

Unified description for κ -deformations of orthogonal groups

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Abstract In this paper we provide universal formulas describing Drinfeld-type quantization of inhomogeneous orthogonal groups determined by a metric tensor of an arbitrary signature living in a spacetime of arbitrary dimension. The metric tensor does not need to be in diagonal form and κ -deformed coproducts are presented in terms of classical generators. It opens the possibility for future applications in deformed general relativity. The formulas depend on the choice of an additional vector field which parametrizes classical r -matrices. Non-equivalent deformations are then labeled by the corresponding type of stability subgroups. For the Lorentzian signature it covers three (non-equivalent) Hopf-algebraic deformations: time-like, space-like (a.k.a. tachyonic) and light-like (a.k.a. light-cone) quantizations of the Poincaré algebra. Finally the existence of the so-called Majid–Ruegg (non-classical) basis is reconsidered.

1 Introduction

Deformations of relativistic symmetries have been fruitful for the description of quantum symmetries governing physics at the Planck scale. Such quantum deformations of spacetime symmetries are described within the Hopf-algebra language and are controlled by classical r -matrices satisfying the classical Yang–Baxter (YB) equation: modified or unmodified one. One of the most interesting deformations, from the point of view of physical applications, the so-called κ -deformation, has been found in [1–4]. The deformation parameter corresponds to the Planck Mass; its inverse defining fundamental length can be considered as a quantum gravity scale. The r -matrix for the κ -deformation of Poincaré algebra is given then by $r = M_{0i} \wedge P^i$ and it satisfies the modified (inhomogeneous) Yang–Baxter equation (MYBE): $[[r, r]] = M_{\mu\nu} \wedge P^\mu \wedge P^\nu$. The κ -Poincaré Hopf algebra con-

stitutes the deformed symmetry of the κ -Minkowski algebra [4, 5] which is a quantum version of the standard Minkowski spacetime. The κ -Minkowski spacetime has been mostly studied in the so-called time-like version of κ -deformation, distinguishing the 'time' coordinate as the quantized one. The r -matrix mentioned above corresponds to this case.¹ Another option is the so-called light-like (null-plane) deformation corresponding to null-vectors, which was firstly considered in [9] (then also in [10, 11]) with quantum-deformed direction on the light cone ($x^+ = x^0 + x^3$) and with the corresponding symmetry the so-called 'null-plane quantum Poincaré Lie algebra'. It was inspired by the central problem of quantum relativistic systems in the Hamiltonian formulation, which has been studied for the null-plane evolution. In this case the information provided by the Poincaré invariance splits into a dynamical and kinematical part which is also the case after the deformation. One of the advantages of the deformation of this type is that it is triangular i.e. it can be described by the classical r -matrix satisfying classical Yang–Baxter equation (CYBE) and the twisting element satisfying two-cocycle condition do exist [12]. Moreover, the differential calculus for the null-plane κ -Minkowski is shown to be bicovariant and four-dimensional [13], which has been proved to be impossible to built for other kinds of κ -deformations (i.e. time- and space-like) [14]. It was also shown [15] that after suitable (nonlinear) change of basis the quantum algebra presented in [9] can be identified with the κ -deformation, given in [16] for the choice of $g_{00} = 0$.

Till now the most popular form of presentation of quantum κ -Poincaré algebra is the one which uses formulas for deformed coproducts found for the first time in [4] (with the primitive energy generator P_0). The corresponding system of generators, known also as Majid–Ruegg or bicrossproduct basis, satisfy classical commutation relations between

^a e-mail: andrzej.borowiec@ift.uni.wroc.pl^b e-mail: pachol@hi.is¹ The corresponding classification of quantum deformations (complete for Lorentz and almost-complete for Poincaré algebras) has been performed in Refs. [6, 7] (see also dual matrix quantum group version in [8]).

Lorentzian generators and deformed ones in the boosts-momenta sector. In contrast our formulas for quantized coproducts are written entirely in the classical Lie algebra basis.² Some formulas for coproducts can be found in different realization-dependent context in [18–25]; see also [26]. The κ -Poincaré algebra combined with the non-orthogonal form of the metric tensor was originally studied in [16] in Majid–Ruegg basis and later with extended analysis, e.g. in [13]. A passage from the Majid–Ruegg into the classical basis, which provides the so-called Drinfeld quantization map, has been a subject of investigations in various context [27–29]. Particularly, the explicit formulas expressing classical basis in terms of bicrossproduct one have been obtained therein. Similarly, the null-plane deformation has been originally obtained and investigated in the basis inherited from the so-called deformation embedding method [9]. The classical Lie algebra basis in this context has not been explored yet.

Our aim in this paper is to provide a unified description for κ -deformed coproducts of classical Poincaré generators characterizing various κ -deformations according to the Zakrzewski classification scheme [6,7]. The formulas depend on the choice of an external vector field τ which parameterizes classical r -matrices. Non-equivalent deformations are then labeled (classified) by the corresponding stability groups of τ .

The metric tensor can take the form of arbitrary symmetric and non-degenerate matrix. For example for the Lorentzian signature (in arbitrary dimension) one can distinguish three different quantizations: time-like, space-like and light-like. The corresponding orbits are characterized by the following stability subgroups in $SO(D - 1, 1)$: $SO(D - 1)$, $ISO(D - 2)$, $SO(D - 2, 1)$ respectively. This form of the unified description is particularly important from the point of view of future applications in deformations of general relativity [30,31] where the metric might be a function of the coordinates [32] and/or in the so-called relative locality where it might live on the momentum space [33].

