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Large-spin expansions of GKP strings

Emmanuel Floratos,^{a,b} George Georgiou^b and Georgios Linardopoulos^{a,b}

^a*Department of Physics, National and Kapodistrian University of Athens, Zografou Campus, 157 84, Athens, Greece*

^b*Institute of Nuclear and Particle Physics, N.C.S.R. “Demokritos”, 153 10, Agia Paraskevi, Greece*

E-mail: mflorato@phys.uoa.gr, georgiou@inp.demokritos.gr, glinard@inp.demokritos.gr

ABSTRACT: We demonstrate that the large-spin expansion of the energy of Gubser-Klebanov-Polyakov (GKP) strings that rotate in $\mathbb{R} \times S^2$ and AdS_3 can be expressed in terms of Lambert’s W-function. We compute the leading, subleading and next-to-subleading series of exponential corrections to the infinite-volume dispersion relation of GKP strings that rotate in $\mathbb{R} \times S^2$. These strings are dual to the long $\mathcal{N} = 4$ SYM operators $\text{Tr} [\Phi Z^m \Phi Z^{J-m}] + \dots$ and provide their scaling dimensions at strong coupling. We also show that the strings obey a short-long (strings) duality. For the folded GKP strings that spin inside AdS_3 and are dual to twist-2 operators, we confirm the known formulas for the leading and next-to-leading coefficients of their anomalous dimensions and derive the corresponding expressions for the next-to-next-to-leading coefficients.

KEYWORDS: AdS-CFT Correspondence, Strong Coupling Expansion

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1 Introduction

According to the standard lore of AdS/CFT correspondence [1–3], the partition functions of type IIB string theory on $\text{AdS}_5 \times \text{S}^5$ and $\mathcal{N} = 4$, $\text{SU}(N)$ super Yang-Mills (SYM) theory are equal. The field-operator correspondence of AdS/CFT states that every operator of $\mathcal{N} = 4$ SYM has a dual string state in IIB string theory on $\text{AdS}_5 \times \text{S}^5$ and every boundary observable (e.g. operator scaling dimensions, correlation functions, scattering amplitudes, etc.) possesses a dual observable in the bulk. Therefore, in order to establish the validity of AdS/CFT [4, 5], the spectra of scaling dimensions of the two theories should match (among other things and at least in the planar limit [6, 7]).

The difficulty with the program of matching the spectra on both sides of the duality has traditionally been linked to the insufficient knowledge of type IIB String Theory on $\text{AdS}_5 \times \text{S}^5$ backgrounds. A breakthrough in this direction was made in 2002 by Gubser, Klebanov and Polyakov (GKP) [8], who proposed to study closed semiclassical strings, spinning, rotating or pulsating in $\text{AdS}_5 \times \text{S}^5$, in order to obtain the (anomalous) scaling dimensions of their dual SYM operators at strong coupling, a regime where all perturbative calculations from the gauge theory side typically break down [9].

The paper of Gubser, Klebanov and Polyakov [8] contains three prototype string settings for which the energy-spin relation is calculated: (I) a folded closed string rigidly rotating at the equator of S^3 of AdS_5 , (II) a folded closed string rigidly rotating around the pole of $\text{S}^2 \subset \text{S}^5$ and (III) a closed string pulsating inside AdS_3 . Each of these string configurations is dual to a gauge-invariant operator of $\mathcal{N} = 4$ SYM, the (anomalous) scaling dimensions of which at strong coupling are given by the energy of the corresponding closed string state.

Case (I) has been thoroughly analyzed over the past years. The key observation of GKP was that the energy minus the spin angular momentum of a long folded closed string that rotates inside AdS_3 , scales as the logarithm of its (large) spin, behavior already familiar from the study of anomalous dimensions of twist-2 Wilson operators in perturbative QCD. Being able to reproduce this calculation within $\mathcal{N} = 4$ SYM, GKP asserted that their string is dual to the twist-2 operators of $\mathcal{N} = 4$ SYM, $\text{Tr}[\mathcal{Z} \mathcal{D}_+^S \mathcal{Z}] + \dots$ ¹

Twist operators of large spin S originally came up in a QCD context where their anomalous scaling dimensions were shown to be responsible for the (logarithmic) violation of Bjorken scaling in deep inelastic scattering (DIS). For twist-2 they have been calculated² at one-loop [11, 12], two-loops [13–16] and three loops [17, 18]. As it turns out, the QCD results may be used to extract the corresponding anomalous dimensions in perturbative $\mathcal{N} = 1, 2, 4$ SYM theory (see e.g. [10]). For $\mathcal{N} = 4$ SYM the anomalous dimensions of twist-2 operators $\text{Tr}[\mathcal{Z} \mathcal{D}_+^S \mathcal{Z}] + \dots$ have thus been calculated to one-loop [19], two-loops [20] and, using transcendentality, to three-loops [21] for large-spin S and weak 't Hooft coupling

¹We denote $\mathcal{Z}, \mathcal{W}, \mathcal{Y}$ the three complex scalars of $\mathcal{N} = 4$ SYM composed out of its six real scalars ϕ . Also $\mathcal{D}_+ = \mathcal{D}_0 + \mathcal{D}_3$, $\mathcal{D}_- = \mathcal{D}_1 + \mathcal{D}_2$ stand for the light-cone derivatives. The dots in a trace operator generally denote terms that are built by permuting trace fields \mathcal{Z} and impurities, $\mathcal{W}, \mathcal{Y}, \mathcal{D}_\pm$.

²For more references and a brief historical perspective, see [10].

λ . To wit, the following logarithmic behavior is obtained, also known as Sudakov scaling:

$$\gamma(S, g) = \Delta - (S + 2) = f(g) \ln S + \dots, \quad g = \frac{\sqrt{\lambda}}{4\pi}, \quad \lambda = g_{\text{YM}}^2 N. \quad (1.1)$$

$f(g)$ is known as the cusp anomalous dimension or the universal scaling function of $\mathcal{N} = 4$ SYM and has a history of its own with analytic expressions obtained by solving the Beisert-Eden-Staudacher (BES) equation at weak [22, 23] and strong coupling [24, 25]. The latter is found to agree at two loops with the cusp anomalous dimension that is calculated from the folded closed AdS₃ string when quantum corrections are included [26–28].

Alternative techniques may provide the anomalous dimensions of twist-2 operators of $\mathcal{N} = 4$ SYM at weak coupling up to five loops. By solving the Baxter equation analytically, three-loop [29] and four-loop [30] expressions are obtained. By including wrapping corrections after three-loops, the anomalous dimensions to four and five-loops have been computed in [31, 32].

At strong coupling, the structure³ of the large-spin expansion of anomalous scaling dimensions of $\text{Tr}[\mathcal{Z} \mathcal{D}_+^S \mathcal{Z}] + \dots$ is identical to the perturbative one (1.1), albeit with different coefficients $f(\sqrt{\lambda})$:

$$\gamma(S, \lambda) = f \ln(S/\sqrt{\lambda}) + \sum_{n=1}^{\infty} f_{nn} \frac{\ln^n(S/\sqrt{\lambda})}{S^n} + \sum_{n=1}^{\infty} f_{nn-1} \frac{\ln^n(S/\sqrt{\lambda})}{S^{n+1}} + \sum_{n=0}^{\infty} \frac{f_n}{S^n} + \dots \quad (1.2)$$

It is a daunting task to calculate all the coefficients of (1.2) at strong coupling. Theoretically, one may obtain them all by means of the thermodynamic Bethe ansatz (TBA) [35]. Alternatively, one may calculate quantum corrections to the AdS₃ GKP-string [36, 37].

In a recent paper, the authors of [38] succeeded in calculating all the classical leading and subleading terms of series (1.2) at strong coupling, by introducing an iterative method that can potentially supply all the classical terms at an arbitrary subleading order. The current work extends the method of [38] to examples other than the classic GKP case (I).⁴ In their original treatment [8], Gubser, Klebanov and Polyakov gave the following formula for the anomalous dimensions of the $\mathcal{N} = 4$ SYM operator that is dual to the $\mathbb{R} \times S^2$ closed and folded string (II):

$$E - J = \frac{2\sqrt{\lambda}}{\pi}, \quad J, \lambda \rightarrow \infty. \quad (1.3)$$

For finite yet large J , this expression receives subleading exponential contributions. In this paper we are interested in calculating all of the subleading in J terms of the series (1.3).

GKP case (II) also turns out to have been extensively studied over the past years. Some initial considerations based on the GKP model [8] may be found in [26, 40]. Closed folded single-spin strings rotating on $\mathbb{R} \times S^2$ can be decomposed into more elementary string theory excitations, known as giant magnons. These are open, single-spin strings rotating

³For recent developments on the structure of 3-point correlators involving twist- J operators both at weak and strong coupling, see [33, 34] and references therein.

⁴At this point let us mention that an interesting generalization of this calculation would be to calculate the anomalous dimensions of twist-2 operators in the context of a generalized Yang-Mills theory, such as [39].

in $S^2 \subset S^5$, identified in 2006 by Hofman and Maldacena [41] as the string theory duals of $\mathcal{N} = 4$ SYM magnon excitations. The energy-spin relation of one giant magnon of angular extent $\Delta\varphi$ is:

$$E - J = \frac{\sqrt{\lambda}}{\pi} \left| \sin \frac{\Delta\varphi}{2} \right|, \quad J, \lambda \rightarrow \infty, \tag{1.4}$$

where $\Delta\varphi = p$ is the dual magnon's momentum. Superimposing two such giant magnons of maximum angular extent $\Delta\varphi = \pi$ gives the GKP formula (1.3).

As we have just mentioned, giant magnons are dual to magnons, the elementary spin chain excitations. The exact infinite-volume magnon dispersion relation, valid at weak and strong coupling, was found by Beisert in [42] by considering the centrally extended superalgebra $\mathfrak{su}(2|2)_c \oplus \mathfrak{su}(2|2)_c \subset \mathfrak{psu}(2, 2|4)$:

$$E - J = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \left(\frac{p}{2} \right)}, \quad J \rightarrow \infty. \tag{1.5}$$

This result reproduces both the string (1.4) and the perturbative gauge-theory results.

On the side of $\mathcal{N} = 4$ SYM, it has long been known that the dilatation operator has the form of an integrable spin chain Hamiltonian⁵ which can be diagonalized by means of the Bethe ansatz (BA) [43]. However, and due to their being asymptotic in nature, the BA equations can only reproduce the correct form of anomalous dimensions only when the length L of the spin chain is infinite or larger than the loop order L . At and above this *critical* loop-order L , virtual particles start circulating around the spin chain (as the range of spin-chain interactions then exceeds its length) and wrapping corrections have to be taken into account.

Indeed, the inefficiency of BA has been noted in both gauge [44] and string theories [45, 46]. The wrapping effects that appear at the critical loop-order have the form of exponentially small corrections to the anomalous dimensions, as noticed in [47, 48]. An important theoretical issue is therefore the calculation of the exact anomalous dimensions of unprotected operators of $\mathcal{N} = 4$ SYM that have finite size L .

To this end, it was proposed in [49] that the thermodynamic Bethe ansatz (TBA) [50]⁶ can correctly account for the wrapping interactions. The Y-system [52] also accounts for wrapping corrections. On the string theory side, one equivalently calculates exponential corrections to the corresponding giant magnon dispersion relation (1.4). The following

⁵The compact $\mathfrak{su}(2)$ sector of $\mathcal{N} = 4$ SYM consists of the single-trace operators $\text{Tr} [Z^L \Phi^{L-M}]$, where $\Phi \in \{\mathcal{W}, \mathcal{Y}\}$. It is dual to strings rotating in $\mathbb{R} \times S^3 \subset \text{AdS}_5 \times S^5$ and its one-loop dilatation operator is given by the Hamiltonian of the ferromagnetic $\text{XXX}_{1/2}$ Heisenberg spin chain. The other closed sector of relevance to our paper is the non-compact $\mathfrak{sl}(2)$ sector composed out of the twist- J operators $\text{Tr} [\mathcal{D}_+^{S_1} \Phi \mathcal{D}_+^{S_2} \Phi \dots \mathcal{D}_+^{S_J} \Phi]$, where $S_1 + S_2 + \dots + S_J = S$ and Φ is any of the three complex scalar fields of $\mathcal{N} = 4$ SYM. This sector is dual to strings rotating in $\text{AdS}_3 \times S^1$ and its dilatation operator is given by the Hamiltonian of the ferromagnetic $\text{XXX}_{-1/2}$ Heisenberg spin chain.

⁶For more, see the review [51].

result was first derived by Arutyunov, Frolov and Zamaklar in [53]:⁷

$$E - J = \frac{\sqrt{\lambda}}{\pi} \left| \sin \frac{p}{2} \right| \left\{ 1 - 4 \sin^2 \frac{p}{2} e^{-2-2\pi J/\sqrt{\lambda}|\sin \frac{p}{2}|} - 8 \sin^2 \frac{p}{2} \left[6 \cos^2 \frac{p}{2} + \frac{1}{2} + \left(6 \cos^2 \frac{p}{2} - 1 \right) \frac{2\pi J}{\sqrt{\lambda}|\sin \frac{p}{2}|} + \cos^2 \frac{p}{2} \left(\frac{2\pi J}{\sqrt{\lambda}|\sin \frac{p}{2}|} \right)^2 \right] e^{-4-4\pi J/\sqrt{\lambda}|\sin \frac{p}{2}|} + \dots \right\}. \quad (1.6)$$

Astolfi, Forini, Grignani and Semenoff have proven in [54] that, when placed upon an orbifold, giant magnons are completely independent of any gauge parameter. The first two terms of the same result (1.6) have been found by the algebraic curve method in [55]. They have also been obtained by using Lüscher’s perturbative method [56, 57] at strong coupling [58–60].

By using the relation between strings on S^2 and the sine-Gordon model [61], Klose and McLoughlin [62] have obtained the following leading terms of the series (1.6):

$$E - J = \frac{\sqrt{\lambda}}{\pi} \sin \frac{p}{2} \left\{ 1 - 4 \sin^2 \frac{p}{2} e^{-L_{\text{eff}}} \left[1 + 2 L_{\text{eff}}^2 \cos^2 \frac{p}{2} e^{-L_{\text{eff}}} + 8 L_{\text{eff}}^4 \cos^4 \frac{p}{2} e^{-2L_{\text{eff}}} + \frac{128}{3} L_{\text{eff}}^6 \cos^6 \frac{p}{2} e^{-3L_{\text{eff}}} + \frac{800}{3} L_{\text{eff}}^8 \cos^8 \frac{p}{2} e^{-4L_{\text{eff}}} + \frac{9216}{5} L_{\text{eff}}^{10} \cos^{10} \frac{p}{2} e^{-5L_{\text{eff}}} + \dots \right] \right\}, \quad (1.7)$$

where $L_{\text{eff}} \equiv L/\sin p/2$ is the effective length and L is the spatial periodicity of the spin chain. Giant magnons have been generalized to β -deformed backgrounds [63, 64], TsT-transformed $\text{AdS}_5 \times S^5$ [65, 66] and $\text{AdS}_4/\text{CFT}_3$ [67, 68].

In this paper we study the dispersion relations of the following (long) operators of $\mathcal{N} = 4$ SYM:

$$\mathcal{O}_S = \text{Tr} [\mathcal{Z} \mathcal{D}_+^S \mathcal{Z}] + \dots \quad \& \quad \mathcal{O}_J = \text{Tr} [\Phi \mathcal{Z}^m \Phi \mathcal{Z}^{J-m}] + \dots, \quad (1.8)$$

for large values of the ’t Hooft coupling constant λ . These operators are dual to cases (I) and (II) of GKP strings that were discussed above. Our work is motivated by the interesting properties of twist-2 operators that originally appeared in the context of deep inelastic scattering in QCD and have been studied intensively ever since. Twist-2 operators are also present in $\mathcal{N} = 4$ super Yang-Mills theory and the AdS/CFT correspondence permits to study them in the strong coupling regime. Our investigation aims to shed light on the structure of the corresponding strong-coupling, large-spin, semiclassical dispersion relations, with a view to gaining insight into their weak-coupling, small-spin and quantum generalizations. A reorganization of these relations might lead to better-recognizable structures that could be useful, especially when one is trying to derive series (1.2)–(1.6) with other integrability methods such as Lüscher corrections, the TBA or the Y-system.

