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Approximating common fixed points of averaged self-mappings with applications to the split feasibility problem and maximal monotone operators in Hilbert spaces

Young-Ye Huang[1] and Chung-Chien Hong[2*]

*Correspondence:
chong@mail.npust.edu.tw
[2]Department of Industrial
Management, National Pingtung
University of Science and
Technology, 1 Shuefu Rd., Neipu,
Pingtung, 91201, Taiwan
Full list of author information is
available at the end of the article

Abstract

In this paper, a modified proximal point algorithm for finding common fixed points of averaged self-mappings in Hilbert spaces is introduced and a strong convergence theorem associated with it is proved. As a consequence, we apply it to study the split feasibility problem, the zero point problem of maximal monotone operators, the minimization problem and the equilibrium problem, and to show that the unique minimum norm solution can be obtained through our algorithm for each of the aforementioned problems. Our results generalize and unify many results that occur in the literature.

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Keywords: averaged mapping; firmly nonexpansive mapping; maximal monotone operator; split feasibility problem; minimization problem; equilibrium problem

1 Introduction

Throughout this paper, $\mathcal{H}$ denotes a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$, I the identity mapping on $\mathcal{H}$, $\mathbb{N}$ the set of all natural numbers and $\mathbb{R}$ the set of all real numbers. For a self-mapping T on $\mathcal{H}$, $F(T)$ denotes the set of all fixed points of T.

Let C and Q be nonempty closed convex subsets of two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, and let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear mapping. The split feasibility problem (SFP) is the problem of finding a point with the property:

$$x^* \in C \quad \text{and} \quad Ax^* \in Q. \tag{1}$$

The SFP was first introduced by Censor and Elfving [1] for modeling inverse problems which arise from phase retrievals and medical image reconstruction. Recently, it has been found that the SFP can also be used to model the intensity-modulated radiation therapy. For details, the readers are referred to Xu [2] and the references therein.

Assume that the SFP has a solution. There are many iterative methods designed to approximate its solutions. The most popular algorithm is the CQ algorithm introduced by Byrne [3, 4]:

It starts with any $x_1 \in \mathcal{H}_1$ and generates a sequence $\{x_n\}$ through the iteration

$$x_{n+1} = P_C\big(I - \gamma A^*(I - P_Q)A\big)x_n, \tag{2}$$

where $\gamma \in (0, \frac{2}{\|A\|^2})$, A^* is the adjoint of A, P_C and P_Q are the metric projections onto C and Q respectively.

The sequence $\{x_n\}$ generated by the *CQ* algorithm (2) converges weakly to a solution of SFP (1), *cf.* [2–4]. Under the assumption that SFP (1) has a solution, it is known that a point $x^* \in \mathcal{H}_1$ solves the SFP (1) if and only if x^* is a fixed point of the operator

$$P_C\big(I - \gamma A^*(I - P_Q)A\big), \tag{3}$$

cf. [2], where Xu also proposed the regularized method

$$x_{n+1} = P_C\big(I - \gamma_n\big(A^*(I - P_Q)A + \alpha_n I\big)\big)x_n, \tag{4}$$

and proved that the sequence $\{x_n\}$ converges strongly to a minimum norm solution of SFP (1) provided the parameters $\{\alpha_n\}$ and $\{\gamma_n\}$ verify some suitable conditions. This regularized method was further investigated by Yao, Jiang and Liou [5], and Yao, Liou and Shahzad [6].

Motivated by the above works, it is desirable to devise an algorithm for approximating a point $x^* \in C$ so that

$$Ax^* \in Q \quad \text{and} \quad Bx^* \in Q, \tag{5}$$

where A, B are two bounded linear mappings from $\mathcal{H}_1$ to $\mathcal{H}_2$.

On the other hand, it has been an interesting topic of finding zero points of maximal monotone operators. A set-valued map $A : \mathcal{H} \to 2^{\mathcal{H}}$ with the domain $\mathcal{D}(A)$ is called monotone if

$$\langle x - y, u - v \rangle \geq 0$$

for all $x, y \in \mathcal{D}(A)$ and for any $u \in A(x)$, $v \in A(y)$, where $\mathcal{D}(A)$ is defined to be

$$\mathcal{D}(A) = \{x \in \mathcal{H} : Ax \neq \varnothing\}.$$

A is said to be maximal monotone if its graph $\{(x, u) : x \in \mathcal{H}, u \in A(x)\}$ is not properly contained in the graph of any other monotone operator. For a positive real number α, we denote by J_α^A the resolvent of a monotone operator A, that is, $J_\alpha^A(x) = (I + \alpha A)^{-1}(x)$ for any $x \in \mathcal{H}$. A point $v \in \mathcal{H}$ is called a zero point of a maximal monotone operator A if $0 \in A(v)$. In the sequel, we shall denote the set of all zero points of A by $A^{-1}0$, which is equal to $F(J_\alpha^A)$ for any $\alpha > 0$. A well-known method to solve this problem is the proximal point algorithm which starts with any initial point $x_1 \in \mathcal{H}$ and then generates the sequence $\{x_n\}$ in $\mathcal{H}$ by

$$x_{n+1} = J_{\alpha_n}^A x_n, \quad n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence of positive real numbers. This algorithm was first introduced by Martinet [7] and then generally studied by Rockafellar [8], who devised the iterative

sequence $\{x_n\}$ by

$$x_{n+1} = J^A_{\alpha_n} x_n + e_n, \quad n \in \mathbb{N}, \tag{6}$$

where $\{e_n\}$ is an error sequence in $\mathcal{H}$. Rockafellar showed that the sequence $\{x_n\}$ generated by (6) converges weakly to an element of $A^{-1}0$ provided that $A^{-1}0 \neq \varnothing$ and $\liminf_{n\to\infty} \alpha_n > 0$. In 1991, Güler [9] established an example showing that the sequence $\{x_n\}$ generated by (6) converges weakly but not strongly. Since then, many authors have conducted research on modifying the sequence in (6) so that the strong convergence is guaranteed, *cf.* [10–19] and the references therein. Recently, Wang and Cui [16] considered the following algorithm:

$$x_{n+1} = a_n u + b_n x_n + c_n J^A_{\alpha_n} x_n + e_n, \quad n \in \mathbb{N}, \tag{7}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are sequences in $(0,1)$ with $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, and $\{e_n\}$ is an error sequence in $\mathcal{H}$. They showed that the sequence $\{x_n\}$ generated by (7) converges strongly to a zero point of A provided the following conditions (i) and (ii) are verified:

(i) $\quad \lim_{n\to\infty} a_n = 0, \quad \sum_{n=1}^{\infty} a_n = \infty, \quad \liminf_{n\to\infty} c_n > 0, \quad \liminf_{n\to\infty} \alpha_n > 0;$

(ii) either $\quad \sum_{n=1}^{\infty} \|e_n\| < \infty \quad$ or $\quad \lim_{n\to\infty} \frac{\|e_n\|}{a_n} = 0.$

This theorem generalizes and unifies many results that occur in the literature, *cf.* [10–12, 18, 20].

For another maximal monotone operator B, we would like to seek appropriate conditions on the coefficient sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ so that the sequence $\{x_n\}$ generated by

$$x_{n+1} = a_n u + b_n J^B_{\beta_n} x_n + c_n J^A_{\alpha_n} x_n + d_n e_n, \quad n \in \mathbb{N}, \tag{8}$$

can converge strongly to a common zero of A and B.

