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Padé approximant related to remarkable inequalities involving trigonometric functions

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Abstract
In this paper we, respectively, give simple proofs of some remarkable trigonometric inequalities, based on the Padé approximation method. We also obtain rational refinements of these inequalities. We are convinced that the Padé approximation method offers a general framework for solving many other similar inequalities.

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1 Introduction and motivation

The starting point of this paper is the following famous inequalities.

The Cusa-Huygens inequalities state that for $x \in (0, \frac{\pi}{2})$,

$$\frac{1+\cos x}{2} < \frac{\sin x}{x} < \frac{2+\cos x}{3}. \tag{1.1}$$

The Wilker inequality asserts that

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 \quad \text{holds for every } x \in \left(0, \frac{\pi}{2}\right). \tag{1.2}$$

The Huygens inequality for sine and tangent functions states that for $x \in (0, \frac{\pi}{2})$,

$$2\sin x + \tan x > 3x. \tag{1.3}$$

The inequality

$$(\cos x)^{\frac{1}{3}} < \frac{\sin x}{x}, \quad 0 < x < \frac{\pi}{2} \tag{1.4}$$

was established by Adamović and Mitrinović (see, *e.g.* [1], p.238).

Another inequality which is of interest to us is the following:

$$\sin x + \tan x > 2x \quad \text{for all } x \in \left(0, \frac{\pi}{2}\right). \tag{1.5}$$

In [2], the following inequality was given:

$$3\frac{x}{\sin x} + \cos x > 4 \quad \text{for every } x \in \left(0, \frac{\pi}{2}\right). \tag{1.6}$$

The classical Jordan inequality states that

$$\frac{\sin x}{x} \geq \frac{2}{\pi} \quad \text{hold for all } x \in \left(0, \frac{\pi}{2}\right). \tag{1.7}$$

This inequality has been further refined in the past year. For example, in [3], Redheffer proposed the inequality

$$\frac{\sin x}{x} \geq \frac{\pi^2 - x^2}{\pi^2 + x^2} \quad \text{for all } x \neq 0. \tag{1.8}$$

A similar result, which involves the rational approximation, has been obtained in [4] as follows:

$$\frac{\sin x}{x} \geq \frac{53}{53 + 9x^2}, \quad 0 \leq x \leq \frac{1}{3}. \tag{1.9}$$

These inequalities were of great interest throughout the research, since they were extended in different forms in the recent past. We refer to [1–15] and closely related references therein. Some of the recent improvements are nice through their symmetric form, but these inequalities have also practical importance, because they provide some bounds for a given expression. We noticed that many of these new inequalities were obtained using Taylor's expansions of some trigonometric functions. That is why in our paper we propose a new approach based on the Padé approximation. It is known that a Padé approximant is the 'best' approximation of a function by a rational function of given order. This method has the somewhat magical property of converting a poorly converging power series into a usually much better behaved rational polynomial. The rational approximation is particularly good for series with alternating terms and poor polynomial convergence. For these reasons, longer versions of optimized rational polynomials are frequently used in computer calculations.

Following the techniques developed and the results obtained in recent work [2, 6, 10, 11], we expect that the Padé approximation method will be useful in solving and refining others problems concerning inequalities.

The Padé approximant [L/M] corresponds to the Taylor series. When it exists, the [L/M] Padé approximant to any power series $A(x) = \sum_{j=0}^{\infty} a_j x^j$ is unique. If $A(x)$ is a transcendental function, then the terms are given by the Taylor series about x_0, $a_n = \frac{1}{n!}A^{(n)}(x_0)$.