We propose a systematic method, based on the Lagrange-Bürmann inversion formula, for the order-by-order inversion of a certain class of functions that are related to elliptic

⁷Actually, this is the gauge-independent result of [54], which coincides with that of AFZ [53] in the temporal gauge.

integrals. Next, we apply this method in order to express the dispersion relations of operators (1.8) as sums of Lambert’s W-functions. The latter are analytic in their arguments and therefore have the property of collecting infinitely many terms that were previously unknown. We emphasize that these previously unknown terms have not yet been obtained by any other integrability method, such as Lüscher corrections (where only the first two terms are known), the TBA or the Y-system. Within the context of AdS/CFT correspondence, our analysis could have repercussions in the dispersion relations of giant magnons, spiky strings, spinning membranes, or strings rotating in β -deformed and TsT-transformed backgrounds. Our technique can also be applied to the solution of the renormalization group equations up to any loop-order. For more, the reader is referred to the discussion section 7.

Our paper is organized as follows. We begin with a brief summary of our findings in section 2. In section 3 we derive the single-spin string solution of GKP that rotates in $\mathbb{R} \times S^2$ by minimizing the energy functional of a generic $\mathbb{R} \times S^2$ string configuration and show that it obeys a duality of the short-long type. Section 4 contains our calculation of the exponential corrections to the anomalous dimensions of long $\mathbb{R} \times S^2$ strings. We start from a 2×2 system of equations

$$\mathcal{E} = d(x) \ln x + h(x) \tag{1.9}$$

$$\mathcal{J} = c(x) \ln x + b(x), \tag{1.10}$$

where \mathcal{E} and \mathcal{J} are the string’s energy and spin, x is a parameter depending on the angular velocity of the string and $d(x)$, $h(x)$, $c(x)$, $b(x)$ are some known power series of x . We invert equation (1.10) for the inverse spin function $x = x(\mathcal{J})$, then plug it back into (1.9) in order to obtain the anomalous dimensions $\gamma \equiv \mathcal{E} - \mathcal{J} = \gamma(\mathcal{J})$. The main thrust of our paper is showing that the dispersion relation $\gamma(\mathcal{J})$ can be written in terms of Lambert’s W-function:

$$W(z) e^{W(z)} = z \Leftrightarrow W(z e^z) = z. \tag{1.11}$$

We obtain formulas for all the leading, subleading and next-to-subleading coefficients for both long folded (angular velocity $\omega > 1$) and fast circular (angular velocity $\omega < 1$) strings in $\mathbb{R} \times S^2$. In sections 5–6, our analysis is repeated for long folded strings spinning (with angular velocity $\omega > 1$) in AdS_3 where again we derive expressions for the leading, subleading and next-to-subleading coefficients of anomalous dimensions. A discussion of our results can be found in section 7. Appendix A is a brief introduction to Lambert’s W-function. Appendix B contains the definitions and some useful formulas of elliptic integrals and functions. In appendix C we briefly consider short/slow-spin strings and in appendix D we have collected our symbolic computations on the long and fast GKP strings of our paper.

2 Summary of results

In this section we summarise our basic results.

2.1 Long strings in $\mathbb{R} \times S^2$

The $\mathbb{R} \times S^2$ string solutions that we will be considering are dual to the following gauge theory operators: $\text{Tr} [\Phi Z^m \Phi Z^{J-m}] + \dots$, where the dots denote appropriate weighted

permutations of the complex scalar fields $\mathcal{Z}, \Phi \in \{\mathcal{W}, \mathcal{Y}\}$ and $J \gg \sqrt{\lambda}$. By computing the energy of the GKP string, we obtain the strong coupling value of scaling dimensions of the aforementioned operators. The result can be expressed in terms of Lambert's function $W(\pm 8\mathcal{J}e^{-2\mathcal{J}-2})$ as follows:

$$\mathcal{E} - \mathcal{J} = 1 - \frac{1}{4\mathcal{J}}(2W + W^2) - \frac{1}{16\mathcal{J}^2}(W^2 + W^3) - \frac{1}{256\mathcal{J}^3} \frac{W^3(11W^2 + 26W + 16)}{1 + W} + \dots, \tag{2.1}$$

- leading terms: $-\frac{1}{4\mathcal{J}}(2W + W^2) = \sum_{n=1}^{\infty} \mathbf{a}_n \mathcal{J}^{n-1} (e^{-2\mathcal{J}-2})^n$.
- subleading terms: $-\frac{1}{16\mathcal{J}^2}(W^2 + W^3) = \sum_{n=2}^{\infty} \mathbf{b}_n \mathcal{J}^{n-2} (e^{-2\mathcal{J}-2})^n$.
- next-to-subleading terms: $-\frac{1}{256\mathcal{J}^3} \frac{W^3(11W^2 + 26W + 16)}{1 + W} = \sum_{n=3}^{\infty} \mathbf{c}_n \mathcal{J}^{n-3} (e^{-2\mathcal{J}-2})^n$.

where $\mathcal{E} \equiv \pi E/2\sqrt{\lambda}$ and $\mathcal{J} \equiv \pi J/2\sqrt{\lambda}$. The plus sign in the argument of Lambert's W-function corresponds to closed and folded strings (angular velocity $\omega > 1$) and the minus sign to circular strings (angular velocity $\omega < 1$). Upon expansion of Lambert's W-function, the second, third and fourth term on the r.h.s. of (2.1) provide three infinite series of coefficients which completely determine the leading, subleading and next-to-subleading contributions to the large- J finite-size corrections to the dispersion relation of a closed folded single-spin string rotating on S^2 . The precise expressions for these infinite series can be found in equations (4.21), (4.31) and (4.37). One can argue that all finite-size corrections (N^k -subleading terms) can be written in terms of Lambert's W-function. Note that all (super-leading) terms of the form $\mathcal{J}^n(e^{-2\mathcal{J}-2})^n$ are absent from expansion (2.1). Effectively, our calculation provides the finite-size corrections to the dispersion relation of a single-magnon state with maximal momentum $p = \pi$, the dual operator of which is:

$$\tilde{\mathcal{O}}_J = \sum_m e^{im\pi} |\dots \mathcal{Z} \mathcal{Z} \Phi(m) \mathcal{Z} \mathcal{Z} \dots\rangle, \quad J \rightarrow \infty. \tag{2.2}$$

These corrections are exactly equal to one-half the finite-size corrections (2.1) to the energy of a closed folded single-spin string that rotates on S^2 .

We have also found a duality between short and long strings. For each solution of energy E and spin J , there exists a dual solution whose energy E' and spin J' are related to the original by equations (3.23)–(3.24).

2.2 Long strings in AdS₃

Secondly, we will consider the classical GKP solution rotating in an AdS₃ subspace of AdS₅. These strings are dual to twist-2 operators which schematically have the following form:

$$\mathcal{O}_S = Tr [\mathcal{Z} \mathcal{D}_+^S \mathcal{Z}] + \dots \tag{2.3}$$

The case of long AdS₃ strings has been extensively studied in the literature because the corresponding field theory operators play a vital role in DIS. The form of their anomalous dimensions admits the following strong coupling expansion:

$$\begin{aligned} \mathcal{E} - \mathcal{S} = \rho_c \ln \mathcal{S} + \sum_{n=0}^{\infty} \sum_{k=0}^n \rho_{(nk)} \frac{\ln^k \mathcal{S}}{\mathcal{S}^n} = \rho_c \ln \mathcal{S} + \rho_0 + \sum_{n=1}^{\infty} \rho_{(nn)} \frac{\ln^n \mathcal{S}}{\mathcal{S}^n} + \sum_{n=2}^{\infty} \rho_{(nn-1)} \frac{\ln^{n-1} \mathcal{S}}{\mathcal{S}^n} + \\ + \sum_{n=3}^{\infty} \rho_{(nn-2)} \frac{\ln^{n-2} \mathcal{S}}{\mathcal{S}^n} + \dots + \frac{\rho_1}{\mathcal{S}} + \frac{\rho_2}{\mathcal{S}^2} + \frac{\rho_3}{\mathcal{S}^3} + \dots, \end{aligned} \quad (2.4)$$

where $\mathcal{S} = \pi S/2\sqrt{\lambda}$. By expressing the above result in terms of Lambert's function $W_{-1}(-e^{-4\mathcal{S}-3/2}/4)$,

$$\begin{aligned} \gamma = -\frac{W_{-1}}{2} - \left(2\mathcal{S} + \frac{5}{4}\right) + \frac{9}{8W_{-1}} - \left[\frac{\mathcal{S}}{2} + \frac{35}{16}\right] \frac{1}{(W_{-1})^2} + \left[\frac{5\mathcal{S}}{2} + \frac{2213}{384}\right] \frac{1}{(W_{-1})^3} - \left[\mathcal{S}^2 + \frac{361\mathcal{S}}{32} + \frac{6665}{384}\right] \cdot \\ \cdot \frac{1}{(W_{-1})^4} + \left[\frac{19\mathcal{S}^2}{2} + \frac{1579\mathcal{S}}{32} + \frac{433501}{7680}\right] \frac{1}{(W_{-1})^5} - \left[\frac{10\mathcal{S}^3}{3} + \frac{259\mathcal{S}^2}{4} + \frac{81.799\mathcal{S}}{384} + \frac{2.963.887}{15.360}\right] \frac{1}{(W_{-1})^6} + \\ + \left[\frac{136\mathcal{S}^3}{3} + \frac{3069\mathcal{S}^2}{8} + \frac{175.481\mathcal{S}}{192} + \frac{2.350.780.111}{3.440.640}\right] \frac{1}{(W_{-1})^7} - \dots, \end{aligned} \quad (2.5)$$

we have analytically calculated the following coefficients:

$$\rho_c = \frac{1}{2}, \quad \rho_0 = 2 \ln 2 - \frac{1}{2}, \quad \rho_1 = \frac{\ln 2}{2} - \frac{1}{8}, \quad \rho_2 = -\frac{\ln^2 2}{4} + \frac{9 \ln 2}{32} - \frac{5}{128}$$

and also the following three infinite series of coefficients:

$$\begin{aligned} \rho_{(mm)} &= \frac{(-1)^{m+1}}{4^m} \frac{1}{2m}, \\ \rho_{(m+1,m)} &= \frac{1}{2} \frac{(-1)^{m+1}}{4^{m+1}} \left[H_m + \frac{m}{4} + 1 - 4 \ln 2 \right] \\ \rho_{(m+2,m)} &= \frac{(-1)^{m+1}}{4^{m+3}} \cdot (m+1) \cdot \left\{ H_{m+1}^2 - H_{m+1}^{(2)} + \frac{1}{2} (m - 16 \ln 2 + 5) \cdot H_{m+1} + \frac{m^2}{24} - \right. \\ &\quad \left. - \left(2 \ln 2 + \frac{1}{24} \right) m + 16 \ln^2 2 - 10 \ln 2 \right\}. \end{aligned}$$

The series $\rho_{(mm)}$ and $\rho_{(m+1,m)}$ were derived for the first time in [38]. The infinite series of the next-to-next-to-leading coefficients $\rho_{(m+2,m)}$ is derived in section 6.

3 Gubser-Klebanov-Polyakov $\mathbb{R} \times S^2$ string

In [8] the following configuration of a string that has its center at the north pole of $S^2 \subset S^5$ and rotates around it was considered:

$$\left\{ t = \kappa\tau, \rho = \theta = \phi_1 = \phi_2 = 0 \right\} \times \left\{ \bar{\theta}_1 = \bar{\theta}_1(\sigma), \bar{\theta}_2 = \kappa\omega\tau, \bar{\phi}_1 = \bar{\phi}_2 = \bar{\phi}_3 = 0 \right\}, \quad (3.1)$$

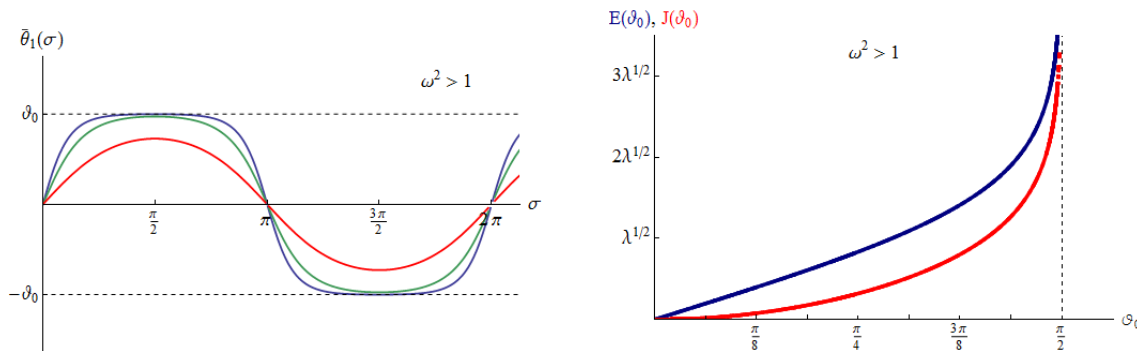


Figure 1. $\bar{\theta} = \bar{\theta}_1(\sigma)$ and energy/spin of the folded closed $\mathbb{R} \times \mathbb{S}^2$ string (3.1) for $\omega > 1$.

where the line element of $\text{AdS}_5 \times \mathbb{S}^5$ is

$$ds^2 = \ell^2 \left[-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho \left(d\theta^2 + \cos^2 \theta d\phi_1^2 + \sin^2 \theta d\phi_2^2 \right) + d\bar{\theta}_1^2 + \cos^2 \bar{\theta}_1 d\bar{\phi}_1^2 + \sin^2 \bar{\theta}_1 \left(d\bar{\theta}_2^2 + \cos^2 \bar{\theta}_2 d\bar{\phi}_2^2 + \sin^2 \bar{\theta}_2 d\bar{\phi}_3^2 \right) \right]. \quad (3.2)$$

We shall now briefly demonstrate that the above GKP solution is unique in the sense that it minimizes the energy of an $\mathbb{R} \times \mathbb{S}^2$ string with a single, constant angular momentum J . Let us consider the following generic ansatz on $\mathbb{R} \times \mathbb{S}^2$:

$$\left\{ t = \kappa\tau, \rho = \theta = \phi_1 = \phi_2 = 0 \right\} \times \left\{ \bar{\theta}_1 = \bar{\theta}_1(\tau, \sigma), \bar{\theta}_2 = \bar{\theta}_2(\tau, \sigma), \bar{\phi}_1 = \bar{\phi}_2 = \bar{\phi}_3 = 0 \right\}, \quad (3.3)$$

which leads to the Polyakov action (in the conformal gauge, $\gamma_{ab} = \eta_{ab}$),

$$\mathcal{S}_P = \frac{\ell^2}{4\pi\alpha'} \int \left(-\dot{t}^2 + \dot{\bar{\theta}}_1^2 - \bar{\theta}'_1{}^2 + \sin^2 \bar{\theta}_1 (\dot{\bar{\theta}}_2^2 - \bar{\theta}'_2{}^2) \right) d\tau d\sigma. \quad (3.4)$$

One can derive the Hamiltonian density of the string in the standard way, namely

$$\mathcal{H} = \frac{\ell^2}{4\pi\alpha'} \left\{ -\kappa^2 + \frac{p_1^2}{4} + \bar{\theta}'_1{}^2 + \frac{p_2^2}{4\sin^2 \bar{\theta}_1} + \bar{\theta}'_2{}^2 \sin^2 \bar{\theta}_1 \right\} = 0, \quad \begin{aligned} p_1 &= 2\dot{\bar{\theta}}_1 \\ p_2 &= 2\dot{\bar{\theta}}_2 \sin^2 \bar{\theta}_1. \end{aligned} \quad (3.5)$$

(3.5) is identical to one of the Virasoro constraints. We want to find the minimum of the energy E under the constraint of fixed angular momentum J . To achieve this we introduce the Lagrange multiplier ω and look for the minimum of the quantity

$$Y \equiv \frac{\ell^2}{4\pi\alpha'} \int d\sigma \cdot 2 \sqrt{\frac{p_1^2}{4} + \bar{\theta}'_1{}^2 + \frac{p_2^2}{4\sin^2 \bar{\theta}_1} + \bar{\theta}'_2{}^2 \sin^2 \bar{\theta}_1} - \omega \left[\frac{\ell^2}{4\pi\alpha'} \int p_2 d\sigma - J \right], \quad (3.6)$$

which is obtained after substituting κ from (3.5) into the integral of the energy, $E = \ell^2/4\pi\alpha' \int d\sigma \cdot 2\kappa$. We find:

$$\frac{\delta Y}{\delta p_1} = 0 \Rightarrow p_1 = 0 \Rightarrow \dot{\bar{\theta}}_1 = 0 \Rightarrow \bar{\theta}_1 = \bar{\theta}_1(\sigma) \quad (3.7)$$

$$\frac{\delta Y}{\delta p_2} = 0 \Rightarrow \dot{\bar{\theta}}_2 = \omega \kappa \Rightarrow \bar{\theta}_2 = \kappa \omega \tau + \tilde{\phi}(\sigma). \quad (3.8)$$

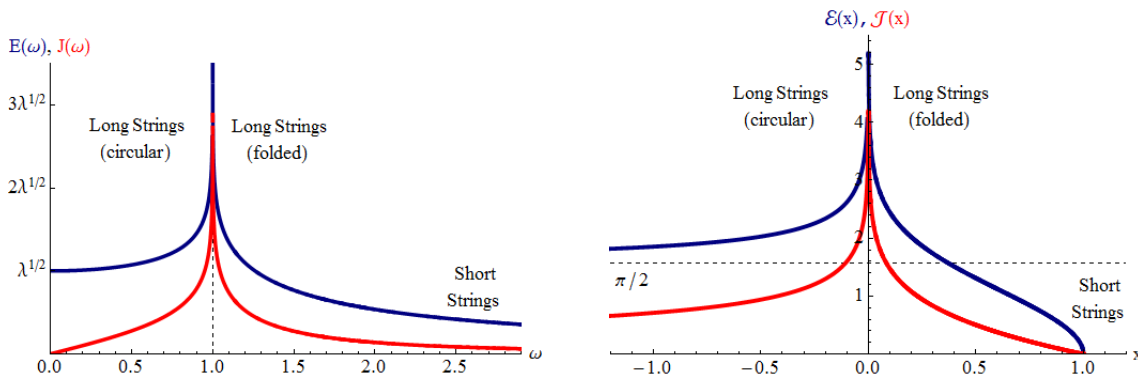


Figure 2. Energy/spin of the closed folded/circular $\mathbb{R} \times S^2$ string as functions of ω and x .

