{j\neq k}_\alpha = \frac{\widetilde{S}^j_\alpha}{\varphi_\eta(\tilde{x}_j)}.$$

Proof. After division by $\varphi_\eta(z)$, the Lax operator $L^\eta_{\widetilde{D}_n}$ in (3.7) acquires the same monodromies as the matrix L in (2.21). We consider residues and constant terms of these operators. We have

$$\frac{L^\eta_{\widetilde{D}_n}}{\varphi_\eta(z)} = \widetilde{S}^i_0 T_0 + \sum_{j\neq i}^{n} \left[\widetilde{S}^j_0 \frac{\varphi_\eta(z-\tilde{x}_j)}{\varphi_\eta(z)} T_0 + \sum_\alpha \left(\widetilde{S}^j_\alpha \frac{\varphi_{\alpha,\eta}(z-\tilde{x}_j)}{\varphi_\eta(z)} + \widetilde{S}^i_\alpha \frac{\varphi_{\alpha,\eta}(z)}{\varphi_\eta(z)} \right) T_\alpha \right]. \tag{3.13}$$

Using relations (B.9), (B.10), and (B.11) yields

$$\begin{aligned} \frac{L^\eta_{\widetilde{D}_n}}{\varphi_\eta(z)} = &\left(\widetilde{S}^i_0 + \sum_{j\neq i}^{n} \frac{1}{\varphi_\eta(\tilde{x}_j)}(E_1(\tilde{x}_j) + E_1(\eta))\widetilde{S}^j_0 \right) T_0 - \\ &- E_1(z+\eta) \sum_{j\neq i}^{n} \frac{\widetilde{S}^j_0}{\varphi_\eta(\tilde{x}_j)} \cdot T_0 + \sum_{j\neq i}^{n} E_1(z-\tilde{x}_j) \cdot \frac{\widetilde{S}^j_0}{\varphi_\eta(\tilde{x}_j)} T_0 + \\ &+ \sum_{\alpha, j\neq i} \varphi_\alpha(z-\tilde{x}_j) \frac{\widetilde{S}^j_\alpha}{\varphi_\eta(\tilde{x}_j)} \cdot T_\alpha + \\ &+ \sum_{\alpha, j\neq i} \varphi_\alpha(z+\eta) \cdot \left(\frac{\varphi_{-\alpha}(\tilde{x}_j)\widetilde{S}^j_\alpha}{\varphi_\eta(\tilde{x}_j)\varphi_\alpha(\eta)} + \frac{\widetilde{S}^i_\alpha}{\varphi_\alpha(\eta)} \right) T_\alpha. \end{aligned}$$

We note that in this case, a new pole occurs at the point $x_b = -\eta$. Comparing with (2.21), we obtain Eq. (3.12).

3.3. Bi-Hamiltonian structure.

We introduce a linear Lie–Poisson bracket on the space $\mathbb{C}^{nN^2}$. For this, we consider n copies of $gl(N, \mathbb{C})$, $\mathfrak{g}^* = gl(N, \mathbb{C}) \oplus \cdots \oplus gl(N, \mathbb{C})$, with the brackets[2]

$$\{S^j_\alpha, S^k_\beta\}_1 = C(\alpha, \beta) S^j_{\alpha+\beta} \delta^{jk}. \tag{3.14}$$

Remark 3. The Lie–Poisson bracket has the r-matrix form

$$\{\widetilde{L}_1(z), \widetilde{L}_2(w)\} = [r(z-w), \widetilde{L}_1(z) + \widetilde{L}_2(w)],$$

where the r-matrix is the same as for the quadratic bracket in (2.14) and $\widetilde{L} = \sum_{j=1}^n \widetilde{L}_j(z)$, Eq. (2.21).

Two Poisson structures are said to be *compatible* (constituting a *Poisson pair*) if their linear combinations are also Poisson structures.

[2]The subscript 1 on the braces in $\{\cdot, \cdot\}_1$ indicates the linear bracket.

Proposition 5. *Linear Poisson brackets* (3.14) *and quadratic Poisson brackets* (3.2)–(3.6) *on the space* $\mathbb{C}^{nN^2}$ *are compatible.*

Proof. We choose a point $x_k \in \widetilde{D}_n$ and replace $\widetilde{S}_0^k$ with $\widetilde{S}_0^k + \lambda$, where $\lambda \in \mathbb{C}$ is a number. Then $\widetilde{S}_0^k + \lambda$ commutes with respect to the Poisson bracket with all of the elements of the quadratic Poisson algebra. We substitute this new variable in (3.9) and (3.11). The change of variables does not spoil the Jacobi identity, and we therefore obtain the Poisson structure

$$\{\widetilde{S}, \widetilde{S}\}_\lambda := \{\widetilde{S}, \widetilde{S}\}_2 + \lambda\{\widetilde{S}, \widetilde{S}\}_1.$$

We consider the terms pertaining to linear brackets,

$$\begin{aligned} &\{\widetilde{S}_\alpha^k, \widetilde{S}_\beta^k\}_1 = F_1 \widetilde{S}_{\alpha+\beta}^j + F_2 \widetilde{S}_{\alpha+\beta}^k, \qquad &&\{\widetilde{S}_\alpha^k, \widetilde{S}_\beta^j\}_1 = G_2 \widetilde{S}_{\alpha+\beta}^j, \\ &\{\widetilde{S}_\alpha^j, \widetilde{S}_\beta^k\}_1 = \widetilde{G}_2 \widetilde{S}_{\alpha+\beta}^j, &&\{\widetilde{S}_\alpha^j, \widetilde{S}_\beta^j\}_1 = H_2 \widetilde{S}_{\alpha+\beta}^j, \end{aligned} \tag{3.15}$$

where the coefficients up to a common factor $C(\alpha, \beta)$ have the forms

$$\begin{aligned} &F_1 = \varphi_{\alpha+\beta,\eta}(x_{kj}), \qquad F_2 = -E_1(\breve{\alpha}) - E_1(\breve{\beta}) - E_1(\eta) + E_1(\breve{\alpha} + \breve{\beta} + \eta), \\ &G_2 = -\varphi_\alpha(x_{kj}), \qquad \widetilde{G}_2 = -\varphi_\beta(x_{kj}), \qquad H_2 = \varphi_{0,-\eta}(x_{kj}), \end{aligned} \tag{3.16}$$

where $x_{kj} = x_k - x_j$. The following lemma completes the proof of the proposition.

