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Planck constant as spectral parameter in integrable systems and KZB equations

A. Levin,[a,b] **M. Olshanetsky**[b,c] **and A. Zotov**[d,b,c]

[a] *NRU HSE, Department of Mathematics, Myasnitskaya str. 20, Moscow, 101000, Russia*
[b] *ITEP, B. Cheremushkinskaya str. 25, Moscow, 117218, Russia*
[c] *MIPT, Inststitutskii per. 9, Dolgoprudny, Moscow region, 141700, Russia*
[d] *Steklov Mathematical Institute RAS, Gubkina str. 8, Moscow, 119991, Russia*

E-mail: alevin@hse.ru, olshanet@itep.ru, zotov@mi.ras.ru

Abstract: We construct special rational gl_N Knizhnik-Zamolodchikov-Bernard (KZB) equations with $\tilde{N}$ punctures by deformation of the corresponding quantum gl_N rational R-matrix. They have two parameters. The limit of the first one brings the model to the ordinary rational KZ equation. Another one is τ. At the level of classical mechanics the deformation parameter τ allows to extend the previously obtained modified Gaudin models to the modified Schlesinger systems. Next, we notice that the identities underlying generic (elliptic) KZB equations follow from some additional relations for the properly normalized R-matrices. The relations are noncommutative analogues of identities for (scalar) elliptic functions. The simplest one is the unitarity condition. The quadratic (in R matrices) relations are generated by noncommutative Fay identities. In particular, one can derive the quantum Yang-Baxter equations from the Fay identities. The cubic relations provide identities for the KZB equations as well as quadratic relations for the classical r-matrices which can be treated as halves of the classical Yang-Baxter equation. At last we discuss the R-matrix valued linear problems which provide $\mathrm{gl}_{\tilde{N}}$ CM models and Painlevé equations via the above mentioned identities. The role of the spectral parameter plays the Planck constant of the quantum R-matrix. When the quantum gl_N R-matrix is scalar ($N = 1$) the linear problem reproduces the Krichever's ansatz for the Lax matrices with spectral parameter for the $\mathrm{gl}_{\tilde{N}}$ CM models. The linear problems for the quantum CM models generalize the KZ equations in the same way as the Lax pairs with spectral parameter generalize those without it.

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1 Introduction

Let V be a finite-dimensional module of the group GL_N. The quantum R-matrix is an operator $R : V \otimes V \to V \otimes V$ satisfying the quantum Yang-Baxter equation [52–57]:

$$R^{\hbar}_{12}(z-w)\, R^{\hbar}_{13}(z)\, R^{\hbar}_{23}(w) = R^{\hbar}_{23}(w)\, R^{\hbar}_{13}(z)\, R^{\hbar}_{12}(z-w)\,, \tag{1.1}$$

where z, w - spectral parameters. We consider a special class of non-dynamical R-matrices which includes Belavin's elliptic gl_N R-matrix and its (nontrivial) degenerations, i.e. z is a local coordinate on the (degenerated) elliptic curve. Let us fix the normalization of $R^{\hbar}$ in the way that the unitarity condition takes the form

$$R^{\hbar}_{12}(z) R^{\hbar}_{21}(z) = 1 \otimes 1 \Phi^{\hbar}(z) \Phi^{\hbar}(-z)\,, \tag{1.2}$$

where $\Phi^\hbar(z)$ is the function defined in the elliptic case[1] as

$$\Phi^\hbar(z) = N\phi(N\hbar, z)\,, \quad \phi(z,u) = \frac{\vartheta'(0)\vartheta(u+z)}{\vartheta(z)\vartheta(u)}\,, \tag{1.3}$$

where $\vartheta(z) = \theta_{11}(z|\tau)$ is the odd Riemann theta-function, τ — elliptic moduli.

We demonstrate here that starting with the R-matrix one can construct different families of classical and quantum integrable system. These constructions are based on two special features of the R-matrices. The first one is the quasi-classical expansion. With the normalization (1.2)–(1.3) it acquires the form:

$$R^\hbar_{12}(z) = \frac{1}{\hbar}\, 1\otimes 1 + r_{12}(z) + \hbar\; m_{12}(z) + O(\hbar^2)\,, \tag{1.4}$$

where $r_{12}(z)$ is the classical r-matrix. It leads to integrable Euler-Arnold gl_N tops[2] and Gaudin systems.

The second is the property of Painlevé-Calogero correspondence, which is equivalent to the heat equation:

$$\partial_\tau R^\hbar_{12}(z) = \partial_z\partial_\hbar R^\hbar_{12}(z) \tag{1.5}$$

The latter leads to the monodromy preserving equations (non-autonomous tops, Schlesinger systems) and the KZB systems.

At last, the main tool is the set of identities for the quantum R-matrices which we introduce below. R-matrix is an operator acting on the tensor product of vector spaces V. Consider a set of points $z_1,\ldots,z_{\tilde N}$ (on the curve where z is a local coordinate). Let

$$R^\hbar_{ab} = R^\hbar(z_a - z_b)\,, \tag{1.6}$$

be the R-matrix acting on the a-th and b-th components of $V^{\otimes\tilde N}$. In our case R-matrices satisfy the following property:

$$R^\hbar_{ab}(z_a - z_b) = -R^{-\hbar}_{ba}(z_b - z_a)\,, \tag{1.7}$$

i.e. the terms of the expansion (1.4) are of definite parity:

$$r_{ab} = -r_{ba}\,, \quad m_{ab} = m_{ba}\,. \tag{1.8}$$

We show that the R-matrices satisfy a set of relations similar to identities for function $\phi(z,u)$ (1.3). In particular, $\phi(z,u)$ satisfies the Fay identity

$$\phi(x, z_{ab})\phi(y, z_{bc}) = \phi(x-y, z_{ab})\phi(y, z_{ac}) + \phi(y-x, z_{bc})\phi(x, z_{ac})\,, \tag{1.9}$$

where $z_{ab} = z_a - z_b$. We notice that the following analogue of the Fay identity holds:

$$R^\hbar_{ab}R^{\hbar'}_{bc} = R^{\hbar'}_{ac}R^{\hbar-\hbar'}_{ab} + R^{\hbar'-\hbar}_{bc}R^\hbar_{ac} \tag{1.10}$$

It will be shown that one can derive the quantum Yang-Baxter equation (1.1) from (1.10).

[1] In the rational case we use $\Phi^\hbar(z) = z^{-1} + \hbar^{-1}$. The trigonometric case will be considered separately.

[2] The integrable tops were previously proved to be related (equivalent) to the (spin) Calogero-Ruijsenaars models by the symplectic Hecke transformations. See. e.g. [35, 37–40].

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While the quantum R-matrix is similar to $\phi(\hbar, z)$ the classical r-matrix is the analogue of function $E_1(z) = \partial_z \log \vartheta(z)$. For example, the following relation holds:

$$(r_{ab} + r_{bc} + r_{ca})^2 = 1_a \otimes 1_b \otimes 1_c \, N^2(\wp(z_a - z_b) + \wp(z_b - z_c) + \wp(z_c - z_a))\,, \tag{1.11}$$

where $\wp(z)$ is the Weierstrass $\wp$-function with moduli τ. It is the analogue of the identity

$$(E_1(z_a - z_b) + E_1(z_b - z_c) + E_1(z_c - z_a))^2 = \wp(z_a - z_b) + \wp(z_b - z_c) + \wp(z_c - z_a)\,. \tag{1.12}$$

Together with (1.11) the classical Yang-Baxter equation

$$[r_{ab}, r_{ac}] + [r_{ac}, r_{bc}] + [r_{ab}, r_{bc}] = 0 \tag{1.13}$$

leads to the following relations:

$$r_{ab}\, r_{ac} - r_{bc}\, r_{ab} + r_{ac}\, r_{bc} = m_{ab} + m_{bc} + m_{ac}\,. \tag{1.14}$$

Difference of (1.14) written for indices a, b, c and a, c, b gives (1.13).

Let us remark that the class of R-matrices we discuss here includes Baxter-Belavin's one [5–7] as the most general. Its trigonometric analogue was found in [4, 16] (we are going to consider it in separate publications). At last the rational case is known from [16, 39, 40, 58–60]. In the simplest cases one gets the ordinary XXZ and XXX Yang's R-matrices. In the rational case the Yang's R-matrix [61] (with normalization (1.2)) is of the form:

$$R^{\hbar,\text{Yang}}_{ab} = \frac{1_a \otimes 1_b}{\hbar} + \frac{P_{ab}}{z_a - z_b}\,, \tag{1.15}$$

where P_{ab} is the permutation operator. We deal with non-trivial deformations of (1.15). In particular, they allow us to define not only KZ but also KZB equations. At the same time the rest of our construction works for ordinary XXX (and XXZ) R-matrices as well.[3]

The purpose of the paper. is twofold. First, we construct the rational analogue of the (elliptic) KZB equations. For this purpose we find τ deformation of the quantum R-matrix suggested in [39, 40]. Second, we show that integrable systems of Calogero-Moser type admit higher rank Lax representations which generalize the Krichever's one [30] in the same way as (1.10) generalize (1.9). The standard (non-diagonal) matrix elements $\phi(\lambda, z_a - z_b)$ are replaced by the quantum R-matrices R^{λ}_{ab}, i.e. the spectral parameter is given by the Planck constant entering R-matrix. Our constructions are independent of specific form of the R-matrix, but based only on the set of identities (such as (1.10), (1.14), (1.5)) which can be verified separately.

Rational KZB equations. Besides the standard trigonometric and rational versions of the elliptic R-matrix there are more sophisticated degenerations. In this paper we consider one of them [39, 40] and show that it leads to some modifications of the standard Gaudin and Schlesinger systems and the KZ (KZB) equations.

[3]It is interesting if similar construction works for Toda-like models which can be obtained from the elliptic systems by nontrivial (Inozemtsev) degenerations.

The Belavin's R-matrix depends on the moduli of the elliptic curve τ. We notice that it satisfies the heat equation (1.5) and treat this equation as Painevé-Calogero property. In [31, 32] it was formulated in the following way: the Lax pair of the CM model satisfies also the monodromy preserving equations and describe the (higher rank) Painlevé equations. We refer to (1.5) as the heat equation because this equation for the function $\phi(\hbar, z)$ follows from the heat equation for ϑ-function $2\partial_\tau\vartheta(z|\tau) = \partial_z^2\vartheta(z|\tau)$.

