

Spacetime entanglement with $f(\mathcal{R})$ gravity

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ABSTRACT: We study the entanglement entropy of a general region in a theory of induced gravity using holographic calculations. In particular we use holographic entanglement entropy prescription of Ryu-Takayanagi in the context of the Randall-Sundrum 2 model while considering general $f(\mathcal{R})$ gravity in the bulk. Showing the leading term is given by the usual Bekenstein-Hawking formula, we confirm the conjecture by Bianchi and Myers for this theory. Moreover, we calculate the first subleading term to entanglement entropy and show they agree with the Wald entropy up to extrinsic curvature terms.

KEYWORDS: AdS-CFT Correspondence, Gauge-gravity correspondence, Classical Theories of Gravity, Models of Quantum Gravity

ARXIV EPRINT: [1403.0951](https://arxiv.org/abs/1403.0951)

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1 Introduction

Dividing a quantum system into two parts, *entanglement entropy* (EE) is a measure of quantum correlations between the two subsystems. EE has appeared in different areas such as condensed matter theory [1], quantum information theory [2, 3] and black hole physics [4–6]. In QFT, the EE between a spatial region A and its complement, at fixed time, is given by the usual von-Neumann entropy: $S = -\text{tr}[\rho_A \log \rho_A]$. Whereas the reduced density matrix ρ_A is obtained by tracing out the degrees of freedom in the complementary region and describes the remaining degrees of freedom in region A . Note that, because of the appearance of divergences in continuum limit, one needs to introduce a UV regulator to make sense of this calculation.

More recently, EE has been widely studied in the AdS/CFT, after an elegant conjecture by Ryu and Takayanagi [7, 8] which relates EE in the boundary to the classical geometry in the bulk through holography. According to this prescription, the EE associated with an entangling surface Σ in the CFT, i.e., on the conformal boundary of an AdS bulk, is determined by evaluating $A/4G$ on an extremal surface σ in the bulk, which extends to the AdS boundary to meet Σ . Here the UV boundary cut-off corresponds to introducing a regulator surface at some finite large r , as shown in figure 1. While the conjecture has passed lots of consistency tests [7, 9–11], e.g., reproduces the same results for 2-dimensional CFT and for the thermal ensemble as well as connection to central charges of CFT, it has been proved recently by Lewkowycz and Maldacena [12].

In a related framework, Randall and Sundrum showed that the standard four-dimensional gravity will arise at long distances on a brane embedded in a warped five-dimensional background [13, 14]. One may construct such a model in arbitrary dimension by taking two copies of $(d + 1)$ -dimensional AdS spacetime, gluing them together along a cut-off surface at some large radius while inserting $(d - 1)$ -brane at this junction.

Here we are motivated with the recent conjecture by Bianchi and Myers proposing that in quantum gravity, EE of a general surface with smooth geometry in a smooth background

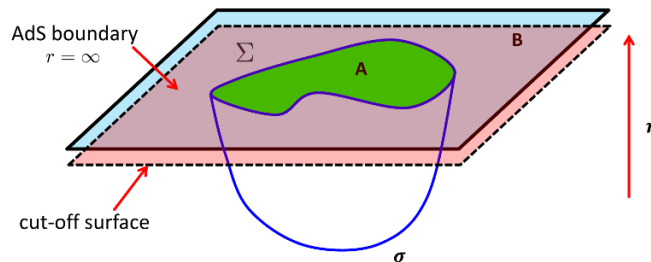


Figure 1. Ryu-Takayanagi surface σ associated to the entangling surface Σ on the AdS boundary.

is finite and the leading term is the usual Bekenstein-Hawking formula. Therefore, in this letter, which is a followup for the previous work with Myers and Smolkin [15], we first briefly introduce the Randall-Sundrum 2 (RS2) model in section 2. In section 3, we use the RS2 model to derive induced gravity on a $(d - 1)$ -brane which is embedded in the $(d + 1)$ -dimensional bulk with $f(\mathcal{R})$ gravity theory. Then in section 4, we use holographic entanglement entropy to study EE of a general surface on this brane. Finally, we summarize the results in section 5.