The universal formulas are followed by the example of orthogonal $D = 1 + (D - 1)$ decomposition, which is suitable for non-null τ . This case allows for the change of system of generators into the well-known Majid–Ruegg (nonlinear) basis. Another example is the null-vector case ($\tau^2 = 0$) which provokes orthogonal $2 + (D - 2)$ decomposition. This (a.k.a. null-plane) case admits additionally Drinfeld twist, due to the fact that r -matrix satisfies CYBE. The universal formulas for coproducts coincide (up to quantum R -matrix) with the twisted ones. Also the partial analog of Majid–Ruegg basis can be found in that case. We finish this paper with some conclusions and perspectives.

² It has been demonstrated in [17] that the classical basis is related with bi-crossed product construction as well.

2 Preliminaries and notation

Let V be a vector space (spacetime) of arbitrary dimension D equipped with the metric tensor g of arbitrary signature. Let denote the (special) orthogonal group of g as $SO(g) = \{\Lambda \in GL(V) : \Lambda^T g \Lambda = g, \det \Lambda = 1\}$ and the corresponding inhomogeneous orthogonal group as $ISO(g)$ —Poincaré group.³ Adopting typical relativistic notation one chooses the basis $\{e_\mu\}_{\mu=0}^{D-1}$ and introduces the components of the metric: $g_{\mu\nu} = g(e_\mu, e_\nu)$.

There always exists an orthonormal basis $\{e_a\}$ and a vierbein matrix ξ_a^μ which diagonalizes the metric and then $g_{\mu\nu} = \xi_\mu^a \xi_\nu^b \eta_{ab}$ with the diagonal elements $\eta_{aa} = \pm 1$. Therefore we are used to write $ISO(p, q)$ in order to distinguish between positive and negative diagonal entries: $p + q = D$.

The Lie algebra $\text{iso}(g)$ as an infinitesimal form of this group admits the Lie algebra basis $\{M_{\mu\nu}, P_\mu\}$ adopted to the basis $\{e_\mu\}_{\mu=0}^{D-1}$ in V . It consists of the familiar commutation relations:

$$[M_{\mu\nu}, M_{\rho\lambda}] = i(g_{\mu\lambda}M_{\nu\rho} - g_{\nu\lambda}M_{\mu\rho} + g_{\nu\rho}M_{\mu\lambda} - g_{\mu\rho}M_{\nu\lambda}), \tag{1}$$

$$[M_{\mu\nu}, P_\rho] = i(g_{\nu\rho}P_\mu - g_{\mu\rho}P_\nu), \quad [P_\mu, P_\lambda] = 0. \tag{2}$$

The universal enveloping algebra $U_{\text{iso}(g)}$ of this Lie algebra can be equipped with a primitive Hopf-algebra structure, which can later be quantized within the Drinfeld formalism. Because of this, one requires the extension to the formal power series $U_{\text{iso}(g)}[[\frac{1}{\kappa}]]$ (see e.g. [34–36] for more details). This associative and unital algebra has a quadratic Casimir element (aka Casimir of mass) defined as $C_g = P_\mu P^\mu = g^{\mu\nu} P_\mu P_\nu = C \in U_{\text{iso}(g)}$ (a central element of $U_{\text{iso}(g)}$) which plays very important role in physics, it represents the mass. It takes a constant numerical value in any irreducible representation.

However, from the point of view of physical applications e.g. at the Planck scale we are interested in quantum deformations. As is well known, quantum groups are quantizations of Poisson–Lie groups determined by Lie bialgebra structures on the corresponding Lie algebras. They are described by classical r -matrices satisfying Yang–Baxter equation. In the case of orthogonal groups an interesting class of r -matrices has been found in [6,7] (see also [37,38]). For any (non-zero) vector $\tau = \tau^\mu e_\mu \in V$ one defines the corresponding r -matrix

$$r_\tau = \tau^\alpha M_{\alpha\mu} \wedge P^\mu \equiv \frac{1}{2} \tau_\perp \Omega, \tag{3}$$

where $\Omega = M_{\mu\nu} \wedge P^\mu \wedge P^\nu$ is known to be the only invariant element in $\wedge^3 \text{iso}(g)$ and τ_\perp is used for contraction with the

³ In fact, we should restrict ourselves to the connected component of unity instead of the full $SO(g)$. For example the special Lorentz group $SO(1, 3) = SO^\uparrow(1, 3) \cup SO^\downarrow(1, 3)$ has two connected components.

vector τ . It appears that the Schouten bracket gives

$$[[r_\tau, r_\tau]] = -g(\tau, \tau) \Omega. \tag{4}$$

This implies two possibilities:

- I. $\tau^2 \equiv g(\tau, \tau) \equiv \tau^\mu \tau_\mu \neq 0$ for which the corresponding r -matrix satisfies MYBE (Modified Yang–Baxter Equation). It will provide the so-called standard (a.k.a. Drinfeld–Jimbo) quantization with the quasi-triangular quantum R -matrix.
- II. $\tau^2 = 0$ (provided non-Euclidean signature) with r_τ satisfying CYBE (Classical Yang–Baxter equation). In this case one deals with the non-standard (a.k.a. twisted) triangular deformation.

