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From correlation functions to scattering amplitudes

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Abstract: We study the correlation functions of half-BPS protected operators in $\mathcal{N}=4$ super-Yang-Mills theory, in the limit where the positions of adjacent operators become light-like separated. We compute the loop corrections by means of Lagrangian insertions. The divergences resulting from the light-cone limit are regularized by changing the dimension of the integration measure over the insertion points. Switching from coordinates to dual momenta, we show that the logarithm of the correlation function is identical with twice the logarithm of the matching MHV gluon scattering amplitude. We present a number of examples of this new relation, at one and two loops.

Keywords: Supersymmetric gauge theory, Extended Supersymmetry, Duality in Gauge Field Theories, AdS-CFT Correspondence

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1 Introduction

One of the most remarkable manifestations of the AdS/CFT correspondence [1–3] in recent years was the duality between planar gluon scattering amplitudes and light-like polygonal Wilson loops. It was first proposed at strong coupling [4, 5] and soon afterwards also observed at weak coupling, first at one loop [6, 7], followed by extensive two-loop tests [8–11]. This duality can be formulated as follows:

$$\ln\left(A_n/A_n^{(0)}\right) = \ln\left(W[C_n]\right) + O(1/N_c) + O(\epsilon)\,. \tag{1.1}$$

Here A_n is the all-order $n-$gluon (color-ordered) MHV scattering amplitude depending on the particle light-like four-momenta p_i (with $p_i^2 = 0$ and $\sum_{i=1}^n p_i = 0$), and $A_n^{(0)}$ is the tree-level amplitude. An essential step in establishing the relation (1.1) is the so-called T-duality transformation from momenta to dual coordinates:

$$p_i = x_i - x_{i+1} \equiv x_{i,i+1}\,, \qquad x_{i,i+1}^2 = 0\,, \qquad x_{i+n} \equiv x_i\,. \tag{1.2}$$

The Wilson loop $W[C_n]$ is defined on a closed polygonal contour C_n in the dual space, with cusps at points x_i and with light-like edges $[x_i, x_{i+1}]$. The duality (1.1) holds in planar $\mathcal{N} = 4$ super-Yang-Mills theory (SYM) and up to terms vanishing when the suitably identified infrared (for amplitudes) and ultraviolet (for Wilson loops) regulators $\epsilon \to 0$.[1]

An important ingredient in this duality is the notion of dual conformal symmetry. This is the natural symmetry of the light-like Wilson loop, becoming anomalous due to the cusp singularities [13–16]. By virtue of the duality (1.1) it is extended to a symmetry of the planar scattering amplitudes of *dynamical* origin. The first evidence for this new symmetry came from the study [17, 18] of the loop momentum integrals appearing in the four-gluon amplitudes up to four (or even five) loops [19–22]. Once rewritten in dual space according to (1.2), they become pseudo-conformal (the infrared regulator breaks the symmetry). This dual conformal symmetry, or rather its anomalous version [9, 23], was instrumental in explaining the so-called BDS ansatz for MHV amplitudes [20].

In the present paper we provide evidence for another duality relation in the planar $\mathcal{N} = 4$ SYM theory, this time between MHV gluon amplitudes and correlation functions of gauge invariant composite operators on the light cone. The operators we consider belong to the class of half-BPS (or "short") scalar operators. They are of the type $\mathcal{O}^{(k)} = \mathrm{Tr}(\phi^k)$, made of the six real scalars ϕ of the $\mathcal{N} = 4$ SYM theory. They carry R-symmetry SU(4) Dynkin labels $[0, k, 0]$ and transform as chiral primaries under the superconformal symmetry PSU$(2, 2|4)$ of the $\mathcal{N} = 4$ theory, with fixed conformal dimension $d = k$. In perturbation theory, such operators do not undergo renormalization and are thus protected to all orders. The best known example is the simplest, bilinear ($k = 2$) operator, belonging to the so-called stress-tensor superconformal multiplet. In the context of the AdS/CFT correspondence, these operators are dual to massive Kaluza-Klein modes in the compactification of type IIB supergravity on an $\mathrm{AdS}_5 \times S^5$ background.

[1] Quite remarkably, the duality between planar amplitudes and light-like Wilson loops also holds in gauge theories with less or no supersymmetry, including QCD. However, in distinction with $\mathcal{N} = 4$ SYM, there the relation (1.1) is satisfied in the high-energy (Regge) limit only [6, 12].

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Correlation functions of half-BPS operators have been the subject of numerous studies. Not only the conformal dimension of the operators, but also their two- and three-point correlation functions are protected [24–28]. The first non-trivial quantum corrections appear in the four-point correlation functions of protected bilinear operators, which have been computed up to two loops in [29–34]. The knowledge of these quantum corrections allowed one to extract the spectrum of anomalous dimensions of the Konishi operator [35], and later on of all twist-two operators up to two loops by means of a conformal operator product expansion (OPE) [36, 37].

Here we propose to look at such correlation functions from a novel point of view. Consider the correlation function of n protected operators

$$G_n = \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\dots\mathcal{O}(x_n)\rangle \,. \tag{1.3}$$

As long as we maintain the points x_i (with $i = 1, \dots, n$) in generic positions, this function is well defined and has conformal symmetry. As a consequence, it is given by a product of free scalar propagators times some (coupling dependent) function of conformal cross-ratios $x_{ij}^2 x_{kl}^2/(x_{ik}^2 x_{jl}^2)$. In perturbation theory this function is expressed in terms of conformally invariant space-time loop integrals. Now, imagine that we wish to take the limit in which neighboring points become light-like separated,[2]

$$x_{i,i+1}^2 \to 0 \,, \qquad x_{i+n} \equiv x_i \,. \tag{1.4}$$

The correlation function G_n becomes singular in this limit. The first problem we have to face are the pole singularities in $G_n^{(0)}$, due to the propagators $1/x_{i,i+1}^2$ connecting two neighboring scalars. Secondly, the loop integrals develop logarithmic light-cone divergences $\sim \ln x_{i,i+1}^2$ when the integration points approach one of the light-like segments $[x_i, x_{i+1}]$. To deal with the first problem, it is sufficient to consider the ratio $G_n/G_n^{(0)}$, in which the pole singularities are removed. The second problem is more serious, it requires introducing an appropriate regularization.

Two possible choices of a regularization procedure are: (i) use the small distances $x_{i,i+1}^2$ as a cutoff; (ii) employ standard dimensional regularization and set $x_{i,i+1}^2 = 0$ from the very beginning. These two regularizations are considered in the parallel publication [39], where it is shown that in both cases the ratio of the correlation function and its tree-level expression reduces to the square of a Wilson loop,

$$\lim_{x_{i,i+1}^2 \to 0} G_n/G_n^{(0)} \propto (W[C_n])^2 \,. \tag{1.5}$$

The exact form of the proportionality factor in the right-hand side of this relation depends on the regularization; for case (ii) it is just 1. Here $W[C_n]$ is the light-like polygonal Wilson loop in the fundamental representation of the gauge group described earlier. Since $W[C_n]$ is dual to the MHV gluon amplitude, eq. (1.1), we expect that the ratio of the correlation functions in the left-hand side of (1.5) is also related to the ratio of amplitudes, $A_n/A_n^{(0)}$.

[2] A similar light-cone limit has extensively been studied in QCD, see, e.g., the review [38].

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The question arises if we could find another, more direct way of establishing the relation between correlation functions and amplitudes without invoking Wilson loops.

In the present paper we propose a scenario which realizes this direct link. It employs an unusual, *dual infrared* dimensional regularization procedure. This may seem surprising, as we have just argued that the singularities of the correlation function occur at short distances. Nevertheless, we can consider the following alternative. We start by computing the loop corrections to the correlation function by means of Lagrangian insertions. This method is well known in field theory and has been further developed in [32, 33, 40] for the two-loop computations of four-point correlation functions, but it has universal applicability. The idea is to interpret the loop corrections to the correlation function as derivatives with respect to the coupling g. For instance, the one-loop correction

$$g^2 \frac{\partial}{\partial g^2} G_n = -i \int d^D x_0 \, \mathcal{G}^{(0)}_{n+1}(x_0; x_1, \ldots, x_n) \tag{1.6}$$

is calculated from the *Born-level* $(n+1)$-point correlation function

$$\mathcal{G}^{(0)}_{n+1}(x_0; x_1, \ldots, x_n) = \langle L(x_0) \mathcal{O}(x_1) \ldots \mathcal{O}(x_n) \rangle \tag{1.7}$$

obtained by inserting the Lagrangian at the extra point x_0. The crucial point here is that the correlation function (1.7) stays well defined (after dividing it by the tree approximation $G^{(0)}_n$) in $D = 4$ dimensions, even if we put the outer points x_i (with $i = 1, \ldots, n$) on the light cone, but keeping the insertion point x_0 in a generic position. The logarithmic singularities originate from the integration over the insertion point in (1.6). Then we propose to regularize this integral by choosing a measure in $D = 4 - 2\epsilon$ dimensions, with $\epsilon < 0$. This unusual regularization is motivated by the analogy between the space-time loop integrals appearing in (1.6), and the momentum loop integrals in the gluon MHV scattering amplitude A_n discussed earlier. The analogy becomes possible after the T-duality transformation (or change of variables in the integrals) (1.2), provided we use the infrared-like regulator above. Still, nothing guarantees at this stage that we will find a result, not only similar, but identical with a scattering amplitude. Yet, rather surprisingly, this is what happens.

We can rephrase the proposed dual infrared regularization in the following way. The Born-level $(n+1)-$point correlation function (1.7) defines the *integrand* of the one-loop correction to the $n-$point correlation function in (1.6). So, the proposal is to compare this integrand to that of the matching amplitude, re-expressed in terms of dual coordinates. Both objects are rational functions of the coordinates (for the correlator) or of the dual momenta (for the amplitude), well defined in $D = 4$ dimensions. The need to introduce infrared dimensional regularization for the amplitude occurs when we start integrating over the loop momenta (the analogs of the integration point x_0 in (1.6)). So, our proposal is to compare the *finite integrands* of the correlator in the light-cone limit and of the amplitude, and not the corresponding *divergent integrals*.

In the present paper we show a number of examples of this new phenomenon. These include all the $n-$point correlation functions (1.3) at one loop, and the four- and five-point correlation function up to two loops. At present we have no explanation why this is so, but if this new duality correlators/amplitudes is confirmed, it will provide the natural

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explanation of the mysterious dual conformal symmetry [17, 18] of the loop momentum integrals in all available gluon amplitude calculations (n gluons at one loop [41], four gluons up to five loops [19–22], five [42] and six [11] gluons up to two loops).

The paper is organized as follows. In section 2 we show a simple example of a correlation function becoming an amplitude in the light-cone limit and we formulate the main idea of the correlators/amplitudes relation. In section 3 we give a more detailed description of the correlation functions of protected half-BPS operators in superspace. We explain how their loop corrections can be obtained by Lagrangian insertions. We then outline the procedure of establishing the duality with amplitudes, in particular the introduction of dual infrared regularization. In section 4 we show in detail how the duality works for $n-$point correlation functions of bilinear half-BPS operators at one loop. In section 5 this is extended to four-point correlation functions up to two loops, and in section 6 to five-point correlation functions up to two loops. Appendix A summarizes some key points of the harmonic superspace formalism which we employ for loop calculations. In appendix B we generalize our findings to four-point correlation functions of half-BPS operators of arbitrary weight.

2 A simple example of the duality correlators/amplitudes

In this section we discuss some general properties of the protected operators in $\mathcal{N}=4$ SYM and of their correlation functions. To illustrate our main idea, we show how the four-point one-loop correlation function is transformed into the four-gluon one-loop amplitude in the light-cone limit.

The gauge invariant composite operators in $\mathcal{N}=4$ SYM can be classified as representations of the superconformal group PSU(2, 2|4) (see, e.g., [43–45] and references therein). There are two basic types of such operators, usually referred to as protected (or "short") and unprotected (or "long"). The former satisfy conditions of BPS shortening, i.e. they are annihilated by a fraction of the supercharges. This, together with the conditions that they are superconformal primaries, implies that they have quantized, or protected conformal dimension equal to their canonical dimension. The long operators correspond to generic superconformal representations, they receive quantum corrections and acquire anomalous dimensions.

In this paper we will consider only operators of the half-BPS type. Their lowest components (or superconformal primaries) are made of the six real scalars in the $\mathcal{N}=4$ vector multiplet, $\phi_{AB}=-\phi_{BA}=\frac{1}{2}\epsilon_{ABCD}\bar{\phi}^{CD}$, where $A,B=1,\ldots,4$ are indices of the fundamental irrep of the R symmetry group SU(4). Generically, they are of the type $\mathcal{O}^{(k)}=\mathrm{Tr}(\phi^k)$, carry SU(4) Dynkin labels $[0,k,0]$ and have fixed conformal dimension $d=k$. The best known example is the simplest, bilinear ($k=2$) operator, belonging to the so-called stress-tensor superconformal multiplet. The top spin state in this multiplet is the stress tensor, while the state of highest dimension is the Lagrangian of the $\mathcal{N}=4$ SYM theory. The lowest dimension state of the multiplet is the bilinear scalar operator

$$\mathcal{O}_{ABCD}=\mathrm{Tr}(\phi_{AB}\phi_{CD})-\frac{1}{12}\epsilon_{ABCD}\mathrm{Tr}(\bar{\phi}^{EF}\phi_{EF}) \tag{2.1}$$

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belonging to the irrep $\mathbf{20'} = [020]$ of SU(4). Here $\phi_{AB} = \phi^a_{AB} t^a$, where t^a are the generators of the fundamental representation of the gauge group SU(N_c), normalized as $\mathrm{tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$. In what follows we always assume the planar limit,

$$a = \frac{g^2 N_c}{8\pi^2}\,, \qquad N_c \to \infty\,. \tag{2.2}$$

Let us consider certain projections of (2.1), namely

$$\mathcal{O} = \mathrm{Tr}(\phi_{12}\phi_{12})\,, \qquad \tilde{\mathcal{O}} = \mathrm{Tr}(\bar{\phi}^{12}\bar{\phi}^{12})\,, \qquad \hat{\mathcal{O}} = 2\,\mathrm{Tr}(\bar{\phi}^{12}\phi_{12}) - \frac{1}{6}\mathrm{Tr}(\bar{\phi}^{EF}\phi_{EF})\,, \tag{2.3}$$

where $\mathcal{O}$ is the (complex) highest-weight state, $\tilde{\mathcal{O}}$ is the conjugate lowest-weight state and $\hat{\mathcal{O}}$ is a real projection. We want to evaluate the correlation function of n such operators. For $n = 2m$ we can take, e.g., m operators $\mathcal{O}$ and m conjugates $\tilde{\mathcal{O}}$ and consider the correlation function

$$G_n = \langle \mathcal{O}(x_1)\tilde{\mathcal{O}}(x_2)\ldots\mathcal{O}(x_{n-1})\tilde{\mathcal{O}}(x_n)\rangle\,. \tag{2.4}$$

For $n = 2m+1$ we can add one operator $\hat{\mathcal{O}}$, replacing (2.4) by

$$G_n = \langle \mathcal{O}(x_1)\tilde{\mathcal{O}}(x_2)\ldots\mathcal{O}(x_{n-2})\tilde{\mathcal{O}}(x_{n-1})\hat{\mathcal{O}}(x_n)\rangle\,. \tag{2.5}$$

Such correlation functions are finite (the operators (2.3) are not renormalized) and conformally covariant, as long as the points x_i are kept apart, $x_i \neq x_j$. If we let two points get close to each other, $x_i \to x_j$, we are dealing with the well-known short distance expansion of the product of operators $\mathcal{O}(x_i)\mathcal{O}(x_j)$ mentioned above. Here we plan to do something else. We wish to take the limit where the neighboring points become light-like separated, without coinciding with each other[3]

$$x^2_{i,i+1} \;\to\; 0\,, \qquad x_i \neq x_{i+1}\,, \qquad (i = 1,\ldots,n)\,, \tag{2.6}$$

(with the cyclic condition $x_{n+1} \equiv x_1$). This limit is singular for two reasons. Firstly, the correlation function develops pole singularities, as can be seen already from the (connected, planar) tree-level approximation

$$G^{(0)}_n = \frac{(2\pi)^{-2n} N_c^2}{x^2_{12}x^2_{23}\ldots x^2_{n1}} + \text{subleading terms}\,. \tag{2.7}$$

By "subleading" we mean terms corresponding to different Wick contractions of the scalar fields ϕ which are less singular in the limit (2.6) (see an illustration in figure 1). This can be remedied by considering the ratio

$$\lim_{x^2_{i,i+1}\to 0} G_n/G^{(0)}_n\,. \tag{2.8}$$

Notice that the limit (2.6) breaks the symmetry of G_n under the exchange of identical operators (e.g., exchanging all points with odd or with even numbers in (2.4)). Instead, it has a cyclic symmetry, $x_i \to x_{i+1}$, and a flip symmetry, $x_i \to x_{n-i+1}$.

[3]The standard OPE is done in the Euclidean regime, where $x^2_{i,i+1} = 0$ implies $x_i = x_{i+1}$. The Minkowski regime allows us to consider the new possibility (2.6).

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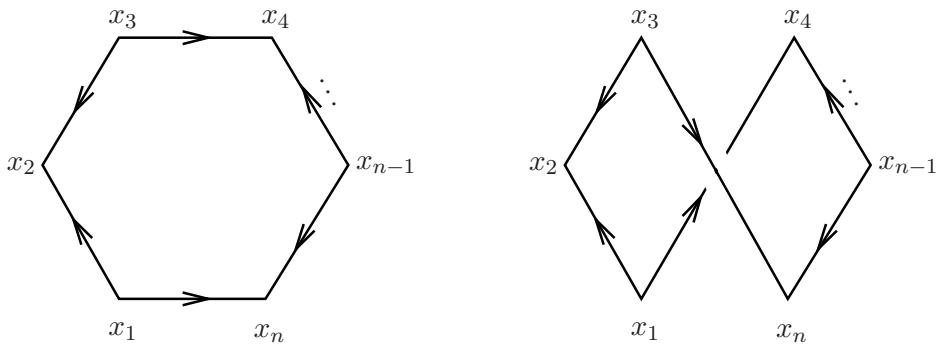

Figure 1. Feynman diagrams of different types contributing to the correlation function (2.4) at tree level. Arrowed lines denote free scalar propagators $\langle\bar{\phi}^{12}(x_i)\phi_{12}(x_j)\rangle$. In the light-cone limit $x^2_{i,i+1}\to 0$ the leading contribution comes from diagram (a), while that of diagram (b) is suppressed by the factor $x^2_{34}x^2_{1n}/(x^2_{3n}x^2_{14})$.

Secondly, the loop integrals develop additional light-cone singularities in the limit (2.6). To illustrate this, let us take a simple example — the four-point correlation function (2.4) at one loop. It has been computed in [29–31] and the result for the ratio $G_4/G_4^{(0)}$ is given by

$$G_4/G_4^{(0)} = 1 + 2a\, x^2_{13}x^2_{24}g(x_i) + O(a^2)\,. \tag{2.9}$$

Here the one-loop integral $g(x_i)$ is defined by

$$g(x_i) = \frac{i}{2\pi^2}\int \frac{d^4x_0}{x^2_{10}x^2_{20}x^2_{30}x^2_{40}}\,. \tag{2.10}$$

As long as the outer points are kept in generic positions, $x^2_{i,i+1}\neq 0$, this integral is finite and conformally covariant in four dimensions. This allows us to write it down as a function of two conformal cross-ratios

$$g(x_i) = \frac{1}{x^2_{13}x^2_{24}}\Phi^{(1)}(u,v)\,,\quad u=\frac{x^2_{14}x^2_{23}}{x^2_{13}x^2_{24}}\,,\quad v=\frac{x^2_{12}x^2_{34}}{x^2_{13}x^2_{24}}\,, \tag{2.11}$$

where the two-variable function $\Phi^{(1)}$ can be found in [46, 47]. The cross-ratios vanish in the limit (2.6) and this function develops a logarithmic singularity, $\Phi^{(1)}(u,v)\sim \ln v\ln u$ as $u,v\to 0$. It originates from the integration in (2.10), when x_0 approaches one of the four light-cone segments $[x_i,x_{i+1}]$.

Therefore, to define the integral (2.10) in the limit (2.6) we have to introduce a regularization. The standard approach is to use dimensional regularization from the very beginning, that is, to repeat the whole calculation that leads to (2.9), but in $D=4-2\epsilon$ dimensions (with $\epsilon>0$). This approach was adopted in [39], where it was shown that the limit (2.8) turns the correlation function into a light-like Wilson loop.