The natural (noncommutative) analogue of ϑ-function is the modification of bundle $\Xi(z,\tau)$. In the elliptic case it was found in [26] in the context of the IRF-Vertex transformation, and then described in [35] (see also [37–40]) as an example of the Symplectic Hecke Correspondence for integrable systems. Its rational analogue was suggested in [3] and was know to be free of τ dependence. Here we explain how to introduce the τ-dependence. We construct the τ deformation of the rational R-matrix based on the heat equation

$$2\partial_\tau\Xi = \partial_z^2\Xi\,. \tag{1.16}$$

The solution provides possibility for construction of the rational analogue of the KZB equations

$$\begin{cases} \hat\nabla_a\psi = 0\,, \quad \nabla_a = \partial_{z_a} + \sum\limits_{c\neq a} \mathfrak{r}^\tau_{ac}(z_a - z_c)\,, \\ \hat\nabla_\tau\psi = 0\,, \quad \nabla_\tau = \partial_\tau + \frac{1}{2}\sum\limits_{b,c} \mathfrak{m}^\tau_{bc}(z_b - z_c)\,, \end{cases} \tag{1.17}$$

where r and m are the terms of the expansion (1.4) and τ indicates the τ-deformation. The system of KZ of KZB equations is known to be related to the quantum (and classical) CM models by the Matsuo-Cherednik construction [17, 46] (see also [47]). Recently relations between CM (and Ruijsenaars-Schneider (RS)) models to quantum spin chains were actively investigated [1, 2, 27].

2. *R*-matrix valued Lax pairs. The Fay type identities (1.10) for the quantum R-matrices allows to suggest extended version of the Krichever's ansatz for CM Lax pairs with spectral parameter [30]. Consider the following block matrix Lax operator

$$\mathcal{L} = \sum_{a,b=1}^{\tilde N} \tilde{\mathrm{E}}_{ab} \otimes \mathcal{L}_{ab} \tag{1.18}$$

where $\tilde{\mathrm{E}}_{ab}$ is the standard basis of $\mathrm{gl}_{\tilde N}$ and

$$\mathcal{L}_{ab} = \delta_{ab} p_a\, 1_a \otimes 1_b + \nu(1-\delta_{ab})R^\hbar_{ab}\,, \quad R^\hbar_{ab} = R^\hbar_{ab}(z_a - z_b)\,. \tag{1.19}$$

When $N = 1$ the gl_N R-matrix reduces to its scalar analogue — function $\phi(z,\hbar)$ and we reproduce the answer from [30] for $\tilde N$-body CM system. Notice that the Planck constant of gl_N R-matrix plays here the role of the spectral parameter for $\mathrm{gl}_{\tilde N}$ CM model. The corresponding M-operator is given in (4.14). The Lax equation $\partial_t\mathcal{L} = [\mathcal{L}, \mathcal{M}]$ is equivalent to dynamics of $\tilde N$ CM particles

$$\ddot z_a = N^2\nu^2 \sum_{b\neq a} \wp'(z_a - z_b)\,. \tag{1.20}$$

In the same way the monodromy preserving equation $\partial_\tau \mathcal{L} - \partial_\hbar \mathcal{M} = [\mathcal{L}, \mathcal{M}]$ leads to the Painlevé equations

$$\partial_\tau^2 z_a = N^2 \nu^2 \sum_{b \neq a} \wp'(z_a - z_b)\,. \tag{1.21}$$

The corresponding linear problem has the form

$$(\partial_\hbar + \mathcal{L})\Psi = 0\,. \tag{1.22}$$

Let us also mention that the linear problem for the quantum version of CM model

$$\hat{\mathcal{L}}\Psi = \Psi\Lambda\,, \quad \hat{\mathcal{L}}_{ab} = \delta_{ab}\,\partial_{z_a}\,1_a \otimes 1_b + \nu(1-\delta_{ab})R^{\hbar}_{ab} \tag{1.23}$$

resembles very much the KZ connections from the first line of (1.17). Equation (1.23) (or (1.22) with $\hat{\mathcal{L}}$) generalizes the first line of (1.17) in the same way as the Lax pairs with spectral parameter generalize those without it. We hope to clarify exact relations between R-matrix valued linear problems and KZB equations in our future papers.

Choosing elliptic, trigonometric or the rational R-matrix we describe the CM models similarly to gl_1 case [30]. Notice that the gl_N R-matrix itself describes gl_N integrable systems such as integrable tops which are gauge equivalent to CM or RS models. Here we use gl_N R-matrices as auxiliary spaces for derivation of $\mathrm{gl}_{\tilde{N}}$ models. The next natural step is to get similar result for the Ruijsenaars-Schneider (quantum) model. In this case we deal with two Planck constants. Our general idea is that the both Planck constants can play different roles, i.e. each of the constants can be either the spectral parameter in a "classical-quantum" $\mathrm{gl}_{\tilde{N}}$ system (of (1.19) type) or the Planck constant in a quantum gl_N system or the relativistic deformation parameter in a classical relativistic gl_N model (see [39, 40]).[4] We hope that this can shed light on numerous dualities in integrable systems mentioned in [19, 43–45, 48, 62–64].

2 From integrable tops to KZB equations

In this section we describe the sequence of steps which leads to the KZB equations [20–22] starting from integrable tops. As it was mentioned above, our consideration is independent on the choice of particular top model. The basic element is the underlying quantum R-matrix [39, 40].

First, we briefly recall the structures underlying integrable tops and proceed to the non-autonomous dynamics. It is described by the monodromy preserving equations. In the same way the Schlesinger system is originated from the corresponding Gaudin model. At last, the KZB equations arise from the quantization of the Schlesinger system [28, 29, 33, 34, 49].

[4] Let us also remark that in [41, 42] we have already found an R-matrix intermediate between the Belavin's and the Felders' one. Her we use a different description. Presumably, the interrelation between different descriptions is given by the Fourier-Mukai type transformation.

2.1 Integrable tops

In [39, 40] we defined the relativistic integrable top by means of the quantum R-matrix. The gl_N Lax matrix is given by

$$L^\eta(z,S) = \mathrm{tr}_2(R^\eta_{12}(z)S_2)\,, \quad S = \operatorname*{Res}_{z=0} L^\eta(z,S)\,, \tag{2.1}$$

where $S = \sum\limits_{i,j=1}^{N} \mathrm{E}_{ij}S_{ij}$ is the gl_N-valued dynamical variable,[5] and $R^\eta_{12}(z)$ is the corresponding quantum non-dynamical R-matrix. It satisfies the quantum Yang-Baxter equation (1.1). The non-relativistic limit ($\eta \to 0$)

$$L^\eta(z,S) = \eta^{-1}\,\frac{\mathrm{tr}S}{N}\,1_{N\times N} + L(z,S) + \eta\,\mathcal{M}(z,S) + O(\eta^2) \tag{2.2}$$

is related to the classical limit ($\hbar \to 0$) (1.4) via (2.1):

$$L(z,S) = \mathrm{tr}_2\left(r_{12}(z)S_2\right)\,, \qquad\qquad S = \operatorname*{Res}_{z=0} L(z,S)\,, \tag{2.3}$$

$$\mathcal{M}(z,S) = \mathrm{tr}_2\left(m_{12}(z)S_2\right)\,. \tag{2.4}$$

The quantity $r_{12}(z)$ in (1.4), (2.3) is the classical r-matrix. It is skew-symmetric (1.8)

$$r_{12}(z) = -r_{21}(-z) \tag{2.5}$$

and satisfies the classical Yang-Baxter equation:

$$[r_{12}(z-w), r_{13}(z)] + [r_{12}(z-w), r_{23}(w)] + [r_{13}(z), r_{23}(w)] = 0\,. \tag{2.6}$$

As it was mentioned in [39, 40] the matrices (2.3), (2.4) appear to be the Lax pair of the non-relativistic top. It means that the Lax equation

$$\partial_t L(z,S) = [L(z,S), \mathcal{M}(z,S)] \tag{2.7}$$

is equivalent to equations of motion

$$\partial_t S = [S, J(S)]\,, \tag{2.8}$$

where the inverse inertia tensor is given by the linear functional

$$J(S) = \mathcal{M}(0,S)\,. \tag{2.9}$$

The equations (2.8) are Hamiltonian with the Hamiltonian function

$$\mathcal{H}^{\mathrm{top}}(S) = \frac{1}{2}\,\mathrm{tr}(S\,J(S)) \tag{2.10}$$

and the Poisson-Lie brackets on gl^*_N

$$\{S_1, S_2\} = [S_2, P_{12}] \tag{2.11}$$

or $\{S_{ij}, S_{kl}\} = \delta_{il}S_{kj} - \delta_{kj}S_{il}$.

[5] $\{\mathrm{E}_{ij},\ i,j = 1\ldots N\}$ is the standard basis in the fundamental representation of gl_N: $(\mathrm{E}_{ij})_{kl} = \delta_{ik}\delta_{jl}$.

2.2 Painlevé-Calogero correspondence and non-autonomous tops

The (classical) Painlevé-Calogero correspondence was suggested in [31, 32]. It claims that the (Krichever's) Lax pair of the elliptic Calogero-Moser model can be also used for the monodromy preserving equations, which describe the higher rank Painlevé equations in the elliptic form.

Let us formulate here the Painlevé-Calogero correspondence in the form of the quantum non-dynamical R-matrix property.