The coefficients are found by setting $A(x) = \frac{p_0 + p_1 x + \cdots + p_L x^L}{1 + q_1 x + \cdots + q_M x^M}$. These give the set of equations

$$\begin{cases} p_0 = a_0, \\ p_1 = a_0 q_1 + a_1, \\ p_2 = a_0 q_2 + a_1 q_1 + a_2, \\ \vdots \\ p_L = a_0 q_L + \cdots + a_{L-1} q_1 + a_L, \\ 0 = a_{L-M+1} q_M + \cdots + a_L q_1 + a_{L+1}, \\ 0 = a_L q_M + \cdots + a_{L+M-1} q_1 + a_{L+M}. \end{cases}$$

Table 1 We present the first orders of Pade approximant for sin x, cos x, tan x

Function	Padé approximant	Associated Taylor polynomials
$\sin x$	$\sin_{[1/2]}(x) = \frac{6x}{6+x^2}$	$x - \frac{x^3}{6}$
$\sin x$	$\sin_{[5/2]}(x) = \frac{2{,}520x-360x^3+11x^5}{2{,}520+60x^2}$	$x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5{,}040}$
$\cos x$	$\cos_{[2/2]}(x) = \frac{12-5x^2}{12+x^2}$	$1 - \frac{x^2}{2} + \frac{x^4}{24}$
$\cos x$	$\cos_{[4/2]}(x) = \frac{120-56x^2+3x^4}{120+4x^2}$	$1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$
$\cos x$	$\cos_{[4/4]}(x) = \frac{1{,}080-480x^2+17x^4}{1{,}080+60x^2+2x^4}$	$1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$
$\tan x$	$\tan_{[3/2]}(x) = \frac{15x-x^3}{15-6x^2}$	$x + \frac{x^3}{3} + \frac{2x^5}{15}$

For example, we consider the Taylor series for sin: $\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + O(x^7)$ and its associated polynomial: $x - \frac{1}{6}x^3 + \frac{1}{120}x^5$.

The Padé approximant $\sin_{[3/3]}(x) = \frac{p_0+p_1x+p_2x^2+p_3x^3}{1+q_1x+q_2x^2+q_3x^3}$ satisfies

$$\left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)\left(1 + q_1x + q_2x^2 + q_3x^3\right) = p_0 + p_1 + p_2x^2 + p_3x^3.$$

We find

$$p_0 = 0, \qquad p_1 = 1, \qquad p_2 = q_1 = 0, \qquad p_3 = -\frac{7}{60}, \qquad q_2 = \frac{1}{20}, \qquad q_3 = 0.$$

Therefore

$$\sin_{[3/3]}(x) = \frac{x - \frac{7}{60}x^3}{1 + \frac{1}{20}x^2} = \frac{60x - 7x^3}{60 + 3x^2}.$$

The first order versions of a few trigonometric functions which we will use are in Table 1.

2 Some lemmas

In order to attain our aim, we first prove several lemmas.

Lemma 2.1 *For every $x \in (0, \frac{\pi}{2})$, one has*

$$\frac{-7x^3 + 60x}{3x^2 + 60} < \sin x < \frac{11x^5 - 360x^3 + 2{,}520x}{60x^2 + 2{,}520}$$

and also

$$\sin x < \frac{6x}{x^2 + 6}.$$

Proof We introduce the function

$$r(x) = \left(3x^2 + 60\right)\sin x - 60x + 7x^3.$$

Easy computation yields

$$r'(x) = 6x\sin x + \left(3x^2 + 60\right)\cos x - 60 + 21x^2,$$

$$r^{(2)}(x) = -54\sin x + 120x\cos x - 3x^2\sin x + 42x,$$

$$r^{(3)}(x) = -42\cos x - 18x\sin x - 3x^2\cos x + 42,$$

$$r^{(4)}(x) = 24\sin x - 24x\cos x + 3x^2\sin x,$$

$$r^{(5)}(x) = 30x\sin x + 3x^2\cos x.$$

Evidently $r^{(5)} > 0$ on $(0, \frac{\pi}{2})$. Then $r^{(4)}$ is strictly increasing on $(0, \frac{\pi}{2})$. As $r^{(4)}(0) = 0$, we get $r^{(4)} > 0$ on $(0, \frac{\pi}{2})$. Continuing the algorithm, finally we obtain $r(x) > 0$ for all $x \in (0, \frac{\pi}{2})$.