Alternatively, we might use the four-dimensional result (2.9) and declare that we simply regularize the integral (2.10) by choosing a measure in $D=4-2\epsilon$ dimensions (with $\epsilon<0$). Notice the change of sign of the regulator — now it looks more like an infrared, rather than

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the natural ultraviolet regulator needed for such short-distance singularities. This unusual choice is motivated by the observation that the one-loop space-time integral (2.10) is the dual space version [17, 18] of the one-loop scalar box momentum integral appearing in the four-gluon MHV amplitude. The latter is given by [20]

$$A_4/A_4^{(0)} = 1 + a\, st I_4^{(1)}(p_i) + O(a^2)\,, \tag{2.12}$$

where $A_4^{(0)}$ is the tree-level amplitude, $s = (p_1 + p_2)^2$, $t = (p_3 + p_4)^2$ are the Mandelstam variables and

$$I_4^{(1)}(p_i) = \frac{2i}{(2\pi)^{2-2\epsilon}} \int \frac{d^D k}{k^2 (k - p_1)^2 (k - p_1 - p_2)^2 (k + p_4)^2} \tag{2.13}$$

is the one-loop scalar box integral. Switching from momenta to dual coordinates, $k = x_1 - x_0$ and $p_i = x_i - x_{i+1}$ (with $x_5 \equiv x_1$), we identify the two integrals:

$$I_4^{(1)}(p_i) = g_\epsilon(x_i)\,, \tag{2.14}$$

where the subscript in g_ϵ indicates that we have changed the measure in (2.10) to $D = 4-2\epsilon$ dimensions (with $\epsilon < 0$). Notice that the light-cone limit (2.8) for the correlation function implies that the momenta $p_i = x_{i,i+1}$ are light-like, $p_i^2 = 0$, as required for the amplitude.

Further, comparing (2.9) and (2.12), we observe a surprisingly simple relation between correlation function and amplitude:

$$\lim_{x_{i,i+1}^2 \to 0} G_4/G_4^{(0)} = \left(A_4/A_4^{(0)}\right)^2 + O(a^2)\,. \tag{2.15}$$

This one-loop exercise suggests the following general recipe for obtaining MHV gluon amplitudes from correlation functions of protected operators:

1. Compute the $n-$point correlation function of protected operators in four dimensions.
2. Change the $D = 4$ integration measure in the loop integrals to $D = 4-2\epsilon$ (with $\epsilon < 0$).
3. Divide the correlation function by its tree-level value and take the light-cone limit $x_{i,i+1}^2 \to 0$.
4. Switch from coordinates to dual momenta. The result should be the square of the $n-$gluon MHV amplitude divided by its tree-level expression.

Of course, the four-points/gluons case at one loop is probably too simple to allow us to jump to the conclusion that such a duality correlator/amplitude is a general phenomenon. It is the purpose of this paper to provide a lot of evidence in favor of this conjecture.

3 Correlation functions of protected operators in superspace and their loop corrections

In this section we explain the role of the Lagrangian insertion procedure in the calculation of loop corrections to correlation functions. Then we formulate the rules of dual infrared dimensional regularization and state the main result of the paper.

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3.1 Correlation functions of protected operators in $\mathcal{N}=2$ harmonic superspace

The one-loop, and even more so the two-loop calculation of the correlation functions are much easier to do in terms of superspace Feynman graphs. Since there exists no off-shell formulation of the $\mathcal{N}=4$ SYM theory suitable for application to perturbation theory, the best compromise is to use the formulation in terms of $\mathcal{N}=2$ superfields in the so-called harmonic superspace [48, 49]. We give a brief summary of this formalism in appendix A. Here we just mention the types of $\mathcal{N}=2$ supermultiplets and superfields that we are dealing with.

The $\mathcal{N}=4$ vector multiplet is decomposed into an $\mathcal{N}=2$ matter multiplet (hypermultiplet) and an $\mathcal{N}=2$ vector (gauge) multiplet. Upon reducing the R symmetry, $\mathrm{SU}(4)\to\mathrm{SU}(2)\times\mathrm{U}(1)$, the six real scalars ϕ_{AB} split into an isodoublet ϕ^i (with $i=1,2$) and a complex singlet φ; the four (chiral) gluinos $\lambda^{A\alpha}$ split into a doublet $\lambda^{i\alpha}$ and two singlets ψ^α, κ^α (and their antichiral conjugates). These fields can be combined to form $\mathcal{N}=2$ superfields. One of them describes the hypermultiplet:

$$q^+(x,\theta^+,\bar\theta^+,u)=\phi^i(x)u_i^+ +\theta^{+\alpha}\psi_\alpha(x)+\bar\theta^+_{\dot\alpha}\bar\kappa^{\dot\alpha}(x)+\dots \tag{3.1}$$

(the dots denote auxiliary and derivative terms). It is a Grassmann analytic (or G-analytic, or half-BPS) superfield in the sense that it depends on half of the Grassmann variables, $\theta^{+\alpha}=\theta^{i\alpha}u_i^+$ and $\bar\theta^{+\dot\alpha}=\bar\theta^{i\dot\alpha}u_i^+$, obtained by projecting the SU(2) doublets $\theta^{i\alpha}$ and $\bar\theta^{i\dot\alpha}$ with an SU(2) harmonic variable u_i^+. The latter, together with its conjugate $u_i^-=(u^{+i})^*$ forms an SU(2) matrix. Notice the presence of both chiral and antichiral odd variables in the hypermultiplet superfield (3.1). In harmonic superspace one can define a special conjugation (denoted by tilde $\tilde{}$), which takes (3.1) to another G-analytic superfield,

$$\tilde q^+(x,\theta^+,\bar\theta^+,u)=\bar\phi^i(x)u_i^+ +\theta^{+\alpha}\kappa_\alpha(x)+\bar\theta^+_{\dot\alpha}\bar\psi^{\dot\alpha}(x)+\dots\ . \tag{3.2}$$

In contrast with the hypermultiplet, the $\mathcal{N}=2$ vector multiplet is described by the chiral field strength (and its antichiral conjugate)

$$W(x,\theta)=\varphi(x)+\theta^{i\alpha}\lambda_{i\alpha}(x)+\theta^{i\alpha}\theta_i^\beta F_{\alpha\beta}(x)+\dots \tag{3.3}$$

containing, in particular, the self-dual part of the gluon field strength $F_{\alpha\beta}=(\sigma^{\mu\nu})_{\alpha\beta}F_{\mu\nu}$.

The protected half-BPS bilinear operators (2.1) in the SU(4) irrep $\mathbf{20'}$ split into a number of irreps of $\mathrm{SU}(2)\times\mathrm{U}(1)$. They can be descried as the lowest ($\theta=0$) components of bilinears made of the above superfields. Three such bilinears are G-analytic superfields:

$$O=\mathrm{Tr}(q^+q^+)\,,\qquad \tilde O=\mathrm{Tr}(\tilde q^+\tilde q^+)\,,\qquad \hat O=\hat{\tilde O}=2\,\mathrm{Tr}(\tilde q^+q^+)\,, \tag{3.4}$$

where all operators are functions of $x,\theta^+,\bar\theta^+,u$. For example, the operator $O(x,\theta^+,\bar\theta^+,u)$ has the bottom component $\mathcal{O}(x,u)=O|_{\theta^+=\bar\theta^+=0}=\mathrm{Tr}(\phi^i(x)\phi^j(x))u_i^+u_j^+$ containing the complex SU(2) triplet $\mathrm{Tr}(\phi^i\phi^j)$. Another, real triplet is the bottom component of the real operator $\hat O=\hat{\tilde O}$, $\hat{\mathcal{O}}(x,u)=\hat O|_{\theta^+=\bar\theta^+=0}=2\,\mathrm{Tr}(\bar\phi^i(x)\phi^j(x))u_i^+u_j^+$. We may say that the harmonic variables serve as a "bookkeeping device" for organizing the fields into SU(2) representations.

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The remaining SU(2) × U(1) projections of the **20′** are described by different types of superfields. Among them, the chiral operator $\mathrm{Tr}(WW)$ plays the prominent role of the $\mathcal{N}=2$ SYM Lagrangian. In this paper we do not consider the rest of the $\mathcal{N}=2$ projections of the **20′**: $\mathrm{Tr}(W\bar{W})$, $\mathrm{Tr}(Wq^+)$, $\mathrm{Tr}(W\tilde{q}^+)$ and conjugates.

The correlation functions of the three G-analytic operators (3.4) will be the main subject of this and the following sections. We are considering n-point correlation functions of the type, e.g.,

$$G_n = \langle \mathcal{O}(x_1,u_1)\tilde{\mathcal{O}}(x_2,u_2)\cdots\mathcal{O}(x_{n-1},u_{n-1})\tilde{\mathcal{O}}(x_n,u_n)\rangle \tag{3.5}$$

for n even, or

$$G_n = \langle \mathcal{O}(x_1,u_1)\tilde{\mathcal{O}}(x_2,u_2)\cdots\mathcal{O}(x_{n-2},u_{n-2})\tilde{\mathcal{O}}(x_{n-1},u_{n-1})\hat{\mathcal{O}}(x_n,u_n)\rangle \tag{3.6}$$

for n odd. At tree level the connected contribution to the correlation function has the form[4]

$$G_n^{(0)} = \frac{N_c^2}{(2\pi)^{2n}}\frac{(12)(23)\cdots(n1)}{x_{12}^2x_{23}^2\cdots x_{n1}^2} + \text{subleading terms}\,, \tag{3.7}$$

where $(r,r+1)$ is a shorthand for the SU(2) invariant but U(1) covariant contraction of the two harmonics with labels r and $r+1$,

$$(r,r+1) = -(r+1,r) = u_r^{+i}\epsilon_{ij}u_{r+1}^{+j}\,. \tag{3.8}$$

In (3.7), as in the expression (2.7) for the tree amplitude for the highest-weight and lowest-weight state projections of the **20′**, we only show the leading singular term in the light-cone limit (2.6). Compared to (2.7), the tree-level expression (3.7) contains additional information about the isotopic SU(2) structure of the correlation function. It is carried by the harmonic variables $u_r^{+\,i}$ (with $i=1,2$ and $r=1,\ldots,n$) at each point.

3.2 Lagrangian insertions

The correlation functions (3.5) and (3.6) are defined by the path integral

$$\begin{aligned} G_n &= \int \mathcal{D}\Phi\; e^{iS_{\mathcal{N}=4\ \mathrm{SYM}}}\;\mathcal{O}(x_1,u_1)\ldots\mathcal{O}(x_n,u_n) \\ &= G_n^{(0)} + g^2G_n^{(1)} + g^4G_n^{(2)} + \ldots\ . \end{aligned} \tag{3.9}$$

Here the $\mathcal{N}=4$ SYM action consists of two parts, the $\mathcal{N}=2$ SYM action and the action of the $\mathcal{N}=2$ hypermultiplet matter coupled to the gauge sector:

$$S_{\mathcal{N}=4\ \mathrm{SYM}} = S_{\mathcal{N}=2\ \mathrm{SYM}} + S_{\mathcal{N}=2\ \mathrm{matter}}\,. \tag{3.10}$$

[4]More precisely, a n-point correlation function may involve m complex operators $\mathcal{O}$ and also m conjugate operators $\tilde{\mathcal{O}}$, the remaining $n-2m$ operators being of the real type $\hat{\mathcal{O}}$. At tree level, such a correlation function equals $c_{nm}\,G_n^{(0)}$, where $c_{n0} = 1+(-1)^n$ and $c_{nm} = (-1)^{n+m}\ : m>0$. In this paper we are interested in the ratio of loop corrections over the corresponding tree, which is universal. The coefficients c_{nm} can thus safely be omitted.

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Instead of computing the loop corrections to G_n directly, we prefer to evaluate the derivative with respect to the coupling. As we show below, it is given by the insertions of the $\mathcal{N}=2$ SYM action

$$S_{\mathcal{N}=2\ \mathrm{SYM}} = \int d^D x d^4\theta\, L_{\mathcal{N}=2\ \mathrm{SYM}}(x,\theta) = \int d^D x d^4\bar\theta\, \bar L_{\mathcal{N}=2\ \mathrm{SYM}}(x,\bar\theta)\,, \tag{3.11}$$

where, after the appropriate rescaling of all the fields of the $\mathcal{N}=2$ vector multiplet,

$$\begin{aligned} L_{\mathcal{N}=2\ \mathrm{SYM}} &= \frac{1}{2g^2}\mathrm{Tr}(W^2) \\ &= \frac{1}{2g^2}\{\mathrm{Tr}(\varphi^2)+\ldots-(\theta)^4\mathrm{Tr}[F_{\alpha\beta}F^{\alpha\beta}+4\phi^i\Box\bar\phi_i+4i\lambda^i\hat\partial\bar\lambda_i+\text{interaction terms}]\} \end{aligned} \tag{3.12}$$

is the $\mathcal{N}=2$ SYM *chiral* Lagrangian. The effect of the rescaling is that the coupling g disappears inside the Lagrangian $L_{\mathcal{N}=2\ \mathrm{SYM}}$, it is only present in front of it, as indicated in (3.12). The coupling also drops out from the interaction of the vector and matter multiplets (see appendix A.3).

Notice that the action in (3.11) has two forms, one chiral, the other antichiral. They are equivalent due to a Bianchi identity stating that the difference between $\mathrm{Tr}(W^2)$ and $\mathrm{Tr}(\bar W^2)$ is a *total (super)space derivative*. At the component level, this is clearly seen from (3.12), where, for instance, the complex combination $F_{\alpha\beta}F^{\alpha\beta} = F^2 + iF\tilde F$ contains the Yang-Mills Lagrangian F^2 and the topological term $iF\tilde F$. We will come back to this important point later on.

Now, we want to differentiate the correlation function with respect to the coupling, which is present only in $S_{\mathcal{N}=2\ \mathrm{SYM}}$ as an overall factor, see (3.12). Thus, the one-loop (order g^2) correction to the correlation function,

$$g^2\frac{\partial}{\partial g^2}\,G_n = -i\int d^D x_0\, \mathcal{G}^{(0)}_{n+1}(x_0;x_1,u_1;\ldots;x_n,u_n) + O(g^4) \tag{3.13}$$

is calculated from the *Born-level* $(n+1)$-point correlation function

$$\mathcal{G}^{(0)}_{n+1}(x_0;x_1,u_1;\ldots;x_n,u_n) = \int d^4\theta_0 \langle L_{\mathcal{N}=2\ \mathrm{SYM}}(x_0,\theta_0)\mathcal{O}(x_1,u_1)\ldots\mathcal{O}(x_n,u_n)\rangle + O(g^4)\,, \tag{3.14}$$

obtained by integrating the Lagrangian insertion over the Grassmann variables at the insertion point, but not over the space-time point x_0. Note that the tree-level correlation function $\mathcal{G}^{(0)}_{n+1}$ is of order $O(g^2)$, because it involves interaction vertices (see section 4 for details).

A very important property of the correlation function $\mathcal{G}^{(0)}_{n+1}$ is its superconformal symmetry. Indeed, it involves the protected operators $\mathcal{O}$ and $L_{\mathcal{N}=2\ \mathrm{SYM}}$ (the latter belongs to the $\mathcal{N}=4$ stress-tensor multiplet), with fixed conformal dimensions 2 and 4, respectively. Such operators are not renormalized and have well-defined conformal properties. This symmetry greatly facilitates the perturbative calculation, as explained in appendix A.

The same procedure can be applied to the higher-order perturbative corrections. Thus, to obtain the correlation function at two loops (order g^4), we compute the derivative

$$\frac{1}{2}\,g^4\left(\frac{\partial}{\partial g^2}\right)^2 G_n = -\frac{1}{2}\int d^D x_0 d^D x_{0'}\, \mathcal{G}^{(0)}_{n+2}(x_0,x_{0'};x_1,u_1;\ldots;x_n,u_n) + O(g^6) \tag{3.15}$$

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in terms of the Born-level $(n+2)$-point correlation function with two Lagrangian insertions (see appendix A.3)

$$\mathcal{G}^{(0)}_{n+2} = \int d^4\theta_0 d^4\theta_{0'} \langle L(x_0,\theta_0) L(x_{0'},\theta_{0'}) \mathcal{O}(x_1,u_1)\dots\mathcal{O}(x_n,u_n)\rangle + O(g^6)\,. \tag{3.16}$$

In conclusion, the Lagrangian insertion procedure reduces the calculation of loop corrections to the correlation function G_n to a tree-level calculation of the correlation function $\mathcal{G}^{(0)}_{n+2}$ with two insertions of the Lagrangian. The main point of our conjecture is that this tree-level correlation function tells us what the *integrand* of the dual MHV amplitude should look like. The precise matching of the two objects is obtain by introducing a dual infrared regulator.

3.3 Outline of the dual infrared regularization procedure and of the duality correlator/amplitude

Our strategy for establishing the relationship between correlation functions and amplitudes is as follows.

We start with the tree-level correlation functions $\mathcal{G}^{(0)}_{n+1}$ and $\mathcal{G}^{(0)}_{n+2}$, defined in eqs. (3.14) and (3.16), respectively. They are computed in $D = 4$ and need no regularization. Then, we put $\mathcal{G}^{(0)}_{n+1}$ and $\mathcal{G}^{(0)}_{n+2}$ on the light cone by setting the adjacent external points at light-like distances, $x^2_{i,i+1} \to 0$ (with $i = 1,\dots,n$). In order to remove the pole singularities, we divide the correlation function with Lagrangian insertions by the tree-level correlation function without insertions (3.7), thus obtaining U(1) chargeless ratios, $\mathcal{G}^{(0)}_{n+1}/G^{(0)}_n$ and $\mathcal{G}^{(0)}_{n+2}/G^{(0)}_n$. After that it becomes safe to set the external points on the light cone, while keeping the insertion points x_0 and $x_{0'}$ in arbitrary positions. We remark that at this stage we still need no regularization.

Next, we perform the integration over the insertion points, thus passing from the tree-level correlation functions with insertions $\mathcal{G}^{(0)}_{n+1}$ and $\mathcal{G}^{(0)}_{n+2}$ to the loop corrections of the correlation function without insertions G_n. Here we are facing logarithmic singularities due to the divergent integrals over x_0 and $x_{0'}$. Divergences arise when the integration points approach a light-like segment $[x_i, x_{i+1}]$. We regularize the integrals by modifying the dimension of the integration measure, $D = 4 - 2\epsilon$, with $\epsilon < 0$. We emphasize that this is *not standard dimensional regularization*, for two reasons: (i) the tree-level correlation function with extra points from the Lagrangian insertions has been computed in $D = 4$ and then put on the light cone, without regularization; (ii) the sign of the regulator ϵ is chosen to match the *infrared divergences of the dual amplitude*, so this is not the usual ultraviolet regulator. We call this "dual infrared regularization".

In order to make contact between the n-point correlation function and the n-gluon amplitude, we identify the momenta with the dual coordinates

$$p_i = x_{i,i+1}\,, \qquad x_{n+1} \equiv x_1\,, \tag{3.17}$$

so that $\sum_1^n p_i = 0$ and $p_i^2 = 0$. The Mandelstam variables are identified with the non-vanishing distances in dual space,

$$s_{ij} = (p_i + p_{i+1} + \ldots + p_j)^2 = x^2_{i,j+1}\,. \tag{3.18}$$

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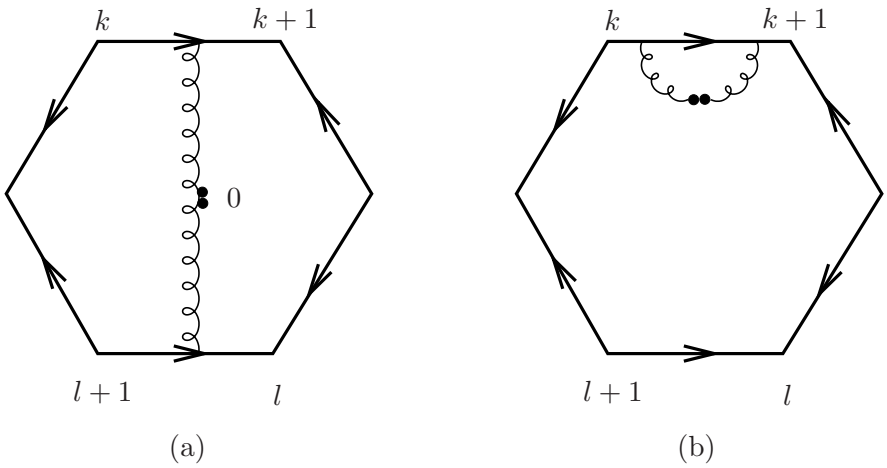

Figure 2. The n-point correlation function with one insertion. The solid and wavy lines are hypermultiplet and gauge propagators, respectively. The double dot denotes the insertion of the $\mathcal{N} = 2$ SYM Lagrangian $\mathrm{Tr}(W^2)$.

The main result of the next three sections is that the above procedure leads to the following duality relation between correlation functions restricted to the light cone and MHV n-gluon scattering amplitudes in planar $\mathcal{N} = 4$ SYM:

$$\lim_{x_{i,i+1}^2 \to 0} \ln\left(G_n/G_n^{(0)}\right) = \ln\left(A_n/A_n^{(0)}\right)^2 + O(\epsilon)\,, \tag{3.19}$$

where $A_n^{(0)}$ is the tree-level amplitude and $O(\epsilon)$ denotes terms that vanish after we remove the regularization. The reason why we formulate the relation in terms of logs will become clear in section 6. In section 4 we demonstrate this duality for any n at one loop, and in sections 5 and 6 we will show it for $n = 4, 5$ up to two loops.