We consider a stability group G_τ of the vector τ , as a subgroup which leaves the vector τ invariant under the natural action of $SO(g)$ in V . Isomorphism classes of stability groups classify the type of orbits. According to the general formalism developed in [6,7] they can also be used to single out the non-equivalent deformations labeled by τ . Regarding the possible orbit types for the non-trivial vector $\tau \neq 0$ in (V, g) , assuming generic (p, q) signature, one can encounter two main situations:

- (A) $\tau^2 \neq 0$ and there is a basis $\{e_0 = \tau, e_i\}_{i=1}^{D-1}$ in V such that $g_{0i} = 0$. This basis provides the so-called $1 + (D - 1)$ orthogonal decomposition. The stability subgroup is a homogeneous orthogonal group $SO(g_{ij})$ in $D - 1$ dimensions. The signature of the metric g_{ij} indicates the orbit type.
- (B) $\tau^2 = 0$ (provided we have a non-Euclidean signature). There is a basis $\{e_0 = \tau, e_{D-1} = \tilde{\tau}, e_a\}_{a=1}^{D-2}$ in V such that $g_{00} = g_{D-1 D-1} = g_{0a} = g_{D-1 a} = 0$ and $g_{D-1 D-1} = g_{D-1 0} = 1$. This basis is called a light-cone basis and it provides the so-called $2 + (D - 2)$ orthogonal splitting. The two-dimensional Lorentzian space with anti-diagonal metric is spanned by two light-like vectors $\tau, \tilde{\tau}$. Again the signature of the metric g_{ab} indicates the orbit type. The stability subgroup is an inhomogeneous orthogonal group $ISO(g_{ab})$ in $D - 2$ dimensions in this case.

In other words if $1 \leq p, q \leq D - 1$ one distinguishes three cases: either $SO(p - 1, q)$ or $SO(p, q - 1)$ or $ISO(p - 1, q - 1)$. Particularly, for the Lorentzian signature one recovers the well-known cases:

- (i) τ is a time-like vector, with Euclidean $G_\tau \cong SO(D - 1)$ as a stability group. It corresponds to the original κ -deformation with $r_\tau = M_{0i} \wedge P^i$;

- (ii) $\tau^2 = 0$, i.e. τ is a null-vector (light-cone deformation). In this case the stability group $G_\tau \cong E(D - 2) \equiv ISO(D - 2)$ is an inhomogeneous Euclidean group;
- (iii) τ is a space-like vector (tachyonic deformation) with $G_\tau \cong SO(D - 2, 1)$ being the $D - 1$ -dimensional Lorentz group.

3 Unified description for κ -deformations

For a given pair $(g_{\mu\nu}, \tau^\lambda)$ the deformed coproducts implemented by the vector τ (in fact, by the corresponding classical r -matrix r_τ), when written in classical generators satisfying the commutation relations (1)–(2)⁴, take the following form:

$$\Delta_\tau (P_\mu) = P_\mu \otimes \Pi_\tau + 1 \otimes P_\mu - \frac{\tau_\mu}{\kappa} P^\alpha \Pi_\tau^{-1} \otimes P_\alpha - \frac{\tau_\mu}{2\kappa^2} C_\tau \Pi_\tau^{-1} \otimes P_\tau, \tag{5}$$

$$\Delta_\tau (M_{\mu\nu}) = M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} + \frac{1}{\kappa} P^\alpha \Pi_\tau^{-1} \otimes (\tau_\nu M_{\alpha\mu} - \tau_\mu M_{\alpha\nu}) - \frac{1}{2\kappa^2} C_\tau \Pi_\tau^{-1} \otimes (\tau_\mu M_{\tau\nu} - \tau_\nu M_{\tau\mu}), \tag{6}$$

where τ_μ denote covariant components of τ with respect to the metric $g_{\mu\nu}$ and $P_\tau = \tau^\mu P_\mu, M_{\tau\lambda} = \tau^\alpha M_{\alpha\lambda}$. In order to preserve a compact form for the formulas (5)–(6) we have also introduced the following notation (extending our previous notation from [39]):

$$\begin{aligned} \Pi_\tau &= \frac{1}{\kappa} P_\tau + \sqrt{1 + \frac{1}{\kappa^2} \tau^2 C}, \\ \Pi_\tau^{-1} &= \frac{\sqrt{1 + \frac{1}{\kappa^2} \tau^2 C} - \frac{1}{\kappa} P_\tau}{1 + \frac{1}{\kappa^2} (\tau^2 C - P_\tau^2)}, \end{aligned} \tag{7}$$

$$\tau^2 C_\tau = \kappa^2 \left(\Pi_\tau + \Pi_\tau^{-1} - 2 + \frac{1}{\kappa^2} (\tau^2 C - P_\tau^2) \Pi_\tau^{-1} \right). \tag{8}$$

The left (and right) hand side of the last equation vanishes when $\tau^2 = 0$. Further calculations give rise to

$$\begin{aligned} P_\tau &= \frac{\kappa}{2} \left(\Pi_\tau - \Pi_\tau^{-1} \left(1 + \frac{1}{\kappa^2} (\tau^2 C - P_\tau^2) \right) \right), \\ \tau^2 C_\tau &= 2\kappa^2 \left(\sqrt{1 + \frac{1}{\kappa^2} \tau^2 C} - 1 \right). \end{aligned} \tag{9}$$

We would like to point out that all formulas considered so far are valid also in the null-vector case, i.e. as $\tau^2 = 0$. (For $\tau \equiv 0$ one recovers the primitive undeformed coproducts). We shall specify later the expression for C_τ for this particular case. At the moment one can observe that the inverse to (9), the formula

⁴ For a realization-dependent version see e.g. [22–26].

$$C = C_\tau \left(1 + \frac{\tau^2}{4\kappa^2} C_\tau \right), \tag{10}$$

strongly suggests $C_\tau = C$ for $\tau^2 = 0$.