Let

$$s(x) = (60x^2 + 2{,}520)\sin x - 11x^5 + 360x^3 - 2{,}520x.$$

Then

$$s'(x) = 120x\sin x + (60x^2 + 2{,}520)\cos x - 55x^4 + 1{,}080x^2 - 2{,}520,$$

$$s^{(2)}(x) = -2{,}400\sin x + 240x\cos x - 60x^2\sin x - 220x^3 + 2{,}160x,$$

$$s^{(3)}(x) = -2{,}160\cos x - 360x\sin x - 60x^2\cos x - 660x^2 + 2{,}160,$$

$$s^{(4)}(x) = 1{,}800\sin x - 480x\cos x + 60x^2\sin x - 1{,}320x,$$

$$s^{(5)}(x) = 1{,}320\cos x + 600x\sin x + 60x^2\cos x - 1{,}320,$$

$$s^{(6)}(x) = -720\sin x + 720x\cos x - 60x^2\sin x,$$

$$s^{(7)}(x) = -840x\sin x - 60x^2\cos x.$$

The function $s^{(7)} < 0$ for all $x \in (0, \frac{\pi}{2})$, therefore $s^{(6)}$ is strictly decreasing on $(0, \frac{\pi}{2})$. As $s^{(6)}(0) = 0$, we get $s^{(6)} < 0$ on $(0, \frac{\pi}{2})$. Using the same arguments, finally we have $s(x) < 0$ for every $x \in (0, \frac{\pi}{2})$.

For proving the last part of the lemma, we consider the function

$$p(x) = (x^2 + 6)\sin x - 6x$$

and its derivatives

$$p'(x) = 2x\sin x + (x^2 + 6)\cos x - 6,$$

$$p^{(2)}(x) = (-x^2 - 4)\sin x + 4x\cos x,$$

$$p^{(3)}(x) = -x^2\cos x - 6x\sin x.$$

We have $p^{(3)} < 0$ on $(0, \frac{\pi}{2})$, hence $p^{(2)}$ is strictly decreasing on $(0, \frac{\pi}{2})$. As $p^{(2)}(0) = 0$, it follows that $p^{(2)} < 0$ on $(0, \frac{\pi}{2})$. As above, finally we deduce that $p < 0$ on $(0, \frac{\pi}{2})$.

This completes the proof. □

Lemma 2.2 *For every* $x \in (0, \frac{\pi}{2})$, *one has*

$$\frac{17x^4 - 480x^2 + 1{,}080}{2x^4 + 60x^2 + 1{,}080} < \cos x < \frac{3x^4 - 56x^2 + 120}{4x^2 + 120}$$

and also

$$\frac{-5x^2+12}{x^2+12} < \cos x.$$

Proof Let

$$f(x) = \left(4x^2+120\right)\cos x - 3x^4 + 56x^2 - 120.$$

We get

$$\begin{aligned}
f'(x) &= 8x\cos x - \left(4x^2+120\right)\sin x - 12x^3 + 112x,\\
f^{(2)}(x) &= \left(-4x^2-112\right)\cos x - 16x\sin x - 36x^2 + 112,\\
f^{(3)}(x) &= -24x\cos x + \left(4x^2+96\right)\sin x - 72x,\\
f^{(4)}(x) &= \left(4x^2+72\right)\cos x + 32x\sin x - 72,\\
f^{(5)}(x) &= -4x^2\sin x - 40\sin x + 40x\cos x,\\
f^{(6)}(x) &= -4x^2\cos x - 48x\sin x.
\end{aligned}$$

Evidently $f^{(6)} < 0$ on $(0, \frac{\pi}{2})$. It follows that $f^{(5)}$ is strictly decreasing on $(0, \frac{\pi}{2})$. As $f^{(5)}(0) = 0$, we get $f^{(5)} < 0$ on $(0, \frac{\pi}{2})$. Using the same algorithm, we finally obtain $f(x) < 0$ for $x \in (0, \frac{\pi}{2})$.