2 Warped spacetime

If string theory is expected to be the UV completion of general relativity, then the existence of extra dimensions is required in order to cure quantum anomalies from stringy gauge symmetries [16]. Although the size and shape of the extra dimensions can not be predicted by string theory, they need to be compactified and sufficiently small so that 4-dimensional spacetime is recovered at low energies. One way to do this is through *warped compactification*, as proposed by Randall and Sundrum [14]. Here, we focus on a version known as the RS2 model.¹ In this set-up a single 3-brane is embedded in a five dimensional bulk given by

$$I_{\text{bulk}} = \frac{1}{16\pi G_5} \int d^4x \int dr \sqrt{-g} \left(\frac{12}{L^2} + \mathcal{R} \right), \tag{2.1}$$

where L is the scale of cosmological constant as well as the curvature radius of AdS_5 spacetime and r is the fifth extra dimension in a bulk. Therefore, one can produce a 4D effective theory on 3-brane from 5D gravitational action in the bulk. According to UV/IR duality, this brane is usually called the UV brane, since in the bulk we are working in the IR regime of gravity theory by considering the region at smaller radius. In fact, in addition to the bulk action there exists the 4D action of the brane which to leading order is the brane tension, looking like the cosmological term contribution, and any other matter fields living on the brane. Indeed, one can think of the 4D metric as it is multiplied by a *warp factor* which is an exponential function of the fifth coordinate, i.e., locally we have the AdS geometry as

$$ds^2 = e^{-2r/L} \eta_{\mu\nu} dx^\mu dx^\nu + dr^2 \tag{2.2}$$

¹There is also the RS1 model [13] in which there exists another brane in the interior of the bulk, i.e., the IR brane, located at a finite distance from the UV brane. Therefore, one can think of the RS2 as a limit of the RS1 where the IR brane has been taken to infinity.

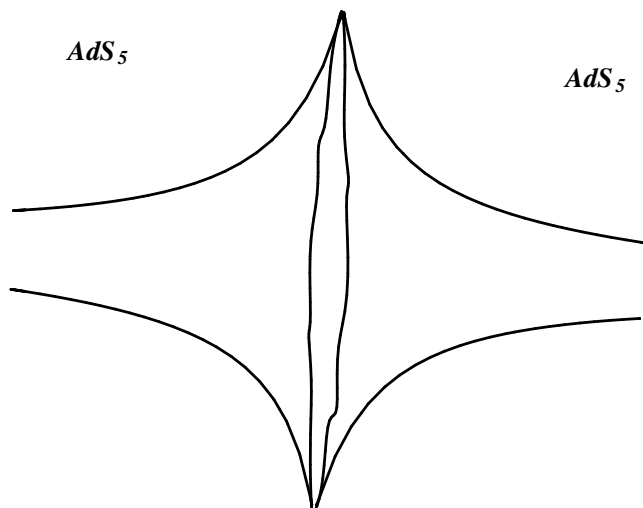


Figure 2. Two copies of AdS_5 spaces have been glued together at some finite radial direction and a 3-brane has been inserted at the junction.

where L is assumed to be large compared to the 5D Planck scale, indicating that the bulk is smoothly curved. Therefore we can trust that metric (2.2), which is the solution to the 5D Einstein equation of the total action, i.e., bulk plus brane, is a valid solution provided the tension of the brane is set to be $T_{\text{brane}} = 3/4\pi LG_5$.

The most important outcome of the model is the 4D effective Newton’s constant in terms of the bulk parameters as

$$G_N = \frac{2G_5}{L} \tag{2.3}$$

which is obtained from 5D action by integrating out the extra dimension. Thus in the RS2 model, gravity is localized in the vicinity of the brane which contains 4D UV theory with finite effective coupling. One can easily extend this construction to arbitrary dimensions to describe a d -dimensional UV theory on the brane from a $d + 1$ -dimensional AdS bulk.