We find that both of problems (5) and (8) can be solved simultaneously in a more general setting. As a matter of fact, any resolvent is firmly nonexpansive and any firmly nonexpansive mapping is $\frac{1}{2}$-averaged, *cf.* [21], which is a special case of λ-averaged mappings (for the definition of λ-averaged mappings, we refer readers to Section 2). Also, as shown in the proof of Theorem 3.6 of [2], for any $\gamma \in \mathbb{R}$ with $0 < \gamma < \frac{2}{\|A\|^2}$, the operator (3) is $\frac{2+\gamma\|A\|^2}{4}$-averaged. It is quite natural to ask whether the sequence $\{x_n\}$ generated by

$$x_{n+1} = a_n u + b_n S_n x_n + c_n T_n x_n + d_n e_n \tag{9}$$

can converge strongly to a point of $\bigcap_{n=1}^{\infty} F(S_n) \cap \bigcap_{n=1}^{\infty} F(T_n)$ provided the coefficient sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ are imposed on appropriate conditions, where for any $n \in \mathbb{N}$, each S_n is μ_n-averaged by G_n, and each T_n is λ_n-averaged by K_n. We shall show in Section 3 that *the sequence $\{x_n\}$ generated by (9) converges strongly to a point of $\bigcap_{n=1}^{\infty} F(S_n) \cap \bigcap_{n=1}^{\infty} F(T_n)$ provided $\bigcap_{n=1}^{\infty} F(S_n) \cap \bigcap_{n=1}^{\infty} F(T_n) \neq \varnothing$ and $\{\mu_n\}$, $\{\lambda_n\}$ and the coefficient sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ verify the conditions:*

(i) $\{\mu_n\}$ and $\{\lambda_n\}$ are convergent sequences in $(0,1)$ with limit $\mu, \lambda \in (0,1)$ respectively;

(ii) there are two nonnegative real-valued functions κ_1 and κ_2 on $\mathbb{N}$ with

$$\|G_m x - x\| + \|K_m x - x\| \leq \kappa_1(m)\|G_n x - x\| + \kappa_2(m)\|K_n x - x\|,$$

$$\forall m \in \mathbb{N}, \forall n \geq m, \forall x \in C;$$

(iii) $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ are sequences in $[0,1]$ with $a_n + b_n + c_n + d_n = 1$ and $a_n \in (0,1)$, $\forall n \in \mathbb{N}$;

(iv) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{d_n}{a_n} = 0$, $\sum_{n=1}^{\infty} a_n = \infty$, $\sum_{n=1}^{\infty} d_n < \infty$;

(v) $\liminf_{n\to\infty} b_n > 0$, $\liminf_{n\to\infty} c_n > 0$.

Based on this main result, we shall deduce many corollaries for averaged mappings in Section 3. Section 4 is devoted to applications. We apply our results in Section 3 to study the split feasibility problem, the zero point problem of maximal monotone operators, the minimization problem and the equilibrium problem, and to show that the unique minimum norm solution can be obtained through our algorithm for each of the aforementioned problems.

2 Preliminaries

In order to facilitate our investigation in Section 3, we recall some basic facts. Let C be a nonempty closed convex subset of $\mathcal{H}$. A mapping $T : C \to \mathcal{H}$ is said to be

(i) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C;$$

(ii) firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C;$$

(iii) λ-averaged by K if

$$T = (1 - \lambda)I + \lambda K$$

for some $\lambda \in (0,1)$ and some nonexpansive mapping K.

If $T : C \to C$ is nonexpansive, then the fixed point set $F(T)$ of T is closed and convex, *cf.* [21]. If $T = (1 - \lambda)I + \lambda K$ is averaged, then T is nonexpansive with $F(T) = F(K)$.

The metric projection P_C from $\mathcal{H}$ onto C is the mapping that assigns each $x \in \mathcal{H}$ the unique point $P_C x$ in C with the property

$$\|x - P_C x\| = \min_{y \in C} \|y - x\|.$$

It is known that P_C is nonexpansive and characterized by the inequality: for any $x \in \mathcal{H}$,

$$\langle x - P_C x, y - P_C x \rangle \leq 0, \quad \forall y \in C. \tag{10}$$

For $\alpha > 0$, the resolvent J_α^A of maximal monotone operator A on $\mathcal{H}$ has the following properties.

Lemma 2.1 *Let A be a maximal monotone operator on $\mathcal{H}$. Then*

(a) J_α^A *is single-valued and firmly nonexpansive*;

(b) $\mathcal{D}(J_\alpha^A) = \mathcal{H}$ *and* $F(J_\alpha^A) = A^{-1}0$;

(c) (The resolvent identity) *for $\mu, \lambda > 0$, the following identity holds*:

$$J_\mu^A x = J_\lambda^A \left(\frac{\lambda}{\mu} x + \left(1 - \frac{\lambda}{\mu} \right) J_\mu^A x \right), \quad \forall x \in \mathcal{H}.$$

We still need some lemmas that will be quoted in the sequel.

Lemma 2.2 *Let $x, y, z \in \mathcal{H}$. Then*

(a) $\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y \rangle$;

(b) *for any $\lambda \in \mathbb{R}$*,

$$\left\| \lambda x + (1 - \lambda)y \right\|^2 = \lambda \|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2;$$

(c) *for $a, b, c \in [0,1]$ with $a + b + c = 1$*,

$$\|ax + by + cz\|^2 = a\|x\|^2 + b\|y\|^2 + c\|z\|^2 - ab\|x - y\|^2 - ac\|x - z\|^2 - bc\|y - z\|^2.$$

Lemma 2.3 (Demiclosedness principle [21]) *Let T be a nonexpansive self-mapping on a nonempty closed convex subset C of $\mathcal{H}$, and suppose that $\{x_n\}$ is a sequence in C such that $\{x_n\}$ converges weakly to some $z \in C$ and $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$. Then $Tz = z$.*

Lemma 2.4 [18] *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \le (1 - \alpha_n)s_n + \alpha_n \mu_n + \nu_n, \quad n \in \mathbb{N},$$

where $\{\alpha_n\}$, $\{\mu_n\}$ and $\{\nu_n\}$ verify the conditions:

(i) $\{\alpha_n\} \subseteq [0,1]$, $\sum_{n=1}^\infty \alpha_n = \infty$;

(ii) $\limsup_{n \to \infty} \mu_n \le 0$;

(iii) $\{\nu_n\} \subseteq [0, \infty)$ *and* $\sum_{n=1}^\infty \nu_n < \infty$.

Then $\lim_{n \to \infty} s_n = 0$.

Lemma 2.5 [22] *Let $\{s_n\}$ be a sequence in $\mathbb{R}$ that does not decrease at infinity in the sense that there exists a subsequence $\{s_{n_i}\}$ such that*

$$s_{n_i} < s_{n_i+1}, \quad \forall i \in \mathbb{N}.$$

For any $k \in \mathbb{N}$, define $m_k = \max\{j \le k : s_j < s_{j+1}\}$. Then $m_k \to \infty$ as $k \to \infty$ and $\max\{s_{m_k}, s_k\} \le s_{m_k+1}, \forall k \in \mathbb{N}$.

3 Strong convergence theorems

To establish a strong convergence theorem for averaged mappings S_n, T_n, $n \in \mathbb{N}$, on $\mathcal{H}$ associated with algorithm (9), we at first need a lemma.

Lemma 3.1 *If $T = (1 - \lambda)I + \lambda K$ is a λ-averaged self-mapping by K on a nonempty closed convex subset C of $\mathcal{H}$ and $p \in F(T)$, then for any $x \in C$, one has*

$$\|Tx - p\|^2 \leq \|x - p\|^2 - \lambda(1 - \lambda)\|x - Kx\|^2.$$

Proof Let x be any point in C. Then, using $Tp = Kp = p$ and the nonexpansiveness of K, we have from Lemma 2.2(b) that

$$
\begin{aligned}
\|Tx - p\|^2 &= \|Tx - Tp\|^2 \\
&= \left\|(1 - \lambda)x + \lambda Kx - \left((1 - \lambda)p + Kp\right)\right\|^2 \\
&= \left\|(1 - \lambda)(x - p) + \lambda(Kx - Kp)\right\|^2 \\
&= (1 - \lambda)\|x - p\|^2 + \lambda\|Kx - Kp\|^2 - \lambda(1 - \lambda)\left\|x - p - (Kx - Kp)\right\|^2 \\
&\leq (1 - \lambda)\|x - p\|^2 + \lambda\|x - p\|^2 - \lambda(1 - \lambda)\|x - Kx\|^2 \\
&= \|x - p\|^2 - \lambda(1 - \lambda)\|x - Kx\|^2. \qquad \square
\end{aligned}
$$