If we consider

$$g(x) = \left(2x^4+60x^2+1{,}080\right)\cos x - 17x^4 + 480x^2 - 1{,}080,$$

then we have

$$\begin{aligned}
g'(x) &= \left(8x^3+120x\right)\cos x - \left(2x^4+60x^2+1{,}080\right)\sin x - 68x^3 + 960x,\\
g^{(2)}(x) &= \left(-2x^4-36x^2-960\right)\cos x - \left(16x^3+240x\right)\sin x - 204x^2 + 960,\\
g^{(3)}(x) &= \left(-24x^3-312x\right)\cos x + \left(2x^4-12x^2+720\right)\sin x - 408x,\\
g^{(4)}(x) &= \left(2x^4-84x^2+408\right)\cos x + \left(32x^3+288x\right)\sin x - 408,\\
g^{(5)}(x) &= \left(40x^3+120x\right)\cos x + \left(-2x^4+180x^2-120\right)\sin x,\\
g^{(6)}(x) &= \left(-2x^4+300x^2\right)\cos x + \left(-48x^3+240x\right)\sin x.
\end{aligned}$$

The positivity of $g^{(6)}$ on $(0, \frac{\pi}{2})$ shows that $g^{(5)}$ is strictly increasing on $(0, \frac{\pi}{2})$. As $g^{(5)}(0) = 0$, we get $g^{(5)} > 0$ on $(0, \frac{\pi}{2})$. Similar arguments lead us to the positivity of g on $(0, \frac{\pi}{2})$.

Next, we get

$$h(x) = \left(x^2+12\right)\cos x + 5x^2 - 12.$$

Elementary calculations reveal that

$$\begin{aligned}
h'(x) &= 2x\cos x - \left(x^2+12\right)\sin x + 10x,\\
h^{(2)}(x) &= \left(-x^2-10\right)\cos x - 4x\sin x + 10,
\end{aligned}$$

$$h^{(3)}(x) = \left(x^2 + 6\right)\sin x - 6x\cos x,$$
$$h^{(4)}(x) = 8x\sin x + x^2\cos x.$$

We remark that $h^{(4)} > 0$ on $(0, \frac{\pi}{2})$.

Using a similar algorithm to above, we find $h > 0$ on $(0, \frac{\pi}{2})$.

This completes the proof of the second lemma. □

Lemma 2.3 *For every* $x \in (0, \frac{\pi}{2})$, *one has*

$$\tan x > \frac{-x^3 + 15x}{-6x^2 + 15}.$$

Remark 2.1 The denominator $-6x^2 + 15$ is positive for all $x \in (0, \sqrt{\frac{5}{2}}) = (0, 1.58)$. Since $x \in (0, \frac{\pi}{2}) = (0, 1.57075)$, the denominator of the function from the right-hand side is positive.

Proof of Lemma 2.3 We introduce the function

$$q(x) = \left(-6x^2 + 15\right)\sin x - \left(15x - x^3\right)\cos x.$$

The derivative is $q'(x) = x[(3 - x^2)\sin x - 3x\cos x]$. Then we consider the function $u(x) = (3 - x^2)\sin x - 3x\cos x$ and its derivative $u'(x) = x(\sin x - x\cos x)$. Again we consider the function $v(x) = \sin x - x\cos x$ and its derivative $v'(x) = x\sin x$. We see that $v' > 0$ on $(0, \frac{\pi}{2})$. As $v(0) = 0$, it follows that $v > 0$ on $(0, \frac{\pi}{2})$, therefore $u'(x) > 0$ for every $x \in (0, \frac{\pi}{2})$.