In the rest of this letter, we use the RS2 construction as a holographic framework to study the entanglement entropy of a general region in induced gravity, i.e., the 4D gravity theory on the brane. To do so, we assume two copies of AdS spacetime, cut each of them at some finite radial distance, glue them together at the cut and then insert a brane at this junction — see figure 2. Therefore according to the RS2 mechanism, we have localized gravity in the vicinity of the brane and induced gravity on the brane with finite coupling. On the other hand, due to AdS/CFT correspondence, we have two copies of strongly coupled CFT living on the brane coupled to induced metric of the brane. Also cutting the AdS geometry at some finite radius corresponds to introducing a finite UV regulator in the dual field theory. Since this cutting surface is also the location of the brane, in order to deal with one length scale on the brane, without loss of generality, we choose the AdS scale in the bulk to be equal to the UV cut-off on the boundary theory where we denote both with δ . The fact that the QFT cut-off matches the AdS curvature scale indicates the bulk geometry is highly curved if δ is small.

3 $f(\mathcal{R})$ gravity in the bulk

We will consider extended theories of gravity in the bulk as a generalization of the usual Einstein-Hilbert action (2.1). In particular, we study the RS2 model with $f(\mathcal{R})$ gravity where f is an arbitrary function of the Ricci scalar [17], as an interesting toy-model. Thus the action for the AdS_{d+1} bulk becomes

$$I_{\text{bulk}} = \frac{1}{16\pi G_{d+1}} \int d^d x d\rho \sqrt{-G} \left[\frac{d(d-1)}{L^2} + f(\mathcal{R}(G)) \right] + I_{\text{surf}} \quad (3.1)$$

where G_{d+1} is the gravitational constant in the bulk, L is the scale of cosmological constant and \mathcal{R} is the curvature scalar in the bulk. The dimensionless coordinate ρ is the extra radial direction in the bulk and x^i are the coordinates along the brane located at $\rho = \rho_c$ whereas $\rho = 0$ would be the boundary of AdS_{d+1} .² Note that the AdS_{d+1} geometry again has the curvature radius δ matching the cut-off in the boundary theory. However, we will see that AdS_{d+1} scale is no longer the cosmological scale, i.e., $\delta \neq L$. To have a well-defined action the proper surface term is of the form [18]

$$I_{\text{surf}} = \frac{1}{8\pi G_{d+1}} \int d^d x \sqrt{-\tilde{g}} \mathcal{K} f'(\mathcal{R})|_{\rho=\rho_c}, \quad (3.2)$$

where \mathcal{K} is the trace of second fundamental form of the metric on the brane and prime denote derivative with respect to \mathcal{R} .

We use Fefferman-Graham gauge [19] for the metric in the bulk which is

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu = \frac{\delta^2}{4} \frac{d\rho^2}{\rho^2} + \frac{1}{\rho} g_{ij}(x, \rho) dx^i dx^j, \quad (3.3)$$

where δ , the curvature radius of AdS_{d+1} , is related to the cosmological constant L and the gravitational couplings implicit in $f(\mathcal{R})$ through the equation of motion in the bulk, i.e.,

$$f'(\mathcal{R}) \mathcal{R}_{\mu\nu} + \left(G_{\mu\nu} \nabla^\sigma \nabla_\sigma - \nabla_\mu \nabla_\nu \right) f'(\mathcal{R}) - \frac{G_{\mu\nu}}{2} \left(f(\mathcal{R}) + \frac{d(d-1)}{L^2} \right) = 0. \quad (3.4)$$

