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On the growth of solutions of a class of second-order complex differential equations

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Abstract

In this paper, we consider the differential equation $f'' + h(z)e^{P(z)}f' + Q(z)f = 0$, where $h(z)$ and $Q(z) \not\equiv 0$ are meromorphic functions, $P(z)$ is a non-constant polynomial. Assume that $Q(z)$ has an infinite deficient value and finitely many Borel directions. We give some conditions on $P(z)$ which guarantee that every solution $f \not\equiv 0$ of the equation has infinite order.

MSC: 34AD20; 30D35

Keywords: complex differential equation; meromorphic function; Borel direction; deficient value; hyper-order

1 Introduction and main results

In this paper, we shall involve the deficient value and the *Borel* direction in investigating the growth of solutions of the second-order linear differential equation

$$f'' + h(z)e^{P(z)}f' + Q(z)f = 0, \qquad (1)$$

where $h(z)$ and $Q(z) \not\equiv 0$ are meromorphic functions, $P(z)$ is a non-constant polynomial. We assume that the reader is familiar with the Nevanlinna theory of meromorphic functions and the basic notions such as $N(r,f)$, $m(r,f)$, $T(r,f)$ and $\delta(r,f)$. For the details, see [1] or [2].

The order σ and the hyper-order σ_2 are defined as follows:

$$\sigma(f) = \limsup_{r\longrightarrow\infty} \frac{\log^+ T(r,f)}{\log r}, \qquad \sigma_2(f) = \limsup_{r\longrightarrow\infty} \frac{\log^+ \log^+ T(r,f)}{\log r}.$$

It is well known that if $A(z) = h(z)e^{P(z)}$ and $B(z) = Q(z)$ are transcendental entire functions in equation (1) and f_1, f_2 are two linearly independent solutions of equation (1), then at least one of f_1, f_2 must have infinite order. Hence, 'most' solutions of equation (1) will have infinite order. On the other hand, there are some equations of the form (1) that possess a solution $f \not\equiv 0$ which has finite order; for example, $f(z) = e^z$ satisfies the equation $f'' + e^{-z}f' - (e^{-z} + 1)f = 0$. Thus the main problem is what condition on $A(z)$ and $B(z)$ can guarantee that every solution $f \not\equiv 0$ of equation (1) has infinite order? There has been much work on this subject. For example, it follows from the work by Gunderson [3], Hellerstein *et al.* [4] that if $A(z)$ and $B(z)$ are entire functions with $\sigma(A) < \sigma(B)$ or $A(z)$ is a polynomial and $B(z)$ is transcendental; or if $\sigma(B) < \sigma(A) \le \frac{1}{2}$, then every solution $f \not\equiv 0$ of equation

(1) has infinite order. Furthermore, if A is an entire function with finite order having a finite deficient value and $B(z)$ is a transcendental entire function with $\mu(B) < \frac{1}{2}$, then every solution $f \not\equiv 0$ of equation (1) has infinite order [5]. More results can be found in [6–9].

However, it seems that there is little work done on equation (1) whose coefficient functions are meromorphic functions. Recently, Wu *et al.* discussed the problem correlating with this in [10]. Now we still consider equation (1) with transcendental meromorphic coefficients and discuss the growth of its meromorphic solutions. We shall also involve the deficient value and the Borel direction in the studies of the oscillation of the second-order complex differential equation. We hope that the relations between the orders of coefficient functions will not be restricted. In general, it would not hold that every solution $f \not\equiv 0$ of equation (1) has infinite order; for example, $f(z) = \frac{e^z}{z}$ satisfies

$$f'' + \frac{e^z}{z^2 - z} f' - \frac{e^z - 2z + 2}{z^2} f = 0$$

and $\sigma(f) = 1 < \infty$.