4 From correlation functions to amplitudes: n points at one loop

In this section we perform the calculation of the one-loop correction to the correlation function G_n according to the procedure of section 3.3.[5] We use the Feynman rules from appendix A.2. The harmonic superspace Feynman diagrams for the correlation function with one extra point corresponding to the Lagrangian insertion are shown in figure 2.[6] They are constructed from the two basic building blocks T and TT described in appendix A.3.2, combined with free hypermultiplet propagators (A.24). The block T shown in figure 3(a) is the supersymmetric analog of the vertex correction $\langle\bar{\phi}_a(x_1)F_b^{\mu\nu}(x_0)\phi_c(x_2)\rangle$ for two scalars and one gauge field strength. The main difference is that at the insertion point we have only the self-dual part of the gauge field strength $F_{\alpha\beta} = (\sigma^{\mu\nu})_{\alpha\beta}F_{\mu\nu}$, as well as the auxiliary field Y^{ij} (see (A.13)); the scalar φ and the gluino $\lambda^{i\alpha}$ from (3.3) do

[5] Our result agrees with earlier calculations of one-loop correlation functions in [29–31, 50, 51].

[6] Notice that the graphs are drawn with a polygonal matter frame. Graphs based on the "zigzag" configurations like in figure 1(b) are suppressed in the light-cone limit (2.6), after dividing out the leading singularity of the tree-level correlation function (2.7).

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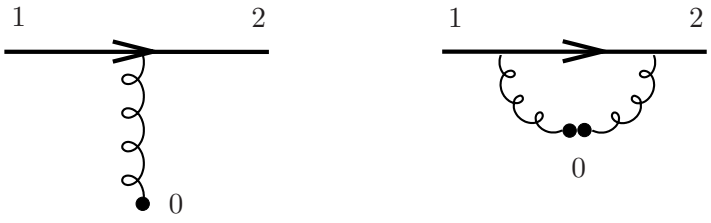

(a) (b)

Figure 3. Building blocks for the graphs in figure 2. The single dot in (a) denotes a W insertion, the double dot in (b) a $\mathrm{Tr}(W^2)$ insertion.

not have a cubic vertex with the external scalars. Since we are doing the calculation in $D = 4$, the integral at the vertex can be easily computed yielding a rational expression. It is obtained from the block (A.47) by setting the external $\theta_1^+ = \theta_2^+ = 0$:

$$\langle \tilde{q}_a^+(x_1,0,u_1) W_b(x_0,\theta_0) q_c^+(x_2,0,u_2)\rangle = -\frac{2ig^2 f_{abc}}{(2\pi)^4}\frac{(12)}{x_{12}^2} i_{12} \tag{4.1}$$

with

$$i_{12} = x_{12}^2 \frac{\theta_{0/1}^+ \cdot \theta_{0/2}^+}{(12) x_{10}^2 x_{20}^2} - \frac{\theta_{0/1}^+ \cdot \theta_{0/1}^-}{x_{10}^2} + \frac{\theta_{0/2}^+ \cdot \theta_{0/2}^-}{x_{20}^2} - \frac{\theta_{0/1}^+ [x_{10}, x_{20}] \theta_{0/2}^+}{(12) x_{10}^2 x_{20}^2}\,. \tag{4.2}$$

Here $\theta_{0/r}^{\pm} = \theta_0^i (u_r)_i^{\pm}$ are the two U(1) projections of the SU(2) doublet θ_0^i with the harmonics at point r and the notation was introduced for $\theta \cdot \theta \equiv \theta^\alpha \theta_a$ and $[x,y]_\alpha{}^\beta \equiv x_{\alpha\dot\alpha} y^{\dot\alpha\beta} - y_{\alpha\dot\alpha} x^{\dot\alpha\beta} = -2i x^\mu y^\nu (\sigma_{\mu\nu})_\alpha^\beta$.

The second building block shown in figure 3(b) is the supersymmetric analog of the propagator correction $\langle \bar\phi(x_k)\phi(x_{k+1})\rangle$, after integration over the insertion point. It is obtained from the TT block (A.48) by identifying the insertion points $0 \equiv 0'$ and by setting $\theta_1^+ = \theta_2^+ = 0$:

$$\langle \tilde{q}_a^+(x_1,0,u_1) \mathrm{Tr}(W^2)(x_0,\theta_0) q_b^+(x_2,0,u_2)\rangle = \frac{2g^4 N_c\, \delta_{ab}}{(2\pi)^6}\frac{(12)}{x_{12}^2} j_{12} \tag{4.3}$$

with

$$j_{12} \equiv -\frac{(1^-2^-)}{(12)}\frac{(\theta_{0/1}^+)^2 (\theta_{0/2}^+)^2}{x_{10}^2 x_{20}^2} = \frac{1}{2}(i_{12})^2\,. \tag{4.4}$$

Summing up all graphs (all sums are cyclic, $k+n \equiv k$) and integrating over the odd variable θ_0 at the insertion point (but not yet over the even x_0), we find

$$\begin{aligned}\mathcal{G}_{n+1} &= \frac{2\, g^2 N_c^3}{(2\pi)^{2(n+2)}}\frac{(12)(23)\cdots(n1)}{x_{12}^2 x_{23}^2 \cdots x_{n1}^2}\int d^4\theta_0 \,\Big(\frac{1}{2}\sum_{k,l=1;\,k\neq l}^{n} i_{k,k+1} i_{l,l+1} + \sum_{k=1}^{n} j_{k,k+1}\Big) \\ &= \frac{a}{2\pi^2}\, G_n^{(0)} \int d^4\theta_0 \,\Big(\sum_{k=1}^{n} i_{k,k+1}\Big)^2\,,\end{aligned} \tag{4.5}$$

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where $G_n^{(0)}$ is the leading singular part of the connected tree-level n-point correlation function defined in (3.7). In deriving (4.5) we have used the identity $j_{kl} = \frac{1}{2}(i_{kl})^2$.

The next step is to set the adjacent external points on the light cone, $x_{k,k+1}^2 = 0$, for the ratio

$$\lim_{x_{i,i+1}^2 \to 0} \mathcal{G}_{n+1}/G_n^{(0)} = \frac{a}{2\pi^2} \lim_{x_{i,i+1}^2 \to 0} \int d^4\theta_0 \left(\sum_{k=1}^{n} i_{k,k+1}\right)^2 . \tag{4.6}$$

We notice that for $x_{12}^2 \to 0$ the first term in i_{12}, eq. (4.2), vanishes while the second and the third terms cancel in the cyclic sum

$$\sum_{k=1}^{n} i_{k,k+1} = -\sum_{k=1}^{n} \frac{\theta_{0/k}^+[x_{k,0}, x_{k+1,0}]\theta_{0/k+1}^+}{(k,k+1)x_{k,0}^2 x_{k+1,0}^2} . \tag{4.7}$$

Then, using the identity

$$\int d^4\theta_0\, \theta_0^{\alpha_1 i_1}\theta_0^{\alpha_2 i_2}\theta_0^{\alpha_3 i_3}\theta_0^{\alpha_4 i_4} = -\frac{1}{4}\left(\epsilon^{i_1 i_2}\epsilon^{i_3 i_4}\epsilon^{\alpha_1\alpha_4}\epsilon^{\alpha_2\alpha_3} - \epsilon^{i_1 i_4}\epsilon^{i_2 i_3}\epsilon^{\alpha_1\alpha_2}\epsilon^{\alpha_3\alpha_4}\right) , \tag{4.8}$$

after some algebra we obtain

$$\begin{aligned}
&\lim_{x_{i,i+1}^2 \to 0} \int d^4\theta_0 \left(\sum_{k=1}^{n} i_{k,k+1}\right)^2 \\
&\quad = \lim_{x_{i,i+1}^2 \to 0} \sum_{k,\,l=1}^{n} \frac{[(k,k+1)(l,l+1)]^{-1}}{x_{k,0}^2 x_{k+1,0}^2 x_{l,0}^2 x_{l+1,0}^2} \int d^4\theta_0\, \theta_{0/k}^+[x_{k,0}, x_{k+1,0}]\theta_{0/k+1}^+\theta_{0/l}^+[x_{l,0}, x_{l+1,0}]\theta_{0/l+1}^+ \\
&\quad = \lim_{x_{i,i+1}^2 \to 0} \sum_{k,\,l=1}^{n} \left\{ 2\frac{(x_{k,0}\cdot x_{l+1,0})(x_{k+1,0}\cdot x_{l,0}) - (x_{k,0}\cdot x_{l,0})(x_{k+1,0}\cdot x_{l+1,0})}{x_{k,0}^2 x_{k+1,0}^2 x_{l,0}^2 x_{l+1,0}^2} \right. \\
&\qquad\qquad \left. + i\frac{\epsilon_{\mu\nu\lambda\rho} x_{k,0}^{\mu} x_{k+1,0}^{\nu} x_{l,0}^{\lambda} x_{l+1,0}^{\rho}}{x_{k,0}^2 x_{k+1,0}^2 x_{l,0}^2 x_{l+1,0}^2} \right\} .
\end{aligned} \tag{4.9}$$

The parity-odd terms in the last line deserve a special comment. They will subsequently be integrated over the insertion point x_0, according to (3.13). The resulting pseudo-scalar integral will depend on the four external points $x_k, x_{k+1}, x_l, x_{l+1}$, which is not sufficient to make a translation invariant pseudo-scalar. But in fact there is another reason why these terms are suppressed by the integral over the insertion point. By inspecting the component content of the inserted Lagrangian (3.12), we see that the pseudo-scalar terms are due to the presence of the topological term $iF\tilde{F}$. Such an insertion is a total space-time derivative with respect to the insertion point, so the integral must vanish. The underlying reason why the loop corrections to the correlation function G_n cannot contain parity odd terms is that the fields ϕ in the operators $\mathcal{O}$ can be treated as true scalars (see appendix A.5). We will come back to this point in section 6.

Further, replacing $2(x_{k,0}\cdot x_{l+1,0}) = x_{k,0}^2 + x_{l+1,0}^2 - x_{k,l+1}^2$, etc. in (4.9) and using the properties of the cyclic sum over k and l, we obtain

$$\lim_{x_{i,i+1}^2 \to 0} \int d^4\theta_0 \left(\sum_{k=1}^{n} i_{k,k+1}\right)^2 = \frac{1}{2} \lim_{x_{i,i+1}^2 \to 0} \sum_{k,\,l=1}^{n} \frac{x_{k,l+1}^2 x_{k+1,l}^2 - x_{kl}^2 x_{k+1,l+1}^2}{x_{k,0}^2 x_{k+1,0}^2 x_{l,0}^2 x_{l+1,0}^2} . \tag{4.10}$$

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Notice that the numerator vanishes when $k = l-1$, l or $l+1$.

Up to now, we have done the entire *tree-level* calculation, including the light-cone limit, in $D = 4$. The last step is to integrate over the insertion point, $\int d^D x_0$, with $D = 4 - 2\epsilon$ and $\epsilon < 0$, in order to regularize the divergences occurring when the integration point approaches any of the light-like segments $[x_i, x_{i+1}]$. We reiterate that this is not the natural UV regularization (for which $\epsilon > 0$), but it is what we call a dual infrared regularization. This yields

$$\lim_{x_{i,i+1}^2 \to 0} \frac{\partial}{\partial a} \left(G_n / G_n^{(0)} \right) \Big|_{a=0} = -\frac{i}{4\pi^2} \lim_{x_{i,i+1}^2 \to 0} \sum_{k,l=1}^{n} \int d^D x_0 \frac{x_{k,l+1}^2 x_{k+1,l}^2 - x_{kl}^2 x_{k+1,l+1}^2}{x_{k,0}^2 x_{k+1,0}^2 x_{l,0}^2 x_{l+1,0}^2} \,. \tag{4.11}$$

To make a comparison with the amplitudes, it is instructive to rewrite the one-loop integral in the right-hand side of (4.11) in terms of dual momenta, eqs. (3.17) and (3.18). Defining the $D-$dimensional loop momenta as $\ell = x_{0,k}$ we find that, for general k and l this *space-time* integral becomes the two-mass easy box *momentum* integral [41]

$$F(p, P, q, Q) = -\frac{i}{4\pi^2} \int \frac{d^D \ell \, (P^2 Q^2 - (p+P)^2 (q+Q)^2)}{\ell^2 (\ell+p)^2 (\ell+p+P)^2 (\ell-Q)^2} \,, \qquad (p^2 = q^2 = 0) \tag{4.12}$$

evaluated for $p = x_{k,k+1}$, $P = x_{k+1,l}$, $q = x_{l,l+1}$ and $Q = x_{l+1,k}$. When rewritten in terms of these functions, the right-hand side of (4.11) coincides with twice the one-loop n-gluon MHV amplitude (see eq. (4.19) in [41])

$$\lim_{x_{i,i+1}^2 \to 0} \frac{\partial}{\partial a} \left(G_n / G_n^{(0)} \right) \Big|_{a=0} = 2 A_n^{(1)} / A_n^{(0)} = 2 \sum F(p, P, q, Q) \,. \tag{4.13}$$

In conclusion, the one-loop $n-$point *correlation function* calculated by means of a Lagrangian insertion and in dual infrared dimensional regularization, reproduces the one-loop $n-$particle *amplitude* expressed in terms of momentum integrals.

5 Four-point correlation functions and four-gluon amplitudes to two loops

In this section we extend the one-loop duality from the previous section to two loops. We show that the logarithm of the correlation function of four half-BPS operators of weight two (bilinears), when put on the light cone using the dual IR regularization procedure, becomes identical with the logarithm of the square of the four-gluon scattering amplitude.

5.1 Four-point correlation functions of bilinear half-BPS operators

Let us consider the correlation function of four protected $\mathcal{N} = 2$ half-BPS complex operators (3.4)

$$G_4 = \langle \mathcal{O}(x_1, u_1) \tilde{\mathcal{O}}(x_2, u_2) \mathcal{O}(x_3, u_3) \tilde{\mathcal{O}}(x_4, u_4) \rangle \,. \tag{5.1}$$

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It has been computed up to two loops in [32, 33][7] (we summarize the computation in appendix A):[8]

$$
\begin{aligned}
G_4 = {} & \frac{N_c^2}{(2\pi)^8}\frac{(12)(23)(34)(41)}{x_{12}^2x_{23}^2x_{34}^2x_{41}^2} + \frac{N_c^2}{(2\pi)^8}\frac{2a\,\mathcal{R}}{x_{12}^2x_{23}^2x_{34}^2x_{41}^2}\; g_0(1,2,3,4) \\
& + \frac{N_c^2}{(2\pi)^8}\frac{2a^2\mathcal{R}}{x_{12}^2x_{23}^2x_{34}^2x_{41}^2}\Big[\frac{1}{2}(x_{12}^2x_{34}^2+x_{13}^2x_{24}^2+x_{14}^2x_{23}^2)(g_0(1,2,3,4))^2 \\
& \qquad + x_{12}^2 h_0(1,2,3;1,2,4) + x_{23}^2 h_0(1,2,3;2,3,4) + x_{34}^2 h_0(1,3,4;2,3,4) \\
& \qquad + x_{41}^2 h_0(1,2,4;1,3,4) + x_{13}^2 h_0(1,2,3;1,3,4) + x_{24}^2 h_0(1,2,4;2,3,4)\Big] + O(a^3)\,,
\end{aligned}
\tag{5.2}
$$

where the one- and two-loop four-dimensional integrals in *coordinate* space are defined by

$$
g_0(1,2,3,4) = c_0\int\frac{d^4x_0}{x_{10}^2x_{20}^2x_{30}^2x_{40}^2}\,, \tag{5.3}
$$

$$
h_0(1,2,3;1,2,4) = c_0^2\int\frac{d^4x_0\,d^4x_{0'}}{(x_{10}^2x_{20}^2x_{30}^2)x_{00'}^2(x_{10'}^2x_{20'}^2x_{40'}^2)}\,, \tag{5.4}
$$

with $c_0 = i/(2\pi^2)$. A characteristic feature of the all-order loop corrections is the presence of the universal harmonic-space-time polynomial prefactor

$$
\mathcal{R} = (12)^2(34)^2x_{14}^2x_{23}^2 + (14)^2(23)^2x_{12}^2x_{34}^2 + (12)(23)(34)(41)\left[x_{13}^2x_{24}^2 - x_{12}^2x_{34}^2 - x_{14}^2x_{23}^2\right]. \tag{5.5}
$$

This phenomenon was first revealed in [52] under the name "partial non-renormalization".

Note that if the external points are in generic positions with $x_{ij}^2 \neq 0$, the integrals above are well defined in $D = 4$ and are manifestly conformally covariant. This allows us to write them as functions of the conformal cross-ratios,

$$
\begin{aligned}
g_0(1,2,3,4) &= \frac{1}{x_{13}^2x_{24}^2}\Phi^{(1)}\left(\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2},\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}\right), \\
h_0(1,2,3;1,2,4) &= \frac{1}{(x_{12}^2)^2x_{34}^2}\Phi^{(2)}\left(\frac{x_{14}^2x_{23}^2}{x_{12}^2x_{34}^2},\frac{x_{13}^2x_{24}^2}{x_{12}^2x_{34}^2}\right),
\end{aligned}
\tag{5.6}
$$

having well-known expressions in terms of classical polylogs (see, e.g., [46, 47]). In what follows we wish to take the light-cone limit $x_{i,i+1}^2 \to 0$, which makes the integrals diverge. We regularize them, mimicking the gluon scattering amplitudes, by modifying the dimension of the integration measure to $D = 4-2\epsilon$ (with $\epsilon < 0$), thus giving up conformal invariance

$$
\begin{aligned}
g(1,2,3,4) &= c_\epsilon\int\frac{d^Dx_0}{x_{10}^2\,x_{20}^2\,x_{30}^2\,x_{40}^2}\,, \\
h(1,2,3;1,2,4) &= c_\epsilon^2\int\frac{d^Dx_0\,d^Dx_{0'}}{(x_{10}^2\,x_{20}^2\,x_{30}^2)\,x_{00'}^2\,(x_{10'}^2\,x_{20'}^2x_{40'}^2)}\,,
\end{aligned}
\tag{5.7}
$$

[7] The terminology of [32, 33] is based on the topology of the Feynman graphs, rather than on the perturbative order of the correlation function. Thus, the lowest order (Born level) contribution is called "one-loop", and the contributions of order $g^{2(\ell-1)}$ are called "ℓ-loop".

[8] The Born approximation in (5.2) has a connected and a disconnected sectors (see [32, 33]), with different color factors. Here we show only the connected contribution to the tree-level correlation function. Also, the color factors are given in the large N_c approximation.

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where the normalization factor $c_\epsilon = 2i/(2\pi)^{2-2\epsilon}$ is introduced to simplify the final expressions for the two-loop corrections.

5.2 Four-gluon amplitudes

We expect that the four-point correlation function (5.1) in the light-cone limit (2.6) is related to the four-gluon planar MHV amplitude. The latter has been computed to two loops in [20] and it has the following form:

$$A_4 = A_4^{(0)} \left[1 + aM^{(1)} + a^2 M^{(2)} + O(a^3)\right] , \tag{5.8}$$

where $A_4^{(0)}$ is the tree-level amplitude and the loop corrections $M^{(1)}$ and $M^{(2)}$ are given by

$$M^{(1)} = stI_4^{(1)}(s,t) , \qquad M^{(2)} = st\Big(sI_4^{(2)}(s,t) + tI_4^{(2)}(t,s)\Big) , \tag{5.9}$$

where $s = (p_1 + p_2)^2$ and $t = (p_2 + p_3)^2$ are the Mandelstam variables. All gluons are outgoing, so that their momenta satisfy the relations $\sum_1^4 p_i = 0$ and $p_i^2 = 0$.