That is, if one inserts the metric (3.3) with $g_{ij} = \eta_{ij}$, i.e., pure AdS space, into (3.4) one obtains

$$\frac{1}{L^2} = -\frac{1}{d(d-1)\delta^2} \left[2df'(\mathcal{R}_0) + \delta^2 f(\mathcal{R}_0) \right], \quad (3.5)$$

where \mathcal{R}_0 is the curvature of AdS_{d+1} spacetime, i.e.,

$$\mathcal{R}_0 = -\frac{d(d+1)}{\delta^2}. \quad (3.6)$$

One can obtain the induced gravity action on the brane by integrating out the extra radial dimension of the bulk action (3.1). To do so, we use a derivative expansion for the metric g_{ij} about the position of the brane of the form

$$g_{ij}(x, \rho) = \overset{(0)}{g}_{ij} + \rho \overset{(1)}{g}_{ij} + \rho^2 \overset{(2)}{g}_{ij} + \dots, \quad (3.7)$$

²The dimensionless coordinate ρ is related to dimensionfull coordinate r in previous section as $\rho = e^{2r/\delta}$.

where $g_{ij}^{(0)}$ is the metric of the AdS boundary at $\rho = 0$ and

$$g_{ij}^{(1)} = -\frac{\delta^2}{d-2} \left(R_{ij}[g^{(0)}] - \frac{g_{ij}^{(0)}}{2(d-1)} R[g^{(0)}] \right), \tag{3.8}$$

$$g_{ij}^{(2)} = \delta^4 \left(k_1 C_{mnkl} C^{mnkl} g_{ij}^{(0)} + k_2 C_{iklm} C_j{}^{klm} \right. \\ \left. + \frac{1}{d-4} \left[\frac{1}{8(d-1)} \nabla_i \nabla_j R - \frac{1}{4(d-2)} \square R_{ij} + \frac{1}{8(d-1)(d-2)} \square R g_{ij}^{(0)} \right. \right. \\ \left. - \frac{1}{2(d-2)} R^{kl} R_{ikjl} + \frac{d-4}{2(d-2)^2} R_i{}^k R_{jk} + \frac{1}{(d-1)(d-2)^2} R R_{ij} \right. \\ \left. + \frac{1}{4(d-2)^2} R^{kl} R_{kl} g_{ij}^{(0)} - \frac{3d}{16(d-1)^2(d-2)^2} R^2 g_{ij}^{(0)} \right], \tag{3.9}$$

with R_{ij} and C_{mnkl} being the Ricci and Weyl tensors associated with the boundary metric $g_{ij}^{(0)}$, respectively [20]. The two constants k_1 and k_2 depend on the type of gravity theory in the bulk. By solving the equation of motion (3.4) for $f(\mathcal{R})$ in the bulk, one explicitly finds $k_1, k_2 = 0$. The latter are most easily determined if one picks a fixed geometry on the boundary for $g_{ij}^{(0)}$ and then plugs the metric (3.3) into (3.4) while using the expansion (3.7).

Using the metric expansion (3.7) one can perform a derivative expansion for the curvature scalar in the bulk; it is a matter of calculation to find³

$$\mathcal{R} = \mathcal{R}_0 + \dots, \tag{3.10}$$

since we are just interested in the terms up to curvature squared, we don't really need to specify ellipsis which are of $\mathcal{O}(\partial^6)$ and higher. Indeed, as it is manifestly shown in [15], the only curvature squared term in the expansion (3.10) has a coefficient depending on the constants k_1 and k_2 . However, this term is absent in the present case with $f(\mathcal{R})$ gravity for which k_1 and k_2 are both zero.

According to scaling symmetry of AdS the position of the brane could be set to $\rho = \rho_c = 1$ without loss of generality, therefore we need to justify that the derivative expansion (3.7) can be reasonably truncated for finite ρ . It could be realized from (3.9) that truncation can be consistently achieved by demanding that the boundary metric $g_{ij}^{(0)}$ is weakly curved on the scale of AdS curvature δ . More precisely, we require

$$\delta^2 R_{ij}{}^{kl}[g^{(0)}] \ll 1. \tag{3.11}$$

and similarly for covariant derivatives of the curvatures. Note that to make sense of this inequality, we are assuming that the expressions are evaluated in an orthonormal frame. Although $g_{ij}^{(0)}$ is not the metric on the brane, the constraint (3.11) is sufficient for the brane to be also smoothly curved. It is evident from the expression for the induced metric on the brane

$$\tilde{g}_{ij} = G_{ij}|_{\rho=1} = g_{ij}^{(0)} + g_{ij}^{(1)} + g_{ij}^{(2)} + \dots, \tag{3.12}$$

that the difference between two metrics is small, i.e., $\tilde{g}_{ij} - g_{ij}^{(0)} \simeq g_{ij}^{(1)} \ll 1$ when (3.11) holds.