Due to similar arguments, finally it follows that $q(x) > 0$ for every $x \in (0, \frac{\pi}{2})$.

The proof of Lemma 2.3 is complete. □

3 Results

In this section we will formulate and prove the announced rational inequalities, using as our main tool the estimates from the above lemmas.

First, we establish new inequalities of the Cusa-Huygens type as follows.

Theorem 3.1 *For every* $0 < x < \frac{\pi}{2}$, *the rational inequalities chain*

$$\frac{1 + \cos x}{2} < \frac{3x^4 - 52x^2 + 240}{8x^2 + 240} < \frac{60 - 7x^2}{60 + 3x^2} < \frac{\sin x}{x}$$
$$< \frac{11x^4 - 360x^2 + 2{,}520}{60x^2 + 2{,}520} < \frac{7x^4 - 120x^2 + 1{,}080}{2x^4 + 60x^2 + 1{,}080} < \frac{2 + \cos x}{3}$$

holds.

Proof The inequalities

$$\frac{1 + \cos x}{2} < \frac{3x^4 - 52x^2 + 240}{8x^2 + 240},$$
$$\frac{60 - 7x^2}{60 + 3x^2} < \frac{\sin x}{x} < \frac{11x^4 - 360x^2 + 2{,}520}{60x^2 + 2{,}520},$$

and

$$\frac{7x^4 - 120x^2 + 1{,}080}{2x^4 + 60x^2 + 1{,}080} < \frac{2 + \cos x}{3}$$

are easy consequences of Lemmas 2.1 and 2.2.

We will calculate the difference

$$\begin{aligned} E(x) &= \frac{60 - 7x^2}{60 + 3x^2} - \frac{3x^4 - 52x^2 + 240}{8x^2 + 240} \\ &= \frac{-9x^6 - 80x^4 + 1{,}200x^2}{(60 + 3x^2)(8x^2 + 240)} \\ &= \frac{-x^2(9x^4 + 80x^2 - 1{,}200)}{(60 + 3x^2)(8x^2 + 240)}. \end{aligned}$$

The function

$$P(x) = 9x^4 + 80x^2 - 1{,}200$$

has the real roots

$$x_1 = -\frac{2}{3}\sqrt{5(\sqrt{31} - 2)} \approx -2.78$$

and

$$x_2 = \frac{2}{3}\sqrt{5(\sqrt{31} - 2)} \approx 2.78.$$

We obtain $P(x) < 0$ for all $x \in (0, \frac{\pi}{2}) \subset (-\frac{2}{3}\sqrt{5(\sqrt{31} - 2)}, \frac{2}{3}\sqrt{5(\sqrt{31} - 2)})$ and therefore $E(x) > 0$ for every $x \in (0, \frac{\pi}{2})$.

We will calculate the difference

$$\begin{aligned} F(x) &= \frac{7x^4 - 120x^2 + 1{,}080}{2x^4 + 60x^2 + 1{,}080} - \frac{11x^4 - 360x^2 + 2{,}520}{60x^2 + 2{,}520} \\ &= \frac{-22x^8 + 480x^6 + 15{,}120x^4}{(2x^4 + 60x^2 + 1{,}080)(60x^2 + 2{,}520)} \\ &= \frac{-x^4(22x^4 - 480x^2 - 15{,}120)}{(2x^4 + 60x^2 + 1{,}080)(60x^2 + 2{,}520)}. \end{aligned}$$

The function

$$Q(x) = 22x^4 - 480x^2 - 15{,}120$$

has the real roots

$$x_1 = -\sqrt{\frac{120 + 6\sqrt{2{,}710}}{11}} \approx -6.26930$$

and

$$x_2 = \sqrt{\frac{120 + 6\sqrt{2{,}710}}{11}} \approx 6.26930.$$

We have $Q(x) < 0$ for all $x \in (0, \frac{\pi}{2}) \subset (-\sqrt{\frac{120+6\sqrt{2{,}710}}{11}}, \sqrt{\frac{120+6\sqrt{2{,}710}}{11}})$ and therefore $F(x) > 0$ for all $x \in (0, \frac{\pi}{2})$. □

Using the Padé approximation method we improve the celebrated Wilker inequality as follows.