Theorem 3.2 *For any $n \in \mathbb{N}$, suppose that $S_n = (1 - \mu_n)I + \mu_n G_n$ and $T_n = (1 - \lambda_n)I + \lambda_n K_n$ are averaged self-mappings on a nonempty closed convex subset C of $\mathcal{H}$ with $\Omega := \bigcap_{n=1}^{\infty} F(S_n) \cap \bigcap_{n=1}^{\infty} F(T_n) \neq \varnothing$, satisfying that*

$$(3.1) \quad \lim_{n \to \infty} \mu_n = \mu \in (0, 1), \qquad \lim_{n \to \infty} \lambda_n = \lambda \in (0, 1);$$

and there are two nonnegative real-valued functions κ_1 and κ_2 on $\mathbb{N}$ with

$$(3.2) \quad \|G_m x - x\| + \|K_m x - x\| \leq \kappa_1(m)\|G_n x - x\| + \kappa_2(m)\|K_n x - x\|,$$

$$\forall m \in \mathbb{N}, \forall n \geq m, \forall x \in C.$$

Suppose further that $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ are sequences in $[0, 1]$ with $a_n + b_n + c_n + d_n = 1$ and $a_n \in (0, 1)$ for all $n \in \mathbb{N}$, and that $\{e_n\}$ and $\{v_n\}$ are two bounded sequences in C. For an arbitrary norm convergent sequence $\{u_n\}$ in C with limit u, start with an arbitrary $x_1 = y_1 \in C$ and define two sequences $\{x_n\}$ and $\{y_n\}$ by

$$x_{n+1} = a_n u + b_n S_n x_n + c_n T_n x_n + d_n e_n;$$

$$y_{n+1} = a_n u_n + b_n S_n y_n + c_n T_n y_n + d_n v_n.$$

Then both of $\{x_n\}$ and $\{y_n\}$ converge strongly to $P_\Omega u$ provided the following conditions are satisfied:

$$(i) \quad \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{d_n}{a_n} = 0, \qquad \sum_{n=1}^{\infty} a_n = \infty, \qquad \sum_{n=1}^{\infty} d_n < \infty;$$

$$(ii) \quad \liminf_{n \to \infty} b_n > 0, \qquad \liminf_{n \to \infty} c_n > 0.$$

Moreover, when every S_n is the identity mapping I, the result still holds without the condition $\liminf_{n \to \infty} b_n > 0$.

Proof Put $p = P_\Omega u$. Firstly, we show that $\{x_n\}$ converges strongly to p. It comes from the nonexpansiveness of S_n and T_n that

$$\|x_{n+1} - p\| = \|a_n(u - p) + b_n(S_n x_n - p) + c_n(T_n x_n - p) + d_n(e_n - p)\|$$
$$= a_n\|u - p\| + (b_n + c_n)\|x_n - p\| + d_n\|e_n - p\|,$$

from which it follows that $\{x_n\}$ is a bounded sequence. Taking into account of Lemma 2.2 and using Lemma 3.1, we get

$$\|x_{n+1} - p\|^2$$
$$= \|a_n(u - p) + b_n(S_n x_n - p) + c_n(T_n x_n - p) + d(e_n - p)\|^2$$
$$\leq \|b_n(S_n x_n - p) + c_n(T_n x_n - p) + d_n(e_n - p)\|^2 + 2a_n\langle u - p, x_{n+1} - p\rangle$$
$$= (1 - a_n)^2\left\|\frac{b_n}{1 - a_n}(S_n x_n - p) + \frac{c_n}{1 - a_n}(T_n x_n - p) + \frac{d_n}{1 - a_n}(e_n - p)\right\|^2$$
$$\quad + 2a_n\langle u - p, x_{n+1} - p\rangle$$
$$\leq (1 - a_n)^2\left(\frac{b_n}{1 - a_n}\|S_n x_n - p\|^2 + \frac{c_n}{1 - a_n}\|T_n x_n - p\|^2 + \frac{d_n}{1 - a_n}\|e_n - p\|^2\right)$$
$$\quad + 2a_n\langle u - p, x_{n+1} - p\rangle$$
$$\leq b_n\|S_n x_n - p\|^2 + c_n\|T_n x_n - p\|^2 + d_n\|e_n - p\|^2 + 2a_n\langle u - p, x_{n+1} - p\rangle$$
$$\leq b_n\big(\|x_n - p\|^2 - \mu_n(1 - \mu_n)\|x_n - G_n x_n\|^2\big)$$
$$\quad + c_n\big(\|x_n - p\|^2 - \lambda_n(1 - \lambda_n)\|x_n - K_n x_n\|^2\big) + d_n\|e_n - p\|^2 + 2a_n\langle u - p, x_{n+1} - p\rangle$$
$$= (b_n + c_n)\|x_n - p\|^2 + d_n\|e_n - p\|^2 + 2a_n\langle u - p, x_{n+1} - p\rangle$$
$$\quad - b_n\mu_n(1 - \mu_n)\|x_n - G_n x_n\|^2 - c_n\lambda_n(1 - \lambda_n)\|x_n - K_n x_n\|^2. \tag{11}$$

We now carry on with the proof by considering the following two cases: (I) $\{\|x_n - p\|\}$ is eventually decreasing, and (II) $\{\|x_n - p\|\}$ is not eventually decreasing.

Case I: Suppose that $\{\|x_n - p\|\}$ is eventually decreasing, that is, there is $N \in \mathbb{N}$ such that $\{\|x_n - p\|\}_{n\geq N}$ is decreasing. In this case, $\lim_{n\to\infty}\|x_n - p\|$ exists in $\mathbb{R}$. By condition (ii), we may assume that there are two $b, c \in (0, 1)$ such that $b \leq b_n$ and $c \leq c_n$ for all $n \in \mathbb{N}$. Then from inequality (11) we have

$$0 \leq b\mu_n(1 - \mu_n)\|x_n - G_n x_n\|^2 + c\lambda_n(1 - \lambda_n)\|x_n - K_n x_n\|^2$$
$$\leq b_n\mu_n(1 - \mu_n)\|x_n - G_n x_n\|^2 + c_n\lambda_n(1 - \lambda_n)\|x_n - K_n x_n\|^2$$
$$\leq (b_n + c_n)\|x_n - p\|^2 - \|x_{n+1} - p\|^2 + d_n\|e_n - p\|^2 + 2a_n\langle u - p, x_{n+1} - p\rangle$$
$$= \big(1 - (a_n + d_n)\big)\|x_n - p\|^2 - \|x_{n+1} - p\|^2 + d_n\|e_n - p\|^2 + 2a_n\langle u - p, x_{n+1} - p\rangle$$

and noting via condition (i) that

$$\lim_{n\to\infty}\big(1 - (a_n + d_n)\big)\|x_n - p\|^2 = \lim_{n\to\infty}\|x_{n+1} - p\|^2 \quad \text{and}$$

$$\lim_{n\to\infty} d_n\|e_n - p\|^2 = \lim_{n\to\infty} 2a_n\langle u - p, x_{n+1} - p\rangle = 0,$$

we conclude that

$$\lim_{n\to\infty} \left(b\mu_n(1-\mu_n)\|x_n - G_n x_n\|^2 + c\lambda_n(1-\lambda_n)\|x_n - K_n x_n\|^2 \right) = 0,$$

which implies that

$$\lim_{n\to\infty} \|x_n - G_n x_n\| = \lim_{n\to\infty} \|x_n - K_n x_n\| = 0. \tag{12}$$

Then from condition (3.2) we deduce for all $m \in \mathbb{N}$ that

$$\lim_{n\to\infty} \|x_n - G_m x_n\| = \lim_{n\to\infty} \|x_n - K_m x_n\| = 0. \tag{13}$$