Definition 1. *Suppose that the quantum R-matrix entering (2.1) depends on some additional parameter τ: $R^{\hbar,\tau}(z) = R(z,\hbar,\tau)$. We say that the R-matrix satisfies the property of the "Painlevé-Calogero correspondence" if the following relation holds:*[6]

$$\partial_\tau R^{\hbar,\tau}(z) = \partial_z \partial_\hbar R^{\hbar,\tau}(z)\,. \tag{2.12}$$

Plugging the expansion (1.4) into (2.12) we get a set of relations. The first non-trivial is

$$\partial_\tau r_{12}^\tau(z) = \partial_z m_{12}^\tau(z)\,, \tag{2.13}$$

where $r_{12}^\tau(z) = r_{12}(z,\tau)$ is the classical r-matrix. An example of the R-matrix with this property is given by the Baxter-Belavin's one [5] (see appendix B). The parameter τ in this example equals $\tau^{ell}/2\pi\imath$, where τ^{ell} is the module of the underlying elliptic curve, and the property (2.13) is due to the heat equation for the theta-functions

$$2\partial_\tau \vartheta(z|\tau) = \partial_z^2 \vartheta(z|\tau)\,. \tag{2.14}$$

From (2.13) and (2.3)–(2.4) it follows that

$$\frac{\partial}{\partial\tau} L^\tau(z,S) = \frac{\partial}{\partial z} \mathcal{M}^\tau(z,S)\,, \tag{2.15}$$

where $L^\tau(z,S) = L(z,S,\tau)$, $\mathcal{M}^\tau(z,S) = \mathcal{M}(z,S,\tau)$. Therefore, we can define the monodromy preserving equations in time τ

$$d_\tau L^\tau(z,S) - \partial_z \mathcal{M}^\tau(z,S) = [L^\tau(z,S), \mathcal{M}^\tau(z,S)]\,, \quad S = S(\tau) \tag{2.16}$$

($d_\tau = \frac{d}{d\tau}$) as the non-autonomous version of the integrable top's equations of motion (2.8):[7]

$$\partial_\tau S = [S, J^\tau(S)]\,. \tag{2.17}$$

Indeed, the total derivative $d_\tau L^\tau(z,S)$ contains both — the partial derivatives by explicit and implicit dependence on τ:

$$d_\tau L^\tau(z,S(\tau)) = d_\tau \mathrm{tr}_2(r_{12}^\tau(z)S_2) = \mathrm{tr}_2\Big((\partial_\tau r_{12}^\tau(z))\,S_2\Big) + \mathrm{tr}_2\Big(r_{12}^\tau(z)\,(\partial_\tau S_2)\Big)\,. \tag{2.18}$$

[6] Notice that the definition depends on the gauge choice.

[7] These models are no more integrable but can be treated as alternative description of (higher) Painlevé equations. See [36] for the example of Painlevé VI.

The first term is cancelled by $\partial_z \mathcal{M}^\tau(z,S)$ (2.15), and we get the same result as in (2.8) following from the Lax equations (2.7). But this time it contains explicit dependence on τ via

$$J^\tau(S) = \mathcal{M}^\tau(0,S)\,. \tag{2.19}$$

Similarly to the autonomous case this system is Hamiltonian (see (2.10)) with

$$\mathcal{H}^\tau(S) = \frac{1}{2}\,\mathrm{tr}(S\,J^\tau(S)) \tag{2.20}$$

and the Poisson brackets are given by (2.11).

Let us keep the notation $\frac{\partial}{\partial\tau}$ (but not ∂_τ) for the partial derivative by only explicit dependence on τ, i.e.

$$\frac{\partial}{\partial\tau} L^\tau(z,S(\tau)) = \mathrm{tr}_2\Big((\partial_\tau r^\tau_{12}(z))\,S_2(\tau)\Big)\,. \tag{2.21}$$

2.3 Gaudin models

The phase space of the Gaudin model [8–11] is the direct product of n coadjoint orbits, i.e. $\tilde N$ copies of S: $S^a \in \mathrm{gl}_N$, $a=1,\dots,\tilde N$ with some fixed eigenvalues. Its Poisson structure

$$\{S^a_1, S^b_2\} = \delta^{ab}\,[S^a_2, P_{12}] \tag{2.22}$$

is the direct sum of (2.11). The Lax matrix has n simple poles at $\{z_a,\ a=1,\dots,\tilde N\}$ with residues S^a. It is given in terms of the top Lax matrix (2.3):

$$L^{\mathrm{G}}(z) = \sum_{a=1}^{\tilde N} L^\tau(z-z_a,S^a) = \sum_{a=1}^{\tilde N} \mathrm{tr}_2\Big(r^\tau_{12}(z-z_a)S^a_2\Big)\,. \tag{2.23}$$

Here we imply the existence of the deformation parameter τ (2.14)–(2.20) from the very beginning in order not to repeat (almost) the same notations with τ and without τ as we made for the top and its non-autonomous version.

We consider the flows corresponding to Hamiltonians

$$h_a = -\sum_{c\neq a}^{\tilde N} \mathrm{tr}\,(S^a\,L^\tau(z_a-z_c,S^c)) = -\sum_{c\neq a}^{\tilde N} \mathrm{tr}_{12}\Big(r^\tau_{12}(z_a-z_c)S^a_1 S^c_2\Big) \tag{2.24}$$

for $a=1,\dots,\tilde N$ and

$$\mathcal{H}_0 = \frac{1}{2}\sum_{b,c=1}^{\tilde N} \mathrm{tr}\,\Big(S^b\,\mathcal{M}^\tau(z_b-z_c,S^c)\Big) = \frac{1}{2}\sum_{b,c=1}^{\tilde N} \mathrm{tr}_{12}\Big(m^\tau_{12}(z_a-z_c)S^b_1 S^c_2\Big)\,. \tag{2.25}$$

Notice that the terms coming from $b=c$ in (2.25) are the top Hamiltonians $\mathcal{H}^\tau(S^c)$ (2.20). The functions (2.24)–(2.25) Poisson commute because (2.22) is equivalent to the classical exchange relations

$$\{L^{\mathrm{G}}_1(z), L^{\mathrm{G}}_2(w)\} = [L^{\mathrm{G}}_1(z) + L^{\mathrm{G}}_2(w), r^\tau_{12}(z-w)]\,. \tag{2.26}$$

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The dynamics generated by (2.24)–(2.25)

$$\begin{cases} \partial_{t_a} S^b = -[S^b, L^\tau(z_a - z_b, S^a)]\,, \quad b \neq a \\ \partial_{t_a} S^a = \sum\limits_{c \neq a}^{n} [S^a, L^\tau(z_c - z_a, S^c)] \end{cases} \tag{2.27}$$

for $a = 1, \dots, \tilde{N}$ and

$$\partial_{t_0} S^a = [S^a, J^\tau(S^a)] + \sum_{c \neq a} [S^a, \mathcal{M}^\tau(z_a - z_c, S^c)] \tag{2.28}$$

possesses the Lax representations

$$\partial_{t_d} L^{\mathrm{G}}(z) = [L^{\mathrm{G}}(z), M^{\mathrm{G},\,d}]\,, \quad d = 0, \dots, \tilde{N} \tag{2.29}$$

where

$$M^{\mathrm{G},\,a}(z) = -L^\tau(z - z_a, S^a)\,, \quad a = 1, \dots, \tilde{N} \tag{2.30}$$

and

$$M^{\mathrm{G},\,0}(z) = \sum_{c=0}^{\tilde{N}} \mathcal{M}^\tau(z - z_c, S^c)\,. \tag{2.31}$$

2.4 Schlesinger systems

Similarly to the description of Painlevé equation in the form of non-autonomous tops let us also represent the Schlesinger system [50, 51] as the non-autonomous Gaudin model.

First, it follows from (2.23) and (2.30) that

$$\frac{\partial}{\partial z_a} L^{\mathrm{G}}(z) = \frac{\partial}{\partial z} M^{\mathrm{G},\,a}(z)\,. \tag{2.32}$$

Secondly, it follows from (2.23), (2.31) and (2.15) that[8]

$$\frac{\partial}{\partial \tau} L^{\mathrm{G}}(z) = \frac{\partial}{\partial z} M^{\mathrm{G},\,0}(z)\,. \tag{2.33}$$

Therefore, the monodromy preserving equations (or compatibility conditions for isomonodromic deformations)

$$\partial_{z_a} L^{\mathrm{G}}(z) - \partial_z M^{\mathrm{G},\,a}(z) = [L^{\mathrm{G}}(z), M^{\mathrm{G},\,a}(z)] \tag{2.34}$$

and

$$\partial_\tau L^{\mathrm{G}}(z) - \partial_z M^{\mathrm{G},\,0}(z) = [L^{\mathrm{G}}(z), M^{\mathrm{G},\,0}(z)] \tag{2.35}$$

generate dynamics in time variables z_a and τ. They have form form of non-autonomous versions of the Gaudin's one (2.27)–(2.28):

$$\begin{cases} \partial_{z_a} S^b = -[S^b, L^\tau(z_a - z_b, S^a)]\,, \quad b \neq a \\ \partial_{z_a} S^a = \sum\limits_{c \neq a}^{\tilde{N}} [S^a, L^\tau(z_c - z_a, S^c)] \end{cases} \tag{2.36}$$

[8] In (2.32) and (2.33) the partial derivatives are taken with respect to explicit dependence on τ or z_a (2.21).

for $a=1,\ldots,\tilde{N}$ and

$$\partial_\tau S^a = [S^a, J^\tau(S^a)] + \sum_{c\neq a} [S^a, \mathcal{M}^\tau(z_a - z_c, S^c)]\,. \tag{2.37}$$

The Hamiltonians (2.24)–(2.25) and the Poisson structure (2.22) are of the same form.[9]

2.5 KZB equations

The relation between KZB equations and the quantum monodromy preserving equations was described in [49] (see also [28, 29, 33, 34]). Let us formulate it using notations of (1.4) with the τ-deformation satisfying (2.13). The KZB equations have form:

$$\begin{cases} \hat{\nabla}_a \psi = 0\,, \\ \hat{\nabla}_\tau \psi = 0\,, \end{cases} \tag{2.38}$$

where

$$\nabla_a = \partial_{z_a} + \sum_{c\neq a} \mathfrak{r}^\tau_{ac}(z_a - z_c)\,, \tag{2.39}$$

$$\nabla_\tau = \partial_\tau + \frac{1}{2}\sum_{b,c} \mathfrak{m}^\tau_{bc}(z_b - z_c)\,. \tag{2.40}$$