Theorem 3.2 *Let* $0 < x < \frac{\pi}{2}$. *Then*

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > \frac{-303x^6 + 5{,}550x^4 - 32{,}400x^2 + 108{,}000}{-54x^6 - 2{,}025x^4 - 16{,}200x^2 + 54{,}000} > 2$$

holds.

Proof Using the results from Lemma 2.1, respectively Lemma 2.3, we have

$$\begin{aligned} \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} &> \frac{(60 - 7x^2)^2}{(60 + 3x^2)^2} + \frac{15 - x^2}{15 - 6x^2} \\ &= \frac{-303x^6 + 5{,}550x^4 - 32{,}400x^2 + 108{,}000}{-54x^6 - 2{,}025x^4 - 16{,}200x^2 + 54{,}000}. \end{aligned}$$

The inequality

$$\frac{-303x^6 + 5{,}550x^4 - 32{,}400x^2 + 108{,}000}{-54x^6 - 2{,}025x^4 - 16{,}200x^2 + 54{,}000} > 2$$

has the equivalent form

$$x^4\left(-195x^2 + 9{,}600\right) > 0.$$

This inequality is obviously true for all $x \in (0, \frac{\pi}{2})$. □

We refine the Huygens inequality for the sine and tangent functions (1.3) as follows.

Theorem 3.3 *For* $0 < x < \frac{\pi}{2}$, *we have*

$$2\frac{\sin x}{x} + \frac{\tan x}{x} > \frac{81x^4 - 945x^2 + 2{,}700}{-18x^4 - 315x^2 + 900} > 3.$$

Proof Using again the results from Lemma 2.1, respectively Lemma 2.3, we deduce

$$\begin{aligned} 2\frac{\sin x}{x} + \frac{\tan x}{x} &> \frac{2(60 - 7x^2)}{60 + 3x^2} + \frac{15 - x^2}{15 - 6x^2} \\ &= \frac{81x^4 - 945x^2 + 2{,}700}{-18x^4 - 315x^2 + 900}. \end{aligned}$$

The inequality

$$\frac{81x^4 - 945x^2 + 2{,}700}{-18x^4 - 315x^2 + 900} > 3$$

has the equivalent form

$$135x^4 > 0,$$

which is true for all $x \in (0, \frac{\pi}{2})$. □

In the following we will refine the inequality (1.4) as follows.

Theorem 3.4 *For every* $0 < x < \frac{\pi}{2}$, *we have*

$$\left(\frac{\sin x}{x}\right)^3 > \left(\frac{60 - 7x^2}{60 + 3x^2}\right)^3 > \frac{3x^4 - 56x^2 + 120}{4x^2 + 120} > \cos x.$$

Proof In the light of the results of the above lemmas, we have only to prove the inequality

$$\left(\frac{60 - 7x^2}{60 + 3x^2}\right)^3 > \frac{3x^4 - 56x^2 + 120}{4x^2 + 120}$$

for every $x \in (0, \frac{\pi}{2})$.

The difference

$$\left(\frac{60 - 7x^2}{60 + 3x^2}\right)^3 - \frac{3x^4 - 56x^2 + 120}{4x^2 + 120}$$

can be re-written as

$$\frac{x^4(-81x^6 - 4{,}720x^4 - 15{,}600x^2 + 1{,}728{,}000)}{(60 + 3x^2)^3(4x^2 + 120)}.$$

The polynomial function $R(x) = -81x^6 - 4{,}720x^4 - 15{,}600x^2 + 1{,}728{,}000$ has the real roots $x_1 \approx -3.9658$ and $x_2 \approx 3.9658$. Therefore $R(x) > 0$ for all $x \in (0, \frac{\pi}{2})$ and also the above difference is positive for all $x \in (0, \frac{\pi}{2})$. □

Using the Padé approximation method, we will sharpen the inequality (1.5) as follows.