Since $\{x_n\}$ is bounded, it has a subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\}$ converges weakly to some $z \in \mathcal{H}$ and

$$\limsup_{n\to\infty} \langle u - p, x_{n+1} - p\rangle = \lim_{k\to\infty} \langle u - p, x_{n_k} - p\rangle = \langle u - p, z - p\rangle \leq 0, \tag{14}$$

where the last inequality follows from (13) since $z \in \Omega$ by Lemma 2.3. Choose $M > 0$ so that $\sup\{\|e_n - p\|^2 + 2\|u - p\|\|x_{n+1} - p\| : n \in \mathbb{N}\} \leq M$. From (11) we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 \\
&\leq \left(1 - (a_n + d_n)\right)\|x_n - p\|^2 + (a_n + d_n)\cdot 2\langle u - p, x_{n+1} - p\rangle \\
&\quad + d_n\left(\|e_n - p\|^2 + 2\|u - p\|\|x_{n+1} - p\|\right) \\
&\leq \left(1 - (a_n + d_n)\right)\|x_n - p\|^2 + (a_n + d_n)\cdot 2\langle u - p, x_{n+1} - p\rangle + d_n M.
\end{aligned} \tag{15}$$

Accordingly, because of (14) and condition (i), we can apply Lemma 2.4 to inequality (15) with $s_n = \|x_n - p\|^2$, $\alpha_n = a_n + d_n$, $\mu_n = 2\langle u - p, x_{n+1} - p\rangle$ and $\nu_n = d_n M$ to conclude that

$$\lim_{n\to\infty} x_n = p.$$

Case II: Suppose that $\{\|x_n - p\|\}$ is not eventually decreasing. In this case, by Lemma 2.5, there exists a nondecreasing sequence $\{m_k\}$ in $\mathbb{N}$ such that $m_k \to \infty$ and

$$\max\left\{\|x_{m_k} - p\|, \|x_k - p\|\right\} \leq \|x_{m_k+1} - p\|, \quad \forall k \in \mathbb{N}. \tag{16}$$

Then it follows from (11) and (16) that

$$\begin{aligned}
\|x_{m_k} - p\|^2 \\
&\leq \|x_{m_k+1} - p\|^2 \\
&\leq (b_{m_k} + c_{m_k})\|x_{m_k} - p\|^2 + d_{m_k}\|e_{m_k} - p\|^2 \\
&\quad + 2a_{m_k}\langle u - p, x_{m_k+1} - p\rangle - b_{m_k}\mu_{m_k}(1-\mu_{m_k})\|x_{m_k} - G_{m_k}x_{m_k}\|^2 \\
&\quad - c_{m_k}\lambda_{m_k}(1-\lambda_{m_k})\|x_{m_k} - K_{m_k}x_{m_k}\|^2.
\end{aligned} \tag{17}$$

Therefore,

$$
\begin{aligned}
0 &\leq b_{m_k}\mu_{m_k}(1-\mu_{m_k})\|x_{m_k}-G_{m_k}x_{m_k}\|^2 + c_{m_k}\lambda_{m_k}(1-\lambda_{m_k})\|x_{m_k}-K_{m_k}x_{m_k}\|^2 \\
&\leq -\big(1-(b_{m_k}+c_{m_k})\big)\|x_{m_k}-p\|^2 + d_{m_k}\|e_{m_k}-p\|^2 + 2a_{m_k}\langle u-p,x_{m_k+1}-p\rangle \\
&= -(a_{m_k}+d_{m_k})\|x_{m_k}-p\|^2 + d_{m_k}\|e_{m_k}-p\|^2 + 2a_{m_k}\langle u-p,x_{m_k+1}-p\rangle,
\end{aligned}
$$

and then proceeding just as in the proof in Case I, we obtain

$$
\lim_{k\to\infty}\|x_{m_k}-G_{m_k}x_{m_k}\| = \lim_{k\to\infty}\|x_{m_k}-K_{m_k}x_{m_k}\| = 0, \tag{18}
$$

which in conjunction with condition (3.2) shows for all m_j that

$$
\lim_{k\to\infty}\|x_{m_k}-G_{m_j}x_{m_k}\| = \lim_{k\to\infty}\|x_{m_k}-K_{m_j}x_{m_k}\| = 0,
$$

and then it follows that

$$
\limsup_{k\to\infty}\langle u-p,x_{m_k+1}-p\rangle \leq 0. \tag{19}
$$

From (17) we have

$$
\big(1-(b_{m_k}+c_{m_k})\big)\|x_{m_k}-p\|^2 \leq d_{m_k}\|e_{m_k}-p\|^2 + 2a_{m_k}\langle u-p,x_{m_k+1}-p\rangle,
$$

and thus

$$
\begin{aligned}
\|x_{m_k}-p\|^2 &\leq \frac{d_{m_k}}{a_{m_k}+d_{m_k}}\|e_{m_k}-p\|^2 + \frac{2a_{m_k}}{a_{m_k}+d_{m_k}}\langle u-p,x_{m_k+1}-p\rangle \\
&\leq \frac{d_{m_k}}{a_{m_k}}\|e_{m_k}-p\|^2 + 2\langle u-p,x_{m_k+1}-p\rangle.
\end{aligned}
$$

Letting $k\to\infty$ and using (19) and condition (i), we obtain

$$
\lim_{k\to\infty}\|x_{m_k}-p\| = 0. \tag{20}
$$

Also, since

$$
\begin{aligned}
\|x_{m_k+1}&-x_{m_k}\| \\
&\leq a_{m_k}\|u-x_{m_k}\| + b_{m_k}\mu_{m_k}\|G_{m_k}x_{m_k}-x_{m_k}\| + c_{m_k}\lambda_{m_k}\|K_{m_k}x_{m_k}-x_{m_k}\| \\
&\quad + d_{m_k}\|e_{m_k}-x_{m_k}\|,
\end{aligned}
$$

which together with (18) implies $\lim_{k\to\infty}\|x_{m_k+1}-x_{m_k}\| = 0$, and so

$$
\lim_{k\to\infty}\|x_{m_k+1}-p\| = 0 \tag{21}
$$

by virtue of (20). Consequently, we conclude $\lim_{k\to\infty}\|x_k-p\| = 0$ via (16) and (21). In addition, note that the condition $\liminf_{n\to\infty}b_n = 0$ is used to establish $\lim_{n\to\infty}\|x_n-G_nx_n\| = 0$

and $\lim_{k \to \infty} \|x_{m_k} - G_{m_k} x_{m_k}\| = 0$ in (12) and (18) respectively. However, both limits hold trivially without this condition provided every S_n is the identity mapping I.

Next, we show that $\{y_n\}$ converges strongly to p too. Applying Lemma 2.4 to the following inequality

$$\|y_{n+1} - x_{n+1}\| \le a_n \|u_n - u\| + (b_n + c_n) \|y_n - x_n\| + d_n \|v_n - e_n\|$$
$$= \big(1 - (a_n + d_n)\big) \|y_n - x_n\| + a_n \|u_n - u\| + d_n \|v_n - e_n\|$$

for all $n \in \mathbb{N}$, we see that $\lim_{n \to \infty} \|y_n - x_n\| = 0$, and hence $\lim_{n \to \infty} y_n = p$ follows. This completes the proof. □

The following lemma is easily proved and so its proof is omitted.

Lemma 3.3 *For any $n \in \mathbb{N}$, suppose that $S_n = (1 - \mu_n)I + \mu_n G_n$ and $T_n = (1 - \lambda_n)I + \lambda_n K_n$ are averaged self-mappings on a nonempty closed convex subset C of $\mathcal{H}$ such that condition (3.1) holds. Then $\{S_n\}$ and $\{T_n\}$ satisfy condition (3.2) if and only if $\{G_n\}$ and $\{K_n\}$ satisfy condition (3.2).*

If the sequence $\{S_n\}$ (resp. $\{T_n\}$) of averaged mappings consists of a single mapping S (resp. T), then $\{S_n\}$ and $\{T_n\}$ obviously verify conditions (3.1) and (3.2), and hence from Lemma 3.3 we have the following corollary.