The one- and two-loop scalar box momentum integrals $I_4^{(1)}$ and $I_4^{(2)}$ in (5.9) are given by

$$I_4^{(1)}(s,t) = c_\epsilon \int \frac{d^D k}{k^2(k-p_1)^2(k-p_1-p_2)^2(k+p_4)^2} , \tag{5.10}$$

$$I_4^{(2)}(s,t) = c_\epsilon^2 \int \frac{d^D k\, d^D l}{k^2(k-p_1)^2(k-p_1-p_2)^2(k+l)^2 l^2 (l-p_4)^2(l-p_3-p_4)^2} . \tag{5.11}$$

These massless integrals are infrared divergent, therefore they are regularized dimensionally, $D = 4 - 2\epsilon$ with $\epsilon < 0$. Switching from momenta to dual coordinates [17, 18], $k = x_1 - x_0$, $l = x_{0'} - x_1$ and $p_i = x_i - x_{i+1}$ (with $x_5 \equiv x_1$), we identify the pseudo-conformal integrals from the correlation function with those from the amplitude (see figure 4):

$$I_4^{(1)}(s,t) = g(1,2,3,4) , \qquad I_4^{(2)}(s,t) = h(1,2,3;1,3,4) , \qquad I_4^{(2)}(t,s) = h(1,2,4;2,3,4) . \tag{5.12}$$

Then, the two-loop expression for the amplitude (5.8) can be rewritten as

$$A_4/A_4^{(0)} = 1 + a\, x_{13}^2 x_{24}^2 g(1,2,3,4) + a^2\, x_{13}^2 x_{24}^2 \big[x_{13}^2\, h(1,2,3;1,3,4) + x_{24}^2\, h(1,2,4;2,3,4)\big] . \tag{5.13}$$

5.3 Duality correlator/amplitude

For the purpose of comparing correlation functions and amplitudes, we define the ratio of the correlation function and its Born-level expression:

$$\begin{aligned} G_4/G_4^{(0)} = 1 + \frac{2\,\mathcal{R}}{(12)(23)(34)(41)} \\ \times \Big\{ a\, g(1,2,3,4) + a^2 \Big[\frac{1}{2}(x_{12}^2 x_{34}^2 + x_{13}^2 x_{24}^2 + x_{14}^2 x_{23}^2)(g(1,2,3,4))^2 \\ + x_{12}^2 h(1,2,3;1,2,4) + x_{23}^2 h(1,2,3;2,3,4) + x_{34}^2 h(1,3,4;2,3,4) \end{aligned} \tag{5.14}$$

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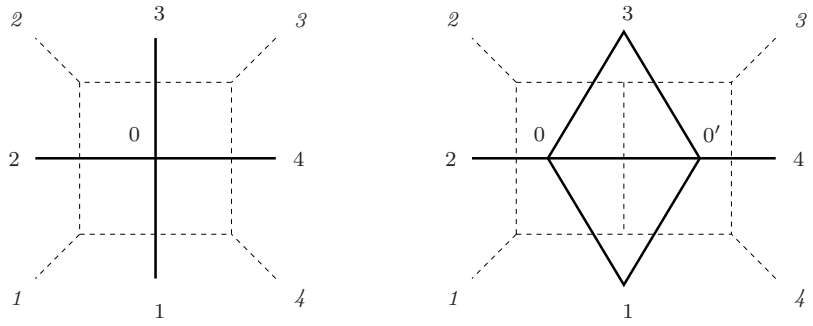

$g(1,2,3,4) \;\leftrightarrow\; I_4^{(1)}(s,t)$ $\qquad$ $h(1,2,3;1,3,4) \;\leftrightarrow\; I_4^{(2)}(s,t)$

Figure 4. One- and two-loop pseudo-conformal integrals contributing to the correlation function G_4, eq. (5.1), and to the amplitude A_4, eq. (5.8). The diagrams with solid lines depict Feynman integrals in $x-$space. The diagrams with dashed lines represent the same integral in the dual momentum space. The straight labels correspond to the points x_i, the slanted labels correspond to the momenta $p_i = x_i - x_{i+1}$.

$$+ x_{41}^2 h(1,2,4;1,3,4) + x_{13}^2 h(1,2,3;1,3,4) + x_{24}^2 h(1,2,4;2,3,4)\Big] + O(a^3)\Big\}\,.$$

Next, we wish to evaluate this ratio on the light cone, i.e. with

$$x_{12}^2 = x_{23}^2 = x_{34}^2 = x_{41}^2 = 0\,. \tag{5.15}$$

From (5.5) we see that the prefactor $\mathcal{R}/[(12)(23)(34)(41)]$ in (5.14) is reduced to $x_{13}^2 x_{24}^2$. Further, most of the two-loop $h-$integrals do not contribute to the right-hand side of (5.14) due to vanishing kinematic prefactors like x_{12}^2. The result is

$$\lim_{x_{i,i+1}^2\to 0} G_4/G_4^{(0)} = 1 + 2a\, x_{13}^2 x_{24}^2 g(1,2,3,4) + a^2\Big[\big(x_{13}^2 x_{24}^2 g(1,2,3,4)\big)^2 + 2x_{13}^2 x_{24}^2\Big(x_{13}^2 h(1,2,3;1,3,4) + x_{24}^2 h(1,2,4;2,3,4)\Big)\Big] + O(a^3)\,. \tag{5.16}$$

Comparing this result with the perturbative expansion of the amplitude, eq. (5.13), we obtain

$$\lim_{x_{i,i+1}^2\to 0} G_4/G_4^{(0)} = \Big(A_4/A_4^{(0)}\Big)^2 + O(a^3)\,, \tag{5.17}$$

confirming the general statement (3.19).

We recall that G_4 was defined in (5.1) as the correlation function of operators (2.3) bilinear in the scalar fields. The duality (5.17) can be extended to correlation functions of half-BPS operators $\mathcal{O} = \mathrm{Tr}(\phi^k)$ of arbitrary weight k, see appendix B.

6 Five-point correlation functions and five-gluon amplitudes to two loops

In this section we extend the previously found duality between dual-IR-regularized correlation functions and gluon MHV amplitudes to the case of five points/gluons. This is

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a rather non-trivial test, in view of the significantly more complicated integrals involved. Also, we explain why the duality should be formulated in terms of logs, rather than the correlator/amplitude themselves. The reason is in the parity-odd (pseudo-scalar sector) of the amplitude, which is reduced to $O(\epsilon)$ terms by taking the log. This is essential for the duality to work, because the correlation function must be a true scalar, as we argue in appendix A.5.

We consider the correlation function of $n = 5$ half-BPS operators of weight $k = 2$,

$$G_5 = \langle \mathcal{O}(1)\tilde{\mathcal{O}}(2)\mathcal{O}(3)\tilde{\mathcal{O}}(4)\hat{\mathcal{O}}(5)\rangle \,, \tag{6.1}$$

where the local scalar operators $\mathcal{O}$, $\tilde{\mathcal{O}}$ and $\hat{\mathcal{O}}$ are the bottom components of the hypermultiplet bilinears

$$O(i) = \mathrm{Tr}(q^+q^+)\,, \qquad \tilde{O}(i) = \mathrm{Tr}(\tilde{q}^+\tilde{q}^+)\,, \qquad \hat{O}(i) = 2\,\mathrm{Tr}(\tilde{q}^+q^+)\,. \tag{6.2}$$

Here the arguments of the operators $i = (x_i, u_i^+)$ denote the set of space-time and harmonic coordinates of the hypermultiplet scalar fields, $q^+(i)|_{\theta=0} = \phi^r(x_i)u_{ir}^+$ and $\tilde{q}^+(i)|_{\theta=0} = \bar{\phi}^r(x_i)u_{ir}^+$ (with $r = 1, 2$). We wish to examine the correlation function (6.1) in the limit

$$x_{i,i+1}^2 \equiv (x_i - x_{i+1})^2 \to 0\,, \qquad x_{i+5} \equiv x_i\,. \tag{6.3}$$

In this limit, the leading asymptotic behavior of the tree-level correlation function is given by the product of free scalar propagators

$$G_5^{(0)} = \frac{N_c^2}{(2\pi)^{10}} \frac{(12)(23)(34)(45)(51)}{x_{12}^2x_{23}^2x_{34}^2x_{45}^2x_{51}^2} + \dots\,, \tag{6.4}$$

where the ellipses denote subleading terms as $x_{i,i+1}^2 \to 0$. Notice that the obvious symmetry of the correlation function (6.1) under the exchange of operators, $1 \leftrightarrow 3$ and $2 \leftrightarrow 4$, is lost in the light-cone limit. At loop level, the correlation function (6.1) turns out to have the same leading singularity as the tree $G_5^{(0)}$. This suggests to study the following ratio in the light-cone limit (6.3),

$$F_G \equiv \lim_{x_{i,i+1}^2 \to 0} \ln\left(G_5/G_5^{(0)}\right) = aF_G^{(1)}(x_i) + a^2F_G^{(2)}(x_i) + O(a^3)\,, \tag{6.5}$$

where $a = g^2N_c/(8\pi^2)$ and $F_G^{(p)}(x_i)$ are scalar functions of x_i only. The rationale for considering the log of the ratio of correlation functions in the left-hand side of (6.5) is that, firstly, it does not receive a contribution at $O(a^0)$ and, secondly, as we will argue below, it has a much simpler form. Computing $F_G^{(1)}(x_i)$ and $F_G^{(2)}(x_i)$ we shall follow the same routine as before, that is, we shall expand $F_G^{(p)}(x_i)$ over a basis of one- and two-loop pseudo-conformal integrals in $D = 4$ dimensions, and then regularize them by modifying the integration measure at the Lagrangian insertion points to $D = 4 - 2\epsilon$ dimensions, $d^4x_0 \to d^{4-2\epsilon}x_0$, with $\epsilon < 0$.

In section 4 we have demonstrated that the one-loop correction in (6.5) is given by

$$F_G^{(1)}(x_i) = x_{13}^2x_{24}^2 g(1, 2, 3, 4) + (\text{cyclic})\,, \tag{6.6}$$

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where $g(1,2,3,4)$ is the one-loop pseudo-conformal "cross" integral in (6.8); (cyclic) means the four non-trivial cyclic permutations, $i \mapsto i+1$, of the points $\{1,2,3,4,5\}$. The two-loop correction $F_G^{(2)}$ is computed in appendix A.4, using the method of double Lagrangian insertions in harmonic superspace. The result is the following expression:

$$\begin{aligned} F_G^{(2)} =& x_{13}^2\, x_{24}^2\, \left[2\, x_{13}^2\, h(1,2,3;1,3,4) + 2\, x_{24}^2\, h(1,2,4;2,4,3) - x_{14}^2\, h(1,2,4;1,3,4)\right] \\ &+ x_{13}^2\, x_{14}^2\, x_{25}^2\, [2\, h(1,2,5;1,3,4) \;-\; h(1,2,3;1,4,5) \;-\; h(1,2,4;1,3,5)] \\ &+ x_{24}^2\, x_{35}^2\, \left[2\, x_{25}^2\, p(1;2,5;3,4) \;-\; x_{24}^2\, p(1;2,4;3,5) \;-\; x_{35}^2\, p(1;3,5;2,4)\right] \\ &- \frac{1}{2}[x_{13}^2 x_{24}^2 g(1,2,3,4)]^2 - x_{13}^2\, x_{24}^2\, g(1,2,3,4)\; x_{13}^2\, x_{25}^2\, g(1,2,3,5) + (\text{cyclic})\,, \end{aligned} \tag{6.7}$$

where the cyclic permutations act on the entire sum of terms. Here we use the dimensionally regularized integrals (5.7) together with

$$p(1;2,3;4,5) = c_\epsilon^2 \int \frac{d^D x_0\, d^D x_{0'}\; x_{10'}^2}{(x_{10}^2\, x_{20}^2\, x_{30}^2)\, x_{00'}^2\, (x_{20'}^2\, x_{30'}^2\, x_{40'}^2\, x_{50'}^2)}\,, \tag{6.8}$$

where $D = 4 - 2\epsilon$ with $\epsilon < 0$ and the normalization factor $c_\epsilon = 2i/(2\pi)^{2-2\epsilon}$ is introduced to simplify the final expressions for the two-loop corrections. Switching to dual momenta $p_i = x_i - x_{i+1}$, we find that the $g-$, $h-$ and $p-$integrals correspond to the scalar box, double-box and penta-box momentum integrals of ref. [42], respectively.

6.1 Five-gluon MHV amplitude

We expect that in the light-cone limit (6.3) the five-point correlation function (6.1) is related to the five-gluon planar MHV amplitude. The latter has been computed to two loops in [42] and it has the form

$$A_5 = A_5^{(0)} \left[1 + a M^{(1)} + a^2 M^{(2)} + O(a^3)\right]\,, \tag{6.9}$$

where $A_5^{(0)}$ is the tree five-gluon MHV amplitude. Here the one- and two-loop corrections are given by the following expressions

$$\begin{aligned} M^{(1)} &= \frac{1}{2} \sum_{\text{cyclic}} s_{12} s_{23} I_a^{(1)} + M_1^{(\text{odd})}\,, \\ M^{(2)} &= \frac{1}{2} \sum_{\text{cyclic}} \left[s_{12}^2 s_{23} I_a^{(2)} + s_{12}^2 s_{15} I_b^{(2)} + s_{12} s_{34} s_{45} I_c^{(2)}\right] + M_2^{(\text{odd})}\,, \end{aligned} \tag{6.10}$$

where $s_{ij} = (p_i + p_j)^2$ is the invariant mass of the gluons with labels i and j. All gluons are considered outgoing, so that their momenta satisfy the relations $\sum_1^5 p_i = 0$ and $p_i^2 = 0$.

Further, $I_a^{(1)}$ is a one-loop and $I_a^{(2)}$, $I_b^{(2)}$, $I_c^{(2)}$ are two-loop planar scalar integrals depending on the gluon momenta, eqs. (5.10) and (5.11). Switching to dual coordinates, $p_i = x_i - x_{i+1}$, these integrals can be expressed in terms of the basis integrals (6.8):

$$\begin{aligned} I_a^{(1)} &= g(1,2,3,4)\,, & I_a^{(2)} &= h(1,3,2;1,3,4)\,, \\ I_b^{(2)} &= h(1,3,2;1,3,5)\,, & I_c^{(2)} &= p(2;1,3;4,5)\,. \end{aligned} \tag{6.11}$$

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Finally, $M^{(1)}_{\rm odd}$ and $M^{(2)}_{\rm odd}$ in (6.10) stand for the parity-odd contributions to the five-gluon MHV amplitude, proportional to the pseudo-scalar $\epsilon(p_1,p_2,p_3,p_4)$. The one-loop parity-odd contribution $M^{(1)}_{\rm odd}$ is of order $O(\epsilon)$, whereas $M^{(2)}_{\rm odd}$ has a simple pole $1/\epsilon$. The residue at this pole is proportional to the product of the one-loop parity-even and parity-odd parts. As a consequence, the parity-odd contribution can be significantly simplified by considering the logarithm of the ratio $A_5/A_5^{(0)}$, as shown in [42]. Namely, we introduce the following ratio function

$$F_A \equiv \ln\left(A_5/A_5^{(0)}\right)^2 = aF_A^{(1)} + a^2F_A^{(2)} + O(a^3)\,, \tag{6.12}$$
$$F_A^{(1)} = 2M^{(1)}\,, \qquad F_A^{(2)} = 2M^{(2)} - (M^{(1)})^2\,.$$

The main advantage of (6.12) is that, unlike the amplitude itself, the parity-odd contributions to $F_A^{(1)}$ and $F_A^{(2)}$ vanish as $\epsilon \to 0$. Replacing $M^{(1)}$ and $M^{(2)}$ in (6.12) by their explicit expressions (6.10) and going to dual coordinates $p_i = x_i - x_{i+1}$, we obtain (up to terms vanishing as $\epsilon \to 0$)

$$F_A^{(1)} = x_{13}^2x_{24}^2 g(1,2,3,4) + (\text{cyclic})\,, \tag{6.13}$$
$$\begin{aligned} F_A^{(2)} =& x_{13}^4x_{24}^2 h(1,2,3;1,3,4) + x_{24}^4x_{13}^2 h(1,2,4;2,3,4) + x_{24}^2x_{25}^2x_{35}^2 p(1;2,5;3,4) \\ &- \frac{1}{4}[x_{13}^2x_{24}^2 g(1,2,3,4)]^2 - \frac{1}{2}x_{13}^2x_{24}^2 g(1,2,3,4)x_{24}^2x_{35}^2 g(2,3,4,5) \\ &- \frac{1}{2}x_{13}^2x_{24}^2 g(1,2,3,4)x_{35}^2x_{14}^2 g(3,4,5,1) + (\text{cyclic})\,. \end{aligned} \tag{6.14}$$

Here g, h and p stand for the dimensionally regularized integrals defined in (6.8).

Although the explicit expressions for individual two-loop integrals in (6.14) are quite complicated, their sum $F_A^{(2)}$ can be found in a closed form thanks to the remarkable duality between MHV amplitudes and light-like Wilson loops [4–7]. This duality allows us to formulate the dual conformal Ward identity for F_A to all loops [9]. For five points it has a unique solution (up to an additive constant) which coincides with the BDS ansatz [20].

6.2 Duality correlator/amplitude

Assuming the duality relation (3.19), the correlation function on the light cone should be related to the scattering amplitude

$$\lim_{x_{i,i+1}^2\to 0} \ln\left(G_5/G_5^{(0)}\right) = \ln\left(A_5/A_5^{(0)}\right)^2 + O(\epsilon)\,. \tag{6.15}$$

In terms of the functions F_G and F_A introduced in (6.5) and (6.12), this relation implies that up to terms vanishing for $\epsilon \to 0$

$$F_A^{(1)} = F_G^{(1)}\,, \qquad F_A^{(2)} = F_G^{(2)}\,, \quad \ldots \tag{6.16}$$

The first of these relations follows immediately from the explicit one-loop expressions (6.13) and (6.6). At two loops, $F_A^{(2)}$ and $F_G^{(2)}$ are given by two seemingly different expressions, eqs. (6.14) and (6.7), respectively. Then, for the duality relation (6.16) to hold, the two-loop integrals in (6.14) and (6.7) should satisfy a very non-trivial identity.

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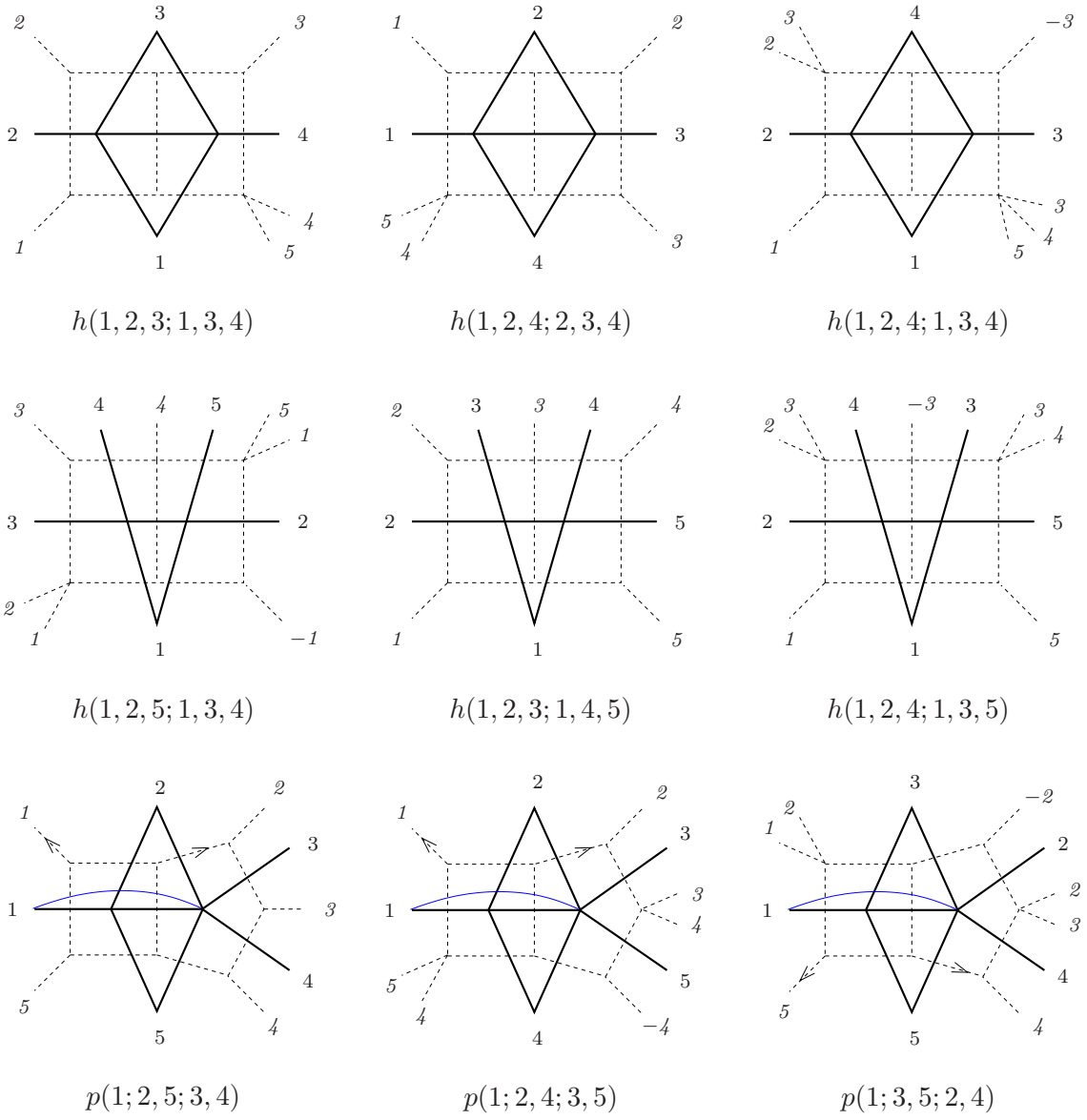

Figure 5. Two-loop pseudo-conformal integrals of different topologies contributing to the correlation function G_5, eq. (6.7), and to the amplitude A_5, eq. (6.14). The diagrams with solid lines depict Feynman integrals in $x-$space. The diagrams with dashed lines represent the same integral in the (dual) momentum space $p_i = x_i - x_{i+1}$. In the latter case, $(-k)$ stands for the particle with momentum $(-p_k)$. Thin solid lines denote numerators in the $x-$integral. In momentum space, this numerator is given by the squared sum of the momenta flowing through the arrowed dashed lines.