To state our theorem, we give some remarks first. Let $P(z) = (\alpha + i\beta)z^n + \cdots$ $(\alpha, \beta \in \mathbb{R})$ be a non-constant polynomial. Denote $\delta(P,\theta) = \alpha \cos n\theta - \beta \sin n\theta$, let $\deg P$ be the degree of $P(z)$, $\Omega(\theta, \varepsilon, r) = \{z : \theta - \varepsilon < \arg z < \theta + \varepsilon, |z| < r\}$. In the following, we give the definition of the *Borel* direction of a meromorphic function $f(z)$.

Definition 1.1 [11] Let $f(z)$ be a meromorphic function in the complex plane with $\sigma(f) = \sigma$ $(0 < \sigma \le \infty)$. A ray $\arg z = \theta$ $(0 \le \theta < 2\pi)$ starting from the origin is called a *Borel* direction of order σ of $f(z)$ if the following equality:

$$\limsup_{r \longrightarrow \infty} \frac{\log n(\Omega(\theta, \varepsilon, r), f = a)}{\log r} = \sigma$$

holds for any real number $\varepsilon > 0$ and every complex number $a \in \mathbb{C} \cup \{\infty\}$ with at most two exceptions.

The main results in this article are stated as follows.

Theorem 1.1 *Let $P(z)$ be a non-constant polynomial with $\deg P = n$, let $h(z)$ be a meromorphic function with $\sigma(h) < n$. Suppose that $Q(z)$ is a finite-order meromorphic function having an infinite deficient value, $Q(z)$ has only finitely many Borel directions: $B_j : \arg z = \theta_j$ $(j = 1, 2, \ldots, q)$. Denote that $\Omega_j = \{z : \theta_j < \arg z < \theta_{j+1}\}, j = 1, 2, \ldots, q$. Suppose that there exists φ_j $(\theta_j < \varphi_j < \theta_{j+1})$ such that $\delta(P, \varphi_j) < 0$ for each angular domain Ω_j. Then every meromorphic solution $f \not\equiv 0$ of equation (1) has infinite order and $\sigma_2(f) \ge \sigma(Q)$.*

Remark 1.1 We apply the theorem to some particular equations. For example, when $Q(z) = g(z)e^{bz}$, where $g(z)$ is a non-constant polynomial and $b \ne -1$. Chen proved [7] that every meromorphic solution $f \not\equiv 0$ of the equation

$$f'' + e^{-z} f' + Q(z) f = 0 \tag{2}$$

has infinite order with $\sigma_2(f) = 1$. Except the case of $\arg b = 0, \pi$, by the theorem, we can get the part of the results above. But this theorem includes more general forms.

From the structure of $E_1 = \{\varphi : \delta(P,\varphi) < 0\}$ and $E_2 = \{\varphi : \delta(P,\varphi) > 0\}$ in $[0, 2\pi)$, we can easily get the following conclusion.

Corollary 1.2 *Let $P(z)$ be a non-constant polynomial with $\deg P = n$, let $h(z)$ be a meromorphic function with $\sigma(h) < n$. Let $Q(z)$ be a transcendental meromorphic function with finite order. If $Q(z)$ has a deficient value ∞ and has only q Borel directions $B_j : \arg z = \theta_j$ $(j = 1, 2, \dots, q)$ that satisfy $\theta_1 < \theta_2 < \cdots < \theta_q < \theta_{q+1}$ $(\theta_{q+1} = \theta_1 + 2\pi)$ and*

$$\omega = \min_{1\le j\le p} \{\theta_{j+1} - \theta_j\} > \frac{\pi}{n}, \tag{3}$$

then every meromorphic solution $f \not\equiv 0$ of equation (1) *has infinite order and $\sigma_2(f) \ge \sigma(Q)$.*

By using the corollary, we see that if $\sigma(h) < n < \deg P$, then every meromorphic solution $f \not\equiv 0$ of the equation

$$f'' + h(z)e^{P(z)}f' + e^{z^n} f = 0$$

has infinite order with $\sigma_2(f) \ge n$.

2 Some lemmas

To prove our theorem, we need the following lemmas.