Corollary 3.4 *Suppose S and T are two averaged self-mappings on a nonempty closed convex subset C of $\mathcal{H}$ with $\Omega = F(S) \cap F(T) \ne \varnothing$, and suppose that $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ are sequences in $[0,1]$ with $a_n + b_n + c_n + d_n = 1$ and $a_n \in (0,1)$ for all $n \in \mathbb{N}$, and $\{e_n\}$ and $\{v_n\}$ are two bounded sequences in C. For an arbitrary norm convergent sequence $\{u_n\}$ in C with limit u, start with an arbitrary $x_1 = y_1 \in C$ and define two sequences $\{x_n\}$ and $\{y_n\}$ by*

$$x_{n+1} = a_n u + b_n S x_n + c_n T x_n + d_n e_n;$$
$$y_{n+1} = a_n u_n + b_n S y_n + c_n T y_n + d_n v_n.$$

Then both of $\{x_n\}$ and $\{y_n\}$ converge strongly to $P_\Omega u$ provided the following conditions are satisfied:

(i) $\quad \lim_{n \to \infty} a_n = \lim_{n \to \infty} \dfrac{d_n}{a_n} = 0, \qquad \sum_{n=1}^{\infty} a_n = \infty, \qquad \sum_{n=1}^{\infty} d_n < \infty;$

(ii) $\quad \liminf_{n \to \infty} b_n > 0, \qquad \liminf_{n \to \infty} c_n > 0.$

Moreover, when S is the identity mapping I, the result still holds without the condition $\liminf_{n \to \infty} b_n > 0$.

Theorem 3.5 *For any $n \in \mathbb{N}$, suppose S_n and T_n are firmly nonexpansive self-mappings on a nonempty closed convex subset C of $\mathcal{H}$ with $\Omega := \bigcap_{n=1}^{\infty} F(S_n) \cap \bigcap_{n=1}^{\infty} F(T_n) \ne \varnothing$, satisfying condition (3.2). Suppose further that $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ are sequences in $[0,1]$ with*

$a_n + b_n + c_n + d_n = 1$ *and* $a_n \in (0,1)$ *for all* $n \in \mathbb{N}$, *and* $\{e_n\}$ *and* $\{v_n\}$ *are two bounded sequences in C. For an arbitrary norm convergent sequence* $\{u_n\}$ *in C with limit u, start with an arbitrary* $x_1 = y_1 \in C$ *and define two sequences* $\{x_n\}$ *and* $\{y_n\}$ *by*

$$x_{n+1} = a_n u + b_n S_n x_n + c_n T_n x_n + d_n e_n;$$

$$y_{n+1} = a_n u_n + b_n S_n y_n + c_n T_n y_n + d_n v_n.$$

Then both of $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $P_\Omega u$ *provided the following conditions are satisfied*:

(i) $\quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{d_n}{a_n} = 0, \qquad \displaystyle\sum_{n=1}^{\infty} a_n = \infty, \qquad \sum_{n=1}^{\infty} d_n < \infty;$

(ii) $\quad \liminf_{n\to\infty} b_n > 0, \qquad \liminf_{n\to\infty} c_n > 0.$

Moreover, when every S_n *is the identity mapping I, the result still holds without the condition* $\liminf_{n\to\infty} b_n > 0$.

Proof Since any firmly nonexpansive mapping is $\frac{1}{2}$-averaged, condition (3.1) holds, and hence by Lemma 3.3 we see that all the requirements of Theorem 3.2 are verified. Therefore, the desired conclusion follows. $\qquad\square$

If $S_n = I$ and $d_n = 0$ for all $n \in \mathbb{N}$ in Theorem 3.2, then we have the following corollary.

Corollary 3.6 *Suppose, for all* $n \in \mathbb{N}$, *that* $T_n = (1 - \lambda_n)I + \lambda_n K_n$ *is an averaged self-mapping on a nonempty closed convex subset C of* $\mathcal{H}$ *with* $\Omega = \bigcap_{n=1}^{\infty} F(T_n) \neq \varnothing$, *and* $\lim_{n\to\infty} \lambda_n = \lambda \in (0,1)$, *and assume that condition* (3.2) *holds for* $\{I\}$ *and* $\{T_n\}$. *Suppose further that* $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *are sequences in* $[0,1]$ *with* $a_n + b_n + c_n = 1$ *and* $a_n \in (0,1)$ *for all* $n \in \mathbb{N}$. *Let* $\{\alpha_n\}$ *be a sequence in* $(0,\infty)$. *For an arbitrary fixed* $u \in C$, *start with an arbitrary* $x_1 \in C$ *and define*

$$x_{n+1} = a_n u + b_n x_n + c_n T_n x_n, \quad n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ *converges strongly to* $P_\Omega u$ *provided the following conditions are satisfied*:

$$\lim_{n\to\infty} a_n = 0, \qquad \sum_{n=1}^{\infty} a_n = \infty, \qquad \liminf_{n\to\infty} c_n > 0.$$

Corollary 3.7 *Suppose, for all* $n \in \mathbb{N}$, *that* $T_n = (1 - \lambda_n)I + \lambda_n K_n$ *is an averaged self-mapping on* $\mathcal{H}$ *with* $\Omega = \bigcap_{n=1}^{\infty} F(T_n) \neq \varnothing$, *and* $\lim_{n\to\infty} \lambda_n = \lambda \in (0,1)$, *and assume that condition* (3.2) *holds for* $\{I\}$ *and* $\{T_n\}$. *Suppose further that* $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *are sequences in* $[0,1]$ *with* $a_n + b_n + c_n = 1$ *and* $a_n \in (0,1)$ *for all* $n \in \mathbb{N}$, *and that* $\{e_n\}$ *is a bounded sequence in* $\mathcal{H}$. *Let* $\{\alpha_n\}$ *be a sequence in* $(0,\infty)$. *For an arbitrary fixed* $u \in \mathcal{H}$, *start with an arbitrary* $x_1 \in \mathcal{H}$ *and define*

$$x_{n+1} = a_n u + b_n x_n + c_n T_n x_n + e_n, \quad n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to $P_\Omega u$ provided the following conditions are satisfied:

(i) $\quad \lim_{n \to \infty} a_n = 0, \qquad \sum_{n=1}^{\infty} a_n = \infty, \qquad \liminf_{n \to \infty} c_n > 0;$

(ii) *either* $\quad \lim_{n \to \infty} \dfrac{\|e_n\|}{a_n} = 0 \quad or \quad \sum_{n=1}^{\infty} \|e_n\| < \infty.$

Proof Put $p = P_\Omega u$. Let $z_1 = x_1$ and define a sequence $\{z_n\}$ iteratively by

$$z_{n+1} = a_n u + b_n z_n + c_n T_n z_n.$$

We have $\lim_{n \to \infty} z_n = p$ by Corollary 3.6. Since

$$\begin{aligned}
\|x_{n+1} - z_{n+1}\| &\le b_n \|x_n - z_n\| + c_n \|T_n x_n - T_n z_n\| + \|e_n\| \\
&\le (b_n + c_n)\|x_n - z_n\| + \|e_n\| \\
&= (1 - a_n)\|x_n - z_n\| + \|e_n\|,
\end{aligned} \qquad (22)$$

the limit $\lim_{n \to \infty} \|x_n - z_n\| = 0$ follows by applying Lemma 2.4 to (22), and thus,

$$\lim_{n \to \infty} x_n = p. \qquad \qquad \square$$

4 Applications

In this section, we shall apply some of the strong convergence theorems in Section 3 to approximate a solution of the split feasibility problem, a common zero of maximal monotone operators, a minimizer of a proper lower semicontinuous convex function, and to study the related equilibrium problem.

Xu [2] transformed SFP (1) to the fixed point problem of the operator (3):

$$P_C\big(I - \gamma A^*(I - P_Q)A\big).$$

He proved Lemma 4.1 below.

Lemma 4.1 [2] *A point $x^* \in \mathcal{H}_1$ solves SFP (1) if and only if x^* is a fixed point of the operator* (3): $P_C(I - \gamma A^*(I - P_Q)A)$.