Lemma. *The linear Poisson algebra in* (3.15) *is equivalent to the direct sum of Lie–Poisson algebras on* $\bigoplus_{l=1}^n gl(N, \mathbb{C})$:

$$\{t_\alpha^j, t_\beta^k\} = C(\alpha, \beta) t_{\alpha+\beta}^j \delta^{jk}. \tag{3.17}$$

Proof. We define

$$\widetilde{S}_\alpha^k = a_\alpha t_\alpha^k + b_\alpha t_\alpha^j, \qquad \widetilde{S}_\alpha^j = H_2 t_\alpha^j.$$

Bracket (3.15) is equivalent to (3.17) with

$$a_\alpha a_\beta = a_{\alpha+\beta} F_2, \qquad b_\alpha = G_2, \qquad b_\alpha b_\beta = F_1 H_2 + b_{\alpha+\beta} F_2. \tag{3.18}$$

We solve these equations. The solution of the first equation, which can be found from (A.10), has the form $a_\alpha = -\varphi_\alpha(\eta)$. We next prove that $b_\alpha = G_2 = -\varphi_\alpha(x_{kj})$ satisfies the last relation in (3.18). For $b_\alpha = -\varphi_\alpha(x_{kj})$, it becomes

$$\begin{aligned} \varphi_\alpha(x_{kj})\varphi_\beta(x_{kj}) = \varphi_{\alpha+\beta,\eta}(x_{kj})\varphi_{0,-\eta}(x_{kj}) +& \\ + \varphi_{\alpha+\beta}(x_{kj})\bigl(E_1(\breve{\alpha}) + E_1(\breve{\beta}) + E_1(\eta) - E_1(\breve{\alpha} + \breve{\beta} + \eta)\bigr)&. \end{aligned}$$

It follows from relation (A.9) that

$$\varphi_{\alpha+\beta,\eta}(x_{kj})\varphi_{0,-\eta}(x_{kj}) = \varphi_{\alpha+\beta}(x_{kj})\bigl(E_1(\breve{\alpha} + \breve{\beta} + \eta) + E_1(-\eta) + E_1(x_{kj}) - E_1(\breve{\alpha} + \breve{\beta} + x_{kj})\bigr),$$

and therefore the last relation in (3.18) is an identity. We thus obtain (3.17).

4. Quantum algebra

4.1. General case. In this section, we consider quantization of the quadratic Poisson algebra in the case where $n > 1$. Let the quantum R-matrix have the form

$$R(z,w) = \sum_{a\in\mathbb{Z}_N^{(2)}} \varphi_a^\hbar(z-w)T_a \otimes T_{-a}, \tag{4.1}$$

where $\varphi_a^\hbar(z) \equiv \varphi_{\hbar,a}(z)$. We note that in contrast to the classical r-matrix, an extra term $\varphi_0^\hbar(z-w)\sigma_0 \otimes \sigma_0$ occurs here. The quantum R-matrix satisfies the quantum Yang–Baxter equation

$$R_{12}(z-w)R_{13}(z)R_{23}(w) = R_{23}(w)R_{13}(z)R_{12}(z-w). \tag{4.2}$$

With the quantum Yang–Baxter equation, the associative algebra is defined as

$$R(z-w)L_1^\hbar(z)L_2^\hbar(w) = L_2^\hbar(w)L_1^\hbar(z)R(z-w). \tag{4.3}$$

The Lax operator in (4.3) with respect to $\hbar$ has the monodromies

$$L^{\hbar+\tau}(z) = \mathbf{e}_N(-z)L^\hbar(z), \qquad L^{\hbar+1}(z) = L^\hbar(z). \tag{4.4}$$

We must therefore assume that the variables S depend on $\hbar$ and x_j. The new variables and the Lax operator in (4.3) then have the forms

$$\begin{aligned} S^j_{\text{new}} &= \widehat{S}^j_0\varphi_0^\hbar(x_j), \\ L^\hbar(z) &= \sum_{j=1}^n\biggl(\widehat{S}^j_0\varphi_0^\hbar(x_j)\varphi_0^\hbar(z-x_j)T_0 + \sum_\alpha \widehat{S}^j_\alpha\varphi_\alpha^\hbar(x_j)\varphi_\alpha^\hbar(z-x_j)T_\alpha\biggr) = \\ &= \sum_{j=1}^n\sum_{a\in\mathbb{Z}_N^{(2)}} \widehat{S}^j_a\varphi_a^\hbar(x_j)\varphi_a^\hbar(z-x_j)T_a. \end{aligned} \tag{4.5}$$

Proposition 6. *The relations in the associative algebra become*

$$\begin{aligned} \sum_c f^\hbar(a,b,c)\cdot \widehat{S}^j_{b+c}\widehat{S}^j_{a-c}\varphi^\hbar_{b+c}(x_j)\varphi^\hbar_{a-c}(x_j)\mathbf{e}_N\biggl(\frac{c\times(a-b)}{2}\biggr) &+ \\ + \sum_c\sum_{k\ne j}\varphi^\hbar_{a-c}(x_j-x_k)\varphi^\hbar_{b+c}(x_j)\varphi^\hbar_{a-c}(x_k) &\times \\ \times\biggl(\widehat{S}^j_{b+c}\widehat{S}^k_{a-c}\mathbf{e}_N\biggl(\frac{c\times(a-b)}{2}\biggr) - \widehat{S}^k_{a-c}\widehat{S}^j_{b+c}\mathbf{e}_N\biggl(\frac{c\times(a-b)}{2}\biggr)\biggr) &= 0, \end{aligned} \tag{4.6}$$

$$\begin{aligned} \sum_c \varphi^\hbar_c(x_j-x_k)\varphi^\hbar_{b+c}(x_k)\varphi^\hbar_{a-c}(x_j) &\times \\ \times\biggl(\widehat{S}^j_{a-c}\widehat{S}^k_{b+c}\mathbf{e}_N\biggl(-\frac{c\times(a-b)}{2}\biggr) - \widehat{S}^k_{b+c}\widehat{S}^j_{a-c}\mathbf{e}_N\biggl(\frac{c\times(a-b)}{2}\biggr)\biggr) &= 0, \end{aligned} \tag{4.7}$$

where $k \ne j$,

$$f^\hbar(a,b,c) = E_1(c+\hbar) - E_1(a-b-c+\hbar) + E_1(a-c+\hbar) - E_1(b+c+\hbar),$$

$a,b,c \in \mathbb{Z}_N^{(2)}$.

Proof. We consider a matrix element $T_a \otimes T_b$. Substituting (4.5) and (4.1) in (4.3), we obtain the expressions

$$\sum_{j,k}\sum_{c,a,b}\varphi_c^\hbar(z-w)\varphi_a^\hbar(z-x_j)\varphi_b^\hbar(w-x_k)\cdot \widehat{S}_a^j\widehat{S}_b^k\varphi_b^\hbar(x_k)\varphi_a^\hbar(x_j)\cdot T_cT_a\otimes T_{-c}T_b =$$
$$= \sum_{j,k}\sum_{c,a,b}\varphi_c^\hbar(z-w)\varphi_a^\hbar(z-x_j)\varphi_b^\hbar(w-x_k)\times$$
$$\times\, \widehat{S}_b^k\widehat{S}_a^j\varphi_b^\hbar(x_k)\varphi_a^\hbar(x_j)\cdot T_aT_c\otimes T_bT_{-c}. \tag{4.8}$$

$$\sum_{j,k}\sum_{c,a,b}\varphi_c^\hbar(z-w)\varphi_a^\hbar(z-x_j)\varphi_b^\hbar(w-x_k)\times$$
$$\times\, \widehat{S}_a^j\widehat{S}_b^k\varphi_b^\hbar(x_k)\varphi_a^\hbar(x_j)\mathbf{e}_N\left(-\frac{c\times(a-b)}{2}\right)\cdot T_{c+a}\otimes T_{-c+b} =$$
$$= \sum_{j,k}\sum_{c,a,b}\varphi_c^\hbar(z-w)\varphi_a^\hbar(z-x_j)\varphi_b^\hbar(w-x_k)\times$$
$$\times\, \widehat{S}_b^k\widehat{S}_a^j\varphi_b^\hbar(x_k)\varphi_a^\hbar(x_j)\mathbf{e}_N\left(\frac{c\times(a-b)}{2}\right)\cdot T_{a+c}\otimes T_{b-c}. \tag{4.9}$$