One can now use the second Virasoro constraint $\dot{\bar{\theta}}_1 \bar{\theta}'_1 + \dot{\bar{\theta}}_2 \bar{\theta}'_2 \sin^2 \bar{\theta}_1 = 0$ and equation (3.7) to conclude that $\bar{\theta}'_2 = 0$ or equivalently $\tilde{\phi}(\sigma) = 0$. Let us mention that the vanishing of the functional derivative $\delta Y / \delta \bar{\theta}_1(\sigma) = 0$ will give the equation of motion for $\bar{\theta}_1$. We thus end up with the GKP solution (3.1), the Polyakov action of which is given by:

$$\mathcal{S}_P = \frac{\ell^2}{4\pi\alpha'} \int \left(-\dot{t}^2 + \dot{\bar{\theta}}_2^2 \sin^2 \bar{\theta}_1 - \bar{\theta}'_1{}^2 \right) d\tau d\sigma = \frac{\ell^2}{4\pi\alpha'} \int \left(-\kappa^2 + \kappa^2 \omega^2 \sin^2 \bar{\theta}_1 - \bar{\theta}'_1{}^2 \right) d\tau d\sigma. \tag{3.9}$$

Setting $\sigma(\vartheta_0) = \pi/2$ for the length of the string, we obtain the following cases, depending on the value of $\omega \neq 1$:

$\omega^2 < 1$. Circular folded string.

$$\bar{\theta}_1(\sigma) = am \left[\kappa \sigma \mid \omega^2 \right], \quad \kappa = \frac{2}{\pi} \cdot \mathbb{K}(\omega^2) \tag{3.10}$$

$$E(\omega) = \frac{2\sqrt{\lambda}}{\pi} \cdot \mathbb{K}(\omega^2) \Rightarrow \mathcal{E} \equiv \frac{\pi E}{2\sqrt{\lambda}} = \mathbb{K}(1 - \tilde{x}) \tag{3.11}$$

$$J(\omega) = \frac{2\sqrt{\lambda}}{\pi\omega} \left[\mathbb{K}(\omega^2) - \mathbb{E}(\omega^2) \right] \Rightarrow \mathcal{J} \equiv \frac{\pi J}{2\sqrt{\lambda}} = \frac{1}{\sqrt{1 - \tilde{x}}} \left[\mathbb{K}(1 - \tilde{x}) - \mathbb{E}(1 - \tilde{x}) \right]. \tag{3.12}$$

$\omega^2 > 1$. Folded closed string.

$$\bar{\theta}_1(\sigma) = am \left[\kappa \sigma \mid \omega^2 \right], \quad \kappa = \frac{2}{\pi\omega} \cdot \mathbb{K}\left(\frac{1}{\omega^2}\right), \quad \omega = \csc \vartheta_0 \tag{3.13}$$

$$E(\omega) = \frac{2\sqrt{\lambda}}{\pi\omega} \cdot \mathbb{K}\left(\frac{1}{\omega^2}\right) \Rightarrow \mathcal{E} \equiv \frac{\pi E}{2\sqrt{\lambda}} = \sqrt{1 - x} \cdot \mathbb{K}(1 - x) \tag{3.14}$$

$$J(\omega) = \frac{2\sqrt{\lambda}}{\pi} \cdot \left[\mathbb{K}\left(\frac{1}{\omega^2}\right) - \mathbb{E}\left(\frac{1}{\omega^2}\right) \right] \Rightarrow \mathcal{J} \equiv \frac{\pi J}{2\sqrt{\lambda}} = \mathbb{K}(1 - x) - \mathbb{E}(1 - x) \tag{3.15}$$

$$\gamma \equiv \mathcal{E} - \mathcal{J} = (\sqrt{1 - x} - 1) \cdot \mathbb{K}(1 - x) + \mathbb{E}(1 - x). \tag{3.16}$$

$x \equiv 1 - 1/\omega^2$ and $\tilde{x} \equiv 1 - \omega^2$ are the complementary parameters of $1/\omega^2$ and ω^2 respectively. In figures 1–2 we have plotted $\bar{\theta}_1 = \bar{\theta}_1(\sigma)$ for various values of $\omega > 1$ as well as the energy and the spin of the $\mathbb{R} \times \mathbb{S}^2$ string as functions of ϑ_0 , ω and x .

For $\omega > 1$, there exist two interesting regimes where one would want to obtain the functional dependence of $E = E(J)$ and the corresponding anomalous dimensions $\gamma = \gamma(J)$, namely the short-string limit $\omega \rightarrow \infty$ and the long-string limit $\omega \rightarrow 1^+$. In what follows we shall be concerned only with the latter. Short strings are briefly treated in appendix C.

3.1 Long folded strings in $\mathbb{R} \times \mathbb{S}^2$: $\omega \rightarrow 1^+$, $J \gg \sqrt{\lambda}$

For long folded strings on \mathbb{S}^2 ($\omega \rightarrow 1^+$), the expansions for the energy and the spin become (cf. appendix B):

$$E = \frac{\sqrt{\lambda}}{\pi^2 \omega} \cdot \sum_{n=0}^{\infty} \left(\frac{\Gamma(n+1/2)}{n!} \right)^2 \left[2\psi(n+1) - 2\psi(n+1/2) - \ln(1-1/\omega^2) \right] \cdot (1-1/\omega^2)^n \quad (3.17)$$

$$J = \frac{\sqrt{\lambda}}{\pi} \cdot \left\{ 4\ln 2 - 2 - \ln(1-1/\omega^2) - \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)\Gamma(n+3/2)}{((n+1)!)^2} \left[2\psi(n+1) - 2\psi(n+1/2) - \ln(1-1/\omega^2) + \frac{2n}{(n+1)(2n+1)} \right] \cdot (1-1/\omega^2)^{n+1} \right\}. \quad (3.18)$$

In terms of the complementary parameter $x \equiv 1 - 1/\omega^2 \rightarrow 0^+$ the above series can be written in compact forms as follows:

$$\mathcal{E} \equiv \frac{\pi E}{2\sqrt{\lambda}} = \sqrt{1-x} \cdot \sum_{n=0}^{\infty} x^n (d_n \ln x + h_n) = - \sum_{n=0}^{\infty} x^n \cdot \sum_{k=0}^n \frac{(2k-3)!!}{(2k)!!} (d_{n-k} \ln x + h_{n-k}) \quad (3.19)$$

$$\mathcal{J} \equiv \frac{\pi J}{2\sqrt{\lambda}} = \sum_{n=0}^{\infty} x^n (c_n \ln x + b_n). \quad (3.20)$$

The coefficients that appear in series (3.19) and (3.20) are given by:⁸

$$d_n = -\frac{1}{2} \left(\frac{(2n-1)!!}{(2n)!!} \right)^2, \quad h_n = -4d_n \cdot (\ln 2 + H_n - H_{2n})$$

$$c_n = -\frac{d_n}{2n-1}, \quad b_n = -4c_n \cdot \left[\ln 2 + H_n - H_{2n} + \frac{1}{2(2n-1)} \right], \quad (3.21)$$

for $n = 0, 1, 2, \dots$

3.2 Short-long strings duality

Following [38], we may now write down a formula that connects the values of conserved charges at the two opposite ends of the closed folded string spectrum, that is “short” strings ($\omega \rightarrow \infty$) and “long” strings ($\omega \rightarrow 1^+$). There’s a known expression between

⁸Some useful values of the double factorial are: $0!! = 1$, $(-1)!! = 1$, $(-3)!! = -1$.

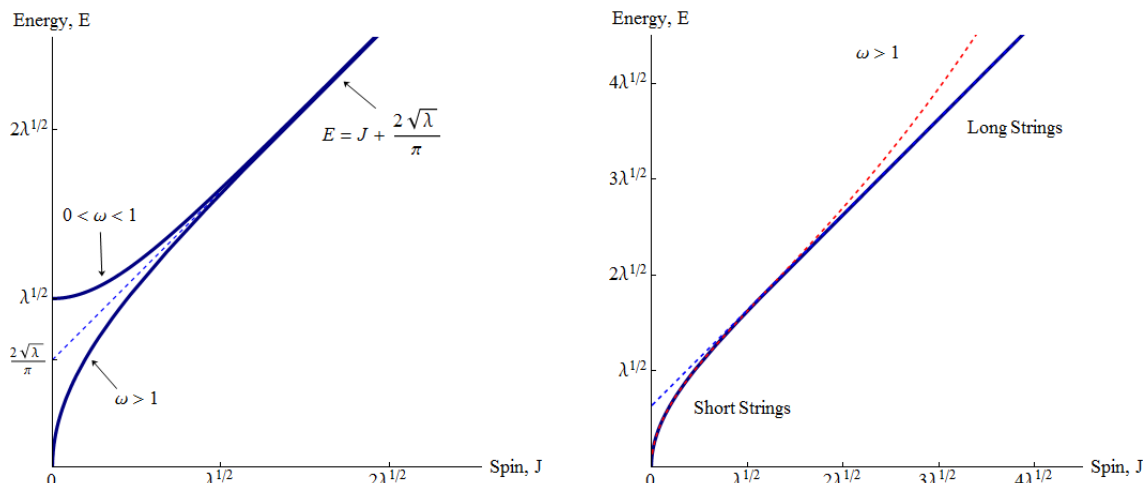


Figure 3. $E = E(J)$ of the folded closed $\mathbb{R} \times S^2$ string.

complete elliptic integrals of the first and second kind, namely Legendre’s relation (see e.g. [69]):

$$\mathbb{E}(k)\mathbb{K}(k') + \mathbb{K}(k)\mathbb{E}(k') - \mathbb{K}(k)\mathbb{K}(k') = \frac{\pi}{2}, \tag{3.22}$$

where the arguments of elliptic integrals, $k = 1/\omega^2$ and $k' = x = 1/\omega'^2$ satisfy $k + k' = 1$. Solving (3.14)–(3.15) for $\mathbb{E}(k)$ and $\mathbb{K}(k)$ and substituting in (3.22), we get the following duality relation between classical folded short and long strings that spin on $S^2 \subset S^5$:

$$\omega \omega' EE' - \omega EJ' - \omega' E'J = \frac{2\lambda}{\pi}, \quad \omega > 1. \tag{3.23}$$

This relation is completely analogous to the one found for closed folded strings that spin inside AdS_3 , (5.18). One could also write (3.23) in terms of $\gamma \equiv E - S$. Plotting the functions $E = E(\omega)$ and $J = J(\omega)$ in a common diagram parametrically, the graph of $E = E(J)$ along with the first 4 terms of its “short” series (C.5) and the first 2 terms of its “long” approximation (D.3) has been obtained in figure 3 (red and blue dashed lines respectively of the plot on the right). Likewise, a similar relation can be formulated for fast and slow circular strings using (3.11)–(3.12):

$$EE' - \omega' EJ' - \omega E'J = \frac{2\lambda}{\pi}, \quad \omega < 1, \tag{3.24}$$

where $\tilde{k} = \omega^2$, $\tilde{k}' = \tilde{x} = \omega'^2$ and $\tilde{k} + \tilde{k}' = 1$.

4 Inverse spin functions and anomalous dimensions on $\mathbb{R} \times \mathbb{S}^2$

4.1 Inverse spin function

We will now follow the method of [38] in order to invert the J-series (3.20) for $x = x(\mathcal{J})$ and obtain $\mathcal{E} = \mathcal{E}(\mathcal{J})$ by substituting $x(\mathcal{J})$ into $\mathcal{E}(x)$. Let us first solve (3.20) for $\ln x$:

$$\mathcal{J} = \sum_{n=0}^{\infty} x^n (c_n \ln x + b_n) \Rightarrow \ln x = \left[\frac{\mathcal{J} - b_0}{c_0} - \sum_{n=1}^{\infty} \frac{b_n}{c_0} x^n \right] \cdot \sum_{n=0}^{\infty} (-1)^n \left(\sum_{k=1}^{\infty} \frac{c_k}{c_0} x^k \right)^n. \quad (4.1)$$

Performing the products between the series and exponentiating, we are led to the following equation that we will eventually have to solve (or invert) for x :

$$x = x_0 \cdot \exp \left[\sum_{n=1}^{\infty} a_n x^n \right] = x_0 \cdot \exp (a_1 x + a_2 x^2 + a_3 x^3 + \dots), \quad (4.2)$$

where

$$x_0 \equiv \exp \left[\frac{\mathcal{J} - b_0}{c_0} \right] = 16 e^{-2\mathcal{J}-2} \quad (4.3)$$

solves (4.1) to lowest order in x and the coefficients a_n are determined from (4.1). One possible way to revert series (4.2) with respect to the variable x , is by means of the Lagrange inversion theorem [69]. In our case it turns out that the function to be inverted has a special form that significantly simplifies the computation of its inverse. The following reversion formula (applied here to the exponential function) is named after J.-L. Lagrange and H.H. Bürmann [70]:

$$x = \sum_{n=1}^{\infty} \frac{x_0^n}{n!} \cdot \left\{ \frac{d^{n-1}}{dz^{n-1}} \exp \left[\sum_{m=1}^{\infty} n a_m z^m \right] \right\}_{z=0}. \quad (4.4)$$

We find

$$x = \sum_{n=1}^{\infty} x_0^n \cdot \sum_{k, j_i=0}^{n-1} \frac{n^k}{n!} \binom{n-1}{j_1, j_2, \dots, j_{n-1}} a_1^{j_1} a_2^{j_2} \dots a_{n-1}^{j_{n-1}}, \quad (4.5)$$

where

$$j_1 + j_2 + \dots + j_{n-1} = k \quad \& \quad j_1 + 2j_2 + \dots + (n-1)j_{n-1} = n-1.$$

Now notice from (4.1) that the a_i 's can only be linear functions of \mathcal{J} , so that the inverse spin function $x = x(\mathcal{J})$ has to be of the form:

$$x = \sum_{n=1}^{\infty} x_0^n \cdot \sum_{k=0}^{n-1} a_{nk} \mathcal{J}^k, \quad (4.6)$$

where a_{nk} are some constants. This follows from the two constraints on the values of j which imply

$$\left. \begin{aligned} j_1 + j_2 + \dots + j_{n-1} &= k \\ j_1 + 2j_2 + \dots + (n-1)j_{n-1} &= n-1 \end{aligned} \right\} \Rightarrow k + j_2 + \dots + (n-2)j_{n-1} = n-1, \quad (4.7)$$

that is the power of J is at most $k \leq n-1$. A second conclusion that can be drawn from these two constraints is that all leading in \mathcal{J} contributions to x are controlled by the leading in \mathcal{J} terms of \mathbf{a}_1 , all subleading \mathcal{J} -contributions to x are controlled by \mathbf{a}_1 and the leading in \mathcal{J} terms of \mathbf{a}_2 , etc., i.e. $x(\mathcal{J})$ has all its coefficients up to $x_0^n \mathcal{J}^{n-m}$ controlled by $\mathbf{a}_1, \dots, \mathbf{a}_{m-1}$, and the leading term of \mathbf{a}_m . To see this, notice from $k + j_2 + \dots + (n-2)j_{n-1} = n-1$ that when some j_m in (4.5) is $j_m \neq 0$ (minimum 1), $k = j_m + \dots + j_{n-1}$ is at most $n-1 - (m-1) = n-m$. This is consistent with our expectations from (4.2), from which the same conclusion about the number of terms that should be kept on the r.h.s., in order to fully determine $x(\mathcal{J})$ up to a given order, is reached.