Moreover, in the proof of Theorem 3.6 of [2], Xu showed the following lemma.

Lemma 4.2 [2] *For any $\gamma \in \mathbb{R}$ with $0 < \gamma < \frac{2}{\|A\|^2}$, the operator* (3): $P_C(I - \gamma A^*(I - P_Q)A)$ *is $\frac{2+\gamma\|A\|^2}{4}$-averaged*.

Invoking Lemmas 4.1 and 4.2, we obtain the theorem below from Corollary 3.4 by putting $S = I$ and $T = P_C(I - \gamma A^*(I - P_Q)A)$ for all $n \in \mathbb{N}$.

Theorem 4.3 *Let C and Q be nonempty closed convex subsets of two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, and let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear mapping. Put $T = P_C(I - $*

$\gamma A^*(I - P_Q)A)$, *where γ satisfies $0 < \gamma < \frac{2}{\|A\|^2}$. Suppose that the solution set Ω of SFP* (1)
*is nonempty, and suppose further that $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ are sequences in $[0,1]$ with
$a_n + b_n + c_n + d_n = 1$ and $a_n \in (0,1)$ for all $n \in \mathbb{N}$, and that $\{e_n\}$ is a bounded sequence in C.
For an arbitrary fixed $u \in C$, start with an arbitrary $x_1 \in C$ and define the sequence $\{x_n\}$ by*

$$x_{n+1} = a_n u + b_n x_n + c_n T x_n + d_n e_n.$$

Then $\{x_n\}$ converges strongly to $P_\Omega u$ provided the following conditions are satisfied:

(i) $\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} \dfrac{d_n}{a_n} = 0, \qquad \sum\limits_{n=1}^{\infty} a_n = \infty, \qquad \sum\limits_{n=1}^{\infty} d_n < \infty;$

(ii) $\liminf\limits_{n \to \infty} c_n > 0.$

When the point u in the above theorem is taken to be 0, we see that the limit point v of
the sequence $\{x_n\}$ is the unique minimum norm solution of SFP (1), that is, $\|v\| = \min\{\hat{x} : \hat{x} \in \Omega\}$.

Here, readers may compare the above theorem with Theorem 3.6 of [2], which says, for
$0 < \gamma < \frac{2}{\|A\|^2}$ and sequence $\{a_n\}$ in $[0, \frac{4}{2 + \gamma \|A\|^2}]$ satisfying

$$\sum_{n=1}^{\infty} a_n \left(\frac{4}{2 + \gamma \|A\|^2} - a_n \right) = \infty,$$

that the sequence $\{x_n\}$ generated by

$$x_{n+1} = (1 - a_n) x_n + a_n P_C \big(I - \gamma A^*(I - P_Q)A \big) x_n$$

converges weakly to a solution of SFP (1) provided the solution set of SFP (1) is nonempty.
It is also interesting to compare Theorem 4.3 with Theorem 5.5 of [2] and Theorem 3.1
of [5]. Our method is different from those in [2] and [5] even in the case of $u = 0$, because
our algorithm contains an error term and uses the operator $P_C(I - \gamma A^*(I - P_Q)A)$ directly
without any regularization.

Theorem 4.4 *Let C and Q be nonempty closed convex subsets of two Hilbert spaces $\mathcal{H}_1$ and
$\mathcal{H}_2$ respectively, and let A, B be bounded linear mappings from $\mathcal{H}_1$ to $\mathcal{H}_2$. Put $S = P_C(I - \gamma B^*(I - P_Q)B)$ and $T = P_C(I - \gamma A^*(I - P_Q)A)$, where γ satisfies $0 < \gamma < \min\{\frac{2}{\|B\|^2}, \frac{2}{\|A\|^2}\}$.
Suppose the solution set Ω of SFP* (5) *is nonempty, and suppose further that $\{a_n\}$, $\{b_n\}$, $\{c_n\}$
and $\{d_n\}$ are sequences in $[0,1]$ with $a_n + b_n + c_n + d_n = 1$, and $a_n \in (0,1)$ for all $n \in \mathbb{N}$, and
that $\{e_n\}$ is a bounded sequence in C. For an arbitrary fixed $u \in C$, start with an arbitrary
$x_1 \in C$ and define the sequence $\{x_n\}$ by*

$$x_{n+1} = a_n u + b_n S x_n + c_n T x_n + d_n e_n.$$

Then $\{x_n\}$ converges strongly to $P_\Omega u$ provided the following conditions are satisfied:

(i) $\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} \dfrac{d_n}{a_n} = 0, \qquad \sum\limits_{n=1}^{\infty} a_n = \infty, \qquad \sum\limits_{n=1}^{\infty} d_n < \infty;$

(ii) $\liminf\limits_{n \to \infty} b_n > 0, \qquad \liminf\limits_{n \to \infty} c_n > 0.$

Proof It is clear that this theorem follows from Lemmas 4.1 and 4.2 and Corollary 3.4.

□

Replacing S_n and T_n in Theorem 3.2 with the resolvents $J_{\beta_n}^B$ and $J_{\alpha_n}^A$ of two maximal monotone operators B and A respectively, we have Theorem 4.5 below.

Theorem 4.5 *Suppose that B and A are two maximal monotone operators on $\mathcal{H}$ with $B^{-1}0 \cap A^{-1}0 \neq \varnothing$, and suppose that $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ are sequences in $[0,1]$ with $a_n + b_n + c_n + d_n = 1$ and $a_n \in (0,1)$ for all $n \in \mathbb{N}$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0,\infty)$, and let $\{e_n\}$ and $\{v_n\}$ be two bounded sequences in $\mathcal{H}$. For an arbitrary norm convergent sequence $\{u_n\}$ in $\mathcal{H}$ with limit u, start with an arbitrary $x_1 = y_1 \in \mathcal{H}$ and define two sequences $\{x_n\}$ and $\{y_n\}$ by*

$$x_{n+1} = a_n u + b_n J_{\beta_n}^B x_n + c_n J_{\alpha_n}^A x_n + d_n e_n;$$

$$y_{n+1} = a_n u_n + b_n J_{\beta_n}^B y_n + c_n J_{\alpha_n}^A y_n + d_n v_n.$$

Then both of the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $P_{B^{-1}0 \cap A^{-1}0} u$ provided the following conditions are satisfied:

(i) $\displaystyle \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{d_n}{a_n} = 0, \qquad \sum_{n=1}^{\infty} a_n = \infty, \qquad \sum_{n=1}^{\infty} d_n < \infty;$

(ii) $\displaystyle \liminf_{n \to \infty} b_n > 0, \qquad \liminf_{n \to \infty} c_n > 0;$

(iii) $\displaystyle \liminf_{n \to \infty} \alpha_n > 0, \qquad \liminf_{n \to \infty} \beta_n > 0.$

Proof Since all the requirements of Theorem 3.2 are satisfied except conditions (3.1) and (3.2), we have to check these two conditions. For any $n \in \mathbb{N}$, let $S_n = J_{\beta_n}^B$ and $T_n = J_{\alpha_n}^A$. By Lemma 2.1(b), we have $B^{-1}0 = F(S_n)$ and $A^{-1}0 = F(T_n)$ for all $n \in \mathbb{N}$. Moreover, since all S_n and T_n are firmly nonexpansive, all of them are $\frac{1}{2}$-averaged, so condition (3.1) is satisfied with $\mu_n = \lambda_n = \frac{1}{2}$ for all $n \in \mathbb{N}$. According to Lemma 3.3, it remains to prove that condition (3.2) holds for $\{J_{\beta_n}^B\}$ and $\{J_{\alpha_n}^A\}$. Since condition (iii) holds, we may assume that there is $\tau \in (0,1)$ such that $\tau < \alpha_n$ and $\tau < \beta_n$ for all $n \in \mathbb{N}$. Let $\kappa_1(n) = 2 + \frac{\alpha_n}{\tau}$ and $\kappa_2(n) = 2 + \frac{\beta_n}{\tau}$. Then, by virtue of the resolvent identity and the nonexpansiveness of $J_{\alpha_m}^A$, one has for all $m \in \mathbb{N}$ that