Following the Drinfeld formalism of quantum groups all equalities presented here are understood in the sense of formal power series in one (undetermined) variable $\frac{1}{\kappa}$, for example⁵

$$\begin{aligned} \sqrt{1 + \frac{1}{\kappa^2} \tau^2 C} &= \sum_{n \geq 0} \frac{(\tau^2)^n}{\kappa^{2n}} \binom{1/2}{n} C^n = 1 \\ &+ \frac{\tau^2 C}{2\kappa^2} + \sum_{n \geq 2} \frac{(-1)^{n-1} (\tau^2)^n (2n-3)!!}{2^n n! \kappa^{2n}} C^n, \end{aligned} \tag{11}$$

where $\binom{1/2}{n} = \frac{1/2(1/2-1)\dots(1/2-n+1)}{n!}$ are binomial coefficients. C_τ is a central element in $U_{\text{ISO}(p,q)}[[\frac{1}{\kappa}]]$ which in the classical limit $\kappa \mapsto \infty$ gives C and, therefore, plays the role of deformed Casimir operator. From the above ones one calculates

$$\Delta_\tau(\Pi_\tau) = \Pi_\tau \otimes \Pi_\tau, \quad \Delta_\tau(\Pi_\tau^{-1}) = \Pi_\tau^{-1} \otimes \Pi_\tau^{-1}, \tag{12}$$

as well as ($\tau^2 \neq 0$)

$$\begin{aligned} \Delta_\tau \left(\sqrt{1 + \frac{1}{\kappa^2} \tau^2 C} \right) &= \sqrt{1 + \frac{1}{\kappa^2} \tau^2 C} \otimes \Pi_\tau \\ &- \frac{1}{\kappa} \Pi_\tau^{-1} \otimes P_\tau + \frac{\tau^2}{\kappa^2} P^\alpha \Pi_\tau^{-1} \otimes P_\alpha - \frac{\tau^2}{\kappa} P_\tau \Pi_\tau^{-1} \otimes P_\tau. \end{aligned} \tag{13}$$

Finally, in order to complete the Hopf-algebra structure we set the classical counit ($\epsilon(1) = 1, \epsilon(P_\lambda) = \epsilon(M_{\mu\nu}) = 0$) and deformed antipodes:

$$\begin{aligned} S_\tau(P_\mu) &= - \left(P_\mu + \frac{\tau_\mu}{\kappa} \left(C + \frac{1}{2\kappa} P_\tau C_\tau \right) \right) \Pi_\tau^{-1}, \\ S_\kappa(\Pi_\tau) &= \Pi_\tau^{-1}, \end{aligned} \tag{14}$$

$$\begin{aligned} S_\tau(M_{\mu\nu}) &= -M_{\mu\nu} + \frac{1}{\kappa} P^\alpha (\tau_\nu M_{\alpha\mu} - \tau_\mu M_{\alpha\nu}) \\ &+ \frac{1}{2\kappa^2} C_\tau (\tau_\nu M_{\tau\mu} - \tau_\mu M_{\tau\nu}). \end{aligned} \tag{15}$$

Moreover, the square of the antipode (14), (15) is given by a similarity transformation $S^2(X) = \Pi_\tau^{D-1} X \Pi_\tau^{1-D}$ (cf. [23,24]). Such deformed Hopf-algebraic structure will be denoted $U_{\text{ISO}(g)}^\tau[[\frac{1}{\kappa}]]$.

Remark 1 It is worth to underline that these universal formulas describe κ -Poincaré Hopf algebra not only in different

Lie algebra basis induced by different basis in the underlying vector space V but also provide the different types of deformations. This can be seen from the well-known formula

$$\lim_{\kappa \rightarrow \infty} \kappa (\Delta_\tau - \Delta_\tau^{\text{op}})(X) = [\Delta_0(X), r_\tau], \tag{16}$$

relating deformed coproducts with the corresponding classical r -matrices. Here $\Delta_0(X) = X \otimes 1 + 1 \otimes X$ denotes primitive (undeformed) coproduct for $X \in \text{iso}(g)$ and Δ^{op} stands for the opposite coproduct with flipped legs. The right hand side of the last equation defines cobracket determining Lie bialgebra structures on $\text{iso}(g)$. Therefore our coproducts can be considered as their quantization. The following further comments are now in order.

Remark 2 One should notice that the expression $\tau^2 C$ is independent of the sign convention for g : the change $g \rightarrow -g$ gives rise to $\tau^2 \rightarrow -\tau^2$ and $C \rightarrow -C$.

Remark 3 Re-scaling at the same time $\tau \rightarrow s\tau$ and $\kappa \rightarrow s\kappa$ for any real parameter s leaves coproducts (5)–(6) invariant. Notice that neither τ nor κ are present in the commutation relations (1)–(2). For this reason (except the case $\tau^2 = 0$) one can assume that the vector τ is normalized, i.e. $\tau^2 = \pm 1$ provided $\tau^2 \neq 0$.

Remark 4 Consider the well-known κ -Minkowski (quantum) algebra \mathcal{M}_τ as a unital associative algebra generated by the noncommutative spacetime coordinate generators \hat{x}^μ modulo the following relations [16]:

$$[\hat{x}^\mu, \hat{x}^\nu] = \frac{i}{\kappa} (\tau^\mu \hat{x}^\nu - \tau^\nu \hat{x}^\mu), \tag{17}$$

where τ^μ is a fixed four-vector from V ; $\mu, \nu = 0, 1, \dots, D-1$. This algebra becomes a Hopf module algebra (see e.g. [40] for necessary definitions) with respect to the κ -deformed Hopf-algebra structure (5)–(6). It means that the relation (17) is preserved under the module action \triangleright , provided classical action of the classical Poincaré generators (1)–(2) on the κ -Minkowski coordinates (17):

$$P_\mu \triangleright \hat{x}^\nu = -i \delta_\mu^\nu, \quad M_{\mu\nu} \triangleright \hat{x}^\rho = -i (\hat{x}_\mu \delta_\nu^\rho - \hat{x}_\nu \delta_\mu^\rho). \tag{18}$$

To this aim one requires the compatibility condition (a.k.a. generalized Leibniz rule):

$$L \triangleright (\hat{x}^\mu \cdot \hat{x}^\nu) = (L_{(1)} \triangleright \hat{x}^\mu) \cdot (L_{(2)} \triangleright \hat{x}^\nu), \tag{19}$$

where, for simplicity, we have used a Sweedler-type notation for the coproduct: $\Delta_\tau(L) = L_{(1)} \otimes L_{(2)}$ for $L \in \{M_{\mu\nu}, P_\rho\}$. Using a smash product construction one can unify spacetime and symmetry generators (see e.g. [40]) into one algebra with quantum Hopf-algebroid structure [41].