Here $\mathfrak{r}^\tau_{ac}$ and $\mathfrak{m}^\tau_{ac}$ are the operators acting by a-th and c-th components of $\mathrm{U}(\mathrm{gl}_N)^{\otimes\tilde{N}}$ (the tensor product of $\tilde{N}$ copies of the universal enveloping algebra). Recall that in classical integrable systems (as well as in the Schlesinger systems) we used the fundamental representation ρ_N of gl_N (see e.g. (2.3)–(2.4)):

$$\begin{aligned} r^\tau_{12}(z) &= \rho_N(\mathfrak{r}^\tau_{12}(z)) = \sum_{i,j,k,l} r^\tau_{ij,kl}\, \mathrm{E}_{ij}\otimes \mathrm{E}_{kl}\,, \\ m^\tau_{12}(z) &= \rho_N(\mathfrak{m}^\tau_{12}(z)) = \sum_{i,j,k,l} m^\tau_{ij,kl}\, \mathrm{E}_{ij}\otimes \mathrm{E}_{kl}\,, \end{aligned} \tag{2.41}$$

The algebra $\mathrm{U}(\mathrm{gl}_N)^{\otimes\tilde{N}}$ can be considered as a quantization of the classical phase space with the Poisson structure (2.22). Indeed, let

$$S^a \;\rightarrow\; \hat{S}^a:\quad \hat{S}^a_{ij} := \mathbf{e}^a_{ji}\,, \tag{2.42}$$

where $\{\mathbf{e}^a_{ij}\}$: $[\mathbf{e}^a_{ij}, \mathbf{e}^a_{kl}] = \delta^{ab}(\mathbf{e}^a_{il}\delta_{kj} - \mathbf{e}^a_{kj}\delta_{il})$ is the standard basis in the a-th component of $\mathrm{U}(\mathrm{gl}_N)^{\otimes\tilde{N}}$. In this notation

$$\mathfrak{r}^\tau_{ab} = \sum_{i,j,k,l} r^\tau_{ij,kl}(z_a - z_b)\, \mathbf{e}^a_{ij}\mathbf{e}^b_{kl} = \sum_{i,j,k,l} r^\tau_{ij,kl}(z_a - z_b)\, \hat{S}^a_{ji}\hat{S}^b_{lk}\,, \tag{2.43}$$

$$\mathfrak{m}^\tau_{ab} = \sum_{i,j,k,l} m^\tau_{ij,kl}(z_a - z_b)\, \mathbf{e}^a_{ij}\mathbf{e}^b_{kl} = \sum_{i,j,k,l} m^\tau_{ij,kl}(z_a - z_b)\, \hat{S}^a_{ji}\hat{S}^b_{lk}\,. \tag{2.44}$$

[9] The elliptic case was considered in [18, 28, 29, 33, 34, 37, 38].

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The fundamental representation is given by $\rho_N(\mathbf{e}^a_{ij}) = 1\otimes\ldots\otimes 1\otimes \mathrm{E}_{ij}\otimes 1\otimes\ldots\otimes 1$, where $(\mathrm{E}_{ij})_{kl} = \delta_{ik}\delta_{jl}$ is on the a-th place. Then r-matrix is an operator acting on the a-th and b-th components of an element of the tensor product $V^{\otimes\tilde{N}}$. The operator is represented by matrix of $N^{\tilde{N}}\times N^{\tilde{N}}$ size because it also contains (as factors) the product of identity operators for the rest of components $\bigotimes\limits_{c\neq a,b} 1_c$. The residue of r-matrix is (up to factor N in (B.11)) the permutation operator replacing a-th and b-th components of an element of the tensor product $V^{\otimes\tilde{N}}$ to which ψ belongs.

Then

$$[\hat{S}^a_0, \hat{S}^b_{0'}] = \delta^{ab}\,[\hat{S}^a_{0'}, P_{00'}]\,,\quad \hat{S}^a = \sum_{i,j=1}^{N}\hat{S}^a_{ij}\,\rho_N(\mathbf{e}^a_{ij}) \tag{2.45}$$

or $[\hat{S}^a_{ij}, \hat{S}^b_{kl}] = \delta^{ab}\left(\hat{S}^a_{kj}\delta_{il} - \hat{S}^a_{il}\delta_{kj}\right)$. The indices $0, 0'$ in (2.45) are the notations for the components of $\left(\rho_N(\mathrm{U}(\mathrm{gl}_N)^{\otimes\tilde{N}})\right)^{\otimes 2}$ — tensor product of auxiliary spaces. To quantize the Hamiltonian (2.25) we also need to fix the ordering. Consider the symmetric (Weyl) ordering

$$\widehat{S^a_{ij}S^b_{kl}} = \frac{1}{2}\left(\hat{S}^a_{ij}\hat{S}^b_{kl} + \hat{S}^b_{kl}\hat{S}^a_{ij}\right). \tag{2.46}$$

Then the KZB connections (2.39)–(2.40) are written in terms of the quantum versions of the classical Hamiltonians h_a and $\mathcal{H}_0$ (2.24)–(2.25):

$$\hat{\nabla}_a = \partial_{z_a} - \hat{h}_a\,,\quad \hat{\nabla}_\tau = \partial_\tau + \hat{\mathcal{H}}_0\,. \tag{2.47}$$

In the same time the KZB equations (2.38) acquire the form of the non-stationary Schrödinger equations in times $z_1,\ldots,z_{\tilde{N}}$ and τ.

The compatibility conditions of KZB equations (2.38)

$$[\hat{\nabla}_a, \hat{\nabla}_b] = 0 \tag{2.48}$$

$$[\hat{\nabla}_a, \hat{\nabla}_\tau] = 0 \tag{2.49}$$

are fulfilled identically.[10] The first one (2.48) follows from the classical Yang-Baxter equation

$$[\mathfrak{r}_{ab}, \mathfrak{r}_{bc}] + [\mathfrak{r}_{bc}, \mathfrak{r}_{ac}] + [\mathfrak{r}_{ab}, \mathfrak{r}_{ac}] = 0\,, \tag{2.50}$$

where $\mathfrak{r}_{ab} = \mathfrak{r}^\tau_{ab}(z_a - z_b)$. The set of identities underlying (2.48) consists of the property (2.13)

$$\partial_\tau \mathfrak{r}_{ab} = \partial_{z_a}\mathfrak{m}_{ab}\,, \tag{2.51}$$

where $\mathfrak{m}_{ab} = \mathfrak{m}^\tau_{ab}(z_a - z_b)$ and

$$\frac{1}{2}\,[\mathfrak{r}_{ab}, \mathfrak{m}_{aa} + \mathfrak{m}_{bb}] + [\mathfrak{r}_{ab}, \mathfrak{m}_{ab}] = 0\,, \tag{2.52}$$

$$[\mathfrak{r}_{ab}, \mathfrak{m}_{bc}] + [\mathfrak{r}_{ab}, \mathfrak{m}_{ac}] + [\mathfrak{r}_{ac}, \mathfrak{m}_{ab}] + [\mathfrak{r}_{ac}, \mathfrak{m}_{bc}] = 0\,. \tag{2.53}$$

[10] This statement was verified directly in different cases. See [23–25, 28, 29] for elliptic examples.

Remark 1. *One can get more identities relating r_{ab} and m_{ab} and higher order terms of expansion (1.4) from the Yang-Baxter equation (1.1) $R^{\hbar,\tau}_{ab}R^{\hbar,\tau}_{ac}R^{\hbar,\tau}_{bc} = R^{\hbar,\tau}_{bc}R^{\hbar,\tau}_{ac}R^{\hbar,\tau}_{ab}$. The first non-trivial identity is (2.50). The next one is*

$$\begin{aligned}&[r_{ab},m_{ac}]+[m_{ab},r_{ac}]+[r_{ab},m_{bc}]+[m_{ab},r_{bc}]+[r_{ac},m_{bc}]+[m_{ac},r_{bc}]+\\&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad+r_{ab}r_{ac}r_{bc}-r_{bc}r_{ac}r_{ab}=0\,,\end{aligned}\tag{2.54}$$

where $r_{ab}=r^{\tau}_{ab}(z_a-z_b)$, $m_{ab}=m^{\tau}_{ab}(z_a-z_b)$.

3 Rational non-autonomous tops and KZB equations

The rational top was first studied for small rank cases in [58–60] by degenerating the elliptic Lax matrix [35]. Later it was constructed for gl_N case using its relation to the rational Calogero-Moser model [3]. The idea was to compute the classical (skew-symmetric non-dynamical) r-matrix as follows:

$$r_{12}(z)=\frac{\partial L_1(z,S)}{\partial S_2}\,,\quad S=\operatorname*{Res}_{z=0}L(z)\,.\tag{3.1}$$

In [39, 40] this relation was extended to the quantum R-matrix by proceeding to the relativistic top:

$$R^{\hbar}_{12}(z)=\frac{\partial L^{\hbar}_1(z,S)}{\partial S_2}\,,\quad S=\operatorname*{Res}_{z=0}L^{\hbar}(z)\,,\tag{3.2}$$

where the classical Lax matrix $L^{\hbar}(z)$ depends on the constant $\hbar$ playing the role of the relativistic deformation parameter. The Lax matrix was found using its relation to the Ruijsenaars-Schneider (RS) model. In the spinless case the gauge transformation relating two models

$$L^{\eta}(z,S)=g(z)L^{\mathrm{RS}}(z,\eta)g^{-1}(z)\tag{3.3}$$

can be written explicitly in terms of the RS particles coordinates q_j: $g(z,q)=\Xi(z,q)D^{-1}$, where[11]

$$\begin{aligned}&\Xi(z,q)=(z+q_j)^{\varrho(i)}\,,\\&\varrho(i)6=i-1\quad\text{for}\quad i\leq N-1;\quad\varrho(N)=N\,.\end{aligned}\tag{3.4}$$

3.1 τ-deformation of quantum rational R-matrix

Our aim is to construct τ-dependent R-matrix satisfying the Painlevé-Calogero property (2.12) starting from the τ-independent one (3.2). The answer follows from (3.8) (see below). It appears that the deformation of the Yang's rational R-matrix suggested in [39, 40] admits this kind of deformation similarly to the elliptic case. The idea is to deform first $\Xi(z)$ (3.4). Let us find $\Xi(z,q|\,\tau)$ satisfying the heat equation

$$2\partial_\tau\Xi(z|\,\tau)=\partial_z^2\,\Xi(z|\,\tau)\tag{3.5}$$

[11]The explicit from of $L^{\mathrm{RS}}(z,\eta)$ as well as diagonal matrix $D_{ij}=\delta_{ij}\prod_{k\neq i}(q_i-q_k)$ is not used in what follows.