$$\left\| J_{\alpha_n}^A x - J_{\alpha_m}^A x \right\| = \left\| J_{\alpha_m}^A \left(\frac{\alpha_m}{\alpha_n} x + \left(1 - \frac{\alpha_m}{\alpha_n} \right) J_{\alpha_n}^A x \right) - J_{\alpha_m}^A x \right\|$$

$$\leq \left| 1 - \frac{\alpha_m}{\alpha_n} \right| \left\| J_{\alpha_n}^A x - x \right\| \leq \left(1 + \frac{\alpha_m}{\alpha_n} \right) \left\| J_{\alpha_n}^A x - x \right\|,$$

and thus

$$\left\| J_{\alpha_m}^A x - x \right\| \leq \left\| J_{\alpha_m}^A x - J_{\alpha_n}^A x \right\| + \left\| J_{\alpha_n}^A x - x \right\|$$

$$\leq \left(1 + \frac{\alpha_m}{\alpha_n} + 1 \right) \left\| J_{\alpha_n}^A x - x \right\|$$

$$\leq \left(2 + \frac{\alpha_m}{\tau} \right) \left\| J_{\alpha_n}^A x - x \right\| = \kappa_1(m) \left\| J_{\alpha_n}^A x - x \right\|, \quad \forall n \geq m, \forall x \in \mathcal{H}.$$

The same argument shows for all $m \in \mathbb{N}$ that

$$\left\| J^B_{\beta_m} x - x \right\| \leq \kappa_2(m) \left\| J^B_{\beta_n} x - x \right\|, \quad \forall n \geq m, \forall x \in \mathcal{H}.$$

Therefore, condition (3.2) is true for $\{J^A_{\alpha_n}\}$ and $\{J^B_{\beta_n}\}$. $\qquad\square$

Putting $T_n = J^A_{\alpha_n}$ in Corollary 3.6 (resp. Corollary 3.7) and noting that $\{I\}$ and $\{J^A_{\alpha_n}\}$ verifies condition (3.2) due to $\liminf_{n\to\infty} \alpha_n > 0$, we obtain the following two corollaries.

Corollary 4.6 *Suppose that A is a maximal monotone operator on $\mathcal{H}$ with $A^{-1}0 \neq \varnothing$, and suppose that $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in $[0,1]$ with $a_n + b_n + c_n = 1$ and $a_n \in (0,1)$ for all $n \in \mathbb{N}$. Let $\{\alpha_n\}$ be a sequence in $(0, \infty)$. For an arbitrary fixed $u \in \mathcal{H}$, choose an arbitrary $x_1 \in \mathcal{H}$ and define*

$$x_{n+1} = a_n u + b_n x_n + c_n J^A_{\alpha_n} x_n, \quad n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to $P_{A^{-1}0} u$ provided the following conditions are satisfied:

$$\lim_{n\to\infty} a_n = 0, \qquad \sum_{n=1}^{\infty} a_n = \infty, \qquad \liminf_{n\to\infty} c_n > 0, \qquad \liminf_{n\to\infty} \alpha_n > 0.$$

Corollary 4.7 [16] *Suppose that A is a maximal monotone operator on $\mathcal{H}$ with $A^{-1}0 \neq \varnothing$, and suppose that $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in $[0,1]$ with $a_n + b_n + c_n = 1$ and $a_n \in (0,1)$ for all $n \in \mathbb{N}$. Let $\{\alpha_n\}$ be a sequence in $(0, \infty)$, and let $\{e_n\}$ be a bounded sequence in $\mathcal{H}$. For an arbitrary fixed $u \in \mathcal{H}$, choose an arbitrary $x_1 \in \mathcal{H}$ and define*

$$x_{n+1} = a_n u + b_n x_n + c_n J^A_{\alpha_n} x_n + e_n, \quad n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to $P_{A^{-1}0} u$ provided the following conditions are satisfied:

(i) $\displaystyle \lim_{n\to\infty} a_n = 0, \qquad \sum_{n=1}^{\infty} a_n = \infty, \qquad \liminf_{n\to\infty} c_n > 0, \qquad \liminf_{n\to\infty} \alpha_n > 0;$

(ii) *either* $\displaystyle \lim_{n\to\infty} \frac{\|e_n\|}{a_n} = 0$ *or* $\displaystyle \sum_{n=1}^{\infty} \|e_n\| < \infty.$

Let $f : \mathcal{H} \to (-\infty, \infty]$ be a proper lower semicontinuous convex function. The set of minimizers of f is defined to be

$$\arg\min_{y\in\mathcal{H}} f(y) = \left\{ z \in \mathcal{H} : f(x) \leq f(y) \text{ for all } y \in \mathcal{H} \right\},$$

and the subdifferential of f is defined as

$$\partial f(x) = \left\{ z \in \mathcal{H} : \langle y - x, z \rangle \leq f(y) - f(x), \forall y \in \mathcal{H} \right\}$$

for all $x \in \mathcal{H}$. As shown in Rockafellar [23], ∂f is a maximal monotone operator. Moreover, one has

$$0 \in \partial f(z) \quad \Leftrightarrow \quad z \in \arg\min_{y \in \mathcal{H}} f(y),$$

that is,

$$(\partial f)^{-1} 0 = \arg\min_{y \in \mathcal{H}} f(y).$$

Hence $\arg\min_{y \in \mathcal{H}} f(y) = F(J_\alpha^{\partial f})$ for any $\alpha > 0$, and then invoking Corollary 4.6, we obtain the following theorem.

Theorem 4.8 *Let $f : \mathcal{H} \to (-\infty, \infty]$ be a proper lower semicontinuous convex function, and suppose that $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in $[0,1]$ with $a_n + b_n + c_n = 1$ and $a_n \in (0,1)$ for all $n \in \mathbb{N}$. Let $\{\alpha_n\}$ be a sequence in $(0,\infty)$, and put $\Omega = \arg\min_{y \in \mathcal{H}} f(y)$. For an arbitrary fixed $u \in \mathcal{H}$, choose an arbitrary $x_1 \in \mathcal{H}$ and define*

$$x_{n+1} = a_n u + b_n x_n + c_n J_{\alpha_n}^{\partial f} x_n. \tag{23}$$

Then the sequence $\{x_n\}$ converges strongly to $P_\Omega u$ provided the following conditions are satisfied:

(i) $\quad \lim_{n\to\infty} a_n = 0, \qquad \sum_{n=1}^{\infty} a_n = \infty;$

(ii) $\quad \liminf_{n\to\infty} c_n > 0, \qquad \liminf_{n\to\infty} \alpha_n > 0.$

For any $n \in \mathbb{N}$, define $g_n : \mathcal{H} \to (-\infty, \infty]$ by

$$g_n(z) = f(z) + \frac{1}{2\alpha_n} \|z - x_n\|^2$$

for all $z \in \mathcal{H}$. Then we have, for any $z \in \mathcal{H}$,

$$\partial g_n(z) = \partial f(z) + \frac{1}{\alpha_n}(z - x_n),$$

cf. [24]. Hence,

$$\begin{aligned}
z \in (\partial g_n)^{-1} 0 \quad &\Leftrightarrow \quad 0 \in \partial g_n(z) \\
&\Leftrightarrow \quad 0 \in \partial f(z) + \frac{1}{\alpha_n}(z - x_n) \\
&\Leftrightarrow \quad x_n \in z + \alpha_n \partial f(z) = (I + \alpha_n \partial f)(z) \\
&\Leftrightarrow \quad z = (I + \alpha_n \partial f)^{-1} x_n = J_{\alpha_n}^{\partial f} x_n.
\end{aligned}$$

This means that $J_{\alpha_n}^{\partial f} x_n = \arg\min_{y \in \mathcal{H}} g_n(y) = \arg\min_{y \in \mathcal{H}} \{f(y) + \frac{1}{2\alpha_n} \|y - x_n\|^2\}$, and thus the iterative scheme (23) is can be replaced with

$$x_{n+1} = a_n u + b_n x_n + c_n \arg\min_{y \in \mathcal{H}} \left\{ f(y) + \frac{1}{2\alpha_n} \|y - x_n\|^2 \right\}.$$

Let $f : C \times C \to \mathbb{R}$. An equilibrium problem is the problem of finding $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0, \quad \forall y \in C,$$

whose solution set is denoted by EP(f). For solving an equilibrium problem, we assume that the function f satisfies the following conditions:

(A1) $f(x, x) = 0, \forall x \in C$;
(A2) f *is monotone, that is,* $f(x, y) + f(y, x) \leq 0, \forall x \in C$;
(A3) *for all* $x, y, z \in C$, $\limsup_{t \downarrow 0} f((1 - t)x + tz, y) \leq f(x, y)$;
(A4) *for all* $x \in C, f(x, \cdot)$ *is convex and lower semicontinuous.*

The following lemma appears implicitly in Blum and Oetti [25] and is proved in detail by Aoyama *et al.* [26], while Lemma 4.10 is Lemma 2.12 of Combettes and Hirstoaga [27].