⁵ For the standard (i.e. Drinfeld–Jimbo-type) deformation one can always switch to the so-called q -analog version with all infinite series hidden in the one additional generator. In the case of κ -Poincaré it is Π_τ^{-1} which solves a specialization problem for κ , for details see [39].

Remark 5 It is well known that the real algebras $U_{\text{iso}(p,q)}$ of different signatures (p, q) can be viewed as different real forms determined on the same complex algebra $U_{\text{iso}(D,C)}$. These real forms are represented by the corresponding $*$ -conjugation. The standard and convenient way to establish appropriate conjugations is by the choice of a Lie algebra basis composed of self-conjugate (Hermitean) elements. One can observe that the basis (1)–(2) is compatible with Hermitean conjugation and it can be used for determining the corresponding real forms. Thus the universal coproducts (6) are compatible with the signature implemented $*$ -conjugations in the following sense:

$$\Delta_\tau(a)^* = \Delta_\tau(a^*); \quad (a \otimes b)^* = a^* \otimes b^*,$$

$$a, b \in U_{\text{iso}(D,C)}.$$

Similarly, the relation (17) can be considered as providing the real structure on the complex module algebra \mathcal{M}_τ provided that the vector τ remains real.

Examples Take the diagonal metric $\eta_{\mu\nu} = \eta^{\mu\nu} = (-, +, +, +)$ of the Lorentzian signature in $D = 4$ dimensions. Three different choices: ${}_1\tau^\mu = (1, 0, 0, 0)$, ${}_1\tau^2 = -1$; ${}_2\tau^\mu = (0, 0, 0, 1)$, ${}_2\tau^2 = 1$ and ${}_3\tau^\mu = (1, 0, 0, 1)$, ${}_3\tau^2 = 0$ provide three different (non-equivalent) Hopf-algebraic structures on $U_{\text{iso}(1,3)}[[\frac{1}{\kappa}]]$: the original κ , tachyonic and light-cone deformations, respectively (see also [11, 15, 16, 42] for earlier works in this context). We shall denote them $U_{\text{iso}(1,3)}^{SO(3)}[[\frac{1}{\kappa}]]$, $U_{\text{iso}(1,3)}^{E(2)}[[\frac{1}{\kappa}]]$ and $U_{\text{iso}(1,3)}^{SO(1,2)}[[\frac{1}{\kappa}]]$ correspondingly. These examples will be treated in more detail in the next subsections.

Yet another example can be considered by taking the diagonal metric $\eta_{\mu\nu} = \eta^{\mu\nu} = (+, -, +, -)$ of neutral (Kleinian) signature in $D = 4$ dimensions. The choice $\tau^\mu = (1, 1, 1, 1)$ ($\tau^2 = 0$) provides a new type of deformation of $U_{\text{iso}(2,2)}$. Equivalently one can take more convenient basis with the metric⁶

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

being a direct product of the two light-cone metrics, and we have $\tau^\mu = (1, 0, 1, 0)$. In this basis one sees Hopf-algebra isomorphism $U_{\text{iso}(2,2)}^\tau[[\frac{1}{\kappa}]] \cong U_{\text{iso}(1,1)}^{1\tau}[[\frac{1}{\kappa}]] \otimes U_{\text{iso}(1,1)}^{1\tau}[[\frac{1}{\kappa}]]$ with $1\tau = (1, 0)$.

⁶ This indicates Lie algebra isomorphism $\text{iso}(2, 2) \cong \text{iso}(1, 1) \oplus \text{iso}(1, 1)$.

4 The orthogonal $D = 1 + (D - 1)$ decomposition versus the Majid–Ruegg basis

Contracting (5)–(6) with τ^μ yields

$$\Delta(P_\tau) = P_\tau \otimes \Pi_\tau + \Pi_\tau^{-1} \otimes P_\tau - \frac{\tau^2}{\kappa} P^\alpha \Pi_\tau^{-1} \otimes P_\alpha - \frac{\tau^2}{2\kappa^2} C_\tau \Pi_\tau^{-1} \otimes P_\tau, \quad (20)$$

$$\Delta_\tau(M_{\tau\nu}) = M_{\tau\nu} \otimes 1 + 1 \otimes M_{\tau\nu} + \frac{1}{\kappa} P^\alpha \Pi_\tau^{-1} \otimes (\tau_\nu M_{\alpha\tau} - \tau^2 M_{\alpha\nu}) - \frac{\tau^2}{2\kappa^2} C_\tau \Pi_\tau^{-1} \otimes M_{\tau\nu}. \quad (21)$$

Let us study the case of $\tau^2 \neq 0$ in more detail (the opposite case will be the subject of our study in the next section). In fact, without the loss of generality, one can assume $\tau^\mu = (1, 0, \dots, 0)$. More exactly, by the choice of the suitable basis $(e_\mu)_{\mu=0}^{D-1}$ in the vector space V with $e_0 = \tau$ and $(e_i)_{i=1}^{D-1}$ being orthogonal to $\tau : g_{00} = \tau^2; g_{0i} = g(e_0, e_i) = 0$. This provides the orthogonal decomposition $(V, g_{\mu\nu}) \cong (\mathbb{R}, g_{00}) \times (V^{D-1}, g_{ij})$. Notice that the $(D - 1)$ dimensional metric g_{ij} does not need to be in the diagonal form.