Lemma 2.1 [12] *Let (f, Γ) denote a pair that consists of a transcendental meromorphic function $f(z)$ and a finite set*

$$\Gamma = \{(k_1, j_1), (k_2, j_2), \dots, (k_q, j_q)\}$$

of distinct pairs of integers that satisfy $k_i > j_i \ge 0$ for $i = 1, 2, \dots, q$. Let $\alpha > 1$ and $\varepsilon > 0$ be given real constants. Then the following three statements hold.

(i) *There exists a set $E_1 \subset [0, 2\pi)$ that has linear measure zero, and there exists a constant $c > 0$ that depends only on α and Γ such that if $\varphi_0 \in [0, 2\pi) - E_1$, then there is a constant $R_0 = R_0(\varphi_0) > 1$ such that for all z satisfying $\arg z = \varphi_0$ and $|z| = r \ge R_0$, and for all $(k, j) \in \Gamma$, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \le c\left(\frac{T(\alpha r, f)}{r} \log^\alpha r \log T(\alpha r, f)\right)^{k-j}. \tag{4}$$

In particular, if $f(z)$ has finite order $\sigma(f)$, then (4) *is replaced by* (5).

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \le |z|^{(k-j)(\sigma(f)-1+\varepsilon)}. \tag{5}$$

(ii) *There exists a set $E_2 \subset (1, \infty)$ that has finite logarithmic measure, and there exists a constant $c > 0$ that depends only on α and Γ such that for all z satisfying $|z| = r \notin E_2 \cup [0, 1]$ and for all $(k, j) \in \Gamma$, the inequality* (4) *holds.*

In particular, if $f(z)$ has finite order $\sigma(f)$, then the inequality (5) *holds.*

(iii) *There exists a set* $E_3 \subset [0,\infty)$ *that has finite linear measure, and there exists a constant* $c > 0$ *that depends only on* α *and* Γ *such that for all* z *satisfying* $|z| = r \notin E_3$ *and for all* $(k,j) \in \Gamma$, *we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \le c\big(T(\alpha r,f)r^{\varepsilon}\log T(\alpha r,f)\big)^{k-j}. \tag{6}$$

In particular, if $f(z)$ *has finite order* $\sigma(f)$, *then* (6) *is replaced by* (7)

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \le |z|^{(k-j)(\sigma(f)+\varepsilon)}. \tag{7}$$

Lemma 2.2 [13] *Suppose that* $g(z) = h(z)e^{P(z)}$, *where* $P(z)$ *is a non-constant polynomial with* $\deg P = n$, *and* $h(z)$ *is a meromorphic function with* $\sigma(h) < n$. *There exists a set* $E_1 \subset [0,2\pi)$ *that has linear measure zero such that for all* $\varphi \in [0,2\pi)\backslash E_1$, *we have*

(i) *If* $\delta(P,\varphi) < 0$, *then there is a constant* $R_0 = R_0(\varphi) > 0$ *such that the inequality*

$$\big|g(re^{i\varphi})\big| < \exp\left\{\frac{1}{2}\delta(P,\varphi)r^n\right\} \tag{8}$$

holds for $r > R_0$.

(ii) *If* $\delta(P,\varphi) > 0$, *then there is a constant* $R_0' = R_0'(\varphi) > 0$ *such that the inequality*

$$\big|g(re^{i\varphi})\big| > \exp\left\{\frac{1}{2}\delta(P,\varphi)r^n\right\} \tag{9}$$

holds for $r > R_0'$.

Lemma 2.3 [2] *Let* $f(z)$ *be a transcendental meromorphic function with finite order* σ, *then there exists a function* $\lambda(r)$ *with the following properties*:

(i) $\lambda(r)$ *is a non-negative and continuous function for* $r \ge 0$ *with* $\lim_{r\longrightarrow\infty}\lambda(r) = \sigma$.