4.2 Anomalous dimensions

Having derived a general formula for $x(\mathcal{J})$, we may express the anomalous scaling dimensions $\gamma = \mathcal{E} - \mathcal{J}$ of the $\mathbb{R} \times \mathbb{S}^2$ spinning closed and folded string as a function of \mathcal{J} :

$$\mathcal{E} - \mathcal{J} = \sum_{n=0}^{\infty} x^n (f_n \ln x + g_n) = \sum_{n=0}^{\infty} x^n \left[A_n + f_n \ln \frac{x}{x_0} \right], \quad \mathcal{E} \equiv \frac{\pi E}{2\sqrt{\lambda}}, \quad \mathcal{J} \equiv \frac{\pi J}{2\sqrt{\lambda}}, \quad (4.8)$$

where

$$f_n \equiv -c_n - \sum_{k=0}^n \frac{(2k-3)!!}{(2k)!!} \cdot d_{n-k}, \quad g_n \equiv -b_n - \sum_{k=0}^n \frac{(2k-3)!!}{(2k)!!} \cdot h_{n-k}, \quad n = 0, 1, 2, \dots \quad (4.9)$$

and the coefficients A_n are defined as:

$$A_n \equiv g_n + f_n \ln x_0 = g_n + 2f_n (2 \ln 2 - \mathcal{J} - 1). \quad (4.10)$$

For large \mathcal{J} , series (4.1) may be inverted for $x = x(\mathcal{J})$ using a computer algebra system. The inverse spin function $x(\mathcal{J})$ may then be plugged into equation (4.8) and give the anomalous dimensions of $\mathcal{N} = 4$ SYM operators $\text{Tr} [\Phi \mathcal{Z}^m \bar{\Phi} \mathcal{Z}^{J-m}] + \dots$ in terms of the (large) R-charge \mathcal{J} . Such a calculation has been performed with Mathematica and both series of the inverse spin function $x = x(\mathcal{J})$ and anomalous dimensions $\gamma = \gamma(\mathcal{J})$ have been obtained (cf. (D.2), (D.3)). It turns out that both expansions contain the following terms:

$$\begin{aligned} \text{Leading Terms (L): } & \mathcal{J}^{n-1} (e^{-2\mathcal{J}-2})^n \\ \text{Next-to-Leading/Subleading Terms (NL): } & \mathcal{J}^{n-2} (e^{-2\mathcal{J}-2})^n \\ \text{NNL Terms: } & \mathcal{J}^{n-3} (e^{-2\mathcal{J}-2})^n \\ & \vdots \end{aligned} \quad (4.11)$$

Note that series (3.19), (3.20) and (4.8) are identical in structure. We may then prove that in order to get $\mathcal{E} - \mathcal{J}$ up to a given subleading order, the inverse spin function $x(\mathcal{J})$ that will be inserted into (4.8) has to be known up to no more than the same order. From (4.2) and (4.6) we write:

$$\ln \frac{x}{x_0} = \sum_{k=1}^{\infty} a_k x^k = a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$x = \sum_{n=1}^{\infty} x_0^n \cdot \sum_{k=0}^{n-1} a_{nk} \mathcal{J}^k = \sum_{n=1}^{\infty} \mathcal{J}^{n-1} x_0^n \cdot \sum_{k=0}^{n-1} \frac{\tilde{a}_{nk}}{\mathcal{J}^k} = \frac{1}{\mathcal{J}} \sum_{n=1}^{\infty} \mathcal{J}^n x_0^n \cdot \sum_{k=0}^{n-1} \frac{\tilde{a}_{nk}}{\mathcal{J}^k}, \quad (4.12)$$

where $a_{nk} = \tilde{a}_{n(n-k-1)}$ are some constants and a_n are linear functions of \mathcal{J} . The last equation follows from (4.6) after some reshuffling. The anomalous dimensions (4.8) are then written as follows:

$$\mathcal{E} - \mathcal{J} = \sum_{n=0}^{\infty} x^n (f_n \ln x + g_n) = \sum_{n=0}^{\infty} x^n \left[A_n + f_n \ln \frac{x}{x_0} \right] = \sum_{n=0}^{\infty} x^n \left[A_n + \sum_{k=1}^{\infty} f_n a_k x^k \right]. \quad (4.13)$$

Since all the leading terms of x^n are of the order $1/\mathcal{J}^n$ and they multiply either A_n or $f_{n-k} \cdot a_k$ in (4.13), which are both linear in \mathcal{J} , we see that the r -th subleading term of $\mathcal{E} - \mathcal{J}$ (which is of the order $1/\mathcal{J}^r$) cannot get any contributions from its x^{r+2} terms. Therefore in order to get precisely the first r -subleading orders of the anomalous dimensions $\mathcal{E} - \mathcal{J}$ ($r = 1$ leading, $r = 2$ subleading, etc.), no more than the first $r + 1$ powers of x need to be retained in (4.13). In addition, the last power of x to be kept in (4.13) (namely x^{r+1}) does not have to be multiplied by coefficients that do not contain \mathcal{J} .

Furthermore, we can see why we need exactly n subleading terms in the expansion of x in order to be able to calculate $\mathcal{E} - \mathcal{J}$ up to the same subleading order n . Keeping less powers inside x would mean that $x \cdot A_1 = -x/4$ in (4.13) essentially misses some of the subleading terms, while terms deeper than $1/\mathcal{J}^n$ into x do not contribute, since there exist no corresponding powers of \mathcal{J} in the expression for $\mathcal{E} - \mathcal{J}$ that can lift them up to the wanted order. All of these observations will become clearer in what follows.

4.3 Leading terms

As a first application of the above, we may calculate the anomalous dimensions to leading order in \mathcal{J} , i.e. the coefficients of the following series:

$$E - J \Big|_{(L)} = \sum_{n=1}^{\infty} \alpha_n \mathcal{J}^{n-1} (e^{-2\mathcal{J}-2})^n. \quad (4.14)$$

In order to be able to do this, x has to be determined up to leading order in \mathcal{J} , i.e. the coefficients of the series

$$x_{(L)} = \sum_{n=1}^{\infty} \alpha_n \mathcal{J}^{n-1} (e^{-2\mathcal{J}-2})^n \quad (4.15)$$

must be computed. To this end, we must keep all the terms that multiply $x^0 = 1$ on the r.h.s. of equation (4.1) and just the leading in \mathcal{J} terms that multiply $x^1 = x$. (4.1) then

becomes:

$$\ln x_{(L)} = \frac{\mathcal{J} - b_0}{c_0} - \frac{c_1}{c_0^2} \mathcal{J} \cdot x_{(L)} \Rightarrow x_0 = x_{(L)} \exp \left[\frac{c_1}{c_0^2} \mathcal{J} \cdot x_{(L)} \right] = x_{(L)} e^{\mathcal{J} \cdot x_{(L)}/2}, \quad (4.16)$$

where $x_0 = 16 e^{-2\mathcal{J}^{-2}}$. This is equation (4.2) for the leading terms of x . We either solve it by the inversion method of the previous section or we could just as well calculate the following iteration:

$$x_{(L)} = x_0 e^{-x_0 \mathcal{J}/2 \cdot e^{-x_0 \mathcal{J}/2 \cdot e^{\dots}}} = x_0 \cdot {}^\infty(e^{-x_0 \mathcal{J}/2}). \quad (4.17)$$

Given the expression for the infinite exponential (see e.g. [71]),

$${}^\infty(e^z) = \frac{W(-z)}{-z} = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} z^{n-1}, \quad (4.18)$$

where $W(z)$ is the principal branch of the Lambert W-function,⁹ we find:

$$x_{(L)} = \frac{2}{\mathcal{J}} W(8 \mathcal{J} e^{-2\mathcal{J}^{-2}}) = \sum_{n=1}^{\infty} \alpha_n \mathcal{J}^{n-1} (e^{-2\mathcal{J}^{-2}})^n, \quad (4.19)$$

where we have defined

$$\alpha_n \equiv (-1)^{n+1} 2^{3n+1} \cdot \frac{n^{n-1}}{n!}. \quad (4.20)$$

As we have explained above, in order to calculate the $E - J$ series to leading order in \mathcal{J} , we have to insert this formula for $x_{(L)}$ into (4.8) and take care as to keep only leading terms. The result is:

$$\begin{aligned} E - J \Big|_{(L)} &= \frac{2\sqrt{\lambda}}{\pi} \left\{ 1 + g_1 x_{(L)} - 2f_2 \mathcal{J} x_{(L)}^2 \right\} = \frac{2\sqrt{\lambda}}{\pi} \left\{ 1 - \frac{x_{(L)}}{4} - \frac{\mathcal{J} x_{(L)}^2}{16} \right\} = \\ &= \frac{2\sqrt{\lambda}}{\pi} \left\{ 1 - \frac{1}{4\mathcal{J}} \left[2 \cdot W(8 \mathcal{J} e^{-2\mathcal{J}^{-2}}) + W^2(8 \mathcal{J} e^{-2\mathcal{J}^{-2}}) \right] \right\} = \\ &= \frac{2\sqrt{\lambda}}{\pi} \left\{ 1 - \frac{1}{16} \sum_{n=1}^{\infty} \left[4\alpha_n + \sum_{k=1}^{n-1} \alpha_k \alpha_{n-k} \right] \cdot \mathcal{J}^{n-1} (e^{-2\mathcal{J}^{-2}})^n \right\}, \quad (4.21) \end{aligned}$$

which contains all the leading-order terms of $E - J$.

⁹We need to choose the principal branch W_0 in order for x to have the correct limiting behavior, i.e. $x \rightarrow 0^+$ as $\mathcal{J} \rightarrow +\infty$. Choosing the W_{-1} branch gives $x \rightarrow -4$, to leading order. For more see also appendix A that deals with the Lambert function.

4.4 Next-to-leading terms

We can go on and calculate all the subleading in \mathcal{J} coefficients of the anomalous dimensions, i.e. obtain an analytic expression for the terms of the following series:

$$E - J \Big|_{(\text{NL})} = \sum_{n=2}^{\infty} \mathfrak{b}_n \mathcal{J}^{n-2} (e^{-2\mathcal{J}-2})^n. \quad (4.22)$$

This time we have to know not only the leading, but also the subleading terms of x in (4.5), namely

$$x_{(\text{NL})} = \sum_{n=2}^{\infty} \beta_n \mathcal{J}^{n-2} (e^{-2\mathcal{J}-2})^n. \quad (4.23)$$

This means that on the r.h.s. of (4.1) we have to keep all the terms that multiply $x^{0,1}$ and only the leading in \mathcal{J} terms that multiply x^2 . Then equation (4.1), precise up to next-to-leading/subleading order, becomes:

$$\begin{aligned} \ln x_{(\text{L+NL+...})} &= \frac{\mathcal{J} - b_0}{c_0} - \frac{\mathcal{J}c_1 + b_1c_0 - b_0c_1}{c_0^2} \cdot x_{(\text{L+NL+...})} + \frac{c_1^2 - c_0c_2}{c_0^3} \mathcal{J} \cdot x_{(\text{L+NL+...})}^2 \Rightarrow \\ &\Rightarrow x_{(\text{L+NL+...})} = x_0 \cdot \exp \left[-\frac{\mathcal{J} + 1}{2} \cdot x_{(\text{L+NL+...})} - \frac{7\mathcal{J}}{32} \cdot x_{(\text{L+NL+...})}^2 \right]. \end{aligned} \quad (4.24)$$

To solve this equation, we invert it for $x_{(\text{L+NL+...})}$ by using the Lagrange-Bürmann formula, as it was done in going from equation (4.2) to (4.5),

$$x_{(\text{L+NL+...})} = \sum_{n=1}^{\infty} \frac{x_0^n}{n!} \sum_{\substack{k, j_1=0 \\ n-1=k+j_1 \\ 0 \leq j_1 \leq k}}^{n-1} (-1)^k n^k \frac{(n-1)!}{(k-j_1)! j_1!} \cdot \left(\frac{\mathcal{J} + 1}{2} \right)^{k-j_1} \left(\frac{7\mathcal{J}}{32} \right)^{j_1}. \quad (4.25)$$

We now have to select and keep only the leading and next-to-leading \mathcal{J} -terms in this expression. Expanding the binomial in powers of \mathcal{J} ,

$$\begin{aligned} \left(\frac{\mathcal{J} + 1}{2} \right)^{k-j_1} \cdot \left(\frac{7\mathcal{J}}{32} \right)^{j_1} &= \left(\frac{1}{2} \right)^{k-j_1} \left(\frac{7\mathcal{J}}{32} \right)^{j_1} \cdot \sum_{m=0}^{k-j_1} \binom{k-j_1}{m} \mathcal{J}^m = \\ &= \frac{7^{j_1}}{2^{k+4j_1}} \cdot \sum_{m=0}^{k-j_1} \binom{k-j_1}{m} \mathcal{J}^{m+j_1} = \frac{7^{j_1}}{2^{k+4j_1}} \cdot \left(\mathcal{J}^{j_1} + \dots + (k-j_1) \mathcal{J}^{k-1} + \mathcal{J}^k \right), \end{aligned}$$

we see that the leading terms $\mathcal{J}^{n-1} x_0^n$, correspond to $m = k = n - 1$, $j_1 = 0$ and we obtain the leading power series (4.19)–(4.20) of the previous section. To get the next-to-leading/subleading terms $\mathcal{J}^{n-2} x_0^n$, we have to put either $k = n - 1$, $j_1 = 0$, $m = n - 2$ or $k = m = n - 2$, $j_1 = 1$ and sum the two terms. We find:

$$x_{(\text{NL})} = \sum_{n=1}^{\infty} \frac{x_0^n}{n!} \cdot \left\{ (-1)^{n-1} n^{n-1} \frac{(n-1)}{2^{n-1}} + (-1)^{n-2} n^{n-2} \frac{7(n-1)(n-2)}{2^{n+2}} \right\} \cdot \mathcal{J}^{n-2}, \quad (4.26)$$

so that the leading and next-to-leading terms of x are given by:

$$x_{(\text{L+NL})} = \sum_{n=1}^{\infty} (\alpha_n \mathcal{J}^{n-1} + \beta_n \mathcal{J}^{n-2}) \cdot (e^{-2\mathcal{J}-2})^n \quad (4.27)$$

with the definition (α_n 's are defined in (4.20)),

$$\beta_n \equiv (-1)^{n+1} 2^{3n-2} \cdot \frac{n^{n-2}}{n!} \cdot (n-1)(n+14). \quad (4.28)$$

Series (4.27) can be written in terms of Lambert's W-function, using formulas (A.7)–(A.12) of appendix A:

$$x_{(\text{L+NL})} = \sum_{n=1}^{\infty} (\alpha_n \mathcal{J}^{n-1} + \beta_n \mathcal{J}^{n-2}) \cdot (e^{-2\mathcal{J}-2})^n = \frac{2}{\mathcal{J}} W - \frac{1}{4\mathcal{J}^2} \frac{W^2(7W+8)}{1+W}, \quad (4.29)$$

where the argument of the W-function is $W(8\mathcal{J}e^{-2\mathcal{J}-2})$. To obtain the leading and next-to-leading coefficients of $E - J$, we insert (4.29) into (4.8) keeping only terms of leading and next-to-leading/subleading order:

$$\begin{aligned} E - J \Big|_{(\text{L+NL})} &= \frac{2\sqrt{\lambda}}{\pi} \left\{ 1 + A_1 (x_{(\text{L})} + x_{(\text{NL})}) - 2f_2 \mathcal{J} x_{(\text{L})}^2 - 4f_2 \mathcal{J} x_{(\text{L})} \cdot x_{(\text{NL})} + \right. \\ &\quad \left. + (g_2 + 2(2\ln 2 - 1)f_2) x_{(\text{L})}^2 - \left(\frac{c_1 f_2}{c_0^2} + 2f_3 \right) \mathcal{J} x_{(\text{L})}^3 \right\} = \quad (4.30) \\ &= \frac{2\sqrt{\lambda}}{\pi} \left\{ 1 - \frac{x_{(\text{L})}}{4} - \frac{x_{(\text{NL})}}{4} - \frac{\mathcal{J}}{16} x_{(\text{L})}^2 - \frac{\mathcal{J}}{8} x_{(\text{L})} \cdot x_{(\text{NL})} - \frac{9}{64} x_{(\text{L})}^2 - \frac{\mathcal{J}}{16} x_{(\text{L})}^3 \right\}. \end{aligned}$$