In the corresponding Lie algebra basis $\{P_\tau, P_i, M_{\tau i}, M_{ij}\}$ the universal coproducts read now as:

$$\Delta_\tau(P_\tau) = P_\tau \otimes \Pi_\tau + \Pi_\tau^{-1} \otimes P_\tau - \frac{\tau^2}{\kappa} P^j \Pi_\tau^{-1} \otimes P_j, \quad (22)$$

$$\Delta_\tau(P_i) = P_i \otimes \Pi_\tau + 1 \otimes P_i, \quad i, j = 1, \dots, D - 1, \quad (23)$$

$$\Delta_\tau(M_{ij}) = M_{ij} \otimes 1 + 1 \otimes M_{ij}, \quad (24)$$

$$\Delta_\tau(M_{\tau i}) = M_{\tau i} \otimes 1 + \Pi_\tau^{-1} \otimes M_{\tau i} + \frac{\tau^2}{\kappa} P^j \Pi_\tau^{-1} \otimes M_{ij}, \quad (25)$$

where τ^2 after normalization can be reduced to ± 1 , here we used the following identity:

$$\left(1 - \frac{\tau^2}{2\kappa^2} C_\tau \Pi_\tau^{-1} - \frac{1}{\kappa} P_\tau \Pi_\tau^{-1}\right) = \Pi_\tau^{-1}.$$

The above reminds one of some formulas from [39].

This enables us to introduce the new system of generators $\{P_\tau, P_i, M_{\tau i}, M_{ij}\} \rightarrow \{\tilde{P}_\tau, \tilde{P}_i, M_{\tau i}, M_{ij}\}$ with

$$\tilde{P}_\tau \doteq \kappa \ln \Pi_\tau, \quad \tilde{P}_i \doteq P_i \Pi_\tau^{-1} \Rightarrow \Pi_\tau = e^{\frac{\tilde{P}_\tau}{\kappa}}, \quad (26)$$

which provides the deformed coproducts in the familiar Majid–Ruegg form

$$\begin{aligned} \Delta_\kappa(\tilde{P}_\tau) &= 1 \otimes \tilde{P}_\tau + \tilde{P}_\tau \otimes 1, \\ \Delta_\kappa(M_{ij}) &= 1 \otimes M_{ij} + M_{ij} \otimes 1, \end{aligned} \tag{27}$$

$$\Delta_\kappa(\tilde{P}_i) = \exp\left(-\frac{\tilde{P}_\tau}{\kappa}\right) \otimes \tilde{P}_i + \tilde{P}_i \otimes 1, \tag{28}$$

$$\begin{aligned} \Delta_\kappa(M_{\tau j}) &= M_{\tau j} \otimes 1 + \exp\left(-\frac{\tilde{P}_\tau}{\kappa}\right) \otimes M_{\tau j} \\ &\quad - \frac{1}{\kappa} \tau^2 \tilde{P}^k \otimes M_{kj}. \end{aligned} \tag{29}$$

The algebraic relations in the Majid–Ruegg basis are

$$\begin{aligned} [M_{\tau i}, \tilde{P}_\tau] &= -i\tau^2 \tilde{P}_i, \quad [M_{ij}, \tilde{P}_k] = i(g_{jk} \tilde{P}_i - g_{ik} \tilde{P}_j), \\ [M_{ij}, \tilde{P}_\tau] &= 0, \end{aligned} \tag{30}$$

$$\begin{aligned} [M_{\tau i}, \tilde{P}_j] &= \frac{i}{2} \kappa g_{ij} \left(1 - \exp\left(-\frac{2\tilde{P}_\tau}{\kappa}\right) - \frac{\tau^2}{\kappa^2} \tilde{P}_i \tilde{P}_j\right) \\ &\quad + \frac{i\tau^2}{\kappa} \tilde{P}_j \tilde{P}_i. \end{aligned} \tag{31}$$

Notice that the Lie algebra of the stability group G_τ consist of the elements $\{M_{ij}\}$ for which the coproduct remains primitive. The expressions (27)–(31) cover all the standard κ -deformations; for the Lorentzian signature they describe both the time-like and the space-like quantizations.

5 The null-plane light-cone deformation and the 2 + (D – 2) decomposition

In the case of light-like deformation, i.e. when $\tau^2 = 0$, one deals with the non-Euclidean geometry $ISO(p, q)$; $p, q \neq 0$. Therefore we shall introduce the most convenient “light-cone” Poincaré generators:

$$\begin{aligned} P_\mu &= (P_+, P_-, P_a), \quad M_{\mu\nu} = (M_{+-}, M_{+a}, M_{-a}, M_{ab}) \\ a, b &= 1, 2 \dots D - 2. \end{aligned} \tag{32}$$

as a basis in the Lie algebra $iso(g_{p,q})$. To this aim we have to decompose the space $V^D = V^2 \times V^{D-2}$, by a suitable choice of basic vectors, into the orthogonal product of the two-dimensional Lorentzian space $\{V^2, g_{AB}\}$ with a $D - 2$ -dimensional one $\{V^{D-2}, g_{ab}\}$: $(A, B = +, -)$, $(a, b = 1, 2 \dots D - 2)$. Moreover, the total metric $g_{\mu\nu} = g_{AB} \times g_{ab}$ becomes a product metric. We choose $g_{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in its anti-diagonal (light-cone) form as well as two null-vectors $\tau^\mu \equiv \tau_+^\mu = (1, 0, \dots, 0)$, $\tilde{\tau}^\mu \equiv \tau_-^\mu = (0, 1, 0 \dots, 0)$: $\tau_+ \tau_- =$