The functions in the left- and right-hand sides here are equivalent because their poles and quasiperiods coincide. After the change of variables $a\to a-c$, $b\to b+c$, the coefficients of the matrix element $T_a\otimes T_b$ become

$$\sum_c \varphi_c^\hbar(z-w)\varphi_{a-c}^\hbar(z-x_j)\varphi_{b+c}^\hbar(w-x_k)\varphi_{b+c}^\hbar(x_k)\varphi_{a-c}^\hbar(x_j)\times$$
$$\times\left(\widehat{S}_{a-c}^j\widehat{S}_{b+c}^k\mathbf{e}_N\left(-\frac{c\times(a-b)}{2}\right)-\widehat{S}_{b+c}^k\widehat{S}_{a-c}^j\mathbf{e}_N\left(\frac{c\times(a-b)}{2}\right)\right)=0. \tag{4.10}$$

We consider two types of these expressions:

$$\sum_c \varphi_c^\hbar(z-w)\varphi_{a-c}^\hbar(z-x_j)\varphi_{b+c}^\hbar(w-x_k)\varphi_{b+c}^\hbar(x_k)\varphi_{a-c}^\hbar(x_j)\times$$
$$\times\left(\widehat{S}_{a-c}^j\widehat{S}_{b+c}^k\mathbf{e}_N\left(-\frac{c\times(a-b)}{2}\right)-\right.$$
$$\left.-\,\widehat{S}_{b+c}^k\widehat{S}_{a-c}^j\mathbf{e}_N\left(\frac{c\times(a-b)}{2}\right)\right)=0,\quad k\neq j,$$
$$\sum_c(\varphi_c^\hbar(z-w)\varphi_{a-c}^\hbar(z-x_j)\varphi_{b+c}^\hbar(w-x_j)-$$
$$-\,\varphi_{a-b-c}^\hbar(z-w)\varphi_{a-c}^\hbar(z-x_j)\varphi_{b+c}^\hbar(w-x_k))\times$$
$$\times\,\widehat{S}_{a-c}^j\widehat{S}_{b+c}^k\varphi_{b+c}^\hbar(x_k)\varphi_{a-c}^\hbar(x_j)\mathbf{e}_N\left(-\frac{c\times(a-b)}{2}\right)=0,\quad k=j. \tag{4.11}$$

The second expression follows after the replacement $c\to a-b-c$. In the limit as $z\to x_j$, $w\to x_j$ and $z\to x_j$, $w\to x_k$, similarly to Sec. 3, we obtain the coefficients that are equal to zero. The proposition is thus proved.

4.2. Quadratic algebra in the case of $GL(2,\mathbb{C})$. We consider the case $N = 2$ in more detail. The quantum R-matrix then has the form

$$R(z,w) = \sum_{a=0}^{3} \varphi_a^\hbar(z-w)\sigma_a \otimes \sigma_a, \tag{4.12}$$

where we use the basis of sigma matrices instead of the T_a.

Proposition 7. *Relations in the associative algebra take the following forms: for $k = j$,*

$$\begin{aligned}
[\widehat{S}^j_\alpha, \widehat{S}^j_\beta]_- = i\varepsilon_{\alpha\beta\gamma} c_1^1(j,j;\alpha,\beta,\gamma)[\widehat{S}^j_\gamma, \widehat{S}^j_0]_+ +\\
+ \sum_{k\neq j} i\varepsilon_{\alpha\beta\gamma}\frac{1}{k_\alpha}(\varphi_\gamma^\hbar(x_{jk})c_1^2(j,k;\alpha,\beta,\gamma)[\widehat{S}^k_\gamma, \widehat{S}^j_0]_+ -\\
- \varphi_0^\hbar(x_{jk})c_1^3(j,k;\alpha,\beta,\gamma)[\widehat{S}^j_\gamma, \widehat{S}^k_0]_+),
\end{aligned} \tag{4.13}$$

$$c_1^1(j,j;\alpha,\beta,\gamma) = \frac{\varphi_\gamma^\hbar(x_j)\varphi_0^\hbar(x_j)}{\varphi_\alpha^\hbar(x_j)\varphi_\beta^\hbar(x_j)}, \qquad c_1^2(j,k;\alpha,\beta,\gamma) = \frac{\varphi_\gamma^\hbar(x_k)\varphi_0^\hbar(x_j)}{\varphi_\alpha^\hbar(x_j)\varphi_\beta^\hbar(x_j)},$$

$$c_1^3(j,k;\alpha,\beta,\gamma) = \frac{\varphi_\gamma^\hbar(x_j)\varphi_0^\hbar(x_k)}{\varphi_\alpha^\hbar(x_j)\varphi_\beta^\hbar(x_j)},$$

$$\begin{aligned}
[\widehat{S}^j_\alpha, \widehat{S}^j_0]_- = i\varepsilon_{\alpha\beta\gamma}\frac{J_\beta - J_\gamma}{J_\alpha} c_2^1(j,j;\alpha,\beta,\gamma)[\widehat{S}^j_\beta, \widehat{S}^j_\gamma]_+ +\\
+ \sum_{k\neq j} i\varepsilon_{\alpha\beta\gamma}\frac{1}{k_\alpha J_\alpha}\big(c_2^2(j,k;\alpha,\beta,\gamma)D(\alpha,\beta)[\widehat{S}^j_\gamma, \widehat{S}^k_\beta]_+ -\\
- c_2^3(j,k;\alpha,\beta,\gamma)D(\alpha,\gamma)[\widehat{S}^k_\gamma, \widehat{S}^j_\beta]_+\big),
\end{aligned} \tag{4.14}$$

$$c_2^1(j,j;\alpha,\beta,\gamma) = \frac{\varphi_\gamma^\hbar(x_j)\varphi_\beta^\hbar(x_j)}{\varphi_\alpha^\hbar(x_j)\varphi_0^\hbar(x_j)}, \qquad c_2^2(j,k;\alpha,\beta,\gamma) = \frac{\varphi_\gamma^\hbar(x_j)\varphi_\beta^\hbar(x_k)}{\varphi_\alpha^\hbar(x_j)\varphi_0^\hbar(x_j)},$$