To begin with, let us examine the topology of the Feynman integrals in the two-loop expression for $F_A^{(2)}$ and $F_G^{(2)}$. By construction, the planar MHV amplitude A_5 receives contributions from planar Feynman diagrams only. However, taking the log of the two-loop amplitude (6.9), we find that $F_A^{(2)}$ contains an admixture of products of one-loop integrals in (6.14). For the correlation function the situation is different. The diagrammatic representation of $F_G^{(2)}$ is shown in figure 5. We observe that the expression for $F_G^{(2)}$ involves

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two-loop integrals of the same type as those contributing to $F_A^{(2)}$, eq. (6.14). Closer examination shows, however, that the external points in these integrals, $x_1, \ldots, x_5$, are not (anti)clock-wise ordered and, therefore, these integrals do not contribute to the planar amplitude $F_A^{(2)}$. Still, they contribute to the two-loop planar correlation function. This is due to the different realizations of the planarity condition for correlation functions and scattering amplitudes. For the latter, the planar diagrams have the topology of a disk, while for the former they have the topology of a sphere (see figure 7 in appendix A.3.2 for an example). Despite of this, the expected duality relation $F_A^{(2)} = F_G^{(2)}$ suggests that the two sums of two-loop integrals of different topologies, eqs. (6.7) and (6.14), are equal to each other.

We can apply the relation $p_i = x_i - x_{i+1}$ to rewrite the Feynman integrals contributing to $F_G^{(2)}$ as conventional momentum integrals, as shown in figure 5. The absence of cyclic ordering of the external points x_i leads to an unusual feature of the resulting momentum integrals: They involve seven external legs including one pair of legs with opposite momenta, p_i and $(-p_i)$.

6.2.1 Soft limit

There exists a simple way to test the duality relations (6.15) and (6.16) by examining them in the special limit $x_4 \to x_5$. For the amplitude, this corresponds, e.g., to the soft limit $p_4 = x_{45} \to 0$ when one of the external gluon momenta vanishes. In this limit, the dimensionally regularized five-gluon MHV amplitude A_5 reduces to the four-gluon amplitude A_4 in such a way that the ratio function (6.12) takes the form

$$\lim_{x_4 \to x_5} F_A = \ln \left(A_4 / A_4^{(0)} \right)^2 . \tag{6.17}$$

For the correlation function (6.1), the limit $x_4 \to x_5$ corresponds to its short distance asymptotic behavior. In this limit, we apply the operator product expansion $\tilde{\mathcal{O}}(4)\hat{\mathcal{O}}(5) \sim \langle \bar{\phi}^+(4)\phi^+(5)\rangle \tilde{\mathcal{O}}(4) + \ldots$, where $\langle \bar{\phi}\phi \rangle$ is a free scalar field propagator in $D = 4 - 2\epsilon$ dimensions, and the dots denote subleading terms. In this way, we find from (6.5) that the ratio of the 5-point correlation functions reduces to the ratio of the 4-point ones $G(1,2,3,4) = \langle \mathcal{O}(1)\tilde{\mathcal{O}}(2)\mathcal{O}(3)\tilde{\mathcal{O}}(4)\rangle$,

$$\lim_{x_4 \to x_5} F_G = \lim_{x_{i,i+1}^2 \to 0} \ln \left(\frac{G(1,2,3,4)}{G^{(0)}(1,2,3,4)} \right) , \tag{6.18}$$

where the light-cone limit in the right-hand side corresponds to $x_{12}^2, x_{23}^2, x_{34}^2, x_{41}^2 \to 0$.

Let us verify (6.17) using the expression for the two-loop ratio function (6.12). We recall that, by definition, the adjacent points in dual space are light-like separated, $x_{i,i+1}^2 = 0$. In the soft limit $x_4 \to x_5$ we find the additional relations $x_{35}^2 \to 0$ and $x_{41}^2 \to 0$. Notice that the dimensionally regularized $g-$, $h-$ and $p-$integrals, eq. (6.8), remain finite as $x_4 \to x_5$. The expression for F_A simplifies because some of these integrals are multiplied by x_{35}^2 and x_{41}^2, so they do not contribute in the soft limit. Thus, the pentabox integral p drops out:

$$\lim_{x_4 \to x_5} F_A^{(2)} = 2\, x_{13}^4 x_{24}^2 h(1,3,2;1,3,4) + 2\, x_{24}^4 x_{13}^2 h(2,4,1;2,4,3) - [x_{13}^2 x_{24}^2 g(1,2,3,4)]^2 . \tag{6.19}$$

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Using eq. (5.13), we verify that the expression in the right-hand side of (6.19) coincides with the two-loop correction to the ratio of four-gluon amplitudes $2\ln(A_4/A_4^{(0)})$, in agreement with (6.17).

In a similar manner, we examine the ratio function $F_G^{(2)}$ and simplify the expression (6.7) in the limit $x_4 \to x_5$. We find that, unlike the case of $F_A^{(2)}$, the pentabox integrals have a non-vanishing contribution which can be expressed in terms of $h-$integrals. For instance,

$$-x_{24}^2\, x_{35}^2 x_{35}^2\, p(1;3,5;2,4) + (\text{cyclic}) \overset{x_4\to x_5}{\to} -b(1,2,3,4) - x_{13}^4 x_{24}^2 h(1,3,2;1,3,4)\,, \quad (6.20)$$

where the notation was introduced for the 'bad' integral $b(1,2,3,4) = x_{24}^4 x_{13}^2 h(2,3,1;2,4,4)$, which does not have an interpretation in terms of two-loop four-particle planar momentum integrals. The same 'bad' integral comes from $h(1,2,5;1,3,4)$ and $p(1;2,4;3,5)$ and their cyclic images. We find that $b(1,2,3,4)$ cancels in the sum of all terms in the right-hand side of (6.7) as $x_4 \to x_5$, leading to

$$\lim_{x_4\to x_5} F_G^{(2)} = 2\, x_{13}^4 x_{24}^2 h(1,3,2;1,3,4) + 2\, x_{24}^4 x_{13}^2 h(2,4,1;2,4,3) - [x_{13}^2 x_{24}^2 g(1,2,3,4)]^2\,. \quad (6.21)$$

Again, we verify that the expression in the right-hand side coincides with the ratio of four-point correlation functions (6.18).

Comparing (6.19) and (6.21), we conclude that the two-loop corrections to F_A and F_G coincide in the soft limit $x_4 \to x_5$,

$$\lim_{x_4\to x_5} \left(F_A^{(2)} - F_G^{(2)}\right) = 0\,. \quad (6.22)$$

In the next subsection we argue that for arbitrary x_i the difference $F_A^{(2)} - F_G^{(2)}$ is a constant. Together with (6.22) this immediately implies that $F_A^{(2)} = F_G^{(2)}$, as announced in (6.16).

6.2.2 Conformal symmetry and proof of the integral identity

As was already mentioned, the duality relation $F_A^{(2)} = F_G^{(2)}$ implies an identity between two-loop integrals of various topologies. Evaluating the difference $F_A^{(2)} - F_G^{(2)}$ with the help of (6.7) and (6.14) and equating it to zero, we formulate the identity that we expect to find as follows:

$$\begin{aligned}
0 = {} & x_{13}^2\, x_{24}^2\, \left[x_{13}^2\, h(1,2,3;1,3,4) + x_{24}^2\, h(1,2,4;2,3,4) - x_{14}^2\, h(1,2,4;1,3,4)\right] \\
& + x_{13}^2\, x_{14}^2\, x_{25}^2\, [2\, h(1,2,5;1,3,4) - h(1,2,3;1,4,5) - h(1,2,4;1,3,5)] \\
& + x_{24}^2\, x_{35}^2\, \left[x_{25}^2\, p(1;2,5;3,4) - x_{24}^2\, p(1;2,4;3,5) - x_{35}^2\, p(1;3,5;2,4)\right] \\
& - \frac{1}{4}[x_{13}^2 x_{24}^2 g(1,2,3,4)]^2 + \frac{1}{2} x_{13}^2 x_{24}^2 g(1,2,3,4) x_{24}^2 x_{35}^2 g(2,3,4,5) \\
& - \frac{1}{2} x_{13}^2 x_{24}^2 g(1,2,3,4) x_{35}^2 x_{14}^2 g(1,3,4,5) + (\text{cyclic})\,.
\end{aligned} \quad (6.23)$$

In the previous subsection we showed that this relation holds in the soft limit.

The proof of (6.23) goes as follows. The Feynman integrals in (6.23) are regularized dimensionally with $D = 4-2\epsilon$. We will first show that the sum of Feynman integrals in the

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right-hand side of (6.23) is finite as $\epsilon \to 0$. This will allow us to remove the regulator, i.e. to restore the $D = 4$ integration measure. Since all integrals we are dealing with are pseudo-conformal, the expression in the right-hand side of (6.23) will thus become an exactly conformally invariant function of $x_1, \ldots, x_5$. Since one cannot construct conformal cross-ratios from the five external points with light-like separated neighbors $x^2_{i,i+1} = 0$, conformal invariance will imply that the right-hand side of (6.23) is a constant. We have already seen that the right-hand side of (6.23) vanishes in the soft limit, so the constant must be zero.

The direct way to prove finiteness is to evaluate the pole part of (6.23) by the Mellin-Barnes (MB) method (see, e.g., [53]) using the package [54] for the ϵ-expansion and evaluation of the representations. At $O(1/\epsilon^4)$ through $O(1/\epsilon^2)$ the program finds only one-parameter integrals which we have analytically evaluated. At $1/\epsilon^4$ and $1/\epsilon^3$ one has to show the cancelation of rational numbers and simple logarithms, respectively, which was in either case immediate. At $1/\epsilon^2$ we needed Landen's identity on dilogarithms, but once again the proof was ultimately simple. With our MB representations, the highest integrals contributing to the simple pole in ϵ were of dimension four (from the pentabox). The precision of the numerical evaluation used by the package [54] was therefore good: For all kinematic points in our (rather general) sample set we obtained values like $0.00(2)/\epsilon$. Within the given accuracy the sum of integrals is seen to be finite.[9] As we have just explained, the finiteness of the integral implies that it equals zero since a non-vanishing constant part is ruled out by the soft limit.

Alternatively, we can show finiteness analytically by combining the sum of integrals in (6.23) into a single integral of the form

$$\text{r.h.s. of eq.(6.23)} = \int \frac{d^D x_0 d^D x_{0'}\, P(x_0, x_{0'}; x_i)}{(x^2_{10} \ldots x^2_{50})(x^2_{10'} \ldots x^2_{50'}) x^2_{00'}}\,, \tag{6.24}$$

where, by construction, the polynomial $P(x_0, x_{0'}, x_i)$ is invariant under cyclic shifts of the external points x_i (with $i = 1, \ldots, 5$) and under the exchange of the integration points $x_0 \leftrightarrow x_{0'}$. Each integral in (6.23) gives a contribution to $P(x_0, x_{0'}, x_i)$ in the form of a product of seven distances x^2_{ij} with various choices of indices i and j. For instance, the third $h-$integral in the first line of (6.23) produces the contribution $(-x^2_{13}x^2_{14}x^2_{24}x^2_{30}x^2_{50}x^2_{20'}x^2_{50'})$, while the contribution of the cross-product of one-loop integrals in the last line of (6.23) looks as $(-\frac{1}{2}x^2_{13}x^2_{24}x^2_{14}x^2_{35}x^2_{50}x^2_{20'}x^2_{00'})$. To save space, we do not present the explicit expression for $P(x_0, x_{0'}; x_i)$. Due to the pseudo-conformal property of the integrals in (6.23), the polynomial $P(x_0, x_{0'}, x_i)$ is covariant under conformal transformations with weight (-4)[10] at points x_0 and $x_{0'}$. Given the weight $(+8)$ of the denominator, and if the integration measures can be made four-dimensional, the integral will be conformally covariant, as stated above.

Let us identify the potential divergences of the integral (6.24). They could only come from the part of the phase space of x_0 and $x_{0'}$, in which some of the propagators in the denominator of (6.24) vanish simultaneously. Notice that upon the change of variables,

[9] An analytical proof of the absence of the simple pole, too, should be feasible given the relatively low dimensionality of the MB-integrals.

[10] We use the standard convention that the scalar propagator has weight $(+1)$.

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$p_1 = x_{12}, \ldots, p_5 = x_{51}$ (with $p_i^2 = 0$) supplemented with $k = x_{10}$ and $k' = x_{10'}$, the integral (6.24) can be rewritten as a conventional two-loop momentum-space Feynman integral with k and k' being the loop momenta and $p_1, \ldots, p_5$ defining the external leg momenta. It is well known that infrared divergences originate from integration over the loop momenta collinear to the light-like momenta of the external legs. For instance, suppose that k^μ is collinear either to p_1^μ and to p_2^μ. To analyze the divergences, it is convenient to employ the so-called Sudakov decomposition of the loop momentum,

$$k^\mu \equiv x_{10}^\mu = \alpha p_1^\mu + \beta p_2^\mu + k_\perp^\mu \,, \tag{6.25}$$

where α, β are scalar variables and $k_\perp^\mu$ are two-dimensional transverse momenta, $(k_\perp p_1) = (k_\perp p_2) = 0$. In terms of Sudakov's variables, the x_0-integral in (6.24) takes the form

$$\int \frac{d^4x_0}{x_{01}^2 x_{20}^2 x_{30}^2} \big[\ldots\big] = \int \frac{x_{13}^2 \, d\alpha d\beta d^2k_\perp \big[\ldots\big]}{(x_{13}^2\alpha\beta - k_\perp^2)(x_{13}^2(1-\alpha)\beta + k_\perp^2)(x_{13}^2\alpha(1-\beta) + k_\perp^2)} \,, \tag{6.26}$$

where $[\ldots]$ denotes the remaining terms in the right-hand side of (6.24).

It is easy to see that the integral (6.26) develops logarithmic divergences originating from the integration region

$$x_{13}^2\alpha\beta - k_\perp^2 = O(\rho^2) \,, \qquad k_\perp = O(\rho) \tag{6.27}$$

with $\rho \to 0$. Depending on the hierarchy between α and β we can distinguish three subregions: For $\alpha, \beta = O(\rho)$ we have $k^\mu = O(\rho)$, for $\alpha = O(\rho^0)$, $\beta = O(\rho^2)$ we have $k^\mu = \alpha p_1^\mu + O(\rho)$ and, finally, for $\alpha = O(\rho^2)$, $\beta = O(\rho^0)$ we have $k^\mu = \beta p_2^\mu + O(\rho)$. In terms of the dual $x-$variables, this corresponds to the limit where the integration point x_0 approaches either the external point, $x_0 \to x_1$, or one of the light-like segments $[x_1, x_2]$ and $[x_2, x_3]$, respectively. Due to the cyclic symmetry of the integral (6.24), divergences are also produced when x_0 approaches the other cusp points x_i and the light-like segments $[x_i, x_{i+1}]$. The same analysis applies to the integral with respect to $x_{0'}$.

Then, for the integral (6.26) to be finite, the expression inside $[\ldots]$ should be finite and, in addition, it should vanish in the region (6.27). This leads in its turn to the condition for the polynomial $P(x_0, x_{0'}, x_i)$ in the right-hand side of (6.24) to vanish sufficiently fast when x_0 and/or $x_{0'}$ approach one the potentially dangerous regions explained above. We return to (6.24) and verify that the polynomial $P(x_0, x_{0'}; x_i)$ indeed satisfies this condition. Thus, the integral (6.24) remains finite as $\epsilon \to 0$. Together with conformal invariance this immediately leads to the identity (6.23).

7 Conclusions

The main result of this paper is the observation that computing the loop corrections to the correlation functions of protected operators by the Lagrangian insertion method, and taking the light-cone limit, gives us the *integrands* of the MHV gluon scattering amplitudes in the dual momentum space. These integrands can be evaluated directly in four dimensions since they are finite and explicitly (dual) conformally covariant. The divergences appear

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in the integrals over the insertions points for the correlation function (UV divergences) or, equivalently, over the loop momenta for the amplitude (IR divergences). In order to achieve an exact matching of the two objects, we need to dimensionally regularize the theory, not as one would do for a correlation function computation in coordinate space, but in the way that is natural for the dual theory in momentum space. We call this dual infrared regularization.

At present we have no understanding of the field-theory mechanism responsible for this surprising relation between two seemingly very different objects. In fact, the surprise is not total, since we already know that MHV amplitudes are dual to light-like Wilson loops. The latter, as shown in [39], are intimately related to correlation functions in the light-cone limit.

Further tests are needed to confirm our conjecture. In particular, a crucial test will be the calculation of the six-point correlation function at two loops and its comparison to the six-gluon amplitude. This is similar to the test of the Wilson loop/amplitude duality at six points and two loops [10, 11]. Before it, one could suspect that the matching of the four- and five-point objects was simply due to (dual) conformal symmetry, which fixes their form completely. As we saw in our five-point two-loop correlation function computation in section 6, conformal symmetry plays an important role here too. However, at six points it will not be sufficient to explain the matching, if it is confirmed.

The Lagrangian insertion procedure described here can give us a simple way to compute the integrands that determine the higher-loop corrections to amplitudes. We would like to emphasize that our one- and two-loop correlation function computations involve a handful of Feynman graphs. This is in striking contrast to the huge number of graphs needed, if one would attempt to do a straightforward amplitude calculation (not using unitarity methods).

The fact that the integrand of the amplitude can be obtained from a tree-level computation raises the hope that one might be able to profit from some hidden enhanced symmetry of the tree-level correlation function. This is motivated by the analogy with the tree-level scattering superamplitudes, which are known to have a dynamical symmetry called dual conformal symmetry [55–57]. If something similar exists also for the correlation functions, it might eventually lead us to the integrability of the loop amplitudes.

A related issue is the observation that the correlation functions we have discussed enjoy the full superconformal symmetry of the $\mathcal{N} = 4$ SYM theory. In this paper we have explored the lowest scalar components of the supersymmetric correlation functions, and we have shown that they match the gluon MHV amplitudes. The question arises if the correlation function defined in superspace has some relation to the superamplitude, and thus possibly to non-MHV amplitudes.

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A Lagrangian insertion procedure in $\mathcal{N} = 2$ harmonic superspace

In this appendix we give a very brief overview of harmonic superspace, in particular of the Feynman rules we need here. More details can be found in [48, 49, 58, 59]. We then summarize the Lagrangian insertion procedure, developed in [32, 33, 60] for the case of four-point correlation functions. Finally, we explain how to adapt this procedure to five (and more) points.

A.1 $\mathcal{N} = 4$ SYM in $\mathcal{N} = 2$ terms

The basic ingredients of the $\mathcal{N} = 4$ SYM theory in $\mathcal{N} = 2$ terms are the hypermultiplet and the super-Yang-Mills (SYM, or vector) multiplet. The $\mathcal{N} = 4$ SYM action consists of two terms:

$$S_{\text{N=4 SYM}} = S_{\text{N=2 SYM}} + S_{\text{HM}} \,. \tag{A.1}$$

Below we give a short description of each multiplet and its action.

A.1.1 $\mathcal{N} = 2$ hypermultiplet in harmonic superspace

The $\mathcal{N} = 2$ massless matter (or hyper)multiplet consists of an R-symmetry SU(2) doublet of complex scalars $\phi^i(x)$ (with $i = 1, 2$) and of two Majorana spinors and SU(2) sunglets $\psi_\alpha(x)$, $\bar{\psi}_{\dot{\alpha}}(x)$ and $\kappa_\alpha(x)$, $\bar{\kappa}_{\dot{\alpha}}(x)$. Their supersymmetry transformations close only on shell. Going off shell requires the introduction of an infinite set of auxiliary fields [48]. This is achieved by extending the space-time by two extra compact dimensions in the form of a sphere S^2. The latter is described in terms of *harmonic variables* $u^{\pm i}$ which form a matrix of SU(2),

$$\| \, u \, \| \in SU(2) : \qquad u^{+i} u^-_i = 1 \,, \qquad \overline{u^{+i}} = u^-_i \equiv \epsilon_{ij} u^{-j} \,, \qquad \epsilon_{12} = -\epsilon^{12} = 1 \,, \tag{A.2}$$

and parametrise the sphere $S^2 \sim \text{SU(2)}/\text{U(1)}$. A harmonic function $f^{(q)}(u^\pm)$ of U(1) charge q is a function of $u^{\pm i}$ invariant under the action of the group SU(2) (which rotates the index i of $u^{\pm i}$) and homogeneous of degree q under the action of the group U(1) (which rotates the index $\pm$ of $u^{\pm i}$). Such functions have infinite harmonic expansions on S^2 whose coefficients are SU(2) tensors (multispinors).

In this framework the hypermultiplet is described by a harmonic superfield $q^+(x, \theta, \bar{\theta}, u)$ of U(1) charge +1 satisfying the Grassmann (or G-)analyticity constraints

$$D^+_\alpha q^+ = \bar{D}^+_{\dot{\alpha}} q^+ = 0 \,, \tag{A.3}$$

where

$$D^+_\alpha = D^i_\alpha u^+_i \,, \qquad \bar{D}^+_{\dot{\alpha}} = \bar{D}^i_{\dot{\alpha}} u^+_i \tag{A.4}$$

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and D^i_α, $\bar{D}^i_{\dot\alpha}$ are the usual supersymmetric spinor derivatives. These constraints can be solved explicitly in the G-analytic basis in superspace

$$x^{\alpha\dot\alpha}_A = x^{\alpha\dot\alpha} - 4i\theta^{\alpha(i}\bar\theta^{\dot\alpha j)}u^+_i u^-_j \ , \quad \theta^\pm_{\alpha,\dot\alpha} = u^\pm_i \theta^i_{\alpha,\dot\alpha} \tag{A.5}$$

where $x^{\alpha\dot\alpha} = x^\mu \sigma^{\alpha\dot\alpha}_\mu$ and (ij) means weighted symmetrization. In this basis q^+ becomes a function of $\theta^+, \bar\theta^+$ only, i.e., a G-analytic superfield $q^+(x_A, \theta^+, \bar\theta^+, u)$.