From this expression we read off the next-to-leading/subleading coefficients as follows (the leading ones are given in (4.21)):

$$\begin{aligned} E - J \Big|_{(\text{NL})} &= -\frac{2\sqrt{\lambda}}{\pi} \left\{ \frac{x_{(\text{NL})}}{4} + \frac{\mathcal{J}}{8} x_{(\text{L})} \cdot x_{(\text{NL})} + \frac{9}{64} x_{(\text{L})}^2 + \frac{\mathcal{J}}{16} x_{(\text{L})}^3 \right\} = -\frac{2\sqrt{\lambda}}{\pi} \frac{1}{16\mathcal{J}^2} (W^2 + W^3) \Rightarrow \\ \Rightarrow E - J \Big|_{(\text{NL})} &= -\frac{\sqrt{\lambda}}{32\pi} \sum_{n=1}^{\infty} \left\{ 16\beta_n + \sum_{k=1}^{n-1} \alpha_k \left[9\alpha_{n-k} + 8\beta_{n-k} \right] + \right. \\ &\quad \left. + 4 \sum_{k,m=1}^{n-2} \alpha_k \alpha_m \alpha_{n-k-m} \right\} \cdot \mathcal{J}^{n-2} (e^{-2\mathcal{J}-2})^n. \quad (4.31) \end{aligned}$$

4.5 NNL terms

Computing higher-order terms in the long-string expansion of $E - J$ is straightforward. Equation (4.5) becomes,

$$x_{(\text{L+NL+NNL+...})} = \sum_{n=1}^{\infty} \frac{x_0^n}{n!} \cdot \sum_{k,j=0}^{n-1} \frac{(-1)^k n^k (n-1)!}{(k-j_1-j_2)! j_1! j_2!} \left(\frac{\mathcal{J}+1}{2} \right)^{k-j_1-j_2} \left(\frac{14\mathcal{J}+9}{64} \right)^{j_1} \left(\frac{15\mathcal{J}}{128} \right)^{j_2}, \quad (4.32)$$

with $n - 1 = k + j_1 + 2j_2$ and $0 \leq j_1 + j_2 \leq k$. Again we have to select and keep only the leading, subleading (NL) and next-to-subleading (NNL) terms the way it was done in the previous subsection and then express the resulting power series with the aid of Lambert's function, by using the formulas of appendix A. We find:

$$x_{(L+NL+NNL)} = \sum_{n=1}^{\infty} (\alpha_n \mathcal{J}^{n-1} + \beta_n \mathcal{J}^{n-2} + \gamma_n \mathcal{J}^{n-3}) \cdot (e^{-2\mathcal{J}-2})^n, \quad (4.33)$$

having defined (α 's and β 's are defined in (4.20)–(4.28)) the γ_n 's as

$$\gamma_n \equiv (-1)^{n+1} 2^{3n-6} \cdot \frac{n^{n-3}}{n!} \cdot (n-1)(n-2)(n^2 + 41n + 228). \quad (4.34)$$

In terms of Lambert's W-function, the inverse spin function $x = x(\mathcal{J})$ (precise up to NNL order) is given by

$$x_{(L+NL+NNL)} = \frac{2}{\mathcal{J}} W - \frac{1}{4\mathcal{J}^2} \frac{W^2(7W+8)}{1+W} + \frac{1}{64\mathcal{J}^3} \frac{W^3(76W^3+269W^2+312W+120)}{(1+W)^3}, \quad (4.35)$$

where the arguments of the W-functions are $W(8\mathcal{J}e^{-2\mathcal{J}-2})$. This expression for x is in turn inserted into (4.8), keeping only terms up to next-to-subleading order. The next-to-subleading (NNL) coefficients of $E - J$ are found by writing

$$\begin{aligned} E - J \Big|_{(NNL)} = & -\frac{2\sqrt{\lambda}}{\pi} \left\{ \frac{x_{(NNL)}}{4} + \frac{9}{32} x_{(L)} \cdot x_{(NL)} + \frac{\mathcal{J}}{16} x_{(NL)}^2 + \frac{\mathcal{J}}{8} x_{(L)} \cdot x_{(NNL)} + \frac{23}{256} x_{(L)}^3 + \right. \\ & \left. + \frac{3\mathcal{J}}{16} x_{(L)}^2 \cdot x_{(NL)} + \frac{111\mathcal{J}}{2048} x_{(L)}^4 \right\} = -\frac{2\sqrt{\lambda}}{\pi} \frac{1}{256\mathcal{J}^3} \frac{W^3(11W^2+26W+16)}{1+W}, \quad (4.36) \end{aligned}$$

which implies,

$$\begin{aligned} E - J \Big|_{(NNL)} = & -\frac{\sqrt{\lambda}}{128\pi} \sum_{n=1}^{\infty} \left\{ 64\gamma_n + 8 \sum_{k=1}^{n-1} \left[9\alpha_k \beta_{n-k} + 2\beta_k \beta_{n-k} + 4\alpha_k \gamma_{n-k} \right] + \sum_{k,m=1}^{n-2} \alpha_k \alpha_m \cdot \right. \\ & \left. \cdot \left[23\alpha_{n-k-m} + 48\beta_{n-k-m} \right] + \frac{111}{8} \sum_{k,m,s=1}^{n-3} \alpha_k \alpha_m \alpha_s \alpha_{n-k-m-s} \right\} \cdot \mathcal{J}^{n-3} (e^{-2\mathcal{J}-2})^n. \quad (4.37) \end{aligned}$$

Our final result for the inverse spin function and the anomalous dimensions of the long $\mathcal{N} = 4$ SYM operators $\text{Tr} [\Phi \mathcal{Z}^m \Phi \mathcal{Z}^{J-m}] + \dots$ at strong 't Hooft coupling and up to next-to-subleading order in large- \mathcal{J} is:

$$x = \frac{2W}{\mathcal{J}} - \frac{1}{4\mathcal{J}^2} \frac{W^2(7W+8)}{1+W} + \frac{1}{64\mathcal{J}^3} \frac{W^3(76W^3+269W^2+312W+120)}{(1+W)^3} + \dots \quad (4.38)$$

$$\mathcal{E} - \mathcal{J} = 1 - \frac{1}{4\mathcal{J}} (2W + W^2) - \frac{1}{16\mathcal{J}^2} (W^2 + W^3) - \frac{1}{256\mathcal{J}^3} \frac{W^3(11W^2+26W+16)}{1+W} + \dots, \quad (4.39)$$

where $\mathcal{E} \equiv \pi E/2\sqrt{\lambda}$, $\mathcal{J} \equiv \pi J/2\sqrt{\lambda}$ and the arguments of the W-functions are $W(8\mathcal{J}e^{-2\mathcal{J}-2})$. Expanding series (4.38) and (4.39) around $\mathcal{J} \rightarrow \infty$, by using the formulas

of appendix A, we find that they completely agree with the ones computed if series (3.15) is inverted with a symbolic calculations program such as Mathematica and the result is plugged into equation (3.14) (cf. appendix D, equations (D.2)–(D.3)).

Likewise, we may keep on going to higher and higher orders in J . Theoretically, we are thus able to obtain all the terms of the long string expansion. We may conjecture that the Lambert W -functions will keep appearing to all orders of x and consequently to all orders of $\mathcal{E} - \mathcal{J}$ as well. To see this, note that equation (4.5) will in general contain a term of the form $n^n/n!$ that multiplies some Laurent polynomial of n , which in turn originates from the multinomial coefficient and the expansion of a_i 's in powers of \mathcal{J} . Using the formulas of appendix A, the resulting series may be expressed in terms of W -functions.

4.6 Fast circular strings in $\mathbb{R} \times \mathbb{S}^2$: $\omega \rightarrow 1^-$, $J \gg \lambda$

Region $\omega < 1$ of GKP strings on the sphere (circular strings) is very similar to the regime $\omega > 1$ that was studied in sections 3 and 4.1–4.5. In this subsection we will briefly derive the corresponding expressions $\tilde{x} = \tilde{x}(\mathcal{J})$ and $\mathcal{E} = \mathcal{E}(\mathcal{J})$ for fast (large- J) circular strings on \mathbb{S}^2 ($\omega \rightarrow 1^-$). Our treatment is very similar to the case $\omega \rightarrow 1^+$:

$$\mathcal{E} \equiv \frac{\pi E}{2\sqrt{\lambda}} = \sum_{n=0}^{\infty} \tilde{x}^n (d_n \ln \tilde{x} + h_n) \tag{4.40}$$

$$\mathcal{J} \equiv \frac{\pi J}{2\sqrt{\lambda}} = \frac{1}{\sqrt{1-\tilde{x}}} \cdot \sum_{n=0}^{\infty} \tilde{x}^n (c_n \ln \tilde{x} + b_n) = \sum_{n=0}^{\infty} \tilde{x}^n \cdot \sum_{k=0}^n \frac{(2k-1)!!}{(2k)!!} (c_{n-k} \ln \tilde{x} + b_{n-k}) \tag{4.41}$$

$$\mathcal{E} - \mathcal{J} = \sum_{n=0}^{\infty} \tilde{x}^n (f_n \ln \tilde{x} + g_n) = \sum_{n=0}^{\infty} \tilde{x}^n \left[A_n + f_n \ln \frac{\tilde{x}}{x_0} \right], \tag{4.42}$$

where the complementary parameter is now $\tilde{x} \equiv 1 - \omega^2 \rightarrow 0^-$, b_n, c_n, d_n, h_n are defined in (3.21),

$$f_n \equiv d_n - \sum_{k=0}^n \frac{(2k-1)!!}{(2k)!!} \cdot c_{n-k}, \quad g_n \equiv h_n - \sum_{k=0}^n \frac{(2k-1)!!}{(2k)!!} \cdot b_{n-k} \tag{4.43}$$

and

$$A_n \equiv g_n + f_n \ln x_0 = g_n + 2f_n (2 \ln 2 - \mathcal{J} - 1), \tag{4.44}$$

while x_0 is defined in (4.3) and $n = 0, 1, 2, \dots$. We find:

$$\tilde{x} = -\frac{2W}{\mathcal{J}} - \frac{1}{4\mathcal{J}^2} \frac{W^2(9W+8)}{1+W} - \frac{1}{64\mathcal{J}^3} \frac{W^3(140W^3+397W^2+376W+120)}{(1+W)^3} + \dots \tag{4.45}$$

$$\mathcal{E} - \mathcal{J} = 1 - \frac{1}{4\mathcal{J}} (2W + W^2) - \frac{1}{16\mathcal{J}^2} (W^2 + W^3) - \frac{1}{256\mathcal{J}^3} \frac{W^3(11W^2 + 26W + 16)}{1+W} + \dots, \tag{4.46}$$

where the argument of the W -function is $W(-8\mathcal{J}e^{-2\mathcal{J}-2})$. Notice that although the inverse spin functions $x(\mathcal{J})$ and $\tilde{x}(\mathcal{J})$ are different for fast folded and circular strings on

S^2 (cf. (4.38), (4.45)), the expressions for the anomalous dimensions in terms of Lambert W-functions coincide (cf. (4.39), (4.46)). Nevertheless, and due to the fact that the arguments of the W-functions have opposite signs in the two cases, the formulas for the anomalous dimensions $\gamma = \gamma(\mathcal{J})$ will have a periodic sign difference (cf. (D.3)–(D.5)). It would be interesting to study the operators that are dual to these circular string states.

5 Gubser-Klebanov-Polyakov AdS₃ string

Let us now consider the Gubser-Klebanov-Polyakov (GKP) folded closed string that rotates at the equator of S^3 of AdS₅ [8]:

$$\left\{ t = \kappa\tau, \rho = \rho(\sigma), \theta = \kappa\omega\tau, \phi_1 = \phi_2 = 0 \right\} \times \left\{ \bar{\theta}_1 = \bar{\theta}_2 = \bar{\phi}_1 = \bar{\phi}_2 = \bar{\phi}_3 = 0 \right\}. \quad (5.1)$$

As in the case of the GKP string rotating in $\mathbb{R} \times S^2$, we will show that the GKP solution in AdS₃ is unique in the sense that it minimizes the energy of an AdS₃ string with a single spin S . Let us again consider the following generic ansatz:

$$\left\{ t = \kappa\tau, \rho = \rho(\tau, \sigma), \theta = \theta(\tau, \sigma), \phi_1 = \phi_2 = 0 \right\} \times \left\{ \bar{\theta}_1 = \bar{\theta}_2 = \bar{\phi}_1 = \bar{\phi}_2 = \bar{\phi}_3 = 0 \right\}. \quad (5.2)$$

The Polyakov action in the conformal gauge ($\gamma_{ab} = \eta_{ab}$) is:

$$S_P = \frac{\ell^2}{4\pi\alpha'} \int \left(-\dot{t}^2 \cosh^2 \rho + \dot{\rho}^2 - \rho'^2 + (\dot{\theta}^2 - \theta'^2) \sinh^2 \rho \right) d\tau d\sigma. \quad (5.3)$$

From this action one can derive the Hamiltonian density of the string that satisfies the first of the Virasoro constraints, namely

$$\mathcal{H} = \frac{\ell^2}{4\pi\alpha'} \left\{ -\kappa^2 \cosh^2 \rho + \frac{p_\rho^2}{4} + \rho'^2 + \frac{p_\theta^2}{4 \sinh^2 \rho} + \theta'^2 \sinh^2 \rho \right\} = 0, \quad \begin{aligned} p_\rho &= 2 \dot{\rho} \\ p_\theta &= 2 \dot{\theta} \sinh^2 \rho. \end{aligned} \quad (5.4)$$

We want to find the minimum of the energy E subject to the constraint of fixed angular momentum S . Introducing the Lagrange multiplier ω , the functional to be minimized is given by:

$$Y \equiv \frac{\ell^2}{4\pi\alpha'} \int d\sigma \cdot 2 \cosh \rho \sqrt{\frac{p_\rho^2}{4} + \rho'^2 + \frac{p_\theta^2}{4 \sinh^2 \rho} + \theta'^2 \sinh^2 \rho} - \omega \left[\frac{\ell^2}{4\pi\alpha'} \int d\sigma p_\theta - S \right], \quad (5.5)$$

where the last formula is obtained by substituting κ from (5.4) into the integral of energy E . Therefore,

$$\frac{\delta Y}{\delta p_\rho} = 0 \Rightarrow p_\rho = 0 \Rightarrow \dot{\rho} = 0 \Rightarrow \rho = \rho(\sigma) \quad (5.6)$$

$$\frac{\delta Y}{\delta p_\theta} = 0 \Rightarrow \dot{\theta} = \kappa\omega \Rightarrow \theta = \kappa\omega\tau + \tilde{\theta}(\sigma). \quad (5.7)$$

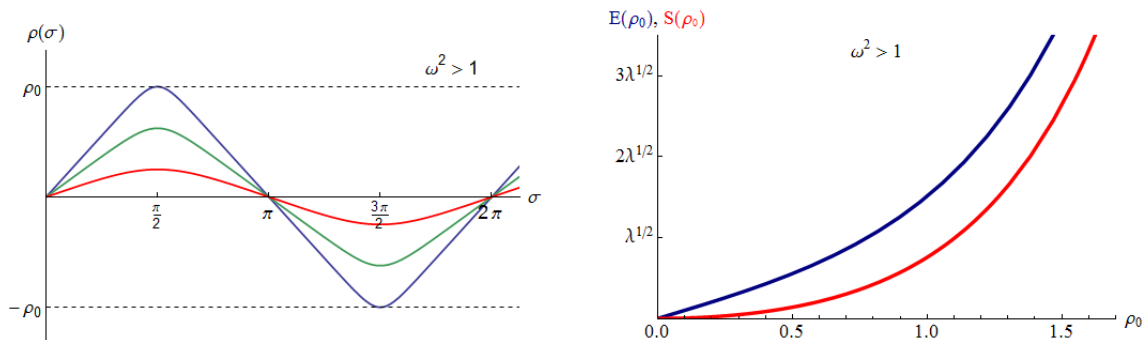


Figure 4. $\rho = \rho(\sigma)$ and energy/spin of the folded closed string AdS₃ string (5.1) for $\omega > 1$.