with the boundary condition

$$\Xi(z|\,0) = \Xi(z)\,. \tag{3.6}$$

Then the R-matrices (3.1), (3.2) constructed by means of $\Xi(z|\,\tau)$ satisfy the property (2.12).[12] The solution of (3.5)–(3.6) is given by

$$\Xi(z|\,\tau) = \exp\left(\frac{\tau}{2}\,\partial_z^2\right)\Xi(z) \tag{3.7}$$

or

$$\Xi(z|\,\tau) = \exp\left(\frac{\tau}{2}\,T\right)\Xi(z)\,, \tag{3.8}$$

where T is the nilpotent operator representing the action of ∂_z^2 on the N-dimensional column-vector $(1, z, z^2, \ldots, z^{N-2}, z^N)^T$. It is $N \times N$ matrix with elements

$$T_{ij} = \begin{cases} j(j+1)\delta_{i-2,j}\,, & i < N\,, \\ j(j+1)\delta_{i-1,j}\,, & i = N\,. \end{cases} \tag{3.9}$$

For example, for $N = 2, 3, 4$ we have:

$$T_{N=2} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}\,, \qquad T_{N=3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 6 & 0 \end{pmatrix}\,, \qquad T_{N=4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 12 & 0 \end{pmatrix}\,. \tag{3.10}$$

Denote

$$\mathcal{T} := \exp\left(\frac{\tau}{2}\,T\right)\,, \tag{3.11}$$

i.e. $\Xi(z|\,\tau) = \mathcal{T}\,\Xi(z|\,0)$. Then for $N = 2, 3, 4$ the operator $\mathcal{T}$ equals

$$\mathcal{T}_{N=2} = \begin{pmatrix} 1 & 0 \\ \tau & 1 \end{pmatrix}\,, \qquad \mathcal{T}_{N=3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3\tau & 1 \end{pmatrix}\,, \qquad \mathcal{T}_{N=4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \tau & 0 & 1 & 0 \\ 6\tau^2 & 0 & 6\tau & 1 \end{pmatrix}\,. \tag{3.12}$$

It follows from (3.1)–(3.3) and (3.8) that τ-deformation of R-matrix is given by the following gauge transformation:

$$R^\hbar(z|\,\tau) = \mathcal{T}_1\mathcal{T}_2\,R^\hbar(z|\,0)\,\mathcal{T}_1^{-1}\mathcal{T}_2^{-1} \tag{3.13}$$

written in terms of (3.11). See appendix A for explicit answer in gl_3 case.

3.2 Rational KZB equations

It follows from (3.13) that

$$\begin{aligned} r^\tau_{ab}(z_a - z_b) &= \mathcal{T}_a\mathcal{T}_b\,r_{ab}(z_a - z_b)\,\mathcal{T}_a^{-1}\mathcal{T}_b^{-1}\,, \\ m^\tau_{ab}(z_a - z_b) &= \mathcal{T}_a\mathcal{T}_b\,m_{ab}(z_a - z_b)\,\mathcal{T}_a^{-1}\mathcal{T}_b^{-1}\,. \end{aligned} \tag{3.14}$$

Then the condition (2.13) is fulfilled as well as (2.51) for (2.43)–(2.44).

[12] It can be also proved directly by using explicit answer for the quantum R-matrix [39, 40].

The Lax pair (2.3)–(2.4) is transformed by not only the gauge transformation since the residue S also changes. From (2.3)–(2.4) and (3.14) we have

$$L(z, S, \tau) = \mathcal{T}\, L(z, \mathcal{T}^{-1} S \mathcal{T}, 0)\, \mathcal{T}^{-1}\,, \tag{3.15}$$

$$\mathcal{M}(z, S, \tau) = \mathcal{T}\, \mathcal{M}(z, \mathcal{T}^{-1} S \mathcal{T}, 0)\, \mathcal{T}^{-1}\,. \tag{3.16}$$

Let us summarize the results:

Proposition 1. *The τ-deformed quantum R-matrix (3.13) satisfies the Painlevé-Calogero property (2.12).*

Proposition 2. *The τ-deformed quantum r and m-matrices (3.14) define the KZB equations (2.38), i.e. the corresponding KZB connections ∇_a (2.39) and ∇_τ (2.40) are compatible (2.48), (2.49).*

The proof is direct. Below we give explicit examples of τ-deformations in the rational case.

3.3 Example: $\mathbf{gl_2}$ case

Quantum R-matrix (satisfying (2.12)):

$$R^{\hbar,\tau}(z) = \begin{pmatrix} \hbar^{-1} + z^{-1} & 0 & 0 & 0 \\ -\hbar - z & \hbar^{-1} & z^{-1} & 0 \\ -\hbar - z & z^{-1} & \hbar^{-1} & 0 \\ -(z+\hbar)(z^2 + z\hbar + \hbar^2 + 4\tau) & \hbar + z & \hbar + z & \hbar^{-1} + z^{-1} \end{pmatrix} \tag{3.17}$$

Classical r-matrix

$$r^{\tau}_{12}(z) = \begin{pmatrix} z^{-1} & 0 & 0 & 0 \\ -z & 0 & z^{-1} & 0 \\ -z & z^{-1} & 0 & 0 \\ -z^3 - 4z\tau & z & z & z^{-1} \end{pmatrix} \tag{3.18}$$

and m-matrix (the next term of expansion of (3.17) in $\hbar$) satisfying (2.13):

$$m^{\tau}_{12}(z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -2z^2 - 4\tau & 1 & 1 & 0 \end{pmatrix} \tag{3.19}$$

The following additional relation holds:

$$-\,\partial_z r^{\tau}_{12}(z) = \frac{P_{12}}{z^2} - \frac{3}{2}\, m^{\tau}_{12}(z) + \frac{1}{2}\, m^{\tau}_{12}(0)\,. \tag{3.20}$$

Non-autonomous top Lax pair and Hamiltonian:

$$L(z, S|\, \tau) = \frac{1}{z} \begin{pmatrix} S_{11} - z^2 S_{12} & S_{12} \\ S_{21} - z^2(S_{11} - S_{22}) - z^4 S_{12} - 4z^2 \tau S_{12} & S_{22} + z^2 S_{12} \end{pmatrix} \tag{3.21}$$

$$\mathcal{M}(z,S|\,\tau) = -\begin{pmatrix} S_{12} & 0 \\ S_{11}-S_{22}+2z^2S_{12}+4\tau S_{12} & -S_{12} \end{pmatrix} \tag{3.22}$$

$$H(S,\tau) = -S_{12}(S_{11}-S_{22}) - 2\tau S_{12}^2\,. \tag{3.23}$$

The Gaudin (or Schlesinger) Hamiltonians:

$$h_a = \sum_{c\neq a}^{\tilde N} h_{a,c}\,, \quad h_{a,c} = -\mathrm{tr}_{12}\left(r_{12}^{\tau}(z_a-z_c)S_1^aS_2^c\right) = \tag{3.24}$$

$$-\frac{\mathrm{tr}(S^aS^c)}{z_a-z_c} + (z_a-z_c)\Big(S_{12}^a(S_{11}^c-S_{22}^c)+S_{12}^c(S_{11}^a-S_{22}^a)+4\tau S_{12}^aS_{12}^c\Big)+(z_a-z_c)^3\,S_{12}^aS_{12}^c\,,$$

$$h_0 = \frac{1}{2}\sum_{b,c=1}^{\tilde N}\mathrm{tr}\left(S^b\,\mathcal{M}(z_b-z_c,S^c)\right) = -\sum_{b,c=1}^{n} S_{12}^b(S_{11}^c-S_{22}^c)+S_{12}^bS_{12}^c\left[(z_b-z_c)^2+2\tau\right]. \tag{3.25}$$

Some similar formulae for gl_3 case are given in the appendix A.

4 Planck constant as spectral parameter

4.1 R-matrix valued Fay identities

In this paragraph we show that the quantum R-matrices satisfy a set of relations which are similar to their scalar analogues — the functions Φ (1.2). It is convenient to discuss the elliptic case (B.5)–(B.14) because the trigonometric and rational versions are obtained by some (nontrivial) degenerations.

The function $\phi(x,z)$ (B.5) (or (B.14)) satisfies the Fay identity:

$$\phi(x,z_{ab})\phi(y,z_{bc}) = \phi(x-y,z_{ab})\phi(y,z_{ac}) + \phi(y-x,z_{bc})\phi(x,z_{ac})\,, \tag{4.1}$$

where $z_{ab} = z_a - z_b$. Let us formulate its noncommutative analogue.

Proposition 3. *The Belavin's R-matrix (B.8) satisfies the following relation:*

$$R_{ab}^{\hbar}R_{bc}^{\hbar'} = R_{ac}^{\hbar'}R_{ab}^{\hbar-\hbar'} + R_{bc}^{\hbar'-\hbar}R_{ac}^{\hbar}\,, \tag{4.2}$$

where $R_{ab}^{\hbar} = R_{ab}^{\hbar}(z_a-z_b)$.