$$c_2^3(j,k;\alpha,\beta,\gamma) = \frac{\varphi_\gamma^\hbar(x_k)\varphi_\beta^\hbar(x_j)}{\varphi_\alpha^\hbar(x_j)\varphi_0^\hbar(x_j)},$$

$$D(\alpha,\beta) = \left(k_\alpha(k_\alpha - k_\beta - \log'\varphi_\alpha^\hbar(x_{jk})) + \log'\frac{\varphi_\alpha^\hbar(x_{jk})}{\varphi_0^\hbar(x_{jk})}(\partial_z - E_1(\hbar))\right)\varphi_\beta^\hbar(x_{jk}),$$

$$\log'\varphi_u^\hbar(x) = \frac{\partial_u(\varphi_u^\hbar(x))}{\varphi_u^\hbar(x)};$$

for $k \neq j$,

$$\begin{aligned}
[\widehat{S}^j_\alpha, \widehat{S}^k_\beta]_- = i\varepsilon_{\alpha\beta\gamma}\frac{1}{k_\gamma}(\varphi_\beta(x_{jk})c_3^1(j,k;\alpha,\beta,\gamma)[\widehat{S}^j_\gamma, \widehat{S}^k_0]_+ -\\
- \varphi_\alpha(x_{jk})c_3^2(j,k;\alpha,\beta,\gamma)[\widehat{S}^k_\gamma, \widehat{S}^j_0]_+),
\end{aligned} \tag{4.15}$$

$$c_3^1(j,k;\alpha,\beta,\gamma) = \frac{\varphi_\gamma^\hbar(x_j)\varphi_0^\hbar(x_k)}{\varphi_\alpha^\hbar(x_j)\varphi_\beta^\hbar(x_k)}, \qquad c_3^2(j,k;\alpha,\beta,\gamma) = \frac{\varphi_\gamma^\hbar(x_k)\varphi_0^\hbar(x_j)}{\varphi_\alpha^\hbar(x_j)\varphi_\beta^\hbar(x_k)},$$

$$[\widehat{S}^j_\alpha, \widehat{S}^k_0]_- = i\varepsilon_{\alpha\beta\gamma}\frac{1}{k_\alpha}(\varphi_\beta(x_{jk})c^1_4(j,k;\alpha,\beta,\gamma)[\widehat{S}^j_\gamma, \widehat{S}^k_\beta]_+ - \\ - \varphi_\alpha(x_{jk})c^2_4(j,k;\alpha,\beta,\gamma)[\widehat{S}^k_\gamma, \widehat{S}^j_\beta]_+), \tag{4.16}$$

$$c^1_4(j,k;\alpha,\beta,\gamma) = \frac{\varphi^\hbar_\gamma(x_j)\varphi^\hbar_\beta(x_k)}{\varphi^\hbar_\alpha(x_j)\varphi^\hbar_0(x_k)}, \qquad c^2_4(j,k;\alpha,\beta,\gamma) = \frac{\varphi^\hbar_\gamma(x_k)\varphi^\hbar_\beta(x_j)}{\varphi^\hbar_\alpha(x_j)\varphi^\hbar_0(x_k)},$$

where $k_\gamma = E_1(\check{\gamma}+\hbar) - E_1(\check{\gamma}) - E_1(\hbar)$, $J_\gamma = E_2(\check{\gamma}+\hbar) - E_2(\hbar)$, *and* $x_{jk} = x_j - x_k$.

Proof. We substitute (4.5) in (4.3) with $N = 2$ and verify the "balance" of the coefficients of the two types of matrix elements $\sigma_\alpha \otimes \sigma_\beta$ and $\sigma_\alpha \otimes \sigma_0$ in the left- and right-hand sides of the equality. We fix these elements and compare the coefficients at the corresponding poles. We obtain the following expressions for the brackets:

$$[\widehat{S}^j_\alpha, \widehat{S}^j_\beta]_- = i\varepsilon_{\alpha\beta\gamma}\frac{\varphi^\hbar_\gamma(x_j)\varphi^\hbar(x_j)}{\varphi^\hbar_\alpha(x_j)\varphi^\hbar_\beta(x_j)}[\widehat{S}^j_\gamma, \widehat{S}^j_0]_+ + \sum_{k\neq j}\frac{\varphi^\hbar_\gamma(x_{jk})}{f^\hbar(\alpha,\beta,0)}\cdot\frac{\varphi^\hbar_\alpha(x_j)\varphi^\hbar_\beta(x_k)}{\varphi^\hbar_\alpha(x_j)\varphi^\hbar_\beta(x_j)}[\widehat{S}^j_\alpha, \widehat{S}^k_\beta]_- - \\ - \frac{\varphi^\hbar_\alpha(x_{jk})}{f^\hbar(\alpha,\beta,0)}\cdot\frac{\varphi^\hbar_\alpha(x_k)\varphi^\hbar_\beta(x_j)}{\varphi^\hbar_\alpha(x_j)\varphi^\hbar_\beta(x_j)}[\widehat{S}^k_\alpha, \widehat{S}^j_\beta]_- + \\ + i\varepsilon_{\alpha\beta\gamma}\frac{\varphi^\hbar_0(x_{jk})}{f^\hbar(\alpha,\beta,0)}\cdot\frac{\varphi^\hbar_\gamma(x_k)\varphi^\hbar_0(x_j)}{\varphi^\hbar_\alpha(x_j)\varphi^\hbar_\beta(x_j)}[\widehat{S}^j_\gamma, \widehat{S}^k_0]_+ - \\ - i\varepsilon_{\alpha\beta\gamma}\frac{\varphi^\hbar_\gamma(x_{jk})}{f^\hbar(\alpha,\beta,0)}\cdot\frac{\varphi^\hbar_\gamma(x_j)\varphi^\hbar_0(x_k)}{\varphi^\hbar_\alpha(x_j)\varphi^\hbar_\beta(x_j)}[\widehat{S}^k_\gamma, \widehat{S}^j_0]_+, \tag{4.17}$$

$$[\widehat{S}^j_\alpha, \widehat{S}^j_0]_- = i\varepsilon_{\alpha\beta\gamma}\frac{J_\beta - J_\gamma}{J_\alpha}\cdot\frac{\varphi^\hbar_\gamma(x_j)\varphi^\hbar_\beta(x_j)}{\varphi^\hbar_\alpha(x_j)\varphi^\hbar_0(x_j)}[\widehat{S}^j_\beta, \widehat{S}^j_\gamma]_+ + \\ + \sum_{k\neq j}\frac{\varphi^\hbar_0(x_{jk})}{J_\alpha}\cdot\frac{\varphi^\hbar_0(x_k)}{\varphi^\hbar_0(x_j)}[\widehat{S}^j_\alpha, \widehat{S}^k_0]_- + \frac{\varphi^\hbar_\alpha(x_{jk})}{J_\alpha}\cdot\frac{\varphi^\hbar_\alpha(x_k)}{\varphi^\hbar_\alpha(x_j)}[\widehat{S}^k_\alpha, \widehat{S}^j_0]_- + \\ + i\varepsilon_{\alpha\beta\gamma}\frac{\varphi^\hbar_\gamma(x_{jk})}{J_\alpha}\cdot\frac{\varphi^\hbar_\gamma(x_k)\varphi^\hbar_\beta(x_j)}{\varphi^\hbar_\alpha(x_j)\varphi^\hbar_0(x_j)}[\widehat{S}^j_\beta, \widehat{S}^k_\gamma]_+ + \\ + i\varepsilon_{\alpha\beta\gamma}\frac{\varphi^\hbar_\beta(x_{jk})}{J_\alpha}\cdot\frac{\varphi^\hbar_\gamma(x_j)\varphi^\hbar_\beta(x_k)}{\varphi^\hbar_\alpha(x_j)\varphi^\hbar_0(x_j)}[\widehat{S}^j_\gamma, \widehat{S}^k_\beta]_+ \tag{4.18}$$