Theorem 3.5 *For* $0 < x < \frac{\pi}{2}$, *the rational inequality*

$$\frac{\sin x}{x} + \frac{\tan x}{x} > \frac{39x^4 - 480x^2 + 1{,}800}{-18x^4 - 315x^2 + 900} > 2$$

holds.

Proof The inequalities from Lemma 2.1, respectively Lemma 2.3, give us the estimate

$$\begin{aligned}\frac{\sin x}{x} + \frac{\tan x}{x} &> \frac{60 - 7x^2}{60 + 3x^2} + \frac{15 - x^2}{15 - 6x^2} \\ &= \frac{39x^4 - 480x^2 + 1{,}800}{-18x^4 - 315x^2 + 900},\end{aligned}$$

which is greater than 2, since $75x^4 + 150x^2 > 0$ for all $x \in (0, \frac{\pi}{2})$. □

Using the Padé approximation for the sine and cosine functions, we obtain an improved version of the inequality (1.6).

Theorem 3.6 *For every $x \in (0, \frac{\pi}{2})$, the rational inequality*

$$3\frac{x}{\sin x} + \cos x > \frac{x^4 + 8x^2 + 96}{2x^2 + 24} > 4$$

holds.

Proof The estimates from the sine and cosine functions obtained in Lemma 2.1, respectively Lemma 2.2, lead to the inequality

$$3\frac{x}{\sin x} + \cos x > \frac{x^2 + 6}{2} + \frac{-5x^2 + 12}{x^2 + 12} = \frac{x^4 + 8x^2 + 96}{2x^2 + 24},$$

which is obviously greater than 4 for all $x \in (0, \frac{\pi}{2})$. □

In the following we will determine a lower rational bound for the sine function.

The sine function

$$\operatorname{sinc}(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0, \\ 1, & x = 0, \end{cases}$$

appears in a variety of applications.

Using the lower rational bound founded in Lemma 2.2, we will provide a refined version of the classical Jordan inequality (1.7):

Theorem 3.7 *The inequality*

$$\operatorname{sinc}(x) > \frac{60 - 7x^2}{60 + 3x^2} > \frac{2}{\pi}$$

is true for all $x \in (0, \sqrt{\frac{60\pi - 120}{7\pi + 6}})$.

Proof We only have to find x such that $\frac{60-7x^2}{60+3x^2} > \frac{2}{\pi}$. Elementary calculations lead to the solution $0 < x < \sqrt{\frac{60\pi - 120}{7\pi + 6}}$. □

We remark that $\sqrt{\frac{60\pi - 120}{7\pi + 6}} \approx 1.5642$ and $\frac{\pi}{2} \approx 1.5707$, therefore our rational refinement of Jordan's inequality gives good results near the origin.

We also remark that $\frac{60-7x^2}{60+3x^2} > \frac{\pi^2 - x^2}{\pi^2 + x^2}$ for $x \in (0, \sqrt{\frac{60-5\pi^2}{2}}) = (0, 2.3081)$, so we improved the Redheffer inequality (1.8) near the origin.

Finally, we see that $\frac{60-7x^2}{60+3x^2} > \frac{53}{53+9x^2}$ for $x \in (0, \sqrt{\frac{10}{63}}) \supset (0, \frac{1}{3})$, so our lower bound for the function sinc offers a better approximation near the origin than the fractional function $\frac{53}{53+9x^2}$.

4 Final remarks

We are convinced that the Padé approximation method is suitable for proving and refining many other inequalities. We also mention that these refinements are of rational type, so these can easily be used in computer calculations.

Competing interests
The author declares that he has no competing interests.

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