³The details of calculation could be found in [15].

In order to calculate the induced gravity action on the brane which is given by

$$I_{\text{ind}} = 2I_{\text{bulk}} + I_{\text{brane}}, \quad (3.13)$$

where the factor of two for the bulk gravitational action is due to the fact that we have two copies of AdS spaces and I_{brane} accounts for any contribution from matter fields localized on the brane as well as brane tension. However, here for simplicity we focus only on the latter, i.e.,

$$I_{\text{brane}} = -T_{\text{brane}} \int d^d x \sqrt{-\tilde{g}}. \quad (3.14)$$

We also need to find the derivative expansions for the extrinsic curvature \mathcal{K}_{ij} at the brane where the outward-pointing unit normal vector is given by $n_\mu = -\sqrt{G_{\rho\rho}}\delta_\mu^\rho$. Therefore, one can easily derive

$$\mathcal{K}_{ij} = \nabla_i n_j|_{\rho=1} = -\frac{\rho}{\delta} \frac{\partial G_{ij}}{\partial \rho}|_{\rho=1} = \frac{1}{\delta} \left(\tilde{g}_{ij} - \sum_{n=1}^{\infty} n \tilde{g}_{ij}^{(n)} \right), \quad (3.15)$$

which up to curvature squared terms yields to

$$\mathcal{K} = \frac{1}{\delta} \left[d + \frac{\delta^2}{2(d-1)} R + \frac{\delta^4}{2(d-1)(d-2)^2} \left(R_{ij} R^{ij} - \frac{d}{4(d-1)} R^2 \right) \right] + \mathcal{O}(\delta^6). \quad (3.16)$$

Note that curvatures in the above expression are constructed from the brane metric \tilde{g}_{ij} .

Finally putting together (3.13), (3.1) and (3.2) while using the derivative expansions for the bulk and brane metrics and curvatures as well as (3.5) and integrating over the radial direction ρ we get

$$I_{\text{ind}} = \int d^d x \sqrt{-\tilde{g}} \left[\frac{R}{16\pi G_N} + \frac{\kappa_1}{2\pi} \left(R_{ij} R^{ij} - \frac{d}{4(d-1)} R^2 \right) + \dots \right]. \quad (3.17)$$

The ellipsis in (3.17) are of order $\mathcal{O}(\delta^6)$ and higher and

$$\frac{1}{G_N} = \frac{2\delta}{d-2} \frac{f'(\mathcal{R}_0)}{G_{d+1}}, \quad \kappa_1 = \frac{\delta^3}{4(d-2)^2(d-4)} \frac{f'(\mathcal{R}_0)}{G_{d+1}}, \quad (3.18)$$

with \mathcal{R}_0 is given by (3.6) and we have tuned the brane tension to be

$$T_{\text{brane}} = \frac{d-1}{4\pi\delta G_{d+1}} f'(\mathcal{R}_0). \quad (3.19)$$

Note that all the curvatures in expression (3.17) are constructed from the brane metric \tilde{g}_{ij} . So far, we have found the effective Newton constant of the brane G_N in terms of the bulk gravitational constant G_{d+1} . Moreover, we have an additional parameter κ_1 on the brane which is expressed in terms of bulk gravity parameters. It is worth to mention that the expression (3.17) for induced action has the same form as previously obtained in [15] for Einstein and Gauss-Bonnet gravity. However, the effective Newton constant G_N and the coupling κ_1 have different definitions in terms of the bulk gravitational couplings.