for $k \neq j$,

$$[\widehat{S}^j_\alpha, \widehat{S}^k_\beta]_- = \frac{\varphi^\hbar_\gamma(x_{jk})}{\varphi^\hbar_0(x_{jk})}\cdot\frac{\varphi^\hbar_\alpha(x_k)\varphi^\hbar_\beta(x_j)}{\varphi^\hbar_\alpha(x_j)\varphi^\hbar_\beta(x_k)}[\widehat{S}^k_\alpha, \widehat{S}^j_\beta]_- + \\ + i\varepsilon_{\alpha\beta\gamma}\frac{\varphi^\hbar_\alpha(x_{jk})}{\varphi^\hbar_0(x_{jk})}\cdot\frac{\varphi^\hbar_\gamma(x_k)\varphi^\hbar_0(x_j)}{\varphi^\hbar_\alpha(x_j)\varphi^\hbar_\beta(x_k)}[\widehat{S}^k_\gamma, \widehat{S}^j_0]_+ - \\ - i\varepsilon_{\alpha\beta\gamma}\frac{\varphi^\hbar_\beta(x_{jk})}{\varphi^\hbar_0(x_{jk})}\cdot\frac{\varphi^\hbar_\gamma(x_j)\varphi^\hbar_0(x_k)}{\varphi^\hbar_\alpha(x_j)\varphi^\hbar_\beta(x_k)}[\widehat{S}^j_\gamma, \widehat{S}^k_0]_+, \tag{4.19}$$

$$[\widehat{S}^j_\alpha, \widehat{S}^k_0]_- = \frac{\varphi^\hbar_\alpha(x_{jk})}{\varphi^\hbar_0(x_{jk})} \cdot \frac{\varphi^\hbar_\alpha(x_k)\varphi^\hbar_0(x_j)}{\varphi^\hbar_\alpha(x_j)\varphi^\hbar_0(x_k)}[\widehat{S}^k_\alpha, \widehat{S}^j_0]_- + \\ + i\varepsilon_{\alpha\beta\gamma}\frac{\varphi^\hbar_\gamma(x_{jk})}{\varphi^\hbar_0(x_{jk})} \cdot \frac{\varphi^\hbar_\beta(x_j)\varphi^\hbar_\gamma(x_k)}{\varphi^\hbar_\alpha(x_j)\varphi^\hbar_0(x_k)}[\widehat{S}^j_\beta, \widehat{S}^k_\gamma]_+ - \\ - i\varepsilon_{\alpha\beta\gamma}\frac{\varphi^\hbar_\beta(x_{jk})}{\varphi^\hbar_0(x_{jk})} \cdot \frac{\varphi^\hbar_\beta(x_k)\varphi^\hbar_\gamma(x_j)}{\varphi^\hbar_\alpha(x_j)\varphi^\hbar_0(x_k)}[\widehat{S}^k_\beta, \widehat{S}^j_\gamma]_+ \tag{4.20}$$

for $k = j$.

We express all the commutators in terms of anticommutators. For the brackets $[\widehat{S}^j_\alpha, \widehat{S}^k_\beta]_-$ and $[\widehat{S}^j_\alpha, \widehat{S}^k_0]_-$, we actually have two additional equations ($j \leftrightarrow k$). Solving the system of six equations, we obtain relations (4.13)–(4.17).

4.3. Quantum determinant. For the group $GL(2, \mathbb{C})$, we prove that the quantum determinant generates central elements of the permutation algebra

$$R_{12}(z_1, z_2)L_1(z_1)L_2(z_2) = L_2(z_2)L_1(z_1)R_{12}(z_1, z_2). \tag{4.21}$$

We consider the case of the classical algebra (3.1)–(3.6). To prove that $\det L(z)$ generates Casimir functions of Poisson structure (3.1)–(3.6) for $GL(N, \mathbb{C})$, we consider each side of the equality

$$\{L_1(z) \cdots L_N(z), L_{N+1}(w)\} = [L_1(z) \cdots L_N(z)L_{N+1}(w), r_{1,N+1}(z, w) + \cdots + r_{N,N+1}(z, w)],$$

where $L_i \in \operatorname{End} V_i$ and $r_{ik} \in \operatorname{End}(V_i \otimes V_k)$ act on $\bigotimes_{i=1}^{N+1} V_i$, $V_i \cong \mathbb{C}^N$ (vector spaces) as linear operators. Obviously, the determinant $\det L(z)$ arises from the subspace $[\bigwedge_{i=1}^N V_i] \otimes V_{N+1}$. On this subspace, the right-hand side of the equality reduces to the expression

$$[\det L(z) \cdot L_{N+1}(w), \operatorname{Tr}_1 r_{1,N+1}(z, w) + \operatorname{Tr}_N r_{N,N+1}(z, w)],$$

where the traces Tr_i are taken over the components in $\operatorname{End} V_i$. All of them vanish for the r-matrix in (2.18). This finishes the proof in the case of the classical algebra.

In the quantum case, the determinant is replaced with the quantum determinant

$$\det_\hbar = \operatorname{tr}(P^- \widehat{L}(z, \hbar) \otimes \widehat{L}(z + 2\hbar, \hbar)),$$

where P^- is the projection on the antisymmetric part of the tensor product

$$P^- a \otimes b = \frac{1}{2}(a \otimes b - b \otimes a).$$

We here consider only the case of 2×2 matrices. The R-matrix

$$R_{12}(z, w) = \sum_{a=0}^{3} \varphi^\eta_a(z - w)\sigma_a \otimes \sigma_a$$

satisfies the important relations

$$R_{12}(z, z + 2\hbar) = 4\frac{\vartheta'(0)}{\vartheta(2\hbar)}P^-, \qquad P^- = \frac{1}{4}\left(1 \otimes 1 - \sum_{\alpha=1}^{3} \sigma_\alpha \otimes \sigma_\alpha\right).$$