Proof. Denote by T_α^a the basis element T_α (B.1) standing on the a-th place in the tensor product $1\otimes\ldots\otimes1\otimes T_\alpha\otimes1\otimes\ldots\otimes1$. It follows from the definition (B.8) and the multiplication rule (B.3) that

$$R_{ab}^{\hbar}R_{bc}^{\hbar'} = \sum_{\alpha,\beta} T_\alpha^a\,T_{\beta-\alpha}^b\,T_{-\beta}^c\,\kappa_{-\alpha,\beta}\,\varphi_\alpha^{\hbar}(z_a-z_b)\varphi_\beta^{\hbar'}(z_b-z_c)\,, \tag{4.3}$$

$$R_{ac}^{\hbar'}R_{ab}^{\hbar-\hbar'} = \sum_{\alpha,\beta} T_\alpha^a\,T_{\beta-\alpha}^b\,T_{-\beta}^c\,\kappa_{\beta,\alpha-\beta}\,\varphi_\beta^{\hbar'}(z_a-z_c)\varphi_{\alpha-\beta}^{\hbar-\hbar'}(z_a-z_b)\,, \tag{4.4}$$

$$R_{bc}^{\hbar'-\hbar}R_{ac}^{\hbar} = \sum_{\alpha,\beta} T_\alpha^a\,T_{\beta-\alpha}^b\,T_{-\beta}^c\,\kappa_{\beta-\alpha,\alpha}\,\varphi_{\beta-\alpha}^{\hbar'-\hbar}(z_b-z_c)\varphi_\alpha^{\hbar}(z_a-z_c)\,, \tag{4.5}$$

Notice that $\kappa_{-\alpha,\beta} = \kappa_{\beta,\alpha-\beta} = \kappa_{\beta-\alpha,\alpha}$ due to (B.4). Then the statement (4.2) follows from (4.1), where $x = \hbar + \omega_\alpha$ and $y = \hbar' + \omega_\beta$. □

Proposition 4. *The quantum Yang-Baxter equation (1.1) follows from (4.2), the property (1.7) and unitarity condition (1.2).*

Proof. Consider (4.2) for $a, b, c = 1, 2, 3$ and $\hbar' = \hbar/2$:

$$R^{\hbar}_{12} R^{\hbar/2}_{23} = R^{\hbar/2}_{13} R^{\hbar/2}_{12} + R^{-\hbar/2}_{23} R^{\hbar}_{13}$$

Replace $\hbar \to 2\hbar$ and multiply this relation by $R^{\hbar}_{23}$ from the left:

$$R^{\hbar}_{23} R^{\hbar}_{13} R^{\hbar}_{12} = R^{\hbar}_{23} R^{2\hbar}_{12} R^{\hbar}_{23} - R^{\hbar}_{23} R^{-\hbar}_{23} R^{2\hbar}_{13} \,. \tag{4.6}$$

Similarly, consider (4.2) for $a, b, c = 1, 3, 2$ and $\hbar' = \hbar/2$, replace $\hbar \to 2\hbar$ and multiply the obtained relation by $R^{\hbar}_{23}$ from the right:

$$R^{\hbar}_{12} R^{\hbar}_{13} R^{\hbar}_{23} = R^{2\hbar}_{13} R^{\hbar}_{32} R^{\hbar}_{23} - R^{-\hbar}_{32} R^{2\hbar}_{12} R^{\hbar}_{23} \,. \tag{4.7}$$

The r.h.s. of (4.6) equals r.h.s. of (4.7) due to the property (1.7) and unitarity condition (1.2). □

Consider the derivative of (4.2) with respect to z_b:

$$R^{\hbar}_{ab} F^{\hbar'}_{bc} - F^{\hbar}_{ab} R^{\hbar'}_{bc} = F^{\hbar'-\hbar}_{bc} R^{\hbar}_{ac} - R^{\hbar'}_{ac} F^{\hbar-\hbar'}_{ab} \,, \tag{4.8}$$

where $F^{\hbar}_{ab}(z) = \partial_z R^{\hbar}_{ab}(z)$. The function $F^{\hbar}_{ab}(z)$ has no singularities at $\hbar = 0$. Therefore, we can put $\hbar = \hbar'$ in (4.8). This gives

$$R^{\hbar}_{ab} F^{\hbar}_{bc} - F^{\hbar}_{ab} R^{\hbar}_{bc} = F^{0}_{bc} R^{\hbar}_{ac} - R^{\hbar}_{ac} F^{0}_{ab} \,, \tag{4.9}$$

The latter equation is analogue of the following identity

$$\begin{gathered} \phi(x, z_{ab}) f(x, z_{bc}) - f(x, z_{ab}) \phi(x, z_{bc}) = \phi(x, z_{ac}) (\wp(z_{ab}) - \wp(z_{bc})) \,, \\ f(x, z_{ab}) = \partial_{z_a} \phi(x, z_{ab}) \end{gathered} \tag{4.10}$$

underlying Lax equations (integrability) of the Calogero-Moser model [12–15, 30].

4.2 *R*-matrix valued linear problem for Calogero-Moser model

Consider the eigenvalue problem

$$\mathcal{L}\Psi = \Psi\Lambda \tag{4.11}$$

for the following block matrix operator

$$\mathcal{L} = \sum_{a,b=1}^{\tilde{N}} \tilde{\mathrm{E}}_{ab} \otimes \mathcal{L}_{ab} \,, \tag{4.12}$$

where $\tilde{\mathrm{E}}_{ab}$ is the standard basis of $\mathrm{gl}_{\tilde{N}}$ and

$$\mathcal{L}_{ab} = \delta_{ab} p_a \, 1_a \otimes 1_b + \nu (1 - \delta_{ab}) R^{\hbar}_{ab} \,, \quad R^{\hbar}_{ab} = R^{\hbar}_{ab}(z_a - z_b) \,. \tag{4.13}$$

It is worth mentioning that in gl_1 case ($N=1$) this operator is the Krichever's Lax matrix with spectral parameter for the Calogero-Moser model [30]. The eigenvalue matrix consists of vectors $\psi_1,\dots,\psi_{\tilde N}$. In the case of quantum CM model ($p_a\to\partial_{z_a}$) equation (4.11) should have well defined limit $\hbar\to 0$ which gives the KZ equations for $\psi_1=\ldots=\psi_{\tilde N}=\psi$.

The spectral parameter in (4.13) is $\hbar$ - the Planck constant. The M-operator is defined as follows:

$$\mathcal{M}_{ab}=\nu\delta_{ab}d_a+\nu(1-\delta_{ab})F^{\hbar}_{ab}+\nu\delta_{ab}\,\mathcal{F}^0\,, \tag{4.14}$$

where

$$F^{\hbar}_{ab}=\partial_{z_a}R^{\hbar}_{ab}(z_a-z_b)\,, \tag{4.15}$$

$$d_a=-\sum_{c:\,c\neq a}^{\tilde N}F^0_{ac}\,,\qquad\qquad F^0_{ac}=F^{\hbar}_{ac}\,|_{\hbar=0}\,, \tag{4.16}$$

$$\mathcal{F}^0=\frac{1}{2}\sum_{b,c:\,b\neq c}^{\tilde N}F^0_{bc}=\sum_{b,c:\,b>c}^{\tilde N}F^0_{bc}\,. \tag{4.17}$$

M-operator (4.14) is also straightforward generalization of the one proposed in [30] except the last term $\mathcal{F}^0$. The latter is not needed in $N=1$ case because in this case it is proportional to the identity matrix.

Proposition 5. *The linear problem*

$$(\partial_t+\mathcal{M})\Psi=0\,,\quad \mathcal{M}=\sum_{a,b=1}^{\tilde N}\tilde{\mathrm{E}}_{ab}\otimes\mathcal{M}_{ab} \tag{4.18}$$

is compatible with (4.11). The compatibility condition is equivalent to dynamics of $\mathrm{gl}_{\tilde N}$ *Calogero-Moser model.*

Proof. The compatibility condition is the Lax equation $\partial_t\mathcal{L}=[\mathcal{L},\mathcal{M}]$. For brevity sake let us denote $\mathcal{L}=p+R$, $\mathcal{M}=d+F+\mathcal{F}^0$. The commutator equals

$$[\mathcal{L},\mathcal{M}]=[p,F]+[R,d]+[R,F]+[R,\mathcal{F}^0]\,. \tag{4.19}$$

The term $[p,F]$ is cancelled by $\partial_t R$ (due to $\dot z_a=p_a$).

Consider the off-diagonal block ac. It has three inputs from

1. from $[R,F]$: $\sum\limits_{b\neq a,c}R^{\hbar}_{ab}F^{\hbar}_{bc}-F^{\hbar}_{ab}R^{\hbar}_{bc}\overset{(4.9)}{=}\sum\limits_{b\neq a,c}F^0_{bc}R^{\hbar}_{ac}-R^{\hbar}_{ac}F^0_{ab}$;
2. from $[R,d]$: $-R^{\hbar}_{ac}\sum\limits_{b\neq c}F^0_{cb}+\sum\limits_{b\neq a}F^0_{ab}R^{\hbar}_{ac}$;
3. from $[R,\mathcal{F}^0]$: $[\mathcal{L}_{ac},\mathcal{F}^0]$.

The sum of the inputs equals zero. We used that $F^0_{ab}=F^0_{ba}$ (due to $F^0_{ab}=\partial_{z_a}r_{ab}(z_a-z_b)$).

On a diagonal block we get equations of motion:

$$\dot p_a=\nu^2\sum_{b\neq a}R^{\hbar}_{ab}F^{\hbar}_{ba}-F^{\hbar}_{ab}R^{\hbar}_{ba}\overset{(4.22)}{=}N^2\nu^2\sum_{b\neq a}\wp'(z_a-z_b)\,. \tag{4.20}$$

$\square$

It is natural to expect that the same receipt works for other root systems (not only gl_N) as well, i.e. one can replace the function $\phi(x,z)$ in the Lax matrix with the corresponding quantum R-matrix.

Denote the off-diagonal part of (4.13) by $\mathcal{L}^0$: $\mathcal{L}^0_{ab} = (1-\delta_{ab})R^\hbar_{ab}$. We conjecture that:[13]

$$\tilde{\mathrm{tr}}((\mathcal{L}^0)^{k+1})_{aa} = \sum_{b_1,\dots,b_k=1}^{\tilde{N}} R^\hbar_{ab_1}\,\dots\,R^\hbar_{b_k a} = 1_1\otimes\dots\otimes 1_{\tilde{N}} \sum_{b_1,\dots,b_k=1}^{\tilde{N}} \Phi^\hbar(z_a-z_{b_1})\dots\Phi^\hbar(z_{b_k}-z_a)\,, \tag{4.21}$$

where $\tilde{\mathrm{tr}}$ denotes the trace over $gl_{\tilde{N}}$ component of $\mathcal{L}$ and the sums do not contain zero arguments (i.e. $b_1\neq a$, $b_2\neq b_1,\dots,b_k\neq a$). Relation (4.21) means that traces of $\mathcal{L}$ (4.12)–(4.13) provides the Hamiltonians of the $\mathrm{gl}_{\tilde{N}}$ Calogero-Moser model (where z_a are coordinates of particles).