4 Entanglement entropy

Our goal is to calculate the leading term and the first subleading term of the entanglement entropy of a general surface with a smooth geometry in a weakly curved background. Therefore, we assume a sufficiently large surface with generic geometry on the RS brane which is weakly curved due to the constraint (3.11). However, in order to calculate the entanglement entropy of such a surface, instead of going through QFT calculations, we are following holographic approach. That is, as shown in figure 1, we extend the general surface $\tilde{\Sigma}$ on the brane into the bulk and then for this bulk surface σ , we apply holographic entanglement entropy formula introduced by Ryu and Takayanagi [7, 8], i.e.,

$$S_\sigma \equiv \min \frac{\mathcal{A}(\sigma)}{4G_{d+1}}. \tag{4.1}$$

However, with $f(\mathcal{R})$ gravity, we need to find the appropriate entropy functional for the bulk surface σ and then extremise the functional to find holographic entanglement entropy [9]. A natural guess with a general covariant Lagrangian $\mathcal{L}(g, R, \nabla R, \dots)$ would be the Wald entropy formula [21–23] as following

$$S_{\text{Wald}} = -2\pi \int_{\text{hypersurface}} d^{d-1} X \sqrt{h} \frac{\partial \mathcal{L}}{\partial R_{ijkl}} \hat{\epsilon}_{ij} \hat{\epsilon}_{kl}, \tag{4.2}$$

where $\hat{\epsilon}_{ij}$ is the volume form in the two dimensional transverse space to the hypersurface. However, this is known not to be correct in general [9]. In general, one must add terms involving the second fundamental forms of the boundary of σ . However there is evidence such terms do not occur for $f(\mathcal{R})$ gravity, e.g., using a novel method called squashed cone, it has been shown in [24] that for the bulk action of the \mathcal{R}^2 form, which is specific form of $f(\mathcal{R})$, no extrinsic curvature appears in the entanglement entropy. Also performing a field redefinition, one can show that $f(\mathcal{R})$ gravity can be transformed into a pure Einstein gravity minimally coupled to matter [25]. For the latter, the entropy functional is simply $A/4G$ and transforming back yields no \mathcal{K} terms. Therefore, we assume in order to obtain the entropy functional associated with the bulk surface σ for $f(\mathcal{R})$ gravity in the bulk, it is enough to use the Wald entropy formula (4.2). This yields

$$S_\sigma = \frac{1}{2G_{d+1}} \int_{\tilde{\Sigma}} d^{d-2} y \int_1^\infty d\rho \sqrt{h} f'(\mathcal{R}), \tag{4.3}$$

where y^i are the coordinates along the entangling surface $\tilde{\Sigma}$ and $h_{\alpha\beta}$ is the induced metric on the codimension-2 surface σ with its components are given as a Taylor series about $\rho = 0$ by

$$h_{\rho\rho} = \frac{\delta^2}{4\rho^2} \left(1 + \overset{(1)}{h}_{\rho\rho\rho} + \dots \right), \quad h_{ab} = \frac{1}{\rho} \left(\overset{(0)}{h}_{ab} + \overset{(1)}{h}_{ab\rho} + \dots \right). \tag{4.4}$$

In the above expression $\overset{(0)}{h}_{ab}$ is the induced metric of the entangling surface Σ on the AdS boundary and the coefficients in the expansion are given by [26]

$$\overset{(1)}{h}_{\rho\rho} = \frac{\delta^2}{(d-2)^2} K^i K^j \overset{(0)}{g}_{ij}, \quad \overset{(1)}{h}_{ab} = \overset{(1)}{g}_{ab} - \frac{\delta^2}{d-2} K^i K^j \overset{(0)}{g}_{ij}, \tag{4.5}$$

where $K^i = \overset{(0)}{h}{}^{ab} K_{ab}^i$ is the trace of the second fundamental form of Σ . It is worth to clarify that the entangling surface is a codimension-2 with a pair of orthonormal vectors n_j^I ($I = 0, 1$) and associated extrinsic curvatures $K_{ab}^I = \nabla_a n_b^I$. Contracting with a normal vector gives $K_{ab}^i = n_j^i K_{ab}^j$, that's why the extrinsic curvatures here and in the following always carry a coordinate index i .