We consider the product $L_1(z_1)L_2(z_2)L_3(w) \in V^{\otimes 3}$. It follows from the Yang–Baxter equation that $R_{12}R_{13}R_{23}L_1L_2L_3 = L_3L_2L_1R_{12}R_{13}R_{23}$. We set $z_2 = z_1 + 2\hbar$, and then we have $P^-_{12}R_{13}R_{23}L_1L_2L_3 = L_3L_2L_1P^-_{12}R_{13}R_{23}$. The most important statement is that $P^-_{12}R_{13}R_{23} \sim P^-_{12} \otimes 1_3$, which follows by direct calculation. To simplify matters, we can use the following identity for $\alpha, \beta, \gamma \sim 1, 2, 3$ up to cyclic permutations:

$$-\varphi^\hbar_0(x)\varphi_\gamma(x-2\hbar) + \varphi^\hbar_\gamma(x)\varphi_0(x-2\hbar) + \varphi^\hbar_\beta(x)\varphi_\alpha(x-2\hbar) + \varphi^\hbar_\alpha(x)\varphi_\beta(x-2\hbar) = 0. \tag{4.22}$$

Because $\mathrm{Tr}_{12}(P^-_{12}L_1L_2) = \mathrm{Tr}_{12}(P^-_{12}L_2L_1)$, we obtain the final result

$$[\mathrm{Tr}_{12}(P^-_{12}L_1(z-2\hbar)L_2(z)), L_3(w)] = 0.$$

4.4. Inhomogeneous algebra and the reflection equation. We consider the case of rank $N = 2$ with four marked points, $n = 4$. First, let the marked points be at $z = 0$ and at the half-periods of Σ_τ,

$$x_0 = 0, \qquad x_1 = \frac{\tau}{2} = \omega_2, \qquad x_2 = \frac{1+\tau}{2} = \omega_1 + \omega_2, \qquad x_3 = \frac{1}{2} = \omega_1.$$

We assume that

$$S^j_\alpha = \delta^j_\alpha \tilde{\nu}_\alpha, \quad j = 1, 2, 3, \tag{4.23}$$

where $S^0_\alpha = S_\alpha$ are arbitrary. This choice is based on the reduction $L(z)L(-z) = \det L(z) \cdot I$.

Let R^- be the quantum vertex R-matrix that arises in the XYZ model. We also introduce the R^+-matrix

$$R^\pm(z, w) = \sum_{a=0}^{3} \varphi^{\hbar/2}_a(z \pm w)\sigma_a \otimes \sigma_a. \tag{4.24}$$

We define the quantum Lax operator

$$\widehat{L}(z) = \widehat{S}_0\phi^\hbar(z)\sigma_0 + \sum_\alpha (\widehat{S}_\alpha\varphi^\hbar_\alpha(z) + \tilde{\nu}_\alpha\varphi^\hbar_\alpha(z-\omega_\alpha))\sigma_\alpha. \tag{4.25}$$

Proposition 8. *Lax operator* (4.26) *satisfies the quantum reflection equation*

$$R^-(z, w)\widehat{L}_1(z)R^+(z, w)\widehat{L}_2(w) = \widehat{L}_2(w)R^+(z, w)\widehat{L}_1(z)R^-(z, w), \tag{4.26}$$

if its components S_a generate the associative algebra with the relations

$$[\tilde{\nu}_\alpha, \tilde{\nu}_\beta] = 0, \qquad [\tilde{\nu}_\alpha, \widehat{S}_a] = 0, \tag{4.27}$$

$$i[\widehat{S}_0, \widehat{S}_\alpha]_+ = [\widehat{S}_\beta, \widehat{S}_\gamma], \tag{4.28}$$

$$[\widehat{S}_\gamma, \widehat{S}_0] = i\frac{K_\beta - K_\alpha}{K_\gamma}[\widehat{S}_\alpha, \widehat{S}_\beta]_+ - 2i\frac{1}{K_\gamma}(\tilde{\nu}_\alpha\rho_\alpha\widehat{S}_\beta - \tilde{\nu}_\beta\rho_\beta\widehat{S}_\alpha), \tag{4.29}$$

where

$$K_\alpha = E_1(\hbar + \check{\alpha}) - E_1(\hbar) - E_1(\check{\alpha}), \qquad \rho_\alpha = -\exp(-2\pi i\check{\alpha}\partial_\tau\check{\alpha})\phi(\check{\alpha} + \hbar, -\check{\alpha}).$$

The proof is based on direct calculation of the left- and right-hand sides of (4.27) (see [9] for the details). If all the $\nu_\alpha = 0$ in relations (4.28)–(4.30), then the associative algebra coincides with the Sklyanin algebra. Therefore, the algebra defined by relations (4.28)–(4.30) is a three-parameter deformation of the Sklyanin algebra.

The elements

$$C_1 = \widehat{S}^2_0 + \sum_\alpha \widehat{S}^2_\alpha, \qquad C_2 = \sum_\alpha \widehat{S}^2_\alpha K_\alpha(K_\alpha - K_\beta - K_\gamma) + 2\tilde{\nu}_\alpha\rho_\alpha K_\alpha\widehat{S}_\alpha$$

belong to the center of generalized Sklyanin algebra (4.28), (4.29). They are the coefficients in the expansion of the quantum determinant

$$\det_\hbar = \mathrm{tr}(P^-\widehat{L}(z, \hbar) \otimes \widehat{L}(z + 2\hbar, \hbar)).$$

Appendix A: Elliptic functions

We set $q = e^{2\pi i\tau}$, where τ is the modular parameter of an elliptic curve E_τ. The basis element is given by the theta function:

$$\vartheta(z\,|\,\tau) = q^{1/8}\sum_{n\in\mathbb{Z}}(-1)^n\mathbf{e}\left(\frac{1}{2}n(n+1)\tau + nz\right), \qquad \mathbf{e}(x) = e^{2\pi i x}.$$

We introduce the Eisenstein functions

$$E_1(z\,|\,\tau) = \partial_z \log\vartheta(z\,|\,\tau), \qquad E_1(z\,|\,\tau) \sim \frac{1}{z} - 2\eta_1 z, \tag{A.1}$$

where

$$\eta_1(\tau) = \frac{24}{2\pi i}\frac{\eta'(\tau)}{\eta(\tau)}, \qquad \eta(\tau) = q^{1/24}\prod_{n>0}(1-q^n),$$

is the Dedekind function and

$$E_2(z\,|\,\tau) = -\partial_z E_1(z\,|\,\tau) = \partial_z^2\log\vartheta(z\,|\,\tau), \qquad E_2(z\,|\,\tau) \sim \frac{1}{z^2} + 2\eta_1. \tag{A.2}$$

The relations

$$\zeta(z,\tau) = E_1(z,\tau) + 2\eta_1(\tau)z, \qquad \wp(z,\tau) = E_2(z,\tau) - 2\eta_1(\tau)$$

hold for the Weierstrass function.