Lemma 4.9 [25, 26] *Let* $f : C \times C \to \mathbb{R}$ *be a function satisfying conditions* (A1)-(A4), *and let* $r > 0$ *and* $x \in \mathcal{H}$. *Then there exists a unique* $z \in C$ *such that*

$$f(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C.$$

Lemma 4.10 [27] *Let* $f : C \times C \to \mathbb{R}$ *be a function satisfying conditions* (A1)-(A4). *For* $r > 0$, *define* $J_r^f : \mathcal{H} \to C$ *by*

$$J_r^f x = \left\{ z \in C : f(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\}$$

for all $x \in \mathcal{H}$. *Then the following hold*:

(a) J_r^f *is single-valued*;
(b) J_r^f *is firmly nonexpansive*;
(c) $F(J_r^f) = \mathrm{EP}(f)$;
(d) EP(f) *is closed and convex.*

We call J_r^f the resolvent of f for $r > 0$. Using Lemmas 4.9 and 4.10, Takahashi *et al.* [15] established the lemma below.

Lemma 4.11 [15] *Let* $f : C \times C \to \mathbb{R}$ *be a function satisfying conditions* (A1)-(A4) *and define a set-valued mapping of* $\mathcal{H}$ *into itself by*

$$G_f(x) = \begin{cases} \{z \in \mathcal{H} : f(x, y) \geq \langle y - x, z \rangle, \forall y \in C\}, & \forall x \in C, \\ \varnothing, & \forall x \notin C. \end{cases}$$

Then the following hold:

(a) G_f *is a maximal monotone operator with* $\mathcal{D}(G_f) \subseteq C$;

(b) $\text{EP}(f) = G_f^{-1}0$;

(c) $J_r^{G_f} x = J_r^f x$ for all $x \in \mathcal{H}$.

Theorem 4.12 *Let C be a nonempty closed convex subset of $\mathcal{H}$ and let $f_i : C \times C \to \mathbb{R}$, $i = 1, 2$, be functions satisfying conditions (A1)-(A4) with $\text{EP}(f_1) \cap \text{EP}(f_2) \neq \varnothing$. Suppose that $\{a_n\}, \{b_n\}, \{c_n\}$ and $\{d_n\}$ are sequences in $[0, 1]$ with $a_n + b_n + c_n + d_n = 1$ and $a_n \in (0, 1)$ for all $n \in \mathbb{N}$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0, \infty)$, and let $\{e_n\}$ be a bounded sequence in $\mathcal{H}$. For an arbitrary fixed $u \in \mathcal{H}$, choose an arbitrary $x_1 \in \mathcal{H}$ and define*

$$x_{n+1} = a_n u + b_n J_{\beta_n}^{f_2} x_n + c_n J_{\alpha_n}^{f_1} x_n + d_n e_n, \quad n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to $P_{\text{EP}(f_1) \cap \text{EP}(f_2)} u$ provided the following conditions are satisfied:

(i) $\displaystyle \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{d_n}{a_n} = 0, \qquad \sum_{n=1}^{\infty} a_n = \infty, \qquad \sum_{n=1}^{\infty} d_n < \infty$;

(ii) $\displaystyle \liminf_{n \to \infty} b_n > 0, \qquad \liminf_{n \to \infty} c_n > 0$;

(iii) $\displaystyle \liminf_{n \to \infty} \alpha_n > 0, \qquad \liminf_{n \to \infty} \beta_n > 0$.

Proof The set-valued mappings G_{f_i} associated with f_i, $i = 1, 2$, defined in Lemma 4.11 are maximal monotone operators with $\mathcal{D}(G_{f_i}) \subseteq C$, and it follows from Lemmas 4.10 and 4.11 that $J_r^{G_{f_i}} = J_r^{f_i}$ and $F(J_r^{G_{f_i}}) = F(J_r^{f_i}) = \text{EP}(f_i) = G_{f_i}^{-1}0$ for any $r > 0$. Putting $B = G_{f_2}$ and $A = G_{f_1}$ in Theorem 4.5, the desired conclusion follows. $\qquad\square$

Here, it is worth mentioning, just as the SFP, that the unique minimum norm solution can be obtained through our algorithm for each of the minimization problem and the equilibrium problem by taking $u = 0$ in Theorems 4.8 and 4.12.

For a nonempty closed convex subset C of $\mathcal{H}$, its indicator function ι_C defined by

$$\iota_C(x) = \begin{cases} 0, & x \in C; \\ \infty, & x \notin C, \end{cases}$$

is a proper lower semicontinuous convex function and its subdifferential $\partial \iota_C$ defined by

$$\partial \iota_C(x) = \left\{ z \in \mathcal{H} : \langle y - x, z \rangle \leq \iota_C(y) - \iota_C(x), \forall y \in \mathcal{H} \right\}$$

is a maximal monotone operator, *cf.* Rockafellar [23]. As shown in Lin and Takahashi [28], the resolvent $J_r^{\partial \iota_C}$ of $\partial \iota_C$ for $r > 0$ is the same as the metric projection P_C, and $(\partial \iota_C)^{-1}0 = C$.

Theorem 4.13 *Let C_i, $i = 1, 2$, be two nonempty closed convex subsets of $\mathcal{H}$ with $C_1 \cap C_2 \neq \varnothing$. Suppose that $\{a_n\}, \{b_n\}, \{c_n\}$ and $\{d_n\}$ are sequences in $[0, 1]$ with $a_n + b_n + c_n + d_n = 1$ and $a_n \in (0, 1)$ for all $n \in \mathbb{N}$. Let $\{e_n\}$ be a bounded sequence in $\mathcal{H}$. For an arbitrary fixed $u \in \mathcal{H}$, choose an arbitrary $x_1 \in \mathcal{H}$ and define*

$$x_{n+1} = a_n u + b_n P_{C_2} x_n + c_n P_{C_1} x_n + d_n e_n, \quad n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to $P_{C_1 \cap C_2} u$ provided the following conditions are satisfied:

(i) $\quad \lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} \dfrac{d_n}{a_n} = 0, \qquad \sum\limits_{n=1}^{\infty} a_n = \infty, \qquad \sum\limits_{n=1}^{\infty} d_n < \infty;$

(ii) $\quad \liminf\limits_{n \to \infty} b_n > 0, \qquad \liminf\limits_{n \to \infty} c_n > 0.$

Proof Putting $A = \partial \iota_{C_1}$ and $B = \partial \iota_{C_2}$ in Theorem 4.5, the desired conclusion follows. $\qquad \square$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to this work. Both authors read and approved the final manuscript.

Author details
[1]Department of Accounting Information, Southern Taiwan University of Science and Technology, 1 Nantai St., Yongkang Dist., Tainan, 71005, Taiwan. [2]Department of Industrial Management, National Pingtung University of Science and Technology, 1 Shuefu Rd., Neipu, Pingtung, 91201, Taiwan.

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