Moreover, since we are using the expansion (4.4) in the vicinity of the brane at $\rho = 1$, then we need to justify that it converges fast enough to be applicable for finite ρ . This requires not only the background geometry is weakly curved as indicated in constraint (3.11) but also the entangling surface $\tilde{\Sigma}$ is smooth. Recall that the surface $\tilde{\Sigma}$ is the intersection of the bulk surface σ and the brane at $\rho = 1$ with its metric given by

$$\tilde{h}_{ab} = h_{ab}|_{\rho=1} = \overset{(0)}{h}_{ab} + \overset{(1)}{h}_{ab} + \dots \tag{4.6}$$

The derivative expansion (4.6) truncates by imposing that the characteristic scale of the extrinsic curvatures is small compared to the AdS scale, i.e.,

$$\delta K_{ab}^i \ll 1, \tag{4.7}$$

whereas the same constraint should be held for covariant derivatives of the extrinsic curvatures which might appear in higher orders. Again, we are assuming that the expressions are evaluated in an orthonormal frame.

Now, in order to evaluate the entanglement entropy associated with $\tilde{\Sigma}$, we plug (4.4) into (4.3), applying Taylor expansion for $f(\mathcal{R})$ and integrating over the radial direction ρ from the location of the brane to infinity we get

$$S_{EE} = \frac{\delta}{2(d-2)G_{d+1}} \int d^{d-2}y \sqrt{\tilde{h}} f'(\mathcal{R}_0) \left[1 + \frac{d-2}{2(d-4)} \overset{(1)}{h}{}_{\rho\rho} + \frac{1}{d-4} \overset{(0)}{h}{}^{ab} \overset{(1)}{h}{}_{ab} + \dots \right], \tag{4.8}$$

where we have used

$$\sqrt{\overset{(0)}{h}} = \sqrt{\tilde{h}} \left(1 - \frac{1}{2} \overset{(0)}{h}{}^{ab} \overset{(1)}{h}{}_{ab} + \dots \right). \tag{4.9}$$

Note that there is no f'' term in (4.8), since curvature squared term is absent in the derivative expansion (3.10).

Finally if we use (4.5) along with the expressions in (3.18) for the effective Newton constant G_N and parameter κ_1 we can rewrite (4.8) as following

$$S_{EE} = \frac{\mathcal{A}(\tilde{\Sigma})}{4G_N} + \kappa_1 \int_{\tilde{\Sigma}} d^{d-2}y \sqrt{\tilde{h}} \left(2R^{ij} \tilde{g}_{ij}^\perp - \frac{d}{d-1} R - K^i K_i \right) + \mathcal{O}(\partial^4). \tag{4.10}$$

It is clear that the leading term is just the area law as it has been already conjectured in [27]. Again one should note that the expression (4.10) for entanglement entropy has the same form as previously obtained in [15] with Einstein and Gauss-Bonnet gravity in the bulk. The only distinction is that the effective Newton constant G_N and the coupling κ_1 are defined differently in terms of bulk parameters.

Moreover, the first subleading term can also teach us an interesting lesson: let's evaluate the Wald entropy associated to the entangling surface $\tilde{\Sigma}$ by directly applying the Wald formula (4.2) for this surface which is a codimension-2 hypersurface on the brane with induced action (3.17). Doing so, one obtains

$$S_{\tilde{\Sigma}} = \frac{\mathcal{A}(\tilde{\Sigma})}{4G_N} + \kappa_1 \int_{\tilde{\Sigma}} d^{d-2}y \sqrt{\tilde{h}} \left(2R^{ij} \tilde{g}_{ij}^{\perp} - \frac{d}{d-1} R \right) + \mathcal{O}(\partial^4), \quad (4.11)$$

where we have used the following identities:

$$\hat{\varepsilon}_{ik} \hat{\varepsilon}_j^k = -\tilde{g}_{ij}^{\perp}, \quad \hat{\varepsilon}_{ij} \hat{\varepsilon}^{ij} = -2. \quad (4.12)$$