As mentioned earlier, the $\mathcal{N} = 2$ supermultiplet can exist off shell because an infinite number of auxiliary fields (coming from the harmonic expansion on S^2) are present. On shell these auxiliary fields are eliminated by the harmonic (or H-)analyticity condition (equation of motion)

$$D^{++}q^+ = 0 \ . \tag{A.6}$$

Here D^{++} is the harmonic derivative on S^2 (the raising operator of the group SU(2) realized on the U(1) charges, $D^{++}u^+ = 0$, $D^{++}u^- = u^+$). In the G-analytic basis (A.5) it becomes a supercovariant operator involving space-time derivatives:

$$D^{++} = u^{+i}\frac{\partial}{\partial u^{-i}} - 4i\theta^{+\alpha}\bar\theta^{+\dot\alpha}\frac{\partial}{\partial x^{\alpha\dot\alpha}_A} \ . \tag{A.7}$$

It is then easy to show that the free on-shell hypermultiplet becomes an "ultrashort" superfield:

$$q^+(x_A, \theta^+, \bar\theta^+, u) = \phi^i(x_A)u^+_i + \theta^{+\alpha}\psi_\alpha(x_A) + \bar\theta^+_{\dot\alpha}\bar\kappa^{\dot\alpha}(x_A) + 4i\theta^+\sigma^\mu\bar\theta^+\partial_\mu\phi^i(x_A)u^-_i \tag{A.8}$$

where the physical scalars ϕ^i and spinors ψ_α, $\bar\kappa^{\dot\alpha}$ satisfy their massless field equations $\Box\phi^i(x) = \not\partial\psi = \bar{\not\partial}\kappa = 0$.

The equation of motion (A.6) can be derived from an action given by an integral over the G-analytic superspace:

$$S_{\rm HM} = -2\int du d^4x_A d^2\theta^+ d^2\bar\theta^+ \ {\rm Tr}\left(\tilde q^+ D^{++} q^+\right) \ . \tag{A.9}$$

Here $\tilde q^+(x_A, \theta^+, \bar\theta^+, u)$ is the conjugate of q^+. The conjugation $\tilde{}$ combines usual complex conjugation with the antipodal map on S^2 in a way to preserve G-analyticity. This action is real (with respect to the $\tilde{}$ conjugation) which can be seen by integrating D^{++} by parts. In this sense the action (A.9) resembles the Dirac action for fermions, although the superfield q^+ is bosonic.

A.1.2 $\mathcal{N} = 2$ SYM multiplet in harmonic superspace

The $\mathcal{N} = 2$ SYM gauge potential is introduced by covariantizing the action (A.9) with respect to a Yang-Mills group with G-analytic parameters $\lambda(x_A, \theta^+, \bar\theta^+, u)$. To this end one replaces the harmonic derivative in (A.9) by the following covariant one:

$$D^{++} \to D^{++} + igV^{++}(x_A, \theta^+, \bar\theta^+, u) \, , \tag{A.10}$$

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where g is the gauge coupling constant. The gauge potential is described by a real ($\widetilde{V}^{++} = V^{++}$) G-analytic superfield of charge +2 (equal to the charge of D^{++}). The matter and gauge superfields are subject to the usual gauge transformations:

$$q^{+\prime} = e^{ig\lambda} q^{+} , \quad V^{++\prime} = -\frac{i}{g} e^{ig\lambda} D^{++} e^{-ig\lambda} + e^{ig\lambda} V^{++} e^{-ig\lambda} , \tag{A.11}$$

so that the covariantized action (A.9)

$$S_{\rm HM/SYM} = -2 \int du\, d^4x_A d^2\theta^+ d^2\bar\theta^+ \ {\rm Tr}(\tilde q^+ D^{++} q^+ + ig\, \tilde q^+ V^{++} q^+) \tag{A.12}$$

is indeed gauge invariant.

In the non-supersymmetric Wess-Zumino gauge the gauge potential has the component expansion

$$\begin{aligned} V^{++}_{\rm WZ}(x_A, \theta^+, \bar\theta^+, u) = {} & -2i\theta^+\sigma^\mu\bar\theta^+ A_\mu(x_A) - i\sqrt{2}(\theta^+)^2\bar\varphi(x_A) + i\sqrt{2}(\bar\theta^+)^2\varphi(x_A) \\ & + 4(\bar\theta^+)^2\theta^{+\alpha}\lambda^i_\alpha(x_A)u^-_i - 4(\theta^+)^2\bar\theta^+_{\dot\alpha}\bar\lambda^{\dot\alpha i}(x_A)u^-_i + 3(\theta^+)^2(\bar\theta^+)^2 Y^{ij}(x_A)u^-_i u^-_j , \end{aligned} \tag{A.13}$$

containing the fields of the $\mathcal{N} = 2$ *off-shell* vector multiplet: the gauge field A_μ, the complex physical scalar φ, the doublet of Majorana gluinos λ^i_α, $\bar\lambda^{\dot\alpha i}$ and the triplet of real auxiliary fields Y^{ij}.

The gauge invariant action for V^{++} can be written down either directly in terms of V^{++} [49, 58, 59], or in terms of the chiral superfield strength $W(x_L, \theta^{i\alpha})$ (or its conjugate antichiral $\bar W(x_R, \bar\theta_{i\dot\alpha})$):

$$S_{\rm N=2\ SYM} = \frac{1}{2g^2}\int d^4x_L d^4\theta\ {\rm Tr}\, W^2 = \frac{1}{2g^2}\int d^4x_R d^4\bar\theta\ {\rm Tr}\, \bar W^2 , \tag{A.14}$$

where

$$x^{\alpha\dot\alpha}_L = x^{\alpha\dot\alpha} - 2i\theta^{i\alpha}\bar\theta^{\dot\alpha}_i \tag{A.15}$$

are the space-time coordinates in the chiral basis, and $x_R = \bar x_L$ are the antichiral ones. In a topologically trivial background these two forms are equivalent (up to a total space-time derivative), due to the Bianchi identity $D^{i\alpha}D^j_\alpha W = \bar D^i_{\dot\alpha}\bar D^{j\dot\alpha}\bar W$.

Unlike the G-analytic potential V^{++}, the field strength $W(x_L, \theta)$ is a chiral superfield which does not depend on the harmonic variable $u^\pm$. It can be expressed as a power series in V^{++}, involving multiple harmonic integrals [61, 62]

$$W = \frac{i}{4} u^+_i u^+_j \bar D^i_{\dot\alpha}\bar D^{j\dot\alpha} \sum_{r=1}^{\infty} \int du_1 \dots du_r \ \frac{(-ig)^r V^{++}(u_1)\dots V^{++}(u_r)}{(u^+u^+_1)(u^+_1u^+_2)\dots(u^+_r u^+)} , \tag{A.16}$$

where $(u^+u^+_1) \equiv u^{+i}\epsilon_{ij}u^{+j}_1$, etc. In terms of component fields we have (in the Abelian case)

$$W = \varphi(x) + \theta^{\alpha i}\lambda_{\alpha i}(x) + \theta^{\alpha i}\theta^{\beta j}(\epsilon_{ij}(\sigma^{\mu\nu})_{\alpha\beta}F_{\mu\nu}(x) + \epsilon_{\alpha\beta}Y_{ij}) + i(\theta^3)^\alpha_i\partial_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha i}(x) + \theta^4\Box\bar\varphi(x) . \tag{A.17}$$

With the hypermultiplet matter in the adjoint representation of the gauge group, the two actions (A.12) and (A.14) describe the $\mathcal{N} = 4$ SYM theory,

$$S_{\rm N=4\ SYM} = S_{\rm N=2\ SYM} + S_{\rm HM/SYM} . \tag{A.18}$$

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As mentioned earlier, the main advantage of the $\mathcal{N}=2$ harmonic superspace formulation is the possibility to quantize the theory in a straightforward and manifestly $\mathcal{N}=2$ supersymmetric way [58, 59]. Further, compared to the $\mathcal{N}=1$ chiral matter superfields, the $\mathcal{N}=2$ hypermultiplet composite operators $O=\mathrm{Tr}(q^+)^k$, etc., need no covariantization, hence no presence of the gauge superfield in the definition of the correlation functions $\langle O\dots\tilde{O}\rangle$. The hypermultiplet matter interacts with the gauge sector only through a single trilinear vertex, which considerably simplifies the Feynman diagrams. The true non-Abelian nature of the theory is encoded in the gauge self-interactions (as well as in the ghost sector, but we do not need it here).

A.2 Feynman rules

In this section we give a subset of the Feynman rules for the combined $\mathcal{N}=2$ matter+gauge system which are sufficient for the one- and two-loop calculations we do in this paper (the complete set can be found in [49, 58, 59]).

The hypermultiplet propagator is the solution to the Green's function equation

$$D_1^{++}\langle\tilde{q}^+(1)q^+(2)\rangle=\delta^4(x_{A1}-x_{A2})\delta^2(\theta_1^+-\theta_2^+)\delta^2(\bar{\theta}_1^+-\bar{\theta}_2^+)\delta(u_1,u_2)\,, \tag{A.19}$$

and is given by

$$\langle\tilde{q}_a^+(1)q_b^+(2)\rangle=\frac{(12)}{4\pi^2\hat{x}_{12}^2}\delta_{ab}=\overset{1,a}{\longrightarrow}\!\!\!\!\!\!\overset{2,b}{}\,. \tag{A.20}$$

Here a,b are color indices and

$$(12)=-(21)=u_1^{+i}\epsilon_{ij}u_2^{+j} \tag{A.21}$$

is a shorthand for the SU(2) invariant but U(1) covariant contraction of the two harmonics. The coordinate difference

$$\hat{x}_{12}^\mu=x_{A1}^\mu-x_{A2}^\mu+\frac{2i}{(12)}[(1^-2)\theta_1^+\sigma^\mu\bar{\theta}_1^+ +(2^-1)\theta_2^+\sigma^\mu\bar{\theta}_2^+ +\theta_1^+\sigma^\mu\bar{\theta}_2^+ +\theta_2^+\sigma^\mu\bar{\theta}_1^+]\,, \tag{A.22}$$

where, e.g., $(1^-2)=u_1^{-i}\epsilon_{ij}u_2^{+j}$, is invariant under the Poincaré supersymmetry transformations in the G-analytic basis (A.5):[11]

$$\delta_Q x_A^{\alpha\dot{\alpha}}=-4iu_i^-(\epsilon^{i\alpha}\bar{\theta}^{+\dot{\alpha}}+\theta^{+\alpha}\bar{\epsilon}^{i\dot{\alpha}})\,,\qquad \delta_Q\theta^{+\alpha,\dot{\alpha}}=u_i^+\epsilon^{i\alpha,\dot{\alpha}}\,,\qquad \delta_Q u_i^\pm=0\,. \tag{A.23}$$

Setting the Grassmann variables in (A.20) to zero, we find the propagator for the physical scalars $\phi_a^i(x)$ projected with harmonics (recall (A.8)):

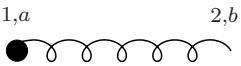
$$\langle\tilde{q}_a^+(1)q_b^+(2)\rangle_{\theta=0}=\langle u_1^{+i}\bar{\phi}_{ia}(x_1)|u_{2j}^+\phi_b^j(x_2)\rangle=\frac{(12)}{4\pi^2x_{12}^2}\delta_{ab}\,. \tag{A.24}$$

The gauge field (gluon) propagator depends on the gauge we have chosen. In our loop calculations we will only need the propagator $\langle W(1)V^{++}(2)\rangle$, having one chiral end (the field strength $W(x_{L1},\theta_1)$) and one G-analytic end (the SYM potential

[11] To check this one makes use of the harmonic cyclic identity $(1^-2)1+(21)1^-+(11^-)2=0$ and of the defining property $(11^-)=1$ (see (A.2)).

$V^{++}(x_{A2},\theta_2^+,\bar\theta_2^+,u_2)$). It is independent of the gauge and has the following form

$$\langle W_a(1)V_b^{++}(2)\rangle = -\frac{g\delta_{ab}}{2\pi^2\widetilde{x}_{12}^2}(\theta_{12})^2 = \quad \text{(A.25)}$$



It involves the coordinate differences

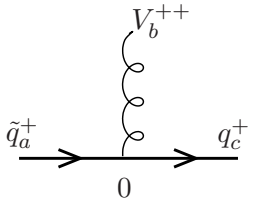
$$\widetilde{x}_{12}^{\alpha\dot\alpha} = x_{L1}^{\alpha\dot\alpha} - x_{A2}^{\alpha\dot\alpha} - 4iu_{2i}^-\theta_1^{i\alpha}\,\bar\theta_2^{+\dot\alpha}\,, \qquad \theta_{12}^\alpha = u_{2i}^+\theta_1^{i\alpha} - \theta_2^{+\alpha} \quad \text{(A.26)}$$

and is invariant under the Poincaré supersymmetry transformations in the chiral basis for x_{L1},

$$\delta_Q x_L^{\alpha\dot\alpha} = -4i\theta^{\alpha i}\bar\epsilon_i^{\dot\alpha}\,, \qquad \delta_Q\theta^{i\alpha} = \epsilon^{i\alpha}\,, \quad \text{(A.27)}$$

and (A.23) in the G-analytic basis for x_{A2}.

Finally, the only interaction vertex we shall need here is the gluon-to-matter coupling which can be read off from the covariantized hypermultiplet action (A.12):



$$= \qquad igf_{bac}\int d^4x_0 du_0 d^4\theta_0^+ \quad \text{(A.28)}$$

The SYM action (A.14), (A.16) is non-polynomial in V^{++}, so it contains infinitely many vertices (but, of course, becomes polynomial in the Wess-Zumino gauge (A.13)). At the one- and two-loop levels that we are studying in this paper only the cubic non-Abelian vertex can appear, but later on we shall see that all such graphs are irrelevant.

A.3 The insertion procedure: four points

Here we illustrate the $\mathcal{N}=2$ Lagrangian insertion procedure described in section 3, on the example of the two-loop four-point correlation function G_4 (5.1) of hypermultiplet bilinears $\mathcal{O} = O_{\theta^+=\bar\theta^+=0}$ with $O = \mathrm{Tr}(q^+)^2$ and $\tilde O = \mathrm{Tr}(\tilde q^+)^2$. In particular, following [32, 33] we explain the important role of $\mathcal{N}=2$ superconformal symmetry, which allows us to drastically simplify the two-loop calculations.

The two-loop (order g^4) corrections to the four-point correlation function G_4 can be obtained by a double insertion of the $\mathcal{N}=2$ SYM chiral action

$$S_{\mathcal{N}=2\ \mathrm{SYM}} = \int \mathrm{d}^4x\mathrm{d}^4\theta\ L(x_L,\theta)\,, \qquad L = \frac{1}{2g^2}\mathrm{Tr}(W(x_L,\theta))^2\,. \quad \text{(A.29)}$$

To see this, we first rescale the gauge potential $V^{++} \to g^{-1}V^{++}$, with the effect that the coupling only appears in front of the SYM action (A.14), but not inside the field strength W (A.16). It also drops out of the gauge/matter coupling (A.12), thus (A.28) loses the explicit g. As another consequence, the gauge propagator (i.e. the inverse of the gauge kinetic term) is scaled up by g^2, which introduces a factor of g^2 in the right-hand side of (A.25).

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Let us write out the perturbative expansion of G_4:

$$G_4 \,=\, G_4^{(0)} + g^2\, G_4^{(1)} + \, g^4\, G_4^{(2)} + \dots \,, \tag{A.30}$$

so that

$$G_4^{(2)} \,=\, \frac{1}{2}\left(\frac{\partial}{\partial g^2}\right)^2_{g=0} G_4\,. \tag{A.31}$$

On the other hand, by considering the effect of the differentiation on the path integral we find

$$\begin{aligned}\frac{1}{2}\,g^4\left(\frac{\partial}{\partial g^2}\right)^2 G_4 \,=\, i\int \mathrm{d}^4x_0\mathrm{d}^4\theta_0\;\langle L(0)O(1)\tilde{O}(2)O(3)\tilde{O}(4)\rangle \\ -\frac{1}{2}\int \mathrm{d}^4x_0\mathrm{d}^4\theta_0\;\mathrm{d}^4x_{0'}\mathrm{d}^4\theta_{0'}\;\langle L(0)L(0')O(1)\tilde{O}(2)O(3)\tilde{O}(4)\rangle\,.\end{aligned} \tag{A.32}$$

The left-hand side in the last formula starts at $O(g^4)$, which leads to a puzzle: The perturbative expansion of the first term on the right-hand side starts at $O(g^2)$, while the second term seemingly starts at $O(g^4)$. The only way to produce a compensating $O(g^2)$ contribution from there is to insert both $L(0), L(0')$ into one gluon line. This means inserting the chiral-to-chiral propagator (it contains just the propagator of the auxiliary field Y in (A.17))

$$\langle W(0)W(0')\rangle = 2ig^2\delta^4(x_{L0}-x_{L0'})\delta^4(\theta_0-\theta_{0'}) \tag{A.33}$$

into that gluon line. Upon performing the chiral superspace integration over point $0'$, the $O(g^2)$ contribution from the single insertion term is identically canceled. This remains true in general: $00'$ contact terms from the double insertion term identically cancel the single insertion term. Returning to $O(g^4)$:

$$G_4^{(2)} \,=\, -\frac{1}{2g^4}\int \mathrm{d}^4x_0\mathrm{d}^4\theta_0\;\mathrm{d}^4x_{0'}\mathrm{d}^4\theta_{0'}\;\langle L(0)L(0')O(1)\tilde{O}(2)O(3)\tilde{O}(4)\rangle^{\mathrm{reg}}_{g^4} \tag{A.34}$$

where the superscript indicates that the $00'$ contact terms are to be omitted. For the class of graphs we find below this simply means not to contract $W(0)$ and $W(0')$. We stress that this $(4+2)-$point correlation function is once again at Born level; it comes with g^4 because there are four Yang-Mills propagators and two explicit factors g^{-2} from the Lagrangian insertions. Below we show that this correlation function with insertions has to be nilpotent, i.e. proportional to θ^8.

A.3.1 Structure of the nilpotent superconformal covariant

The most important feature of the new six-point correlation function is its superconformal symmetry. Indeed, it involves gauge-invariant composite operators, O, $\tilde{O}$ and L. As explained in section 3, all of these operators are particular projections of the $\mathcal{N}=4$ half-BPS protected operator in the $\mathbf{20'}$ of SU(4). As such, they need no renormalization and have well-defined superconformal properties. As long as we keep the end points in this correlation function apart, nothing can break the $\mathcal{N}=2$ superconformal symmetry of the theory. This imposes rather strong constraints on the general form of the correlation function.

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The $\mathcal{N}=2$ superconformal algebra has an $\mathrm{SU}(2)\times\mathrm{U}(1)$ automorphism group (R symmetry). The U(1) factor (to be distinguished form the harmonic $\mathrm{U}(1)\subset\mathrm{SU}(2)$) acts only on the odd superspace variables, $R[\theta]=1/2$, $R[\bar\theta]=-1/2$. From the SYM action (A.29) we deduce $R[L]=2$ and $R[W]=1$. At the same time, the hypermultiplets q^+ and $\tilde q^+$ have no R charge, as follows from the action (A.9), but they carry harmonic charge 1. This implies that the six-point correlation function carries harmonic U(1) charges 2 at points 1 to 4, and a total R charge 4 at points 0 and $0'$. Since the chiral θ_α (at both the insertion and external points) are the only superspace coordinates with positive R charge 1/2, we conclude that the correlation function can be written in the factorized form

$$\langle LLO\tilde O O\tilde O\rangle = \Theta(x,\theta,u)\times f(x,u) + \bar\theta\text{-terms}\,, \tag{A.35}$$

where Θ is a particular nilpotent six-point superconformal covariant, homogeneous in θ of degree 8, and thus carrying the whole R charge. The antichiral odd variables $\bar\theta$ can only come from the external points, but we are ultimately interested only in the lowest components $\mathcal{O}=O_{\theta^+=\bar\theta^+=0}$, so we can ignore the $\bar\theta$ terms in (A.35). So, the essential information about the six-point correlation function (A.35) is contained in the function $f(x,u)$ without R charge. Below we will show that this function is in fact harmonic independent, the harmonic U(1) charge being carried by Θ.