(ii) $\lambda(r)$ *is a differentiable function for all* r *in* $(0,\infty)$ *with at most countable exceptions and* $\lim_{r\longrightarrow\infty}\lambda'(r)\log r = 0$.

(iii) *The inequality* $r^{\lambda(r)} \ge T(r,f)$ *holds for all sufficiently large* r, *and there exists a sequence* r_n *with* $r_n \to \infty$ *satisfying* $r_n^{\lambda(r_n)} = T(r_n,f)$.

We shall call the function $\lambda(r)$ *the proximate order of* $f(z)$, *and the function* $U(r) = r^{\lambda(r)}$ *the type function of* $f(z)$.

Lemma 2.4 [2] *Let* $f(z)$ *be a transcendental meromorphic function with order* σ $(0 < \sigma < \infty)$. $B_1 : \arg z = \varphi_1$ *and* $B_2 : \arg z = \varphi_2$ $(0 \le \varphi_1 < \varphi_2 \le 2\pi + \varphi_1)$ *are two half rays starting from the origin, and* f *has no Borel direction in the angular domain* $\varphi_1 < \arg z < \varphi_2$. *Suppose that there exists a sequence* r_n *with* $r_n \to \infty$ $(n \to \infty)$ *and a complex number* a_0 $(a_0 \in \mathbb{C} \cup \infty)$ *such that the following inequality*:

$$\begin{cases} \log\frac{1}{|f(r_ne^{i\varphi})-a_0|} > r_n^{\sigma-\varepsilon}, & a_0 \ne \infty, \\ \log|f(r_ne^{i\varphi})| > r_n^{\sigma-\varepsilon}, & a_0 = \infty, \end{cases} \tag{10}$$

holds for any given constant $\varepsilon > 0$ *and all sufficiently large* n *in some rays* $\arg z = \varphi$, *where* $\varphi_1 < \varphi < \varphi_2$. *We denote the arc* $A_n = \{r_ne^{i\varphi} : \varphi_1 < \varphi < \varphi_2\}$ *and the angular set* $E_{n'}$ *such that* *any* $\varphi \in E_{n'}$ *satisfies the inequality* (10). *If there exists a constant* $K_1 > 0$ (*not dependent on* ε) *such that* $\operatorname{meas} E_{n'} > K_1$, *then we can get a list of curve segment* L_n *satisfying the following two conditions for any given* K_2 ($K_2 > 0$) *and sufficiently small* $\alpha > 0$:

(i) L_n *lies in the area of* $\varphi_1 + 8\alpha \le \arg z \le \varphi_2 - 8\alpha$, $r_{n-1} \le |z| \le r_n$, *whose end points respectively for* $r_n e^{i(\varphi_1+\varphi_j')}$ *and* $r_n e^{i(\varphi_2-\varphi_j')}$ ($8\alpha \le \varphi_j' \le 9\alpha$), *and we have the following inequality*:

$$\operatorname{meas}\{\varphi : r_n e^{i\varphi} \in A_n - L_n\} < K_2. \tag{11}$$

(ii) *For any positive number* $\eta > 0$, *the inequality*

$$\begin{cases} \log \frac{1}{|f(z)-a_0|} > r_n^{\sigma-\eta}, & a_0 \ne \infty, \\ \log |f(z)| > r_n^{\sigma-\eta}, & a_0 = \infty, \end{cases} \tag{12}$$

holds for sufficiently large n *and* $z \in L_n$.