Now comparing (4.10) with (4.11), it is evident that the entanglement entropy for a general surface agrees to the Wald entropy up to the extrinsic curvature terms. In fact, if the entangling surface is a *Killing horizon*, for which extrinsic curvatures are vanishing, then both entropies coincide. However, for a general entangling surface, the Wald entropy does not give the whole entanglement entropy for the surface; there are some contributions to the entanglement entropy from non vanishing extrinsic curvature terms which they do not appear in the Wald entropy. Indeed, the fact that the entanglement entropy cannot be completely extracted from the Wald formula has been recently studied in [15, 24, 28–30].

5 Discussion

In this letter we constructed the RS2 model by taking two copies of AdS spacetime and gluing them together along a cut-off surface at some large radius and inserting a brane at this junction. Therefore, the standard gravity will arise at long distances on the brane as induced gravity: using a FG expansion around the brane and integrating out the extra radial direction, we derived the induced action (3.14) on the brane embedded in the bulk described by the classical action (3.1) known as $f(\mathcal{R})$ gravity. As a result, we obtained the the effective Newton constant G_N and a new coupling κ_1 on the brane in terms of the bulk parameters in equation (3.18). However, since the brane is located at some finite radial direction, to make sense of the derivative expansion in our calculations, we demanded background geometry is weakly curved compared to the AdS scale by imposing constraint (3.11).

To calculate the entanglement entropy associated with a general surface $\tilde{\Sigma}$ on the brane, we used the RT holographic prescription (4.1) in the context of the RS2 model. In particular, we argued that the entropy functional (4.3) associated to the bulk surface σ , is the one we need to extremise in the RT formula (4.1) for $f(\mathcal{R})$ gravity in the bulk. Note that the holographic surface σ is an extension of the surface $\tilde{\Sigma}$ into the bulk. Therefore, in order to obtain the EE of $\tilde{\Sigma}$, we need to integrate over the extra holographic direction. Again, we used derivative expansion to integrate out the radial coordinate and to ensure the convergence of the expansion, we demanded not only a smooth geometry for the ambient metric but also for the entangling surface. In other word, along with a constraint (3.11) for the intrinsic curvature of the boundary metric, we imposed constraint (4.7) for the extrinsic curvature of the entangling surface on the boundary.

Carrying all the calculations, we finally obtained expression (4.10) for a general entangling surface indicating that EE of any region surrounded by a smooth entangling surface is *finite* and the leading contribution is given precisely by the Bekenstein-Hawking area law [31–34]. Hence this model confirmed a conjecture by Bianchi and Myers [27]. We also calculated the first leading corrections to the area law and found that EE coincides with the Wald entropy if the entangling surface is a Killing horizon but for a general surface in addition to the Wald entropy, there are terms dependent on the extrinsic curvature of the entangling surface. It is easy to see that in the special case where $f(\mathcal{R}) = \mathcal{R}$, which corresponds to have an Einstein gravity in the bulk, we reproduce all the results previously obtained in [15]. Furthermore, we get the same form of R^2 term in the action and $R - K^2$ term in the EE as we previously obtained in [15], however, with different gravity theory in the bulk. In fact, the type of theory in the bulk manifests itself just in the effective Newton constant G_N and the coupling κ_1 which are expressed in terms of bulk parameters.

Acknowledgments

Many thanks to Rob Myers, Misha Smolkin, Nima Doroud and Sergey Solodukhin for encouragement and useful discussions. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research & Innovation. I also acknowledge support from Ontario Graduate Scholarship and President’s Graduate Scholarship awarded by University of Waterloo 2013–2014.

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