The structure of the nilpotent covariant Θ is determined by superconformal symmetry combined with the G-analytic nature of the four external points and the chiral nature of the two insertion points. In addition to Poincaré supersymmetry (parameters ϵ^i_α, $\bar\epsilon^{\dot\alpha}_i$) we need to consider special conformal supersymmetry (parameters η^i_α, $\bar\eta^{\dot\alpha}_i$). In the chiral basis (A.15) we have

$$\delta x_L^{\alpha\dot\alpha} = -4i\theta^{i\alpha}\bar\epsilon_i^{\dot\alpha} - 4i\theta^{i\alpha}x_L^{\beta\dot\alpha}\eta_{\beta i}\,,\qquad \delta\theta^{i\alpha}=\epsilon^{i\alpha}+x_L^{\alpha\dot\beta}\bar\eta^i_{\dot\beta}+O(\theta^2)\,, \tag{A.36}$$

while in the G-analytic basis (A.5) we find

$$\begin{aligned}
\delta x_A^{\alpha\dot\alpha} &= -4iu_i^-(\epsilon^{i\alpha}\bar\theta^{+\dot\alpha}+\theta^{+\alpha}\bar\epsilon^{i\dot\alpha}) + 4i(x_A^{\alpha\dot\beta}\bar\theta^{+\dot\alpha}\bar\eta^i_{\dot\beta} - x_A^{\beta\dot\alpha}\theta^{+\alpha}\eta^i_\beta)u_i^-\,,\\
\delta\theta^{+\alpha} &= u_i^+\epsilon^{i\alpha} + x_A^{\alpha\dot\beta}\bar\eta^i_{\dot\beta}u_i^+ + O(\theta^2)\,,\qquad \delta\bar\theta^{+\dot\alpha} = u_i^+\bar\epsilon^{i\dot\alpha} - x_A^{\beta\dot\alpha}\eta^i_\beta u_i^+ + O(\bar\theta^2)\,,\\
\delta u_i^+ &= 4i(\theta^{+\alpha}\eta^j_\alpha + \bar\eta^j_{\dot\alpha}\bar\theta^{+\dot\alpha})u_j^+u_i^-\,,\qquad \delta u_i^- = 0\,.
\end{aligned} \tag{A.37}$$

Now, the covariant Θ is homogeneous in θ of degree 8. The inhomogeneous part of $\delta\theta$,

$$\delta\theta^{i\alpha}=\epsilon^{i\alpha}+x_L^{\alpha\dot\beta}\bar\eta^i_{\dot\beta}\,,\qquad \delta\theta^{+\alpha}=u_i^+\epsilon^{i\alpha}+x_A^{\alpha\dot\beta}\bar\eta^i_{\dot\beta}u_i^+\,, \tag{A.38}$$

would lower this degree, unless we find combinations of θ's which are invariant (to lowest order in θ, $\bar\theta$) under $\mathcal{N}=2$ superconformal supersymmetry. Such combinations are

$$\xi_{r\dot\alpha} = \rho_{r\dot\alpha}-\sigma_{r\dot\alpha}\,,\qquad r=1,\dots,4 \tag{A.39}$$

with

$$\rho_{r\dot\alpha}=(\theta_r^+-\theta_0^iu_{ri}^+)^\alpha(x_{r0})_{\alpha\dot\alpha}x_{r0}^{-2}\,,\qquad \sigma_{r\dot\alpha}=(\theta_r^+-\theta_{0'}^iu_{ri}^+)^\alpha(x_{r0'})_{\alpha\dot\alpha}x_{r0'}^{-2}\,. \tag{A.40}$$

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Their total number is 8, and we wish to construct the nilpotent covariant Θ of degree 8. We conclude that the leading term of Θ must involve all of the variables ξ_r:

$$\Theta = \xi_1^2\xi_2^2\xi_3^2\xi_4^2\,. \tag{A.41}$$

The aim of our two-loop calculation is to determine the factor $f(x,u)$ in the six-point correlation function (A.35). Since we are only interested in the lowest component of the four-point correlation function $\langle\mathcal{O}\tilde{\mathcal{O}}\mathcal{O}\tilde{\mathcal{O}}\rangle = \langle O\tilde{O}O\tilde{O}\rangle_{\theta^+=\bar\theta^+=0}$, we can set all the external θs to zero, $\theta_r^+ = 0$, $r = 1,\dots,4$. In this case Θ is rather simple [32, 33]:

$$\Theta|_{\theta^+=0} = \theta_0^4\theta_{0'}^4\,\frac{(x_{00'}^2)^2\mathcal{R}}{\prod_{r=1}^4 x_{r0}^2x_{r0'}^2}\,, \tag{A.42}$$

where

$$\mathcal{R} = (12)^2(34)^2x_{14}^2x_{23}^2 + (14)^2(23)^2x_{12}^2x_{34}^2 + (12)(23)(34)(41)\left[x_{13}^2x_{24}^2 - x_{12}^2x_{34}^2 - x_{14}^2x_{23}^2\right]. \tag{A.43}$$

Finally, substituting everything into the double-insertion formula (A.32) and performing the trivial chiral integrations over $\theta_{0,0'}$, we obtain the two-loop correlation function

$$\langle\mathcal{O}\tilde{\mathcal{O}}\mathcal{O}\tilde{\mathcal{O}}\rangle = \langle O\tilde{O}O\tilde{O}\rangle_{\theta^+=\bar\theta^+=0} = \mathcal{R}\int\frac{\mathrm{d}^4x_0\mathrm{d}^4x_{0'}}{\prod_{r=1}^4 x_{r0}^2x_{r0'}^2}(x_{00'}^2)^2\,f(x,u)\,. \tag{A.44}$$

Notice the characteristic presence of the polynomial prefactor $\mathcal{R}$. As shown in [52, 63, 64], this factorization of the loop corrections is a universal feature, called "partial non-renormalization".

A.3.2 Feynman graphs. Harmonic analyticity

Now, the practical question is how to compute $f(x,u)$ from the corresponding set of two-loop Feynman diagrams. It turns out that instead of setting $\theta^+ = 0$, as required in the final expression (A.44), it is much more convenient to do the computations with $\theta_{0,0'} = 0$. The knowledge of the complete Θ (A.41) allows us to easily switch from one of these forms to the other. The new form of Θ is even simpler, yielding

$$\langle LLO\tilde{O}O\tilde{O}\rangle_{\theta_{0,0'}=\bar\theta^+=0} = (x_{00'}^2)^4\prod_{r=1}^4\frac{(\theta_r^+)^2}{x_{r0}^2x_{r0'}^2}\,f(x,u)\,. \tag{A.45}$$

Then it is clear that in working out the expressions for the various Feynman graphs we can concentrate only on the terms with the maximal number of external θ^+. In particular, at order g^4 this choice removes all graphs which contain non-Abelian interaction vertices. For example, the non-Abelian gluon subgraph in figure 6(c) vanishes because it has two chiral ends at the insertion points 0 and $0'$ and one G-analytic end (the gluon without insertion); after setting $\theta_{0,0'} = 0$ we are left with too few chiral θ^+s at the G-analytic gluon end to supply the required R charge 2. Similarly, the block in figure 6(d) has three chiral ends (in fact, only two, points $0'$ and $0''$ should be identified) and two G-analytic

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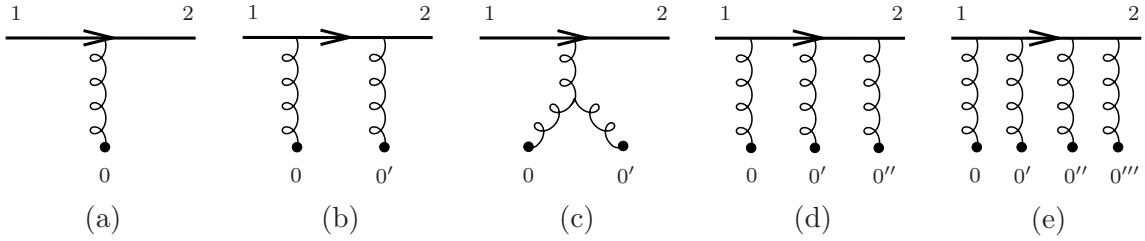

Figure 6. Building blocks of the Feynman graphs.

ends; once again, the G-analytic θ^+s cannot provide the required R charge 3. The same applies to the block in figure 6(e).

As a result of all these simplifications our task is reduced to listing all *tree level* Feynman graphs made out of the two building blocks T and TT in figures 6(a) and (b). They are calculated with the Feynman rules from section A.2. Although these blocks contain gauge/matter interaction vertices (A.28) and hence space-time integrals, the latter are easily done using the identities

$$\Box_1 \int \frac{d^4x_3}{x_{13}^2 x_{23}^2 x_{03}^2} = \frac{4i\pi^2}{x_{12}^2 x_{10}^2}\,, \qquad \partial_1^{[\mu}\partial_2^{\nu]} \int \frac{d^4x_3}{x_{13}^2 x_{23}^2 x_{03}^2} = -4i\pi^2 \frac{x_{10}^{[\mu} x_{20}^{\nu]}}{x_{12}^2 x_{10}^2 x_{20}^2}\,, \tag{A.46}$$

producing very simple *rational* space-time functions [32, 33, 60]:

$$\langle \tilde{q}_a^+(1) W_b(0) q_c^+(2)\rangle = -\frac{2ig^2 f_{abc}}{(2\pi)^4\, x_{12}^2}\left[(21^-)\rho_1^2 + (12^-)\rho_2^2 - 2(\rho_1\rho_2)\right] \tag{A.47}$$

$$\langle \tilde{q}_a^+(1) W_b(0) W_d(0') q_e^+(2)\rangle = -\frac{4g^4 f_{abc} f_{cde}}{(2\pi)^6\, x_{12}^2}\,(1^-2^-)\rho_1^2\,\sigma_2^2\,, \tag{A.48}$$

where ρ and σ were defined in (A.40).

Notice the characteristic presence of negative-charged harmonics in both expressions (A.47) and (A.48). This has to do with the important issue of harmonic analyticity [65]. In an interacting theory the hypermultiplet satisfies its equation of motion (A.6) with a covariant harmonic derivative, $D^{++}q^+ + ig[V^{++}, q^+] = 0$. The gauge-invariant composite operators $O_k = \mathrm{Tr}(q^+)^k$ satisfy the same equation with a flat harmonic derivative, $D^{++}O_k = 0$. As explained above, the harmonic derivative is the raising operator of SU(2). So, O_k corresponds to the highest-weight state of an SU(2) irrep of weight k (a $(k+1)$-plet).

In practice, this means that the $n-$point correlation function is annihilated by the harmonic derivative D^{++} at each point,

$$D_r^{++}\langle O_k \dots O_k\rangle = 0\,, \qquad r = 1, \dots, n\,. \tag{A.49}$$

Since $D^{++}u^+ = 0$ and $D^{++}u^- = u^+$, this implies that the correlation function is a polynomial in u_r^+, $r = 1, \dots, n$, homogeneous of degree k at each point, and no dependence on u^- is allowed.

Clearly, the expressions for the building blocks T (A.47) and TT (A.48) are not harmonic analytic because of the presence of u_1^- and u_2^-. This, however, is not a problem:

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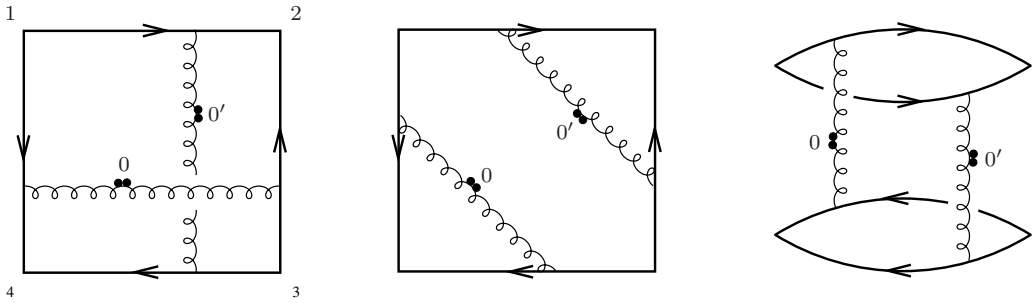

Figure 7. Graphs surviving the identification of harmonics.

The various building blocks or even complete Feynman graphs are not expected to be harmonic analytic, much like they are not conformal and gauge invariants. It is only the sum of all graphs that has these properties. Indeed, it can be shown that by summing up all graphs made from the T and TT blocks, all negative-charged harmonics drop out. To see this one uses the harmonic cyclic identity, e.g.,

$$(12)(1^-2^-) - (12^-)(1^-2) = u_1^{+i}u_1^{-j}u_2^{+k}u_2^{-l}(\epsilon_{ik}\epsilon_{jl} - \epsilon_{il}\epsilon_{jk}) = (11^-)(22^-) = 1\,, \qquad \text{(A.50)}$$

as a consequence of the defining property $u^{+i}u_i^- = 1$ (A.2). In practice, the use of the cyclic identity is cumbersome when there are u's from too many different points. But we can do better, by completely sidestepping this issue.

We can profit from the expected harmonic analyticity of the final result to greatly simplify our graph calculations. Let us come back to the correlation function of four operators $O \equiv O_2$ with two Lagrangian insertions, calculated at $\theta_0 = \theta_{0'} = 0$, see (A.45). The Lagrangian has no harmonic U(1) charge, hence $\langle LLO\tilde{O}O\tilde{O}\rangle$ should have charges +2 at each external point. From (A.45) we see that the nilpotent factor already carries the necessary charges, thus making the function $f(x,u)$ chargeless. Harmonic analyticity then implies that this function is *harmonic independent*. This allows us to compute the correlation function (A.45) with all four harmonic variables identified,

$$u_1^\pm = u_2^\pm = u_3^\pm = u_4^\pm\,. \qquad \text{(A.51)}$$

This simple trick eliminates a number of irrelevant Feynman graphs, namely, all graphs with at least one free hypermultiplet line, since the hypermultiplet propagator (A.24) vanishes if $u_1^+ = u_2^+$. Among them we find the graphs with the blocks from figure 6(c)-(e), for which we already gave a different reason why they do not contribute. In addition, the identification of harmonics eliminates the graphs with TT blocks. This leaves only the three graphs shown in figure 7 (plus point permutations):
Notice that despite the appearance, the third graph is planar. This has to do with the fact that unlike the Green's functions of elementary fields, which are planar on a disk, the correlation functions (Green's functions) of composite operators have the topology of a sphere. This peculiar property of the correlation functions was mentioned in section 6.2.

Each of the graphs in figure 7 is a product of T blocks (A.47) evaluated at $\theta_0 = \theta_{0'} = 0$. Thus, the calculation of the harmonic-independent function $f(x)$ in (A.45) is reduced to

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elementary algebra. Then $f(x)$ is substituted in (A.44), leading to the final result (5.2) from [32, 33].

This procedure is a very efficient tool for calculating not only four-point, but also n-point correlation functions of half-BPS operators made of hypermultiplets. In the next subsection we show how to adapt the procedure to five points.

A.4 The insertion procedure: five points

Here we evaluate the correlation function

$$G_{5+2} = \langle L(0)\, L(0')\, \mathcal{O}(1)\, \tilde{O}(2)\, O(3)\, \tilde{O}(4)\, \hat{O}(5)\rangle\,, \tag{A.52}$$

with $O = \mathrm{Tr}(q^+)^2, \tilde{O} = \mathrm{Tr}(\tilde{q}^+)^2, \hat{O} = 2\,\mathrm{Tr}(\tilde{q}^+ q^+), L = 1/(2g^2)\,\mathrm{Tr}(W)^2$, in order to study the two-loop corrections to the correlation function

$$G_5 = \langle O(1)\, \tilde{O}(2)\, O(3)\, \tilde{O}(4)\, \hat{O}(5)\rangle\,, \tag{A.53}$$

discussed in section 6. We will use the technology developed in [32, 33] and reviewed in appendix A.3 for the four-point case $G_4 = \langle O(1)\, \tilde{O}(2)\, O(3)\, \tilde{O}(4)\rangle$.

As before, we compute G_{5+2} in $D = 4$. It remains finite in the light-cone limit

$$x_{12}^2 = x_{23}^2 = x_{34}^2 = x_{45}^2 = x_{51}^2 \to 0\,. \tag{A.54}$$

The divergences of G_5 in this limit arise from the integration over the insertion points x_0, $x_{0'}$ which we will eventually regularize by the IR prescription of changing only the dimension of the integration measure at these points.

Repeating the superconformal argument of appendix A.3.1, we can claim that the leading term in the θ expansion of the correlation function G_{5+2} will factorize into a nilpotent covariant $\Theta \sim \theta^8$ and some function $f(x,u)$ of the bosonic variables, see (A.35). Once again, Θ will depend only on the invariant variables ξ_r (A.39) (with $r = 1,\ldots,5$). This time, however, we have 10 such variables, while the degree of homogeneity of Θ is still 8. Unlike the four-point case, where the nilpotent structure (A.41) was unique, now we can have two different choices:

$$\begin{aligned} A_5 &= \xi_1^2\,\xi_2^2\,\xi_3^2\,\xi_4^2\, f_5(x,u) \\ B_{45} &= \xi_1^2\,\xi_2^2\,\xi_3^2\,\xi_{4\dot\alpha}\,\xi_{5\dot\beta}\, f_{45}^{\dot\alpha\dot\beta}(x,u)\,, \end{aligned} \tag{A.55}$$

and their point permutations. A graph calculation is needed to fix the coefficient functions $f(x,u)$. The knowledge that only A_r, B_{rs} can occur is very useful, though, if combined with the property of harmonic analyticity, namely the fact that the full gauge invariant correlation function only depends on u_r^+ and that it does so in a polynomial fashion (see appendix A.3.2).

Let us consider the covariants A_r, B_{rs} at $\theta_0 = \theta_{0'} = 0$, in other words, only the terms involving θ_r^+ from the outer points. Now, θ_r^+ carries U(1) charge 1 at point r, and so does u_r^+. Hence the spinor part of the covariant A_5 has charge 2 at points 1,2,3,4, whereas it is chargeless at point 5. Since all five outer operators $O, \tilde{O}, \hat{O}$ carry charge 2, the coefficient function

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$f_5(x,u)$ has to have charge 2 at point 5 and zero at all other points. To be harmonic analytic it must be an SU(2) invariant polynomial in the u_r^+ with the correct charges. The only such invariant is trivially zero, $(55) = \epsilon^{ij} u_{5j}^+ u_{5i}^+ = 0$. This argument rules out all A_r covariants.

Further, in B_{45} the odd variables ξ carry charge 2 at points 1,2,3 and charge 1 at points 4,5. Due to harmonic analyticity, the harmonic dependence of the bosonic factor can only be of the form $f_{45}^{\dot\alpha\dot\beta}(x,u) = (45) f_{45}^{\dot\alpha\dot\beta}(x)$. Thus, the sum of the contributions of the graphs to this covariant will be

$$B_{45} \;=\; \xi_1^2\,\xi_2^2\,\xi_3^2\,\xi_{4\dot\alpha}\,\xi_{5\dot\beta}\,(45)\,f_{45}^{\dot\alpha\dot\beta}(x)\,. \tag{A.56}$$

Individual graphs do contain non-analytic terms, but we need not go through all the details of how they cancel out. Instead, we can apply the powerful trick of identifying the harmonic variables, as we did in the four-point case in appendix A.3.2.

For each covariant B_{rs} (with $r,s = 1,\ldots,5$) we know that the harmonic dependence of the coefficient function will eventually be given by just (rs). This result clearly does not change if all harmonics, except for $u_r^\pm$, are put equal to $u_s^\pm$. As a convention, if $r<s$ we will keep $u_{ri}^\pm$ aside and identify all other harmonics with $u_{si}^\pm$. We will obtain the correct result if this is done consistently for any contribution to the spinor structure pertaining to the given covariant (A.56).

This manoeuvre drastically simplifies the use of the harmonic cyclic identity. For example equation (A.50) reduces to $0-(-1) = 1$ if both $u_{1i}^\pm$, $u_{2i}^\pm$ are sent to $u_{5i}^\pm$. But there are more far-reaching consequences for the supergraphs at $\theta_0 = \theta_{0'} = 0$.

- Any diagram with more than one free line (i.e. a hypermultiplet propagator between two outer points) is put to zero. Spinor structures relating to the A_r covariants may be discarded immediately. For the B_{rs} type contributions the suggested identification of the harmonics will send at least one of the numerators of the free lines to $(ss) = 0$.

- As a consequence, we only need to take into account graphs built out of four T blocks and one free hypermultiplet line.

- The light-cone limit singles out diagrams in which the gluon lines connect to the pentagon frame 123451 of matter lines, because the T blocks with outer ends r,s have an explicit propagator factor $1/x_{rs}^2$. This eliminates graphs with disconnected matter frames, like the third graph in figure 7, since they lack the required light-cone singularity. For the same reason, graphs with a connected "zigzag" frame like in figure 1(b) are not allowed.

All of these simplifications leave us with a very small number of graphs shown in figure 8 (notice that the middle graph is planar, like the two others).
The resulting algorithm for evaluating these graphs is as follows:

- For any of the diagrams in this set multiply out the terms of the four T blocks and classify them according to the various θ_r^+ structures. Discard terms with four $(\theta_r^+)^2$ (A type) but keep those related to the B type covariants. Sum over all graphs.