For $k=1$ (4.21) follows from the unitarity condition:

$$\sum_b R^\hbar_{ab}\,R^\hbar_{ba} = 1_a\otimes 1_b\sum_b \Phi^\hbar(z_a-z_b)\Phi^\hbar(z_b-z_a) = N^2\wp(N\hbar) - N^2\wp(z_a-z_b)\,. \tag{4.22}$$

For $k=2$ and $\tilde{N}=3$ we have

$$R^\hbar_{ab}\,R^\hbar_{bc}\,R^\hbar_{ca} + R^\hbar_{ac}\,R^\hbar_{cb}\,R^\hbar_{ba} = 1_a\otimes 1_b\otimes 1_c\left(\Phi^\hbar(z_{ab})\Phi^\hbar(z_{bc})\Phi^\hbar(z_{ca}) + \Phi^\hbar(z_{ac})\Phi^\hbar(z_{cb})\Phi^\hbar(z_{ba})\right) \tag{4.23}$$

($z_{ab}=z_a-z_b$) or, in particular

$$R^\hbar_{12}\,R^\hbar_{23}\,R^\hbar_{31} + R^\hbar_{13}\,R^\hbar_{32}\,R^\hbar_{21} = 1\otimes 1\otimes 1\left(\Phi^\hbar(z_{12})\Phi^\hbar(z_{23})\Phi^\hbar(z_{31}) + \Phi^\hbar(z_{13})\Phi^\hbar(z_{32})\Phi^\hbar(z_{21})\right) \tag{4.24}$$

The function in the r.h.s. of (4.24) equals

$$\Phi^\hbar(z_{12})\Phi^\hbar(z_{23})\Phi^\hbar(z_{31}) + \Phi^\hbar(z_{13})\Phi^\hbar(z_{32})\Phi^\hbar(z_{21}) = \begin{cases} -N^3\wp'(\hbar) \text{ in elliptic case}\,, \\ 2/\hbar^3 \text{ in rational case}\,. \end{cases} \tag{4.25}$$

4.3 Half of the classical Yang-Baxter equation

Consider the unitarity condition $R^\hbar_{ab}\,R^\hbar_{ba} = \Phi^\hbar(z_{ab})\Phi^\hbar(z_{ba})$. Its expansion in the $\hbar^0$ order gives

$$r^2_{ab} - 2m_{ab} = 1_a\otimes 1_b\,N^2\wp(z_a-z_b)\,. \tag{4.26}$$

Here $r_{ab}=r^\tau_{ab}(z_a-z_b)$, $m_{ab}=m^\tau_{ab}(z_a-z_b)$. Next, consider (4.23)–(4.25). In the $\hbar^1$ order it provides the following relation between r and m matrices:

$$[r_{ab},r_{bc}]_+ + [r_{bc},r_{ca}]_+ + [r_{ab},r_{ca}]_+ + 2(m_{ab}+m_{bc}+m_{ac}) = 0\,, \tag{4.27}$$

where $[*,*]_+$ is the anticommutator $[A,B]_+ := AB+BA$. Using the classical Yang-Baxter equation

$$[r_{ab},r_{ac}] + [r_{ac},r_{bc}] + [r_{ab},r_{bc}] = 0 \tag{4.28}$$

[13] The proof will be given elsewhere.

we can combine (4.27) and (4.28) into two "halves" of the classical Yang-Baxter equation:

$$r_{ab}\, r_{ac} - r_{bc}\, r_{ab} + r_{ac}\, r_{bc} = m_{ab} + m_{bc} + m_{ac} \tag{4.29}$$

and

$$r_{ac}\, r_{ab} - r_{ab}\, r_{bc} + r_{bc}\, r_{ac} = m_{ab} + m_{bc} + m_{ac}\,. \tag{4.30}$$

The difference of (4.29) and (4.30) gives (4.28) while the sum leads to (4.27).

In the light of (4.26) the expansion $R^\hbar(z) = \hbar^{-1} + r(z) + \hbar\, m(z)$ is similar to the expansion (B.9). Indeed, using (4.26) we have

$$R^\hbar_{ab}(z) = \frac{1}{\hbar} 1_a \otimes 1_b + r_{ab} + \hbar\, m_{ab} + \ldots = \frac{1}{\hbar} 1_a \otimes 1_b + r_{ab} + \frac{\hbar}{2}\left(r^2_{ab} - N^2 \wp(z_{ab})\right) + \ldots\,. \tag{4.31}$$

In the same time (4.27) can be re-written as

$$(r_{ab} + r_{bc} + r_{ca})^2 = 1_a \otimes 1_b \otimes 1_c\, N^2(\wp(z_a - z_b) + \wp(z_b - z_c) + \wp(z_c - z_a)) \tag{4.32}$$

using (4.26). It is an analogue of the elliptic functions identity

$$(E_1(z_a - z_b) + E_1(z_b - z_c) + E_1(z_c - z_a))^2 = \wp(z_a - z_b) + \wp(z_b - z_c) + \wp(z_c - z_a)\,. \tag{4.33}$$

4.4 Identities for KZB equations

It follows from (4.26) that

$$[r_{ab}, m_{ab}] = 0\,. \tag{4.34}$$

This is equation (2.52) written in the fundamental representation (in this case m_{aa} are some scalar operators). Equation (2.53) keeps its form in the fundamental representation. Let us prove it.

Proposition 6. *The following identities holds true:*

$$[r_{ab}, m_{ac} + m_{bc}] + [r_{ac}, m_{ab} + m_{bc}] = 0\,, \tag{4.35}$$

$$[r_{bc}, m_{ab} - m_{ac}] + r_{ab} r_{bc} r_{ac} - r_{ac} r_{bc} r_{ab} = 0\,. \tag{4.36}$$

The first one underlies the compatibility of KZB equations. See (2.53).

Proof. Consider the Yang-Baxter equation $R^\hbar_{ca} R^\hbar_{cb} R^\hbar_{ab} = R^\hbar_{ab} R^\hbar_{cb} R^\hbar_{ca}$ in the $\hbar^0$ order. It is given by the sum of (4.35) and (4.36). Consider also (4.23) in the $\hbar^0$ order. It is given by the difference of (4.35) and (4.36). □

The identities (4.26)–(4.27) allow also to get the following Matsuo-Cherednik's like [17, 46] statement:

Proposition 7. *Consider the* gl_N *KZB equations for* $\tilde{N}$ *punctures:*

$$\nabla_i \psi = 0\,, \quad \nabla_i = \partial_i + \nu \sum_{j:j\neq i} r^\tau_{ij}(z_i - z_j)\,, \tag{4.37}$$

for $i = 1, \dots, \tilde{N}$ *and*[14]

$$\nabla_\tau \psi = 0\,, \quad \nabla_i = \partial_\tau + \frac{\nu}{2} \sum_{j \neq k} m^\tau_{jk}(z_j - z_k)\,, \tag{4.38}$$

where r^τ_{ij} *and* m^τ_{ij} *are the coefficients of the expansion (1.4) and* ν *is a free constant. Then the conformal block satisfies the following equation:*

$$\left(\tilde{N}\nu\partial_\tau + \frac{1}{2}\Delta\right)\psi = \left(-\nu\sum_{i<j}\partial_i r^\tau_{ij} - \frac{1}{2}\tilde{N}\nu^2\sum_j m^\tau_{jj} + \nu^2 N^2 \sum_{i<j} 1_i \otimes 1_j\, \wp(z_i - z_j)\right)\psi \tag{4.39}$$

where $\Delta = \sum\limits_i \partial_i^2$ *and* $m^\tau_{jj} = m^\tau_{jj}(0)$ *are scalar operators depending on* τ.

Proof. Let us omit the dependence on τ, i.e. $r^\tau_{ij} := r_{ij}$.

$$\partial_i^2 \psi = \left(-\nu \sum_{j:j\neq i} \partial_i r_{ij} + \nu^2 \left(\sum_{j:j\neq i} r_{ij}\right)^2\right)\psi\,. \tag{4.40}$$

Summing up equations (4.40) for $i = 1 \dots \tilde{N}$ we get

$$\frac{1}{2}\Delta\,\psi = \left(-\nu\sum_{i<j}\partial_i r_{ij} + \nu^2 \sum_{i<j} r^2_{ij} + \frac{1}{2}\nu^2 \sum_k \sum_{i<j} [r_{ki}, r_{kj}]_+\right)\psi \tag{4.41}$$

Let us transform the last sum using identity (4.27):

$$\begin{aligned} \frac{1}{2}\sum_k\sum_{i<j}[r_{ki}, r_{kj}]_+ &= -\frac{1}{2}\sum_{k<i<j} [r_{ki}, r_{ij}]_+ + [r_{ij}, r_{jk}]_+ + [r_{jk}, r_{ki}]_+ \\ &\overset{(4.27)}{=} \sum_{k<i<j}(m_{ki} + m_{kj} + m_{ij}) = (\tilde{N} - 2)\sum_{i<j} m_{ij} \end{aligned} \tag{4.42}$$

Plugging it into the r.h.s. of (4.41) and using (4.26) we obtain:

$$\begin{aligned} \frac{1}{2}\Delta\,\psi &= \left(-\nu\sum_{i<j}\partial_i r_{ij} + \nu^2\sum_{i<j} r^2_{ij} + (\tilde{N}-2)\,\nu^2 \sum_{i<j} m_{ij}\right)\psi \\ &\overset{(4.26)}{=} \left(-\nu\sum_{i<j}\partial_i r_{ij} + \nu^2 N^2 \sum_{i<j} 1_i \otimes 1_j\, \wp(z_i - z_j) + \tilde{N}\nu^2 \sum_{i<j} m_{ij}\right)\psi\,. \end{aligned} \tag{4.43}$$

The Proposition result (4.39) follows from (4.43) and (4.38). $\square$

[14] The summation of indices runs over $1 \dots \tilde{N}$. Here and elsewhere we shall omit the limits of summation when it can be done without ambiguity.

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4.5 Painlevé equations

The block matrix Lax pair (4.13), (4.14) can be also used for description of the Painlevé equations likewise it was done in [31, 32] in $N = 1$ case, i.e. the result of Proposition 5 is naturally generalized to the following one:

Proposition 8. *Consider the linear problem*

$$\begin{cases} (\partial_\hbar + \mathcal{L})\Psi = 0\,, \\ (\partial_\tau + \mathcal{M})\Psi = 0\,, \end{cases} \tag{4.44}$$

where $\mathcal{L}$ and $\mathcal{M}$ are defined by (4.13), (4.14). The compatibility condition

$$\partial_\tau \mathcal{L} - \partial_\hbar \mathcal{M} = [\mathcal{L}, \mathcal{M}] \tag{4.45}$$

is equivalent to $\mathrm{gl}_{\tilde{N}}$ Painlevé equations

$$\partial_\tau^2 z_a = N^2\nu^2 \sum_{b\neq a} \wp'(z_a - z_b|\tau)\,. \tag{4.46}$$

The proof repeats the one for the Proposition 5. Additionally one should use the property (2.12) of the Painlevé-Calogero correspondence.