Combining the second Virasoro constraint $\dot{\rho}\rho' + \dot{\theta}\theta' \sinh^2 \rho = 0$ with equation (5.6), we conclude that $\theta' = 0$, i.e. $\tilde{\theta}(\sigma) = 0$. Again, the vanishing of the functional derivative $\delta Y/\delta \rho(\sigma) = 0$ will give the equation of motion for ρ . Thus the GKP solution (5.1) follows from the minimization of the string energy in AdS₃. Its Polyakov action reads:

$$\begin{aligned} \mathcal{S}_P &= \frac{\ell^2}{4\pi\alpha'} \int \left(-\dot{t}^2 \cosh^2 \rho - \rho'^2 + \dot{\theta}^2 \sinh^2 \rho \right) d\tau d\sigma = \\ &= \frac{\ell^2}{4\pi\alpha'} \int \left(-\kappa^2 \cosh^2 \rho - \rho'^2 + \kappa^2 \omega^2 \sinh^2 \rho \right) d\tau d\sigma, \end{aligned} \tag{5.8}$$

where $\ell^4/\alpha'^2 = \lambda$ is the 't Hooft coupling. The string essentially contains four segments extending between $\rho = 0$ and $\rho = \rho_0$. κ is a factor needed to fix $\sigma(\rho_0) = \pi/2$. The conserved charges that correspond to the two cyclic coordinates t and θ of the action (5.8) both diverge when $\omega < 1$. For $\omega > 1$ we obtain:

$\omega^2 > 1$. Folded closed string.

$$\rho(\sigma) = \operatorname{arctanh} \left[\frac{1}{\omega} \operatorname{sn} \left(\kappa \omega \sigma \left| \frac{1}{\omega^2} \right. \right) \right], \quad \kappa = \frac{2}{\pi \omega} \cdot \mathbb{K} \left(\frac{1}{\omega^2} \right), \quad \omega = \coth \rho_0 \tag{5.9}$$

$$E(\omega) = \frac{2\sqrt{\lambda}}{\pi} \frac{\omega}{\omega^2 - 1} \cdot \mathbb{E} \left(\frac{1}{\omega^2} \right) \Rightarrow \mathcal{E} \equiv \frac{\pi E}{2\sqrt{\lambda}} = \frac{\sqrt{1-x}}{x} \cdot \mathbb{E}(1-x) \tag{5.10}$$

$$S(\omega) = \frac{2\sqrt{\lambda}}{\pi} \left[\frac{\omega^2}{\omega^2 - 1} \mathbb{E} \left(\frac{1}{\omega^2} \right) - \mathbb{K} \left(\frac{1}{\omega^2} \right) \right] \Rightarrow \mathcal{S} \equiv \frac{\pi S}{2\sqrt{\lambda}} = \frac{1}{x} \mathbb{E}(1-x) - \mathbb{K}(1-x) \tag{5.11}$$

$$\gamma \equiv \mathcal{E} - \mathcal{S} = \frac{\sqrt{1-x} - 1}{x} \cdot \mathbb{E}(1-x) + \mathbb{K}(1-x), \tag{5.12}$$

where $x \equiv 1 - 1/\omega^2$ is the complementary parameter of $1/\omega^2$. The plot of $\rho(\sigma)$ for various values of the angular velocity $\omega > 1$ as well as the string's energy and spin as functions of ρ_0 , ω and x may be found in figures 4-5. In figure 6 we have plotted the energy of the string as a function of its spin $E = E(S)$ parametrically, along with the first 4 terms of its “short” series (red dashed line, equation (C.16)) and its leading “long” approximation (blue dashed line, equations (6.29)–(6.30)).

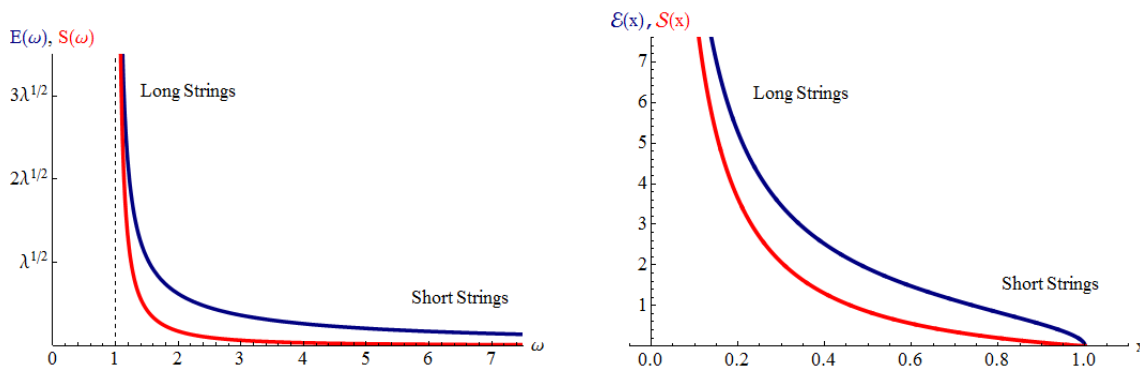


Figure 5. Energy and spin of the folded closed AdS₃ string as functions of $\omega > 1$ and $x > 0$.

5.1 Long strings: $\omega \rightarrow 1^+$, $S \gg \sqrt{\lambda}$

In the long strings-regime, $\omega \rightarrow 1^+$ ($S \gg \lambda$), the formulas for the energy and spin have the following expansions:

$$E = \frac{2\sqrt{\lambda}}{\pi\omega} \cdot \left\{ \frac{\omega^2}{\omega^2 - 1} + \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)\Gamma(n+3/2)}{n!(n+1)!} \left[2\psi(n+1) - 2\psi(n+1/2) - \ln(1-1/\omega^2) - \frac{1}{(n+1)(2n+1)} \right] \cdot (1-1/\omega^2)^n \right\} \quad (5.13)$$

$$S = \frac{2\sqrt{\lambda}}{\pi} \cdot \left\{ \frac{\omega^2}{\omega^2 - 1} - \frac{1}{4\pi} \sum_{n=0}^{\infty} \frac{(\Gamma(n+1/2))^2}{n!(n+1)!} \left[2\psi(n+1) - 2\psi(n+1/2) - \ln(1-1/\omega^2) + \frac{1}{n+1} \right] \cdot (1-1/\omega^2)^n \right\}. \quad (5.14)$$

The two series may also be written in terms of the complementary parameter $x \equiv 1 - 1/\omega^2 \rightarrow 0^+$:

$$\begin{aligned} \mathcal{E} \equiv \frac{\pi E}{2\sqrt{\lambda}} &= \sqrt{1-x} \cdot \left\{ \frac{1}{x} + \sum_{n=0}^{\infty} x^n (d_n \ln x + h_n) \right\} = \\ &= \frac{1}{x} - \sum_{n=0}^{\infty} x^n \cdot \left\{ \frac{(2n-1)!!}{(2n+2)!!} + \sum_{k=0}^n \frac{(2k-3)!!}{(2k)!!} (d_{n-k} \ln x + h_{n-k}) \right\} \end{aligned} \quad (5.15)$$

$$\mathcal{S} \equiv \frac{\pi S}{2\sqrt{\lambda}} = \frac{1}{x} + \sum_{n=0}^{\infty} x^n (c_n \ln x + b_n). \quad (5.16)$$

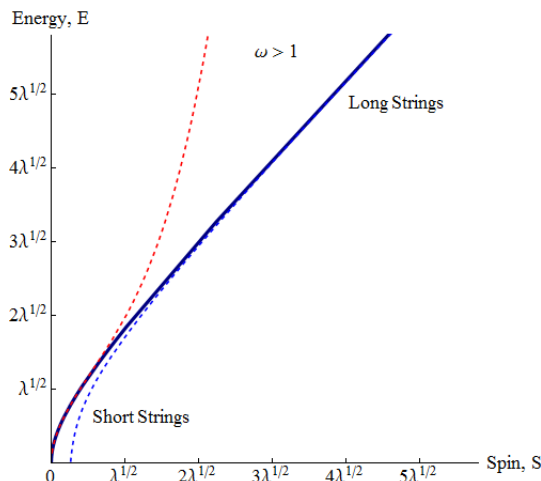


Figure 6. Energy vs spin of the folded closed AdS₃ string for $\omega > 1$.

The coefficients that appear in series (5.15) and (5.16) are given by:¹⁰

$$d_n = -\frac{1}{4} \left(\frac{(2n-1)!!}{(2n)!!} \right)^2 \cdot \frac{2n+1}{n+1}, \quad h_n = -d_n \cdot \left[4 \ln 2 + 4(H_n - H_{2n}) + \frac{1}{n+1} - \frac{2}{2n+1} \right]$$

$$c_n = -\frac{d_n}{2n+1}, \quad b_n = -c_n \cdot \left[4 \ln 2 + 4(H_n - H_{2n}) + \frac{1}{n+1} \right], \quad (5.17)$$

where $n = 0, 1, 2, \dots$

5.2 Short-long strings duality

Solving (5.10) and (5.11) for $\mathbb{E}(k)$ and $\mathbb{K}(k)$ and substituting into Legendre’s relation (3.22), we get the following formula between classical folded short and long strings spinning in AdS₃ \subset AdS₅ [38]:

$$\frac{1}{\omega} E S' + \frac{1}{\omega'} E' S - S S' = \frac{2\lambda}{\pi}, \quad (5.18)$$

where the respective arguments of the primed and unprimed charges are $k = 1/\omega^2$ and $k' = x = 1/\omega'^2$, and satisfy $k + k' = 1$. Again we see that large values of $\omega' \rightarrow \infty$ (“short” strings) correspond to values of $\omega \rightarrow 1^+$ near unity (“long” strings) and (5.18) provides a map between the corresponding energies and spins.¹¹

¹⁰As a reminder, $0!! = 1$, $(-1)!! = 1$, $(-3)!! = -1$.

¹¹This subsection has been included here for reasons of completeness. For more, see [38]. One may also formulate short-long dualities that mix the charges of the AdS₃ and $\mathbb{R} \times S^2$ strings:

$$\frac{\omega'}{\omega} \mathcal{E}'_1 \mathcal{E}_2 + \frac{\omega}{\omega'} \mathcal{E}_1 \mathcal{E}'_2 - \omega \omega' \mathcal{E}_2 \mathcal{E}'_2 = \frac{\pi}{2} \quad \& \quad \mathcal{S}_1 \mathcal{J}'_2 + \mathcal{S}'_1 \mathcal{J}_2 + \mathcal{S}_1 \mathcal{S}'_1 = \frac{\pi}{2} \quad (5.19)$$

$$\frac{1}{\omega'} \mathcal{S}_1 \mathcal{E}'_2 + \frac{1}{\omega} \mathcal{S}'_1 \mathcal{E}_2 = \frac{\pi}{2} \quad \& \quad \frac{1}{\omega \omega'} \mathcal{E}_1 \mathcal{E}'_1 - \mathcal{J}_2 \mathcal{J}'_2 = \frac{\pi}{2}, \quad (5.20)$$

where the index 1 refers to the string energy and spin in AdS₃ and index 2 denotes the corresponding charges in $\mathbb{R} \times S^2$.

6 Inverse spin functions and anomalous dimensions on AdS₃

6.1 Inverse spin function

Following the method of subsection 4.1, we will now invert series (5.16) for $x = x(\mathcal{S})$. First solve (5.16) for $\ln x$:

$$\begin{aligned} \mathcal{S} &= \frac{1}{x} + \sum_{n=0}^{\infty} x^n (c_n \ln x + b_n) \Rightarrow \\ \Rightarrow \ln x &= \left[-\frac{1}{c_0 x} + \frac{\mathcal{S} - b_0}{c_0} - \sum_{n=1}^{\infty} \frac{b_n}{c_0} x^n \right] \cdot \sum_{n=0}^{\infty} (-1)^n \left(\sum_{k=1}^{\infty} \frac{c_k}{c_0} x^k \right)^n. \end{aligned} \quad (6.1)$$

This essentially reduces to an equation of the following form (cf. equation (4.2)):

$$x = x_0 \cdot \exp \left[\frac{a_0}{x} + \sum_{n=1}^{\infty} a_n x^n \right] = x_0 \cdot \exp \left(\frac{a_0}{x} + a_1 x + a_2 x^2 + a_3 x^3 + \dots \right), \quad (6.2)$$

where the a_n 's are linear functions of \mathcal{S} determined from (6.1) ($a_0 = -c_0^{-1} = -4$), and x_0 is defined as:

$$x_0 \equiv \exp \left[\frac{\mathcal{S} - b_0}{c_0} + \frac{c_1}{c_0^2} \right] = 16 e^{4\mathcal{S}+3/2}. \quad (6.3)$$

Equation (6.2), that gives the inverse spin function of a string that rotates in AdS₃ differs significantly from the corresponding inverse spin function in S⁵, (4.2). For one, it contains the $1/x$ term but also x_0 is now an increasing function of \mathcal{S} . The point is that we cannot solve equation (6.2) by the algorithm of subsection 4.1. Instead, a slightly different procedure has to be applied. Suppose x^* solves the following equation:

$$x^* = x_0 \cdot e^{a_0/x^*} \Rightarrow x^* = \frac{a_0}{W(a_0/x_0)} = x_0 \cdot e^{W(a_0/x_0)}, \quad (6.4)$$

where $W(z)$ is again the Lambert W-function (see (A.1) in appendix A). Essentially x^* is the leading-order solution of equation (6.2).¹² Now set:

$$x = x^* \cdot e^v, \quad (6.5)$$

with $v \rightarrow 0$, and substitute it into equation (6.2). We obtain, using also (6.4):

$$v - \frac{a_0}{x^*} \sum_{k=1}^{\infty} (-1)^k \frac{v^k}{k!} - \sum_{n=1}^{\infty} a_n (x^*)^n e^{nv} = 0. \quad (6.6)$$

This series may be inverted for v using the standard series inversion. First we expand the exponential in (6.6):

$$\left(1 + \frac{a_0}{x^*} - \sum_{k=1}^{\infty} k a_k (x^*)^k \right) v - \sum_{n=2}^{\infty} \left[(-1)^n \frac{a_0}{x^*} + \sum_{k=1}^{\infty} k^n a_k (x^*)^k \right] \frac{v^n}{n!} = \sum_{n=1}^{\infty} a_n (x^*)^n. \quad (6.7)$$

¹²Observe though that since we solve (6.1) in the region where $\mathcal{S} \rightarrow +\infty$, $x \rightarrow 0^+$ and $a_0 < 0$, $x_0 \rightarrow +\infty$, we must choose the W_{-1} branch of Lambert's function. We'll return to this issue later.

The inverse series is a power series in x^* . It can be obtained by using a symbolic calculations package such as Mathematica. The result is:

$$v = \frac{a_1}{a_0} (x^*)^2 + \left[\frac{a_2}{a_0} - \frac{a_1}{a_0^2} \right] (x^*)^3 + \left[\frac{a_1}{a_0^3} + \frac{3a_1^2 - 2a_2}{2a_0^2} + \frac{a_3}{a_0} \right] (x^*)^4 + \dots \quad (6.8)$$

Then x in (6.5) is given by

$$x = x^* + \frac{a_1}{a_0} (x^*)^3 + \left[\frac{a_2}{a_0} - \frac{a_1}{a_0^2} \right] (x^*)^4 + \left[\frac{a_3}{a_0} + \frac{2a_1^2 - a_2}{a_0^2} + \frac{a_1}{a_0^3} \right] (x^*)^5 + \dots \quad (6.9)$$

and $1/x$ by

$$\frac{1}{x} = \frac{1}{x^*} - \frac{a_1}{a_0} x^* - \left[\frac{a_2}{a_0} - \frac{a_1}{a_0^2} \right] (x^*)^2 - \left[\frac{a_3}{a_0} + \frac{a_1^2 - a_2}{a_0^2} + \frac{a_1}{a_0^3} \right] (x^*)^3 + \dots \quad (6.10)$$

We may now expand (6.1), obtain the coefficients a_n and, by substituting them into (6.9)–(6.10), find the corresponding expressions for x and $1/x$ (only the first few terms are shown):

$$x = x^* + \left(\frac{S}{8} + \frac{3}{64} \right) (x^*)^3 + \left(\frac{S}{16} + \frac{23}{1024} \right) (x^*)^4 + \left(\frac{S^2}{32} + \frac{55S}{1024} + \frac{349}{24576} \right) (x^*)^5 + \dots \quad (6.11)$$

$$\frac{1}{x} = \frac{1}{x^*} - \left(\frac{S}{8} + \frac{3}{64} \right) x^* - \left(\frac{S}{16} + \frac{23}{1024} \right) (x^*)^2 - \left(\frac{S^2}{64} + \frac{43S}{1024} + \frac{295}{24576} \right) (x^*)^3 + \dots \quad (6.12)$$