We introduce the higher Eisenstein functions

$$E_j(z) = \frac{(-1)^j}{(j-1)!}\partial^{(j-2)}E_2(z), \quad j > 2. \tag{A.3}$$

Let

$$\phi(u,z) = \frac{\vartheta(u+z)\vartheta'(0)}{\vartheta(u)\vartheta(z)}. \tag{A.4}$$

Then $\phi(u,z) = \phi(z,u)$ and $\phi(-u,-z) = -\phi(u,z)$. The function ϕ has a pole at the point $z = 0$ and

$$\phi(u,z) = \frac{1}{z} + E_1(u) + \frac{z}{2}(E_1^2(u) - \wp(u)) + \ldots, \tag{A.5}$$

$$\partial_u\phi(u,z) = \phi(u,z)(E_1(u+z) - E_1(u)), \tag{A.6}$$

$$\partial_z\phi(u,z) = \phi(u,z)(E_1(u+z) - E_1(z)), \tag{A.7}$$

$$\lim_{z\to 0}\log\partial_u\phi(u,z) = -E_2(u).$$

The function ϕ satisfies the heat-kernel equation

$$\partial_\tau\phi(u,w) - \frac{1}{2\pi i}\partial_u\partial_w\phi(u,w) = 0.$$

The quasiperiods of the functions introduced above are

$$\vartheta(z+1) = -\vartheta(z), \qquad \vartheta(z+\tau) = -q^{-1/2}e^{-2\pi i z}\vartheta(z),$$

$$E_1(z+1) = E_1(z), \qquad E_1(z+\tau) = E_1(z) - 2\pi i,$$

$$E_2(z+1) = E_2(z), \qquad E_2(z+\tau) = E_2(z),$$

$$\phi(u,z+1) = \phi(u,z), \qquad \phi(u,z+\tau) = e^{-2\pi i u}\phi(u,z),$$

$$\partial_u\phi(u,z+1) = \partial_u\phi(u,z), \qquad \partial_u\phi(u,z+\tau) = e^{-2\pi i u}\partial_u\phi(u,z) - 2\pi i\phi(u,z).$$

The Fay identity holds:

$$\phi(u_1,z_1)\phi(u_2,z_2)-\phi(u_1+u_2,z_1)\phi(u_2,z_2-z_1)-\phi(u_1+u_2,z_2)\phi(u_1,z_1-z_2)=0. \tag{A.8}$$

From (A.7) and (A.8), we obtain

$$\phi(u_1,z)\phi(u_2,z)=\phi(u_1+u_2,z)(E_1(u_1)+E_1(u_2)-E_1(u_1+u_2+z)+E_1(z)). \tag{A.9}$$

Special cases of identity (A.8) amount to the functional equations

$$\begin{aligned}
&\phi(u,z)\partial_v\phi(v,z)-\phi(v,z)\partial_u\phi(u,z)=(E_2(v)-E_2(u))\phi(u+v,z),\\
&\phi(u,z_1)\phi(-u,z_2)=\phi(u,z_1-z_2)(-E_1(z_1)+E_1(z_2)-\\
&\qquad\qquad -E_1(u)+E_1(u+z_1-z_2))=\\
&\qquad\qquad =\phi(u,z_1-z_2)(-E_1(z_1)+E_1(z_2)+\partial_u\phi(u,z_2-z_1)),\\
&\phi(u,z)\phi(-u,z)=E_2(z)-E_2(u),
\end{aligned} \tag{A.10}$$

$$\begin{aligned}
&\phi(v,z-w)\phi(u_1-v,z)\phi(u_2+v,w)-\phi(u_1-u_2-v,z-w)\phi(u_2+v,z)\phi(u_1-v,w)=\\
&\qquad =\phi(u_1,z)\phi(u_2,w)f(u_1,u_2,v),
\end{aligned} \tag{A.11}$$

where

$$f(u_1,u_2,v)=E_1(v)-E_1(u_1-u_2-v)+E_1(u_1-v)-E_1(u_2+v). \tag{A.12}$$

Function (A.12) can be rewritten as

$$f(u_1,u_2,v)=-\frac{\vartheta'(0)\vartheta(u_1)\vartheta(u_2)\vartheta(u_2-u_1+2v)}{\vartheta(u_1-v)\vartheta(u_2+v)\vartheta(u_2-u_1+v)\vartheta(v)}.$$

Using (A.1), (A.2), and (A.5), we can deduce some important special cases from (A.11). In the first case, with $v=u_1$ (or $v=-u_2$), we obtain Fay identity (A.8). In the second case, with $u_1=0$ (or $u_2=u$), we have

$$\begin{aligned}
&\phi(v,z-w)\phi(-v,z)\phi(u+v,w)-\phi(-u-v,z-w)\phi(u+v,z)\phi(-v,w)=\\
&\qquad =\phi(u_1,z)(E_2(u+v)-E_2(v)).
\end{aligned}$$

If $u_2\to -v$, then relation (A.11) with $u_1=\alpha$ and $u_2=\beta$ in the first nontrivial order becomes

$$\begin{aligned}
&\phi(-\beta,z-w)E_1(w)\phi(\alpha+\beta,z)-\phi(\alpha,z-w)E_1(z)\phi(\alpha+\beta,w)=\\
&\qquad =\phi(\alpha,z)\phi(\beta,w)(E_1(\alpha)+E_1(\beta)-E_1(\alpha+\beta)).
\end{aligned}$$

Appendix B: The Lie algebra $sl(N,\mathbb{C})$ and elliptic functions

We introduce the notation

$$\begin{aligned}
&\mathbf{e}_N(z)=e^{2\pi iz/N},\\
&Q=\mathrm{diag}(\mathbf{e}_N(1),\dots,\mathbf{e}_N(m),\dots,1),\\
&\Lambda=\delta_{j,j+1},\qquad j=1,\dots,N \mod N.
\end{aligned}$$

Let

$$\mathbb{Z}_N^{(2)} = (\mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z}), \qquad \widetilde{\mathbb{Z}}_N^{(2)} = \mathbb{Z}_N^{(2)} \setminus (0,0)$$

be two-dimensional lattices of the respective orders N^2 and N^2-1. The matrices $Q^{a_1}\Lambda^{a_2}$, $a=(a_1,a_2)\in\mathbb{Z}_N^{(2)}$, induce a basis in the group $GL(N,\mathbb{C})$, and $Q^{\alpha_1}\Lambda^{\alpha_2}$, $\alpha=(\alpha_1,\alpha_2)\in\widetilde{\mathbb{Z}}_N^{(2)}$, induce a basis in the Lie algebra $sl(N,\mathbb{C})$. We consider the representation $\mathbb{Z}_N^{(2)}$ in $GL(N,\mathbb{C})$ in detail:

$$a \to T_a = \frac{N}{2\pi i}\mathbf{e}_N\left(\frac{a_1a_2}{2}\right)Q^{a_1}\Lambda^{a_2}, \tag{B.1}$$

$$T_aT_b = \frac{N}{2\pi i}\mathbf{e}_N\left(-\frac{a\times b}{2}\right)T_{a+b}, \qquad a\times b = a_1b_2 - a_2b_1, \tag{B.2}$$

where $N\mathbf{e}_N(-a\times b/2)/(2\pi i)$ is a nontrivial 2-cocycle in $H^2(\mathbb{Z}_N^{(2)},\mathbb{Z}_{2N})$. The matrices T_α, $\alpha\in\widetilde{\mathbb{Z}}_N^{(2)}$, induce a basis in $sl(N,\mathbb{C})$. It follows from (B.2) that

$$[T_\alpha, T_\beta] = \mathbf{C}(\alpha,\beta)T_{\alpha+\beta}, \tag{B.3}$$

where $\mathbf{C}(\alpha,\beta) = (N/\pi)\sin(\pi(\alpha\times\beta)/N)$ are the structure constants of $sl(N,\mathbb{C})$.