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A gl_3 (rational) case

Undeformed gl_3 quantum R-matrix:

$$R^\hbar(z) = \tag{A.1}$$

$$\left(\begin{array}{ccc}
\hbar^{-1} + z^{-1} & 0 & 0 \\
1 & \hbar^{-1} & 0 \\
2\,\hbar^2 + 3\,z\hbar + 2\,z^2 & -3\,\hbar - 3\,z & \hbar^{-1} \\
-1 & z^{-1} & 0 \\
2\,\hbar + 2\,z & 0 & 0 \\
2\,z^3 + 3\,z\hbar^2 + 2\,\hbar^3 + 3\,z^2\hbar & -3\,\hbar^2 - 3\,z\hbar - z^2 & 1 \\
-2\,\hbar^2 - 3\,z\hbar - 2\,z^2 & -3\,\hbar - 3\,z & z^{-1} \\
2\,z^3 + 3\,z\hbar^2 + 2\,\hbar^3 + 3\,z^2\hbar & 3\,z^2 + 3\,z\hbar + \hbar^2 & -1 \\
2\,\hbar^5 + 3\,z^4\hbar + 3\,z^2\hbar^3 + 2\,z^5 + 3\,z\hbar^4 + 3\,z^3\hbar^2 & 3\,z^4 - 3\,\hbar^4 - 3\,z\hbar^3 + 3\,z^3\hbar & -z^2 + \hbar^2
\end{array}\right.$$

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$$
\left.\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
z^{-1} & 0 & 0 & 0 & 0 & 0 \\
-3\,\hbar - 3\,z & 3 & 0 & z^{-1} & 0 & 0 \\
\hbar^{-1} & 0 & 0 & 0 & 0 & 0 \\
0 & \hbar^{-1} + z^{-1} & 0 & 0 & 0 & 0 \\
-3\,z\hbar - 3\,z^2 - \hbar^2 & 0 & \hbar^{-1} & 1 & z^{-1} & 0 \\
-3\,\hbar - 3\,z & -3 & 0 & \hbar^{-1} & 0 & 0 \\
z^2 + 3\,\hbar^2 + 3\,z\hbar & 0 & z^{-1} & -1 & \hbar^{-1} & 0 \\
3\,z\hbar^3 + 3\,\hbar^4 - 3\,z^3\hbar - 3\,z^4 & -6\,\hbar^3 - 6\,z^3 - 9\,z\hbar^2 - 9\,z^2\hbar & 3\,z + 3\,\hbar & -\hbar^2 + z^2 & 3\,z + 3\,\hbar & \hbar^{-1} + z^{-1}
\end{array}\right)
$$

The τ-deformation generated by (3.13) with $\mathcal{T}_{N=3}$ from (3.12) yields

$$
\begin{aligned}
&R^{\hbar}(z|\,\tau) = R^{\hbar}(z|\,0) + \delta R^{\hbar,\tau}(z)\,, \\
&\delta R^{\hbar,\tau}(z) = \\
&= 3\tau \times \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2\,\hbar + 2\,z & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2\,\hbar + 2\,z & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
2(z+\hbar)(2\,z^2 + z\,\hbar + 2\,\hbar^2 + 3\,\tau) & 3\,z^2 - 3\,\hbar^2 & 0 & -3\,z^2 + 3\,\hbar^2 & -6\,z - 6\,\hbar & 0 & 0 & 0 & 0
\end{pmatrix}
\end{aligned}
\tag{A.2}
$$

Classical *tau*-deformed r and m-matrix:

$$
\begin{aligned}
&r^{\tau}(z) = \\
&\begin{pmatrix}
z^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & z^{-1} & 0 & 0 & 0 & 0 & 0 \\
3\,\tau + 2\,z^2 & -3\,z & 0 & -3\,z & 3 & 0 & z^{-1} & 0 & 0 \\
-1 & z^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2\,z & 0 & 0 & 0 & z^{-1} & 0 & 0 & 0 & 0 \\
2\,z^3 + 6\,\tau\,z & -z^2 - 3\,\tau & 1 & -3\,z^2 - 3\,\tau & 0 & 0 & 1 & z^{-1} & 0 \\
-3\,\tau - 2\,z^2 & -3\,z & z^{-1} & -3\,z & -3 & 0 & 0 & 0 & 0 \\
2\,z^3 + 6\,\tau\,z & 3\,z^2 + 3\,\tau & -1 & z^2 + 3\,\tau & 0 & z^{-1} & -1 & 0 & 0 \\
18\,\tau^2 z + 12\,\tau\,z^3 + 2\,z^5 & 9\,\tau\,z^2 + 3\,z^4 & -z^2 & -9\,\tau\,z^2 - 3\,z^4 & -6\,z^3 - 18\,\tau\,z & 3\,z & z^2 & 3\,z & z^{-1}
\end{pmatrix}
\end{aligned}
\tag{A.3}
$$

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$$
m^{\tau}(z) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3\,z & -3 & 0 & -3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3\,z^2 + 6\,\tau & -3\,z & 0 & -3\,z & 0 & 0 & 0 & 0 & 0 \\
-3\,z & -3 & 0 & -3 & 0 & 0 & 0 & 0 & 0 \\
3\,z^2 + 6\,\tau & 3\,z & 0 & 3\,z & 0 & 0 & 0 & 0 & 0 \\
18\,\tau^2 + 18\,\tau\,z^2 + 3\,z^4 & 3\,z^3 & 0 & -3\,z^3 & -9\,z^2 - 18\,\tau & 3 & 0 & 3 & 0
\end{pmatrix}
$$

The Lax pair for τ-deformed (autonomous or non-autonomous) rational top can be found from (2.3)–(2.4). It describes dynamics generated by the following Hamiltonian:

$$
H = S_{12}^2 - 3S_{11}S_{23} + 3S_{33}S_{23} - 3S_{13}S_{21} + 6\tau S_{12}S_{13} - 9\tau S_{23}^2 + 9\tau^2 S_{13}^2\,. \tag{A.4}
$$

B Belavin's R-matrix

Consider the following basis in gl_N (some details can be found in [37, 38]):

$$
T_a = T_{a_1 a_2} = \exp\left(\frac{\pi\imath}{N}\, a_1 a_2\right) Q^{a_1}\Lambda^{a_2}\,, \tag{B.1}
$$

where $a_1\,, a_2 \in \mathbb{Z}_N$ and

$$
Q_{kl} = \delta_{kl}\exp\left(\frac{2\pi i}{N}k\right)\,, \quad \Lambda_{kl} = \delta_{k-l+1=0\,\mathrm{mod}\,N}\,, \quad k,l = 1,\dots,N\,. \tag{B.2}
$$

The multiplication is defied by the following relation:

$$
T_{a_1 a_2} T_{b_1 b_2} = \kappa_{a,b}\, T_{a_1+b_1, a_2+b_2}\,, \tag{B.3}
$$

where

$$
\kappa_{a,b} = \exp\left(\frac{\pi\imath}{N}(b_1 a_2 - b_2 a_1)\right)\,. \tag{B.4}
$$

For the odd Riemann theta function $\vartheta(z) = \vartheta(z|\tau)$

$$
\phi(z,u) = \frac{\vartheta'(0)\vartheta(u+z)}{\vartheta(z)\vartheta(u)}\,, \tag{B.5}
$$

$$
\varphi_a(z) = \exp(2\pi\imath z\partial_\tau\omega_a)\phi(z,\omega_a)\,, \qquad \omega_a = \frac{a_1 + a_2\tau}{N}\,, \tag{B.6}
$$

$$
\varphi_a^{\hbar}(z) = \exp(2\pi\imath z\partial_\tau\omega_a)\phi(z,\omega_a + \hbar)\,. \tag{B.7}
$$

The Belavin's R-matrix [6, 7] can be defined as

$$
R_{12}^{\hbar}(z) = \sum_{\alpha\in\mathbb{Z}_N\times\mathbb{Z}_N} \varphi_\alpha^{\hbar}(z)\, T_\alpha \otimes T_{-\alpha}\,. \tag{B.8}
$$

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The local behavior of $\phi(\hbar, z)$ (B.5) near $\hbar = 0$ is give by

$$\phi(\hbar, z) = \frac{1}{\hbar} + E_1(z) + \frac{\hbar}{2}\left(E_1^2(z) - \wp(z)\right) + \dots \,, \tag{B.9}$$

where

$$E_1(z) = \partial_z \log \vartheta(z) \tag{B.10}$$

and $\wp(z)$ is the Weierstrass $\wp$-function. Therefore, expansion (1.4) of (B.8) gives

$$r_{12}(z) = E_1(z)\, 1 \otimes 1 + \sum_{\alpha \neq 0} \varphi_\alpha(z)\, T_\alpha \otimes T_{-\alpha}\,, \tag{B.11}$$

$$m_{12}(z) = \frac{E_1^2(z) - \wp(z)}{2}\, 1 \otimes 1 + \sum_{\alpha \neq 0} f_\alpha(z)\, T_\alpha \otimes T_{-\alpha}\,, \tag{B.12}$$

where

$$f_a(z) = \exp(2\pi\imath z \partial_\tau \omega_a) \partial_u \phi(z, u)\,|_{u=\omega_\alpha}\,. \tag{B.13}$$

The function Φ entering the unitarity condition (1.2) equals

$$\Phi^\hbar(z) = N\phi(N\hbar, z)\,. \tag{B.14}$$

Notice that the residue of the R-matrix (B.8) at $z = 0$ equals NP_{12}, where $P_{12} = N^{-1}\sum_a T_a \otimes T_{-a}$ is the permutation operator.

It follows from the heat equation for function (B.7)

$$\partial_\tau \varphi_a^\hbar(z) = \partial_z \partial_\hbar \varphi_a^\hbar(z) \tag{B.15}$$

that the R-matrix (B.8) satisfies the property (2.12):

$$\partial_\tau R_{ab}^\hbar = \partial_z \partial_\hbar R_{ab}^\hbar\,. \tag{B.16}$$

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