1 in order to obtain the convenient light-cone basis in the space of the Lie algebra generators (32). This algebra consists of the following (non-vanishing) commutators:

$$[M_{+a}, M_{-b}] = -i(M_{ab} + g_{ab} M_{+-}), \quad [M_{\pm a}, M_{\pm b}] = 0, \tag{33}$$

$$[M_{\pm a}, M_{bc}] = i(g_{ab} M_{\pm c} - g_{ac} M_{\pm b}),$$

$$[M_{+-}, M_{\pm a}] = \pm i M_{\pm a}, \tag{34}$$

$$[M_{+-}, P_\pm] = \pm i P_\pm, \quad [M_{\pm a}, P_b] = i g_{ab} P_\pm, \tag{35}$$

$$[M_{\pm a}, P_\pm] = [M_{+-}, P_a] = 0, \quad [M_{\pm a}, P_\mp] = -i P_a, \tag{36}$$

together with the standard commutation relations within the $D - 2$ dimensional sector (M_{ab}, P_a, g_{ab}) , cf. (1)–(2). The universal formula for the coalgebra structure, in this case, reduces to

$$\begin{aligned} \Delta_\tau(M) &= M \otimes 1 + 1 \otimes M \quad \text{for } M \in \{M_{+a}, M_{ab}\} \\ \Delta_\tau(P) &= P \otimes \Pi_+ + 1 \otimes P \quad \text{for } P \in \{P_+, P_a\}, \end{aligned} \tag{37}$$

$$\begin{aligned} \Delta_\tau(P_-) &= P_- \otimes \Pi_+ + \Pi_+^{-1} \otimes P_- \\ &\quad - \frac{1}{\kappa} \left(P_- + \frac{1}{2\kappa} C_+\right) \Pi_+^{-1} \otimes P_+ - \frac{1}{\kappa} P^a \Pi_+^{-1} \otimes P_a, \end{aligned} \tag{38}$$

$$\begin{aligned} \Delta_\tau(M_{+-}) &= M_{+-} \otimes 1 + \Pi_+^{-1} \otimes M_{+-} \\ &\quad - \frac{1}{\kappa} P^a \Pi_+^{-1} \otimes M_{+a}, \end{aligned} \tag{39}$$

$$\begin{aligned} \Delta_\tau(M_{-a}) &= M_{-a} \otimes 1 + \Pi_+^{-1} \otimes M_{-a} \\ &\quad - \frac{1}{\kappa} \left(P_- + \frac{1}{2\kappa} C_+\right) \Pi_+^{-1} \otimes M_{+a} - \frac{1}{\kappa} P^b \Pi_+^{-1} \otimes M_{ba}, \end{aligned} \tag{40}$$

where $\Pi_+ \doteq 1 + \frac{1}{\kappa} P_+$ and $(1 - \frac{1}{\kappa} P_+ \Pi_+^{-1}) = (\Pi_+ - \frac{1}{\kappa} P_+) \Pi_+^{-1} = \Pi_+^{-1}$ and C_+ is still to be determined. The Lie subalgebra corresponding to the stability group of τ_+ consists of $iso(p - 1, q - 1) = \text{gen}\{M_{ab}, M_{+b}\}$, i.e. the generators with the primitive coproducts.

On the other hand, the classical r -matrix corresponding to the vector τ_+ reads

$$r_{LC} = M_{+-} \wedge P_+ + M_{+a} \wedge P^a.$$

Since $\tau_+^2 = 0$ it satisfies the CYB equation and generates the non-standard (triangular) deformation. Its construction involves two Abelian $D - 1$ -dimensional subalgebras $\Gamma_+ = \text{gen}\{M_{+-}, P^a\}$ and $\Gamma_- = \text{gen}\{P_+, M_{+a}\}$ satisfying certain cross-commutation relations (cf. formulas (35)–(36) and Ref. [12]). The corresponding twisting element has the

following form:⁷

$$\begin{aligned} \mathcal{F} &= \exp(-iM_{+-} \otimes \ln \Pi_+) \exp\left(-\frac{i}{\kappa}M_{+a} \otimes P^a \Pi_+^{-1}\right) \\ &= \exp\left(-\frac{i}{\kappa}M_{+a} \otimes P^a\right) \exp(-iM_{+-} \otimes \ln \Pi_+). \end{aligned} \tag{41}$$

We are now in a position to calculate coproducts directly from the twist by making use of the similarity transformation

$$\Delta_{LC}(X) = \mathcal{F}\Delta_0(X)\mathcal{F}^{-1}, \tag{42}$$

where $\Delta_0(X) = X \otimes 1 + 1 \otimes X$ denotes as before the primitive (undeformed) coproducts. After performing the involved calculations it turns out that

$$\mathcal{R}\Delta_{LC}(X)\mathcal{R}^{-1} = \Delta_{LC}^{\text{op}}(X) = \Delta_{\tau}(X), \tag{43}$$

where $\mathcal{R} = \mathcal{F}_{21}\mathcal{F}^{-1}$ is a triangular quantum R -matrix, provided $C_+ = C$ as suggested by the formula (10). (Note that in the light-cone basis one has $C = 2P_+P_- + P^a P_a$ and $P_- + \frac{1}{2\kappa}C = P_- \Pi_+ + \frac{1}{2\kappa}P^a P_a$). In other words, the formulas (37)–(39) and (42) describe in a different way the same Hopf-algebraic structure.