For $N=2$, the basis T_α is proportional to the basis of Pauli matrices,

$$T_{(1,0)} = \frac{1}{\pi i}\sigma_3, \qquad T_{(0,1)} = \frac{1}{\pi i}\sigma_1, \qquad T_{(1,1)} = \frac{1}{\pi i}\sigma_2.$$

The Lie coalgebra $\mathfrak{g}^* = sl(N,\mathbb{C})$ has the dual basis

$$\mathfrak{g}^* = \left\{\mathbf{S} = \sum_{\widetilde{\mathbb{Z}}_N^{(2)}} S_\gamma t^\gamma\right\}, \qquad t^\gamma = \frac{2\pi i}{N^2}T_{-\gamma}, \qquad \langle T_\alpha t^\beta\rangle = \delta_\alpha^{-\beta}.$$

It follows from (B.3) that $\mathfrak{g}^*$ is a Poisson manifold endowed with the bracket

$$\{S_\alpha, S_\beta\} = \mathbf{C}(\alpha,\beta)S_{\alpha+\beta}.$$

The coadjoint action in these bases has the form

$$\mathrm{ad}^*_{T_\alpha} t^\beta = \mathbf{C}(\alpha,\beta)t^{\alpha+\beta}.$$

Let $\check{\gamma} = (\gamma_1+\gamma_2\tau)/N$. We introduce constants on $\widetilde{\mathbb{Z}}^{(2)}$,

$$\vartheta(\check{\gamma}) = \vartheta\left(\frac{\gamma_1+\gamma_2\tau}{N}\right), \qquad E_1(\check{\gamma}) = E_1\left(\frac{\gamma_1+\gamma_2\tau}{N}\right), \qquad E_2(\check{\gamma}) = E_2\left(\frac{\gamma_1+\gamma_2\tau}{N}\right), \tag{B.4}$$

and the functions

$$\phi_\gamma(z) = \phi(\check{\gamma}, z),$$

$$\varphi_\gamma(z) = \mathbf{e}_N(\gamma_2 z)\phi_\gamma(z), \tag{B.5}$$

$$\varphi_{\gamma,\eta}(z) = \mathbf{e}_N(\gamma_2 z)\phi\left(\eta + \frac{\gamma_1+\gamma_2\tau}{N}, z\right). \tag{B.6}$$

These functions have the quasiperiods

$$\varphi_\gamma(z+1) = \mathbf{e}_N(\gamma_2)\varphi_\gamma(z), \qquad \varphi_\gamma(z+\tau) = \mathbf{e}_N(-\gamma_1)\varphi_\gamma(z), \tag{B.7}$$

$$\varphi_{\gamma,\eta}(z+1) = \mathbf{e}_N(\gamma_2)\varphi_{\gamma,\eta}(z), \qquad \varphi_{\gamma,\eta}(z+\tau) = \mathbf{e}_N(-\gamma_1-\eta)\varphi_{\gamma,\eta}(z). \tag{B.8}$$

Important relations for these functions are given by

$$\frac{\varphi_\eta(z_1-z_2)}{\varphi_\eta(z_1)} = \frac{1}{\varphi_\eta(z_2)}(E_1(z_2)+E_1(\eta)+E_1(z_1-z_2)-E_1(z_1+\eta)), \tag{B.9}$$

$$\frac{\varphi_{\alpha,\eta}(z_1-z_2)}{\varphi_\eta(z_1)} = \frac{1}{\varphi_\eta(z_2)}\varphi_\alpha(z_1-z_2) + \frac{\varphi_{-\alpha}(z_2)}{\varphi_\eta(z_2)\varphi_\alpha(\eta)}\varphi_\alpha(z_1+\eta), \tag{B.10}$$

$$\frac{\varphi_{\alpha,\eta}(z)}{\varphi_\eta(z)} = \frac{\varphi_\alpha(z+\eta)}{\varphi_\alpha(\eta)}. \tag{B.11}$$

Another important relation in the case $N=2$ is

$$k_\gamma f^\hbar(\gamma,\alpha,0) = J_\gamma = E_2(\gamma+\hbar) - E_2(\hbar),$$

where

$$k_\gamma = E_1(\gamma+\hbar) - E_1(\gamma) - E_1(\hbar).$$

We briefly comment on this formula. From (A.11) and (A.12), we have

$$(E_1(\gamma+\hbar)-E_1(\gamma)-E_1(\hbar))(E_1(\alpha+\hbar)+E_1(-\beta+\hbar)-E_1(\gamma+\hbar)-E_1(\hbar)) = E_2(\gamma+\hbar)-E_2(\hbar),$$

where $\alpha-\beta=\gamma$. The functions in the right- and left-hand sides have coincident poles ($\hbar=0$, $\hbar=-\gamma$) and zeros ($\hbar=-\gamma/2$), and these functions are therefore equivalent.

We define the function

$$f_\gamma(z) = \mathbf{e}_N(\gamma_2 z)\partial_u\phi(u,z)\big|_{u=\breve{\gamma}} = \varphi_\gamma(z)(E_1(\breve{\gamma}+z)-E_1(\breve{\gamma})).$$

It follows from (A.6) that

$$\begin{aligned} &f_\gamma(z) = \varphi_\gamma(z)(E_1(\breve{\gamma}+z)-E_1(\breve{\gamma})), \\ &f(\alpha,\beta,\gamma) = E_1(\breve{\gamma}) - E_1(\breve{\alpha}-\breve{\beta}-\breve{\gamma}) + E_1(\breve{\alpha}-\breve{\gamma}) - E_1(\breve{\beta}-\breve{\gamma}) \end{aligned} \tag{B.12}$$

(see (A.12)). Equation (A.4) implies that

$$\begin{aligned} &\varphi_\gamma(z+1) = \mathbf{e}_N(\gamma_2)\varphi_\gamma(z), \qquad \varphi_\gamma(z+\tau) = \mathbf{e}_N(-\gamma_1)\varphi_\gamma(z), \\ &f_\gamma(z+1) = \mathbf{e}_N(\gamma_2)f_\gamma(z), \qquad f_\gamma(z+\tau) = \mathbf{e}_N(-\gamma_1)f_\gamma(z) - 2\pi i\varphi_\gamma(z). \end{aligned} \tag{B.13}$$

A modified version of (A.11) is given by

$$\varphi_\gamma(z-x_j)\varphi_{-\gamma}(z-x_k) = \varphi_\gamma(x_k-x_j)(E_1(z-x_k)-E_1(z-x_j)) - f_\gamma(x_k-x_j).$$

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