6.2 Anomalous dimensions

Just as for the case of strings on S^2 , the anomalous scaling dimensions $\gamma = \mathcal{E} - \mathcal{S}$ of the AdS_3 string may be expressed as a function of \mathcal{S} :

$$\mathcal{E} - \mathcal{S} = \sum_{n=0}^{\infty} x^n (f_n \ln x + g_n) = \sum_{n=0}^{\infty} x^n \left[A_n + f_n \ln \frac{x}{x_0} \right], \quad \mathcal{E} \equiv \frac{\pi E}{2\sqrt{\lambda}}, \quad \mathcal{S} \equiv \frac{\pi S}{2\sqrt{\lambda}}, \quad (6.13)$$

where,

$$x_0 \equiv \exp \left[\frac{\mathcal{S} - b_0}{c_0} + \frac{c_1}{c_0^2} \right] = 16 e^{4\mathcal{S}+3/2} \quad (6.14)$$

and we also define

$$f_n \equiv -c_n - \sum_{k=0}^n \frac{(2k-3)!!}{(2k)!!} \cdot d_{n-k}$$

$$g_n \equiv -b_n - \frac{(2n-1)!!}{(2n+2)!!} - \sum_{k=0}^n \frac{(2k-3)!!}{(2k)!!} \cdot h_{n-k}, \quad n = 0, 1, 2, \dots \quad (6.15)$$

The coefficients A_n are defined as:

$$A_n \equiv g_n + f_n \ln x_0 = g_n + f_n \left(4 \ln 2 + 4\mathcal{S} + \frac{3}{2} \right). \quad (6.16)$$

For later purposes, it will be useful to obtain the expression of the anomalous dimensions $\gamma = \mathcal{E} - \mathcal{S}$ in terms of x^* . We first insert (6.2) into (6.13):

$$\mathcal{E} - \mathcal{S} = \frac{a_0 f_0}{x} + A_0 + \sum_{n=1}^{\infty} x^n \left[A_n + a_0 f_{n+1} + \sum_{k=0}^{n-1} f_{n-k-1} a_{k+1} \right]. \quad (6.17)$$

Subsequently we plug series (6.11) and (6.12) into (6.17), which leads us to the following result (for simplicity, we omit higher-order terms):

$$\begin{aligned} \mathcal{E} - \mathcal{S} = & \frac{2}{x^*} - \left(2\mathcal{S} + \frac{5}{4} \right) - \frac{9x^*}{32} - \left(\frac{\mathcal{S}}{32} + \frac{35}{256} \right) (x^*)^2 - \left(\frac{5\mathcal{S}}{128} + \frac{2.213}{24.576} \right) (x^*)^3 - \\ & - \left(\frac{\mathcal{S}^2}{256} + \frac{361\mathcal{S}}{8.192} + \frac{6.665}{98.304} \right) (x^*)^4 - \left(\frac{19\mathcal{S}^2}{2.048} + \frac{1.579\mathcal{S}}{32.768} + \frac{433.501}{7.864.320} \right) (x^*)^5 + \dots \end{aligned} \quad (6.18)$$

At this point, a note analogous to that made for strings rotating in $\mathbb{R} \times \mathbb{S}^2$, concerning the structure of the large-spin expansion of the inverse spin function and the anomalous dimensions of strings spinning in AdS_3 , should be made. Consider the following kinds of terms ($n = 0, 1, 2, \dots$):

$$\begin{aligned} \text{Leading Terms (L): } & \frac{\ln^n \mathcal{S}}{\mathcal{S}^n} \\ \text{Next-to-Leading/Subleading Terms (NL): } & \frac{\ln^n \mathcal{S}}{\mathcal{S}^{n+1}} \\ \text{NNL Terms: } & \frac{\ln^n \mathcal{S}}{\mathcal{S}^{n+2}} \\ & \vdots \end{aligned} \quad (6.19)$$

In what follows we shall find that, in the expansion of the inverse spin function $x = x(\mathcal{S})$, all the leading terms $\ln^n \mathcal{S} / \mathcal{S}^n$ are absent whereas, in the anomalous dimensions $\gamma = \mathcal{E} - \mathcal{S}$, all of the above terms are present. Therefore the large-spin expansion of γ assumes the following form:

$$\begin{aligned} \mathcal{E} - \mathcal{S} = & \rho_c \ln \mathcal{S} + \sum_{n=0}^{\infty} \sum_{k=0}^n \rho_{(nk)} \frac{\ln^k \mathcal{S}}{\mathcal{S}^n} = \rho_c \ln \mathcal{S} + \rho_0 + \sum_{n=1}^{\infty} \rho_{(nn)} \frac{\ln^n \mathcal{S}}{\mathcal{S}^n} + \sum_{n=2}^{\infty} \rho_{(nn-1)} \frac{\ln^{n-1} \mathcal{S}}{\mathcal{S}^n} + \\ & + \sum_{n=3}^{\infty} \rho_{(nn-2)} \frac{\ln^{n-2} \mathcal{S}}{\mathcal{S}^n} + \dots + \frac{\rho_1}{\mathcal{S}} + \frac{\rho_2}{\mathcal{S}^2} + \frac{\rho_3}{\mathcal{S}^3} + \dots \end{aligned} \quad (6.20)$$

Note also the presence of the exceptional (super-) leading term $f \ln \mathcal{S}$. Equivalently we may write:

$$\begin{aligned}
 E - S &= f \ln \left(S/\sqrt{\lambda} \right) + \sum_{n=0}^{\infty} \sum_{k=0}^n f_{(nk)} \frac{\ln^k \left(S/\sqrt{\lambda} \right)}{S^n} = f \ln \left(S/\sqrt{\lambda} \right) + f_0 + \\
 &+ \sum_{n=1}^{\infty} f_{(nn)} \frac{\ln^n \left(S/\sqrt{\lambda} \right)}{S^n} + \sum_{n=2}^{\infty} f_{(nn-1)} \frac{\ln^{n-1} \left(S/\sqrt{\lambda} \right)}{S^n} + \sum_{n=3}^{\infty} f_{(nn-2)} \frac{\ln^{n-2} \left(S/\sqrt{\lambda} \right)}{S^n} + \\
 &+ \dots + \frac{f_1}{S} + \frac{f_2}{S^2} + \frac{f_3}{S^3} + \dots, \tag{6.21}
 \end{aligned}$$

where

$$\begin{aligned}
 \rho_c &= \frac{\pi f}{2\sqrt{\lambda}}, \quad \rho_0 = \frac{\pi}{2\sqrt{\lambda}} \left(f_0 + f \ln \frac{2}{\pi} \right), \quad \rho_1 = \left(\frac{\pi}{2\sqrt{\lambda}} \right)^2 \left(f_1 + f_{11} \ln \frac{2}{\pi} \right) \\
 \rho_2 &= \left(\frac{\pi}{2\sqrt{\lambda}} \right)^3 \left(f_2 + f_{21} \ln \frac{2}{\pi} + f_{22} \ln^2 \frac{2}{\pi} \right) \\
 \rho_3 &= \left(\frac{\pi}{2\sqrt{\lambda}} \right)^4 \left(f_3 + f_{31} \ln \frac{2}{\pi} + f_{32} \ln^2 \frac{2}{\pi} + f_{33} \ln^3 \frac{2}{\pi} \right) \\
 \rho_{(nn)} &= \left(\frac{\pi}{2\sqrt{\lambda}} \right)^{n+1} \cdot f_{(nn)}, \quad \rho_{(nn-1)} = \left(\frac{\pi}{2\sqrt{\lambda}} \right)^{n+1} \left(f_{(nn-1)} + n f_{(nn)} \ln \frac{2}{\pi} \right) \\
 \rho_{(nn-2)} &= \left(\frac{\pi}{2\sqrt{\lambda}} \right)^{n+1} \left(f_{(nn-2)} + (n-1) f_{(nn-1)} \ln \frac{2}{\pi} + \frac{n(n-1)}{2} \ln^2 \frac{2}{\pi} \right). \tag{6.22}
 \end{aligned}$$

6.3 Leading terms

We may now employ formula (6.4) in order to calculate the leading in \mathcal{S} terms of series (6.13):

$$x^* = x_0 \cdot e^{W_{-1}(a_0/x_0)} = \frac{a_0}{W_{-1}(a_0/x_0)} = \frac{-4}{W_{-1}[-e^{-4\mathcal{S}-3/2}/4]}. \tag{6.23}$$

Note that, as $\mathcal{S} \rightarrow +\infty$ we should have $x^* \rightarrow 0^+$, behavior that is only possible if $W \rightarrow -\infty$, i.e. in the W_{-1} branch of Lambert's function (on the contrary $W_0 \rightarrow 0^-$ for $\mathcal{S} \rightarrow +\infty$, which would cause x^* to blow up as $x^* \rightarrow +\infty$, cf. figure 8). Using expansion (A.3) of the W-function, we may obtain the inverse spin function $x = x(\mathcal{S})$ up to leading order. Let us first calculate $1/x^*$:

$$\begin{aligned}
 \frac{1}{x^*} &= \mathcal{S} + \frac{\ln \mathcal{S}}{4} + \ln 2 + \frac{3}{8} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n} \left(\frac{4 \ln 2 + 3}{8\mathcal{S}} \right)^n - \sum_{n,q=0}^{\infty} \sum_{m=1}^{\infty} \sum_{p=0}^m \frac{(-1)^m}{4^{n+m+1} m!} \begin{bmatrix} n+m \\ n+1 \end{bmatrix} \\
 &\cdot \binom{-n-m}{q} \binom{m}{p} \frac{\ln^p \mathcal{S}}{\mathcal{S}^{n+m}} \left(2 \ln 2 - \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left(\frac{4 \ln 2 + 3}{8\mathcal{S}} \right)^k \right)^{m-p} \left(\frac{4 \ln 2 + 3}{8\mathcal{S}} \right)^q \tag{6.24}
 \end{aligned}$$

where, for large \mathcal{S} , we have written

$$\ln \left[4\mathcal{S} + 2 \ln 2 + \frac{3}{2} \right] = \ln \mathcal{S} + 2 \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{4 \ln 2 + 3}{8 \mathcal{S}} \right)^n \quad (6.25)$$

and the unsigned Stirling numbers of the first kind $\left[\begin{smallmatrix} n+m \\ n+1 \end{smallmatrix} \right]$ are defined in appendix A.

To obtain x^* , we expand the inverse of series (6.24):

$$\begin{aligned} x^* = \frac{1}{\mathcal{S}} \cdot & \left\{ 1 + \frac{\ln \mathcal{S}}{4\mathcal{S}} + \left(\ln 2 + \frac{3}{8} \right) \frac{1}{\mathcal{S}} + 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{(4 \ln 2 + 3)^n}{8 \mathcal{S}^{n+1}} - \sum_{n,q=0}^{\infty} \sum_{m=1}^{\infty} \sum_{p=0}^m \frac{(-1)^m}{4^{n+m+1} m!} \right. \\ & \cdot \left[\begin{smallmatrix} n+m \\ n+1 \end{smallmatrix} \right] \binom{-n-m}{q} \binom{m}{p} \frac{\ln^p \mathcal{S}}{\mathcal{S}^{n+m+1}} \left(2 \ln 2 - \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left(\frac{4 \ln 2 + 3}{8 \mathcal{S}} \right)^k \right)^{m-p} \\ & \left. \cdot \left(\frac{4 \ln 2 + 3}{8 \mathcal{S}} \right)^q \right\}^{-1}. \end{aligned} \quad (6.26)$$

A few comments are in order, before proceeding to the calculation of the corresponding series for the anomalous dimensions. First observe that series (6.24) for $1/x^*$ contains all kinds of small terms (i.e. terms $\rightarrow 0$ as $\mathcal{S} \rightarrow \infty$): leading terms $\ln^n \mathcal{S}/\mathcal{S}^n$, subleading terms $\ln^n \mathcal{S}/\mathcal{S}^{n+1}$, next-to-subleading terms $\ln^n \mathcal{S}/\mathcal{S}^{n+2}$, etc. up to $1/\mathcal{S}^n$ terms ($n = 1, 2, 3 \dots$). Series (6.26) for x^* on the contrary, does not contain leading terms. It is to be expected that $(x^*)^2$ will not contain leading and subleading terms, in $(x^*)^3$ leading, subleading and next-to-subleading terms will be absent, etc. Therefore, to calculate the $E - S$ series up to leading order in \mathcal{S} , we need no more than the first two terms of (6.18),

$$\mathcal{E} - \mathcal{S} \Big|_{(L+\dots)} = \frac{2}{x^*} - \left(2\mathcal{S} + \frac{5}{4} \right), \quad (6.27)$$

since the other terms of series (6.18) only contribute to NL orders. We obtain,

$$\begin{aligned} \mathcal{E} - \mathcal{S} \Big|_{(L+\dots)} = & \frac{1}{2} \ln \mathcal{S} + \left(2 \ln 2 - \frac{1}{2} \right) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n} \left(\frac{4 \ln 2 + 3}{8 \mathcal{S}} \right)^n - \frac{1}{2} \sum_{n,q=0}^{\infty} \sum_{m=1}^{\infty} \sum_{p=0}^m \frac{(-1)^m}{4^{n+m}} \\ & \cdot \frac{1}{m!} \left[\begin{smallmatrix} n+m \\ n+1 \end{smallmatrix} \right] \binom{-n-m}{q} \binom{m}{p} \frac{\ln^p \mathcal{S}}{\mathcal{S}^{n+m}} \left(\frac{4 \ln 2 + 3}{8 \mathcal{S}} \right)^q \left[2 \ln 2 - \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \right. \\ & \left. \cdot \left(\frac{4 \ln 2 + 3}{8 \mathcal{S}} \right)^k \right]^{m-p}. \end{aligned} \quad (6.28)$$

For $p = m$ and $n = q = 0$ we read off the coefficients of the leading terms:

$$\rho_{(mm)} = -\frac{1}{2} \frac{(-1)^m}{4^m m!} \cdot \left[\begin{smallmatrix} m \\ 1 \end{smallmatrix} \right] \binom{-m}{0} \binom{m}{m} = \frac{(-1)^{m+1}}{4^m} \frac{1}{2m}, \quad (6.29)$$

which agrees with the results of [38] (we have used the first property of unsigned Stirling numbers in (A.5)). Also,

$$\rho_c = \frac{1}{2} \quad \& \quad \rho_0 = 2 \ln 2 - \frac{1}{2}. \quad (6.30)$$

6.4 Next-to-leading terms

To extract the subleading coefficients, we have to keep the following terms of series (6.18)

$$\mathcal{E} - \mathcal{S} \Big|_{(L+NL+\dots)} = \frac{2}{x^*} - \left(2\mathcal{S} + \frac{5}{4}\right) - \frac{9x^*}{32} - \frac{\mathcal{S}}{32} (x^*)^2 \quad (6.31)$$

and read off the terms that contribute to subleading order from equations (6.24)–(6.26). The result is (with the aid of (A.5)):

$$\rho_{(m+1,m)} = \frac{1}{2} \frac{(-1)^{m+1}}{4^{m+1}} \left[H_m + \frac{m}{4} + 1 - 4 \ln 2 \right], \quad (6.32)$$

in conformity with [38]. We also confirm the coefficient ρ_1 in expansion (6.20):

$$\rho_1 = \frac{\ln 2}{2} - \frac{1}{8}. \quad (6.33)$$

6.5 NNL terms

We may also go to higher subleading orders, e.g. the NNL one. In this case the terms of (6.18) that contribute are:

$$\begin{aligned} \mathcal{E} - \mathcal{S} \Big|_{L+NL+NNL+\dots} &= \frac{2}{x^*} - \left(2\mathcal{S} + \frac{5}{4}\right) - \frac{9x^*}{32} - \left(\frac{\mathcal{S}}{32} + \frac{35}{256}\right) (x^*)^2 - \frac{5\mathcal{S}}{128} (x^*)^3 \\ &\quad - \frac{\mathcal{S}^2}{256} (x^*)^4. \end{aligned} \quad (6.34)$$

We now can read off all the NNL terms (using property (A.5) of unsigned Stirling numbers):

$$\begin{aligned} \rho_{(m+2,m)} &= \frac{(-1)^{m+1}}{4^{m+3}} \cdot (m+1) \cdot \left\{ H_{m+1}^2 - H_{m+1}^{(2)} + \frac{1}{2} (m - 16 \ln 2 + 5) \cdot H_{m+1} + \frac{m^2}{24} - \right. \\ &\quad \left. - \left(2 \ln 2 + \frac{1}{24}\right) m + 16 \ln^2 2 - 10 \ln 2 \right\}. \end{aligned} \quad (6.35)$$

For the coefficient ρ_2 of (6.20) we find:

$$\rho_2 = -\frac{\ln^2 2}{4} + \frac{9 \ln 2}{32} - \frac{5}{128}. \quad (6.36)$$

The above results agree completely with the (first three) terms of the series (6.21) that were computed in [36, 72].