Lemma 2.5 *Let* $f(z)$ *be a transcendental meromorphic function with order* σ *having an infinite deficient value. If* $f(z)$ *has q Borel directions*, $B_j : \arg z = \theta_j$ ($j = 1, 2, \ldots, q$), *and these half-rays divide the whole complex plane into q angular domains*, $\Omega_j = \{z : \theta_j < \arg z < \theta_{j+1}\}$, $j = 1, 2, \ldots, q$, $\theta_{q+1} = \theta_1 + 2\pi$, *then for any given constant* $\eta > 0$ *and* $\xi > 0$, *there exists an angular domain* Ω_{j_0} *at least and a sequence* R_n *with* $R_n \to \infty$ ($n \to \infty$) *such that the following inequality*:

$$\operatorname{meas}(E_{n,j_0}) \ge \theta_{j_0+1} - \theta_{j_0} - \xi \tag{13}$$

holds for all sufficiently large n, *where*

$$E_{n,j_0} = \left\{\varphi \in (\theta_{j_0}, \theta_{j_0+1}) : \log\left|f\left(R_n e^{i\varphi}\right)\right| > R_n^{\sigma-\eta}\right\}.$$

Proof Let $\lambda(r)$ be a proximate order of $f(z)$ with a type function $U(r) = r^{\lambda(r)}$. According to the properties of $\lambda(r)$ of Lemma 2.3, there exists a sequence r_n with $r_n \to \infty$ satisfying $\lim_{r_n \to \infty} \frac{T(r_n,f)}{U(r_n)} = 1$. Let b_ν ($\nu = 1, 2, \ldots, n\ (3r_n, f = \infty)$) be all the poles of $f(z)$ in $|z| \le 3r_n$. For every r_n, by the Boutroux-Cartan theorem [2], we have

$$\prod_{\nu=1}^{n(3r_n,f=\infty)} |z - b_\nu| > (hr_n)^{n(3r_n,f=\infty)}, \tag{14}$$

except for a set of points that can be enclosed in a finite number of disks (γ_n) with the sum of total radius not exceeding $2ehr_n$. Set $h = \frac{1}{5e}$. Then, for every integer n, we can choose $R_n \in [r_n, 2r_n]$ satisfying $\{z : |z| = R_n\} \cap (\gamma_n) = \emptyset$. By the Poisson-Jensen formula and (14), for any z satisfying $|z| = R_n$, we have

$$\begin{aligned} \log\left|f(z)\right| &\le \frac{3r_n + 2r_n}{3r_n - 2r_n} m(3r_n, f) + \sum_{\nu=1}^{n(3r_n,f=\infty)} \log\left|\frac{(3r_n)^2 - \overline{b_\nu} z}{3r_n(z - b_\nu)}\right| \\ &\le 5m(3r_n, f) + \sum_{\nu=1}^{n(3r_n,f=\infty)} \log\left|\frac{15r_n^2}{3r_n(z - b_\nu)}\right| \end{aligned}$$

$$\begin{aligned}
&\leq 5m(3r_n,f) + n(3r_n,f=\infty)\log 5r_n - \log \prod_{\nu=1}^{n(3r_n,f=\infty)} |z-b_\nu| \\
&\leq 5m(3r_n,f) + n(3r_n,f=\infty)\left(\log 5r_n + \log \frac{1}{hr_n}\right) \\
&= 5m(3r_n,f) + n(3r_n,f=\infty)\log 25e \\
&\leq 5m(3r_n,f) + \frac{\log 25e}{\log \frac{4}{3}} N(4r_n,f) \\
&\leq KT(4r_n,f),
\end{aligned}$$

where $K = 5 + \frac{\log 25e}{\log \frac{4}{3}}$.

We denote

$$E_n =: E\left\{\varphi : 0 \leq \varphi < 2\pi, \log^+\left|f\left(R_n e^{i\varphi}\right)\right| > \frac{1}{2}m(R_n,f)\right\}.$$

And then, we have

$$\begin{aligned}
m(R_n,f) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|f\left(R_n e^{i\varphi}\right)\right| d\varphi \\
&= \frac{1}{2\pi}\int_{E_n} \log^+\left|f\left(R_n e^{i\varphi}\right)\right| d\varphi + \frac{1}{2\pi}\int_{[0,2\pi)\setminus E_n} \log^+\left|f\left(R_n e^{i\varphi}\right)\right| d\varphi \\
&\leq \frac{K}{2\pi} T(4r_n,f)\operatorname{meas} E_n + \frac{1}{2}m(R_n,f).
\end{aligned}$$