Another observation is that for coproducts $\Delta_{LC}(X)$ one can introduce a partial analog of the Majid–Ruegg basis (observed before in [13]). Indeed, setting $\tilde{P}_+ = \ln \Pi_+$, $\tilde{P}_a = P_a \Pi_+^{-1}$ one has a primitive coproduct for \tilde{P}_+ and $M \in \{M_{+a}, M_{ab}\}$ as in (27) and for \tilde{P}_a and M_{+-} as in (28) and (29), respectively. As far as algebra is concerned we get

$$\begin{aligned} [M_{+-}, \tilde{P}_+] &= i \left(1 - \exp\left(-\frac{\tilde{P}_+}{\kappa}\right)\right), \quad [M_{+a}, \tilde{P}_b] \\ &= i g_{ab\kappa} \left(1 - \exp\left(-\frac{\tilde{P}_+}{\kappa}\right)\right), \end{aligned} \tag{44}$$

$$\begin{aligned} [M_{+a}, \tilde{P}_+] &= 0, \quad [M_{+-}, \tilde{P}_a] = i \tilde{P}_a \left(1 - \exp\left(-\frac{\tilde{P}_+}{\kappa}\right)\right), \\ [M_{-a}, \tilde{P}_+] &= -\frac{i}{\kappa} \tilde{P}_a, \end{aligned} \tag{45}$$

with the rest of commutators staying classical as in (33)–(34). The only generator which does not fit into this Majid–Ruegg scheme is P_- .

6 Conclusions and perspectives

In the study of Poincaré algebra, from the point of view of physical applications, one focuses on the representation

⁷ It is called an extended Jordanian twist since it enlarges the basic Jordanian twist $\exp(-iM_{+-} \otimes \ln \Pi_+)$ (see [43] for details).

theory and the value of Casimir operator $C = P^2$. We believe that physical objects are represented by time- or light-like four-momentum. In this paper we have considered a coordinate analog of such four-vector in the κ -deformed case and we have shown that it is possible to consider analogous (time-, space- and light-like) cases which in fact parameterize the deformation. The universal formulas for the deformed Poincaré algebra depend on the choice of the additional vector field τ and allow one to consider three cases of deformations all being the symmetry of the corresponding noncommutative $\kappa(\tau)$ -Minkowski spacetimes (17). These non-equivalent deformations are classified by the stability groups of τ . In other words we have presented a class of κ -deformations of orthogonal groups $SO(g)$ in the way that they explicitly depend on the choice of normalized four-vector τ .

From the point of view of physical applications we can focus here on four dimensions with the metric of Lorentzian signature. Deformed Hopf algebra describes a symmetry of quantized spacetime. In the most studied case the vector τ is time-like which corresponds to κ -Minkowski spacetime algebra with noncommutativity between time and space coordinates. Therefore such (time-like) vector can be identified with a preferred direction which can be interpreted as a four-velocity of the preferred observer. Corresponding 3 + 1 decomposition provides a preferred frame. We have shown that in such frame the utilization of Majid–Ruegg coproducts is fully justified. It is also known that theories with preferred spacetime direction violate Lorentz invariance. In fact, the Lorentzian symmetry should be reduced to the stability subgroup for which the coproducts remain undeformed.

Also general relativity models with a preferred direction (Einstein-æther) are currently under debate (see e.g. [44] and references therein). Astrophysical data indicates as well that the universe has a preferred (primordial) direction imprinted on the microwave background [45,46]. It has been already demonstrated that noncommutative effects turn out to be helpful for the explanation of this fact [47–49].

Alternatively, one can want to restore full Lorentz covariance allowing to transform the vector τ . Indeed, the Lorentz transformation: $\tau^\mu \rightarrow \tilde{\tau}^\mu = \Lambda^\mu_\nu \tau^\nu$ does not change the orbit type. Therefore it preserves the deformation type. Such scenario is similar to that one which encounters in Special Relativity which admits a class of preferred observers (frames)—the inertial ones. Conversely, any two inertial observers are connected by the Lorentz transformation. The same reasoning allows one to restore full diffeomorphism invariance on a curved background.

Another nice feature of our approach comes from the fact that the metric tensor g determining the orthogonal group does not need to be in its canonical (diagonal) form. It implies that the deformation can be executed in an arbitrary coordi-

nate system. In particular, on a curved manifold when the components of the metric g representing gravitational field as well as the components of τ are point-wise dependent. In such a case the usage of Majid–Ruegg bases is not, in general, allowed unless some stronger assumptions (e.g. global hyperbolicity, foliation, etc.) are taken into account. The unified description seems to be particularly useful from the point of view of applications in deformations of general relativity. A deformation in the geometric setting has been under investigation for quite some time as an alternative to the quantization of gravity [30,31]. For example one can follow the most recent proposition to consider gravitational and cosmological models induced directly from noncommutative, i.e. quantum spacetime [32]. In this approach the metric is taken to be a function of coordinates. The unified description already suitable for the arbitrary metric tensor could be generalized to include the metric as a function of coordinates belonging to the center of the algebra (17). Such generalization would require suitable modification of the Poincaré algebra (2) (see e.g. [50–52]). However, the quantum deformation would be still described by the classical r -matrix r_τ and would imply two possibilities for the deformation, i.e. $\tau^2 \neq 0$ or $\tau^2 = 0$. This could allow to obtain more gravitational and cosmological models induced by (17). Another application could be found in the so-called relative locality where the metric might live on the momentum space [33]. The so-called relative-locality effects were already investigated in the time-like case [53,54]. The unified description, however, opens a way to also consider the light-like one. In a similar fashion as used in [55] one could define momenta realizations compatible with light-like deformation, i.e. twisted realizations and proceed with the relative-locality formulation [53]. Moreover, the unified description proposed in this paper might be of use in the formulation of quantum field theory on Lie algebraic type of noncommutative spacetimes; see e.g. [56,57].

It is to be stressed that the twist (41) satisfies two-cocycle conditions which make the light-cone deformation triangular. In contrast, it has been shown recently [58] that the light-like and space-like deformations cannot be reached by a similarity transformation of type (42) even if one abandons the cocycle conditions.

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