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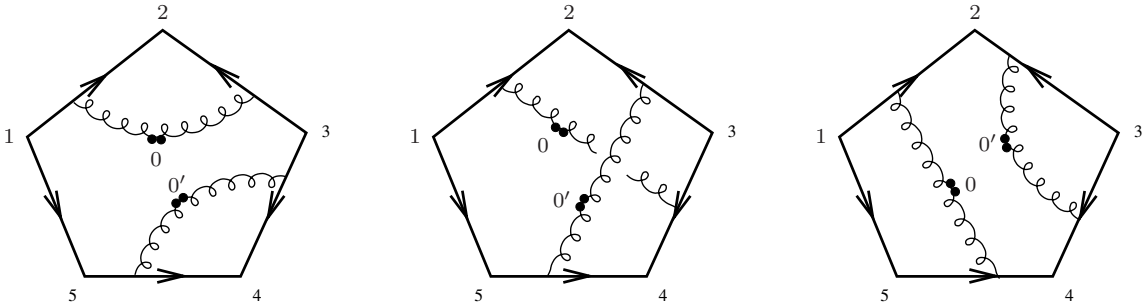

Figure 8. Graphs contributing to G_{5+2}. The permutations of the insertion points 0, 0′ and the cyclic permutations of the external points should be added.

- In each term identify the harmonics according to the spinor structure; by way of example for $(\theta_1^+)^2(\theta_2^+)^2(\theta_3^+)^2\theta_{4\alpha}^+\theta_{5\beta}^+$ we put $u_{1i}^\pm = u_{2i}^\pm = u_{3i}^\pm = u_{5i}^\pm$. The contributions of some diagrams may vanish in doing so, in other cases the harmonics will reduce to the simple factor (45).

- Reconstruct the full covariants. The result of the procedure is the entire leading term of the correlation function G_{5+2} in which the harmonics are not identified any longer. We can now switch to the "opposite end" of the expression by putting all $\theta_r^+ = 0$, while restoring the spinors from the insertion points. We observe that this step produces $(12)(23)(34)(45)(51)\,\theta_0^4\,\theta_{0'}^4\,g_0$, i.e. in the light-cone limit no other SU(2) channel is present.

The reconstruction is in fact elementary. From the definitions (A.39), (A.40) we find

$$\theta_r^{+\alpha} \;=\; \frac{x_{r0}^2\,x_{r0'}^2}{x_{00'}^2}\left(\frac{x_{r0}^{\alpha\dot\alpha}}{x_{r0}^2}-\frac{x_{r0'}^{\alpha\dot\alpha}}{x_{r0'}^2}\right)\xi_{r\dot\alpha}|_{\theta_0=\theta_{0'}=0}\,. \tag{A.57}$$

By this formula we can unambiguously upgrade every θ_r^+ from the graph calculation to the invariant combination ξ_r. Next we note that

$$\frac{x_{r0}^2\,x_{r0'}^2}{x_{00'}^2}\left(\frac{x_{r0}^{\alpha\dot\alpha}}{x_{r0}^2}-\frac{x_{r0'}^{\alpha\dot\alpha}}{x_{r0'}^2}\right)\xi_{r\dot\alpha}|_{\theta_r^+=0} \;=\; x_{r0}^{\alpha\dot\alpha}\,\lambda_{0'\dot\alpha}^{r+} \;-\; x_{r0'}^{\alpha\dot\alpha}\,\lambda_{0\dot\alpha}^{r+} \tag{A.58}$$

where

$$\lambda_{t\dot\alpha}^{r+} \;=\; \frac{x_{00'\,\dot\alpha\alpha}}{x_{00'}^2}\,\theta_t^{\alpha i}u_{ri}^+\,,\qquad t\in\{0,0'\}\,. \tag{A.59}$$

In order to complete the task we must collect the u-projected λ variables into $\theta_0^4\,\theta_{0'}^4$ and a harmonic factor. To this end we use the identity

$$\begin{aligned}\lambda_{0\dot\alpha}^{1+}\,\lambda_{0\dot\beta}^{2+}\,\lambda_{0\dot\gamma}^{3+}\,\lambda_{0\dot\delta}^{4+} \;&=\; \frac{1}{(12)^2}\,\lambda_{0\dot\alpha}^{1+}\,\lambda_{0\dot\beta}^{2+}\,\Big((13)\lambda_{0\dot\gamma}^{2+}-(23)\lambda_{0\dot\gamma}^{1+}\Big)\Big((14)\lambda_{0\dot\delta}^{2+}-(24)\lambda_{0\dot\delta}^{1+}\Big)\\ &=\; \frac{1}{4\,(x_{00'}^2)^2}\Big(\epsilon_{\dot\alpha\dot\gamma}\,\epsilon_{\dot\beta\dot\delta}\,(14)(23)-\epsilon_{\dot\alpha\dot\delta}\,\epsilon_{\dot\beta\dot\gamma}\,(13)(24)\Big)\;\theta_0^4\,,\end{aligned} \tag{A.60}$$

and its special cases where some points coincide. Beyond the factor (rs) in the coefficient function of the B_{rs} covariants, the conversion to $\theta_0^4\,\theta_{0'}^4$ produces four further harmonic

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factors (ij) which now carry the remaining U(1) charges. The harmonic dependence remains manifestly analytic. Out of the u_r^+ from the five outer points one can construct six independent polynomials carrying charge 2 at every point. We stress that it is not obvious that the sum of graphs produces only one channel in the light-cone limit, namely (12)(23)(34)(45)(51).

In summary, we have explained how the evaluation of the correlation function G_{n+2} is reduced to algebraic manipulations by the insertion procedure combined with superconformal symmetry and harmonic analyticity. As in the four-point case, no integral needs to be done once the T block is known. Nevertheless, the amount of algebra is fairly large, so that we have resorted to a *Mathematica* script. It remains to restore the integrations over the insertion points $x_0, x_{0'}$ with the IR-modified measure. The scalar (parity-even) part of the result for G_5 has the concise form displayed in eq. (6.7).

A special comment is due here on the pseudo-scalar (parity-odd) part of the correlation function. Our calculation of G_{5+2} does indeed produce such a (rather complicated) part. But this does not mean that G_5 will have a parity-odd part. It must drop out after the integration over the insertion points x_0, $x_{0'}$. The explanation is given in appendix A.5.

A.5 Parity properties of the scalars in the $\mathcal{N}=2$ theory

Here, following ref. [66], we argue that there exists a parity assignment for the fields of the $\mathcal{N}=2$ vector and hypermultiplets, such that the hypermultiplet scalars are true scalars (not pseudo). With this assignment, all our operators made of hypermultiplets are scalars, and their correlation functions should not contain a parity-odd part.

The components of the two multiplets are contained in the G-analytic superfields q^+, eq. (A.8), and V^{++}, eq. (A.13). The parity assignments of ref. [66] for the superspace coordinates are

$$P: \qquad x_0' = x_0\,, \quad \vec{x}' = -\vec{x}\,, \quad (\theta_\alpha^+)' = \bar{\theta}^{+\dot{\alpha}}\,, \quad (\bar{\theta}_{\dot{\alpha}}^+)' = -\theta^{+\alpha}\,, \quad (u_i^\pm)' = u_i^\pm\,, \tag{A.61}$$

while the superfields remain *inert*,

$$q^{+\prime}(x',\theta',u') = q^+(x,\theta,u)\,, \qquad V^{++\prime}(x',\theta',u') = V^{++}(x,\theta,u)\,. \tag{A.62}$$

In terms of the bosonic physical fields, these assignments imply that the hypermultiplet scalars ϕ^i are *true scalars*, the gluon A_μ is a polar vector, while the complex vector multiplet scalar φ is a mixture of a true and a pseudo-scalar. In addition, the fermion fields transform in an unusual way: the hyperinos transform into each other (up to signs), $\psi \leftrightarrow \bar{\kappa}$, and the gluinos $\lambda^i \leftrightarrow \bar{\lambda}^i$. The latter relation means that, e.g., $\lambda^1 \leftrightarrow \bar{\lambda}^1 = \bar{\lambda}_2 = (\lambda^2)^*$, which again differs from the traditional assignment $\lambda^i \leftrightarrow (\lambda^i)^*$.

These rules can be tested for consistency by inspecting the Yukawa couplings in the $\mathcal{N}=4$ Lagrangian. The gauge/matter coupling reads

$$\int du d^4x d^4\theta^+ \ \mathrm{Tr}(V^{++}[\tilde{q}^+, q^+]) \ \Rightarrow \ \mathrm{Tr}[\phi^i(\{\kappa,\lambda_i\} + \{\bar{\psi},\bar{\lambda}_i\})] + \text{c.c.} \tag{A.63}$$

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We see that the above assignments allow ϕ^i to stay inert under parity. At the same time, the Yukawa coupling from the gauge sector is

$$\int d^4x d^4\theta \ \mathrm{Tr}(W^2) \ \Rightarrow \ \int d^4x \mathrm{Tr}\left(\bar{\varphi}\{\lambda^i, \lambda_i\} + \varphi\{\bar{\lambda}_i, \bar{\lambda}^i\}\right) . \tag{A.64}$$

Here the combination of fermions accompanying each boson is chiral, therefore $\varphi \leftrightarrow \bar{\varphi}$.

The above parity assignments mean that the hypermultiplet composite operators of the type $\mathcal{O} = \mathrm{Tr}(q^+)^k|_{\theta=0} = \mathrm{Tr}(\phi^{i_1}(x)\dots\phi^{i_k}(x))u^+_{i_1}\dots u^+_{i_k}$ are all true scalars. Thus, the correlation functions $\langle\mathcal{O}\tilde{\mathcal{O}}\mathcal{O}\tilde{\mathcal{O}}\rangle$ that we are considering cannot have a parity-odd part.

The reason why we see such a part in the correlation functions G_{n+1} and G_{n+2} was explained after eq. (4.9). It is due to the insertion of the complex (chiral) form of the SYM Lagrangian. In it we find pseudo-scalar terms, for example $iF\tilde{F}$, which are responsible for the parity-odd part in the correlation functions with insertions. But at the final stage of the calculation, the integration over the insertion point will eliminate all such terms, which are total derivatives. Indeed, we have already encountered this phenomenon in section 4. The correlation function with one insertion G_{n+1} has the form (4.9), where we clearly see a pseudo-scalar in the last line. However, it drops out after the integration over the insertion point.

B Four-point correlation functions of operators of weight k

Let us consider the four-point correlation function of protected half-BPS operators of weight k. At the lowest level of the θ expansion they are built from k elementary scalar fields, $\mathrm{Tr}(\phi^k)$. Such correlation functions are discussed in detail in [63, 64], using the method of [32, 33]. We consider the following $\mathcal{N} = 2$ hypermultiplet projection

$$G_{4;k} = \langle\mathcal{O}^k(1)\tilde{\mathcal{O}}^k(2)\mathcal{O}^k(3)\tilde{\mathcal{O}}^k(4)\rangle \tag{B.1}$$

where in this appendix $\mathcal{O}^k = 2^{k/2}\mathrm{Tr}(q^+)^k$.

The (connected, planar) tree-level correlation function is given by (see figure 9)

$$G^{(0)}_{4;k} = k^4 N_c^{2k-2} \sum_{m=1}^{k-1} X^m Y^{k-m} , \tag{B.2}$$

where the notation was introduced for the harmonic and space-time propagator factors

$$X = \frac{(12)(34)}{(2\pi)^4 x_{12}^2 x_{34}^2} , \quad Y = \frac{(14)(32)}{(2\pi)^4 x_{14}^2 x_{23}^2} . \tag{B.3}$$

The loop corrections have the general *factorized* form

$$G^{(\mathrm{loops})}_{4;k} = \mathcal{R}' \sum_{m=0}^{k-2} \mathcal{F}^k_m(x) X^m Y^{k-m-2} \tag{B.4}$$

with the polynomial prefactor

$$\mathcal{R}' = sX^2 + (t-s-1)XY + Y^2 \tag{B.5}$$

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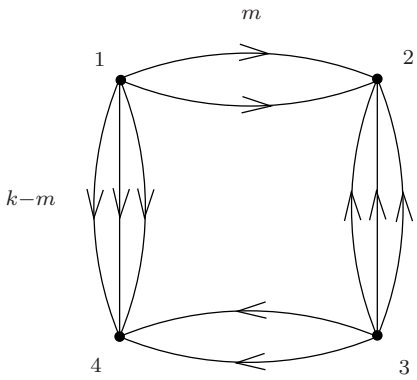

Figure 9. Tree graphs for operators of weight k.

involving the conformal cross-ratios

$$s = \frac{x_{12}^2 x_{34}^2}{x_{14}^2 x_{23}^2}\,, \quad t = \frac{x_{13}^2 x_{24}^2}{x_{14}^2 x_{23}^2}\,. \tag{B.6}$$

The coefficient functions $\mathcal{F}(x)$ will be specified below.

The result of [63] for the one-loop correction in the planar limit is

$$G_{4;k}^{(1)} = k^4 N_c^{2k-2} \mathcal{R}'\ 2a\, x_{14}^2 x_{23}^2\, g(1234) \sum_{m=1}^{k-1} X^{m-1} Y^{k-m-1}\,, \tag{B.7}$$

where the one-loop box integral $g(1234)$ is defined in (5.7). We see that in this case the general amplitude (B.4) becomes completely degenerate, with all $\mathcal{F}_m^k(x) \propto x_{14}^2 x_{23}^2\, g(1234)$. Going to the light-cone limit $x_{12}^2 = x_{23}^2 = x_{34}^2 = x_{41}^2 \to 0$, we find that s remains finite while $t \to \infty$, hence

$$\mathcal{R}' \to tXY\,. \tag{B.8}$$

As a consequence, eq. (B.7) simplifies to

$$G_{4;k}^{(1)} = k^4 N_c^{2k-2}\ 2a\, x_{13}^2 x_{24}^2\, g(1234) \sum_{m=1}^{k-1} X^m Y^{k-m} + \text{subleading terms}\,. \tag{B.9}$$

Dividing this expression by the tree-level correlation function (B.2), we obtain the same ratio as in the case $k = 2$ at one loop, see (5.16). Thus, the ratio does not depend on the value of k, up to one loop.

The same pattern is found at two loops. According to [64], the two-loop coefficient functions in (B.4) are

$$\begin{aligned}\mathcal{F}_m^k = \frac{g^4}{(8\pi^2)^2}\{ &\left[C_m^d x_{13}^2 x_{24}^2 + (2sC_m^b - C_m^d) x_{14}^2 x_{23}^2 + \left(2C_m^b/s - C_m^d\right) x_{12}^2 x_{34}^2\right] [g(1234)]^2 \\ &+ (C_m^c - C_m^d) 2[x_{13}^2 h(123;134) + x_{24}^2 h(124;234)]\end{aligned}$$

$$+ (C_m^d - C_m^{a_1})2[x_{14}^2 h(124;134) + x_{23}^2 h(123;234)]$$
$$+ (C_m^d - C_m^{a_2})2[x_{12}^2 h(123;124) + x_{34}^2 h(134;234)]\big\}\,, \tag{B.10}$$

with the two-loop integral h defined in (5.7) and with color factors C given in [64]. Going to the light-cone limit, only the first term from the first line and the term from the second line survive. Further, in the planar limit the remaining color factors simplify to (see [64])

$$C_m^d = k^4 N_c^{2k}\,, \qquad C_m^c = 2k^4 N_c^{2k}\,. \tag{B.11}$$

As a result, we find that on the light cone all the relevant two-loop coefficients become equal,

$$\mathcal{F}_m^k(x) = a^2\, k^4 N_c^{2k-2}\left\{(x_{13}^2 x_{24}^2)^2[g(1234)]^2 + 2x_{13}^2 x_{24}^2[x_{13}^2 h(123;134) + x_{24}^2 h(124;234)]\right\}\,. \tag{B.12}$$

Substituting this result into (B.4) and taking account of (B.8), we obtain the two-loop correction

$$G_{4;k}^{(2)} = \big\{(x_{13}^2 x_{24}^2)^2[g(1234)]^2 + 2x_{13}^2 x_{24}^2[x_{13}^2 h(123;134) + x_{24}^2 h(124;234)]\big\}$$
$$\times\, a^2\, k^4 N_c^{2k-2} \sum_{m=1}^{k-1} X^m Y^{k-m} + \text{subleading terms}\,. \tag{B.13}$$

Finally, dividing by the tree-level correlation function (B.2), we obtain the same ratio as in the case $k = 2$ at two loops, see (5.16). This confirms that the ratio does not depend on the value of k up to two loops.

In conclusion, we can claim that the duality relation

$$\lim_{x_{i,i+1}^2 \to 0} G_{4;k}/G_{4;k}^{(0)} = \left(\mathcal{A}_{4;k}/\mathcal{A}_{4;k}^{(0)}\right)^2 + O(a^3) \tag{B.14}$$

holds for any weight k of the half-BPS operators.

B.0.1 Relation to Wilson loops

From our analysis it follows that the correlation function in the planar limit has the following universal form on the light-cone

$$\langle q^{k_1}(1)\tilde{q}^{k_2}(2)q^{k_3}(3)\tilde{q}^{k_4}(4)\rangle = \langle q^{k_1}(1)\tilde{q}^{k_2}(2)q^{k_3}(3)\tilde{q}^{k_4}(4)\rangle^{(0)}\,[W(x_1,x_2,x_3,x_4)]^2\,, \tag{B.15}$$

where $W(x_i)$ is a light-like Wilson loop in the fundamental representation of $SU(N_c)$ evaluated along the contour $[x_1,x_2]\cup[x_2,x_3]\cup[x_3,x_4]\cup[x_4,x_1]$,

$$W(x_i) = \frac{1}{N_c}\langle 0|\,\mathrm{Tr\,P}\exp\left(ig\oint_{\square} dx\cdot A(x)\right)|0\rangle\,. \tag{B.16}$$

The diagrammatic derivation of the above relation is shown in figure 10. As explained in the parallel paper [39], each propagator connecting a pair of adjacent points x_i and x_{i+1} is approximated by a free propagator multiplied by a Wilson line along the segment $[x_i,x_{i+1}]$, evaluated in the adjoint representation, $W_{\text{adj}}[x_i,x_{i+1}]$. Then, the vertex at point

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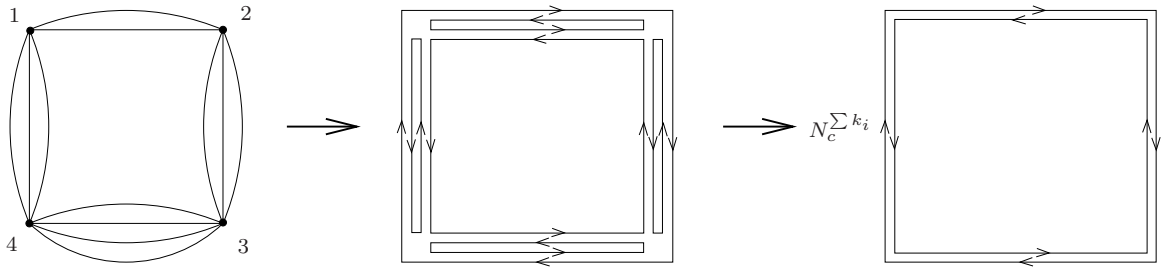

Figure 10. In the planar limit each propagator is replaced by a pair of arrowed lines and the vertex is replaced as shown in figure 11. Each line with an arrow corresponds to a Wilson line.

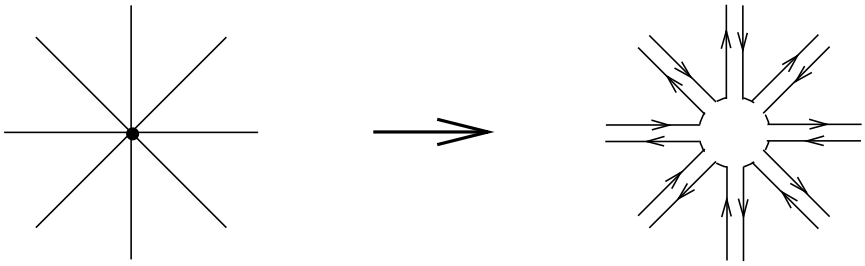

Figure 11. Each vertex q^k in the planar limit is replaced by k pairs of arrowed lines.

x_i contains k_i Wilson lines with their color indices contracted to ensure that the total color charge is zero. The Wilson lines in the adjoint and in the fundamental representations are related to each other as

$$(W_{\rm adj}[x_i, x_{i+1}])_{ab} t^b = W_{\rm fund}[x_i, x_{i+1}] t^a W_{\rm fund}[x_{i+1}, x_i]\,, \tag{B.17}$$

or equivalently (for the gauge group U(N))

$$(W_{\rm fund}[x_i, x_{i+1}])_{ij} (W_{\rm fund}[x_{i+1}, x_i])_{kl} = (t^a)_{kj} (W_{\rm adj}[x_i, x_{i+1}])_{ab} (t^b)_{il}\,. \tag{B.18}$$

In the multi-color limit, we can use the last identity to replace a Wilson line in the adjoint representation by a pair of two parallel fundamental Wilson lines with opposite orientations. This is shown in the middle panel of figure 10. In this way, we obtain a collection of closed cycles. We observe that all cycles but two have a backtrack shape, i.e. the corresponding contour encircles a zero area. We denote such a contour by $C \cup C^{-1}$. Only two cycles go through all cusp points x_i with different orientations. Notice that the Wilson lines satisfy the unitarity condition

$$W_C (W_C)^\dagger = W_C W_{C^{-1}} = 1\,. \tag{B.19}$$

As a consequence, each backtrack cycle reduces to 1 and we arrive at the right-hand side panel in figure 10. It contains only two cycles, each corresponding to a Wilson loop in the fundamental representation.

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