Hence

$$m(R_n,f) \leq \frac{K}{\pi} T(4r_n,f)\operatorname{meas} E_n. \tag{15}$$

In addition, since $\delta = \delta(\infty,f) > 0$, there exists a constant $N_0 > 0$ such that the inequality

$$m(R_n,f) > \frac{\delta}{2} T(R_n,f) \geq \frac{\delta}{2} T(r_n,f) \geq \frac{\delta}{4} U(r_n) \tag{16}$$

holds for all $n > N_0$.

According to the properties of $U(r)$, we have

$$T(4r_n,f) < 2U(4r_n) < 4^{\sigma+1} U(r_n). \tag{17}$$

From (15)-(17), we get

$$\operatorname{meas} E_n \geq \frac{\delta\pi}{K 4^{\sigma+2}}.$$

Since the whole complex plane is divided into q angular domains and there is no *Borel* direction in them, the circle $|z| = R_n$ is also divided into q arcs: $A_{nj} : \{R_n e^{i\varphi} : \theta_j < \varphi < \theta_{j+1}\}$ $(j = 1, 2, \ldots, q)$.

Obviously, we have

$$\operatorname{meas} E_n = \sum_{j=1}^{q} \operatorname{meas} E_{nj},$$

where $E_{nj} = \{\varphi : \theta_j < \varphi < \theta_{j+1}, \log|f(R_n)e^{i\varphi}| > \frac{1}{2}m(R_n,f)\}$. Hence, by (16) and the properties of $U(r_n)$, for any given $\varepsilon > 0$, there exist $j_0 \in \{1,2,\ldots,q\}$ and a sequence R_n with $R_n \to \infty$ $(n \to \infty)$ (otherwise, we use the subsequence R_{n_0} instead of R_n) such that the following inequality:

$$\operatorname{meas}\{\varphi : \theta_{j_0} < \varphi < \theta_{j_0+1}, \log|f(R_n e^{i\varphi})| > R_n^{\sigma-\varepsilon}\} \geq \frac{\delta\pi}{qK4^{\sigma+2}}$$

holds for all sufficiently large n.

We choose $K_1 = \frac{\delta\pi}{qK4^{\sigma+2}}$, $K_2 = \xi$. By Lemma 2.4, for all sufficiently large n, there exists a curve L_{n,j_0} such that (11) and (12) hold. So, for any given $\eta > 0$, we have

$$\begin{aligned}
&\operatorname{meas}\{\varphi : \theta_{j_0} < \varphi < \theta_{j_0+1}, \log|f(R_n e^{i\varphi})| > R_n^{\sigma-\eta}\} \\
&\quad \geq \operatorname{meas}\{\varphi : R_n e^{i\varphi} \in A_{n,j_0} \cap L_{n,j_0}\} \\
&\quad = \operatorname{meas}\{\varphi : R_n e^{i\varphi} \in A_{n,j_0}\} - \operatorname{meas}\{\varphi : R_n e^{i\varphi} \in A_{n,j_0} - L_{n,j_0}\} \\
&\quad \geq \theta_{j_0+1} - \theta_{j_0} - \xi.
\end{aligned}$$

The proof of Lemma 2.5 is completed. □

3 Proof of Theorem 1.1

Proof Suppose that $f \not\equiv 0$ is a meromorphic solution of equation (1) with $\sigma(f) < \infty$. We shall seek for a contradiction. From equation (1), we have the following inequality:

$$|Q(z)| \leq \left|\frac{f''(z)}{f(z)}\right| + \left|\frac{f'(z)}{f(z)}\right| |h(z)e^{P(z)}|. \tag{18}$$

By Lemma 2.1(i), there exists a set $E_1 \subset [0,2\pi)$ of measure zero and $R_0 > 0$ such that the following inequality:

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq |z|^{2\sigma(f)}, \quad j = 1,2, \tag{19}$$

holds for all $z = |z|e^{i\varphi}$ with $\varphi \notin E_1$ and $|z| > R_0$.

Suppose that $P(z) = az^n + \cdots$, where $a = |a|e^{i\theta}$. We have $E = \{\varphi : \delta(P,\varphi) < 0\} = \bigcup_{i=1}^{n}(\frac{(4i-3)\pi-2\theta}{2n}, \frac{(4i-1)\pi-2\theta}{2n})$ by calculation. By Lemma 2.2, there exists a set $E_2 \subset [0,2\pi)$ of measure zero and $R_0' > 0$ such that the following inequality:

$$|h(z)e^{P(z)}| < \exp\left\{\frac{1}{2}\delta(P,\varphi)r^n\right\} \tag{20}$$

holds for all $z = re^{i\varphi}$ satisfying $r > R_0'$ and $\varphi \in E \setminus E_2$.

Denote that $\Omega_j = \{z : \theta_j < \arg z < \theta_{j+1}\}$, $j = 1, 2, \ldots, q$. Applying Lemma 2.5 to $Q(z)$, then for any given constants $\eta > 0$ and $\xi > 0$, there exists an angular domain Ω_{j_0} and a sequence r_m with $r_m \to \infty$ $(m \to \infty)$ such that (13) holds for all sufficiently large m.

On the other hand, since there exists φ_{j_0} in Ω_{j_0} such that $\delta(P, \varphi_{j_0}) < 0$ by the supposition of the theorem, we can get an interval $[\theta_1', \theta_2'] \subset \Omega_{j_0}$ such that (20) holds for all $z = re^{i\varphi}$ satisfying $r > R_0'$ and $\varphi \in [\theta_1', \theta_2'] \backslash E_2$. Now, let $\xi = \frac{\theta_2' - \theta_1'}{2}$. For each sufficiently large m, we can choose $\varphi_m \in [\theta_1', \theta_2'] \backslash (E_1 \cup E_2)$ such that (19), (20) and the inequality

$$\log\left|Q(z_m)\right| > r_m^{\sigma(Q)-\eta} \tag{21}$$

hold for $z_m = r_m e^{i\varphi_n}$. Let $\eta = \frac{\sigma(Q)}{2}$. Hence, from (18)-(21), we get

$$\exp r_m^{\eta} \le r_m^{2\sigma(f)} \left(1 + \exp\left\{\frac{1}{2}\delta(P, \varphi_m) r_m^n\right\}\right). \tag{22}$$

Obviously, when m is sufficiently large, this is a contradiction.

Next, we will prove $\sigma_2(f) \ge \sigma(Q)$.

By using Lemma 2.1, there exist a set $E_3 \subset [0, 2\pi)$ of measure zero and two constants $B > 0$ and $R_0'' > 0$ such that for all z satisfying $|z| = r > R_0''$ and $\arg z \notin E_3$, the following inequality holds:

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \le BT(2r, f)^4. \tag{23}$$

Hence, for each sufficiently large m, we can choose $\varphi_m' \in [\theta_1', \theta_2'] \backslash (E_2 \cup E_3)$ such that (20), (21) and (23) hold for $z_m = r_m e^{i\varphi_m'}$. From (18), (20), (21) and (23), we get

$$\exp r_m^{\sigma(Q)-\eta} \le BT(2r, f)^4 \left(1 + \exp\left\{\frac{1}{2}\delta\left(P, \varphi_m'\right) r_m^m\right\}\right). \tag{24}$$

Thus

$$\limsup_{m \to +\infty} \frac{\log^+ \log^+ T(r, f)}{\log r} \ge \sigma(Q) - \eta.$$

As η can be arbitrary small, we have $\sigma_2(f) \ge \sigma(Q)$.

The proof of the theorem is completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
CFY and XQL completed the main part of this article, CFY, XQL and HYX corrected the main theorems. All authors read and approved the final manuscript.

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