6.6 Higher-order terms

Similarly, we may get going to higher and higher orders. Using Mathematica we find e.g. (cf. (D.7)),

$$\rho_3 = \frac{\ln^3 2}{6} - \frac{3 \ln^2 2}{8} + \frac{11 \ln 2}{64} - \frac{7}{384}$$

$$\rho_4 = -\frac{\ln^4 2}{8} + \frac{43 \ln^3 2}{96} - \frac{51 \ln^2 2}{128} + \frac{937 \ln 2}{8192} - \frac{1919}{196.608}$$

$$\rho_5 = \frac{\ln^5 2}{10} - \frac{49 \ln^4 2}{96} + \frac{23 \ln^3 2}{32} - \frac{777 \ln^2 2}{2048} + \frac{1963 \ln 2}{24.576} - \frac{7423}{1.310.720}$$

$$\rho_6 = -\frac{\ln^6 2}{12} + \frac{17 \ln^5 2}{30} - \frac{581 \ln^4 2}{512} + \frac{11.401 \ln^3 2}{12.288} - \frac{67.715 \ln^2 2}{196.608} + \frac{30.085 \ln 2}{524.288} - \frac{218.431}{62.914.560}.$$

We may also express formulas (6.11)–(6.18) for the inverse spin function $x(\mathcal{S})$ and the anomalous dimensions $\gamma = \mathcal{E} - \mathcal{J}$ of the long operators $\text{Tr}[\mathcal{Z} \mathcal{D}_+^{\mathcal{S}} \mathcal{Z}] + \dots$ of $\mathcal{N} = 4$ SYM at strong 't Hooft coupling in terms of Lambert's W-function. Plugging equation (6.23) into (6.11) and (6.18) we find, respectively (for simplicity, only the first few terms are displayed):

$$\begin{aligned} x = & -\frac{4}{W_{-1}} - \frac{8\mathcal{S} + 3}{(W_{-1})^3} + \left[16\mathcal{S} + \frac{23}{4}\right] \frac{1}{(W_{-1})^4} - \left[32\mathcal{S}^2 + 55\mathcal{S} + \frac{349}{24}\right] \frac{1}{(W_{-1})^5} + \left[152\mathcal{S}^2 + \right. \\ & \left. + \frac{711\mathcal{S}}{4} + \frac{3745}{96}\right] \frac{1}{(W_{-1})^6} - \left[160\mathcal{S}^3 + 704\mathcal{S}^2 + \frac{4765\mathcal{S}}{8} + \frac{26.659}{240}\right] \frac{1}{(W_{-1})^7} + \\ & + \left[\frac{3728\mathcal{S}^3}{3} + \frac{6077\mathcal{S}^2}{2} + \frac{48.955\mathcal{S}}{24} + \frac{2.543.083}{7680}\right] \frac{1}{(W_{-1})^8} - \dots, \end{aligned} \quad (6.37)$$

$$\begin{aligned} \mathcal{E} - \mathcal{S} = & -\frac{W_{-1}}{2} - \left(2\mathcal{S} + \frac{5}{4}\right) + \frac{9}{8W_{-1}} - \left[\frac{\mathcal{S}}{2} + \frac{35}{16}\right] \frac{1}{(W_{-1})^2} + \left[\frac{5\mathcal{S}}{2} + \frac{2213}{384}\right] \frac{1}{(W_{-1})^3} - \\ & - \left[\mathcal{S}^2 + \frac{361\mathcal{S}}{32} + \frac{6665}{384}\right] \frac{1}{(W_{-1})^4} + \left[\frac{19\mathcal{S}^2}{2} + \frac{1579\mathcal{S}}{32} + \frac{433501}{7680}\right] \frac{1}{(W_{-1})^5} - \\ & - \left[\frac{10\mathcal{S}^3}{3} + \frac{259\mathcal{S}^2}{4} + \frac{81.799\mathcal{S}}{384} + \frac{2.963.887}{15.360}\right] \frac{1}{(W_{-1})^6} + \left[\frac{136\mathcal{S}^3}{3} + \frac{3069\mathcal{S}^2}{8} + \right. \\ & \left. + \frac{175.481\mathcal{S}}{192} + \frac{2.350.780.111}{3.440.640}\right] \frac{1}{(W_{-1})^7} - \dots, \end{aligned} \quad (6.38)$$

where the arguments of the W-functions are $W_{-1}(-e^{-4\mathcal{S}-3/2}/4)$ and $\mathcal{E} \equiv \pi E/2\sqrt{\lambda}$, $\mathcal{S} \equiv \pi S/2\sqrt{\lambda}$. Just as for strings in $\mathbb{R} \times \mathbb{S}^2$, the terms of both series (6.37) and (6.38) are organized in decreasing order of importance, e.g. in (6.38) the first two terms contain all the leading coefficients, the first four terms all of the subleading coefficients, and so on. Our results can be verified with a direct calculation in Mathematica (see (D.6), (D.7)).

7 Discussion

We have computed the large-spin expansion of anomalous dimensions of two single-trace operators of $\mathcal{N} = 4$ super Yang-Mills theory at strong coupling, namely

$$\mathcal{O}_J = \text{Tr} [\Phi \mathcal{Z}^m \Phi \mathcal{Z}^{J-m}] + \dots \quad \& \quad \mathcal{O}_S = \text{Tr} [\mathcal{Z} \mathcal{D}_+^S \mathcal{Z}] + \dots, \quad (7.1)$$

where Φ is any scalar field of $\mathcal{N} = 4$ SYM that is not used to build \mathcal{Z} . According to the AdS/CFT correspondence, the anomalous dimensions of the operators under consideration are given by the dispersion relation of their dual $\text{AdS}_5 \times \text{S}^5$ strings. Operator \mathcal{O}_J is dual to single-spin strings that rotate inside $\mathbb{R} \times \text{S}^2$, while \mathcal{O}_S is dual to single-spin strings that rotate inside AdS_3 . We have expressed the anomalous dimensions γ of the above (long) gauge-theory operators in terms of Lambert's W-function. This results in more compact expressions for γ 's and simplify the derivation of the corresponding series. We have also found a duality between short and long strings. For each solution of energy E and spin J , there exists a dual solution whose energy E' and spin J' are related to the original by equations (3.23)–(3.24). These relations are purely classical ($\lambda \gg 1$) but it would be interesting to see if and how they can be promoted to the quantum level.

The inversion of elliptic integrals and Jacobi elliptic functions w.r.t. the *parameter* m , is an active field of research in computational mathematics.¹³ It seems though that due to the presence of the logarithmic singularity at $m = 1$ (cf. appendix B), no significant progress in actually calculating these inverses has yet been made. In our case, we have noticed that equation (4.2) may be solved by the Lagrange-Bürmann formula and that the result may then be expressed in terms of Lambert's W-function. In the case of AdS_3 a slightly different process had to be followed, as the equation to be inverted (6.2) is more difficult, because of the additional $1/x$ term on the r.h.s.¹⁴ The presence of this $1/x$ term, in fact leads to the appearance of the branch W_{-1} of Lambert's function instead of W_0 and subsequently to logarithmic instead of exponential corrections for the inverse spin function and the anomalous dimensions.

Equations (4.38)–(4.39) and (6.37)–(6.38) that we have found, are only valid up to a certain subleading order. It would be interesting if generalizations to all subleading orders via a general formula or a recursive process could be found. As we have already said, we believe that the Lambert functions will keep appearing to all subleading orders. One could also consider changing branches in Lambert's W-functions. Changing between the Lambert branches W_0 and W_{-1} means that the inverse spin function either blows up (i.e. $x \rightarrow \pm\infty$) or in general does not have the desired behavior $x \rightarrow 0$. Conversely, we saw that a single sign flip in the argument of Lambert's function (cf. (4.39), (4.46)) carries us from folded ($\omega > 1$) to circular ($\omega < 1$) strings on S^2 and vice-versa. This could be the manifestation of a more profound link between the two domains. It is also possible that the Lambert function formalism could help make apparent the symmetries that are

¹³See e.g. [73].

¹⁴This difficulty partly accounts for the somewhat unconventional organization of our paper, discussing the GKP configuration (II) before (I).

hidden inside the large-spin expansions (e.g. one such symmetry could be the *near conjugate symmetry* $W_k(\bar{z}) = \overline{W_{-k}(z)}$).

Our results from the study of GKP strings on AdS (6.38) point out that the anomalous dimensions of long twist-2 operators of $\mathcal{N} = 4$ SYM theory at strong coupling may be expressed in terms of Lambert's W-function. This could also be the case at weak coupling. For the cousin theory of QCD, long twist operators (responsible for scaling violations in DIS) could also admit an analogous Lambert parametrization. Note that the exact 3-loop running coupling constant of QCD has already been found to be expressible in terms of Lambert's W-function [74–77]:

$$\alpha_s(Q^2) = \frac{-\pi/c}{1 - c_2/c^2 + W(z)}, \tag{7.2}$$

where c_2 is a renormalization scheme-dependent constant and

$$\beta_0 \equiv \frac{1}{4} \left(11 - \frac{2}{3} n_f \right), \quad c \equiv \frac{1}{4\beta_0} \left[102 - \frac{38}{3} n_f \right], \quad z \equiv -\frac{1}{c} \exp \left[-1 + \frac{c_2}{c^2} - \frac{\beta_0 t}{c} \right], \quad t \equiv \ln \left(\frac{Q^2}{\Lambda^2} \right).$$

It would be surprising if the anomalous dimensions of long, twist QCD operators didn't have anything to do with W (at least in strong coupling).¹⁵ Thermal backgrounds, as well as dilaton geometries within holographic frameworks [78] could also be susceptible to analogous W-formulations since, in the language of holography, Einstein's equations are nothing more than RG equations and these have also been solved in terms of Lambert's W-function [79, 80]. Our analysis suggests that a generic feature of the solution of the RG equations at any loop-order is that they can be expressed in terms of Lambert's W-function (to see this compare equation (6.2) with the integral of the generic RG-equation $\beta(x) = \mu^2 dx/d\mu^2 = -x^2 \sum \beta_n x^n$).

All of our expressions for long/fast strings have been verified with Mathematica (see appendix D). We have also considered short strings (appendix C). Since the elliptic integrals do not have a logarithmic singularity for eccentricities smaller than one, $\mathcal{E} = \mathcal{E}(\mathcal{J})$ and $\mathcal{E} = \mathcal{E}(\mathcal{S})$ can be obtained by simple series reversion in Mathematica. It is possible that these short series also afford an analysis into sums of W-functions. Compact forms for the short series would be rather interesting because of the relationship that short spinning strings bear to closed strings rotating in flat spacetimes. In this case $\mathcal{E} = \sqrt{\pi \mathcal{S}}$, which is just the first term of the short series (either in $\mathbb{R} \times \mathbb{S}^2$ or AdS₃). In figure 7 we have plotted in a common diagram the energy as a function of spin, for all these three cases of folded closed GKP strings: (3.11)–(3.12) and (3.14)–(3.15) for strings on $\mathbb{R} \times \mathbb{S}^2$, (5.10)–(5.11) for strings in AdS₃ and $\mathcal{E} = \sqrt{\pi \mathcal{S}}$ for strings rotating in flat spacetime.

It is known that both GKP strings (I) and (II) that we have studied in our paper, can be formed by the superposition of other elementary string-excitations, spiky strings and giant magnons (GMs) respectively. It would be then natural to inquire whether our findings have any consequences whatsoever to the computation of the general dispersion relations of these elementary excitations.

¹⁵Note that the branch of the W-function in the QCD case (7.2) depends on the number of flavors n_f . For $c > 0 \Leftrightarrow z < 0$ the relevant branch is W_{-1} , while for $c < 0 \Leftrightarrow z > 0$ the branch is W_0 [76].

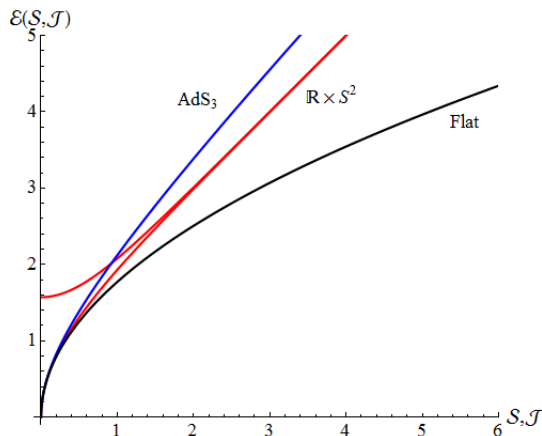


Figure 7. $\mathcal{E} = \mathcal{E}(S, \mathcal{J})$ for GKP strings in AdS_3 , S^2 and flat spacetimes.

As we have already mentioned in the introduction, strings on $\mathbb{R} \times S^2$ are the sum of two giant magnons of maximum momentum $p = \pi$ and angular momentum $J/2$. The all-loop, infinite-volume Beisert dispersion relation (1.5) [42], converges to the Hofman-Maldacena [41] relation (1.4) at strong coupling and receives finite-size corrections as the volume is gradually decreased. Arutyunov, Frolov and Zamaklar (AFZ) [53] and Astolfi, Forini, Grignani and Semenoff [54] have calculated the finite-size corrections to the giant magnon (1.6), while Klose and McLoughlin [62] have provided the leading terms (1.7).

Our result (4.39) completely agrees with formula (3.20) of AFZ [53].¹⁶ We have also found that the finite-size corrections of giant magnons can be expressed in terms of Lambert’s function $W(-16\mathcal{J}^2 \cot^2(p/2) e^{-2\mathcal{J} \csc p/2-2})$ as follows:¹⁷

$$\begin{aligned}
 \mathcal{E} - \mathcal{J} = & \sin \frac{p}{2} + \frac{1}{4\mathcal{J}^2} \tan^2 \frac{p}{2} \sin^3 \frac{p}{2} \left[W + \frac{W^2}{2} \right] - \frac{1}{16\mathcal{J}^3} \tan^4 \frac{p}{2} \sin^2 \frac{p}{2} \left[(3 \cos p + 2) W^2 + \right. \\
 & \left. + \frac{1}{6} (5 \cos p + 11) W^3 \right] - \frac{1}{512\mathcal{J}^4} \tan^6 \frac{p}{2} \sin \frac{p}{2} \left\{ (7 \cos p - 3)^2 \frac{W^2}{1+W} - \right. \\
 & \left. - \frac{1}{2} (25 \cos 2p - 188 \cos p - 13) W^2 - \frac{1}{2} (47 \cos 2p + 196 \cos p - 19) W^3 - \right. \\
 & \left. - \frac{1}{3} (13 \cos 2p + 90 \cos p + 137) W^4 \right\} + \dots
 \end{aligned} \tag{7.3}$$

Upon expanding (7.3) we recover formulas (5.14) of AFZ [53] and (39) of Astolfi-Forini-Grignani-Semenoff [54]. Especially for the above formula, we have set $\mathcal{E} \equiv \pi E/\sqrt{\lambda}$ and

¹⁶In comparing our results to those of AFZ one should note a difference in the definition of $\mathcal{J} \equiv \pi J/2\sqrt{\lambda}$, given in AFZ as $\mathcal{J}_{\text{AFZ}} \equiv 2\pi J/\sqrt{\lambda}$.

¹⁷Our result is valid in the conformal gauge of the string Polyakov action ($\gamma_{ab} = \eta_{ab}$) and the static time gauge, $t = \tau$. The details of the derivation will be given in a forthcoming publication.

$\mathcal{J} \equiv \pi J/\sqrt{\lambda}$. The Klose-McLoughlin series (1.7) is retrieved by letting $L_{\text{eff}} = 2\mathcal{J} \csc p/2$:

$$E - J = \frac{\sqrt{\lambda}}{\pi} \sin \frac{p}{2} \left\{ 1 + L_{\text{eff}}^{-2} \tan^2 \frac{p}{2} \left(W + \frac{1}{2} W^2 \right) \right\}, \quad (7.4)$$

so that the arguments of the W-functions are $W(-4L_{\text{eff}}^2 \cos^2(p/2) e^{-L_{\text{eff}}})$. We should note in passing that the Lüscher corrections that were first calculated in [58] (equation (6) *ibid.*), completely agree with AFZ and therefore our results agree with [58] too. One could also consider further extending these findings to the GMs of ABJM Theory.

Similar considerations should also apply to spiky strings [81], since strings on AdS_3 can be thought of as 2-spike Kruczenski strings. One may keep thinking more and more applications of the Lambert formalism. For example the expression of finite-size corrections to the energy of GMs in γ -deformed backgrounds¹⁸ [63] is very reminiscent of the undeformed ones (1.6):

$$E - J = \frac{\sqrt{\lambda}}{\pi} \sin \frac{p}{2} \left\{ 1 - 4 \sin^2 \frac{p}{2} \cos \Xi e^{-2-2\pi J/\sqrt{\lambda} \sin \frac{p}{2}} + \dots \right\}, \quad \Xi \equiv \frac{2\pi(n_2 - \beta J)}{2^{3/2} \cos^3 p/4}, \quad (7.5)$$

where n_2 is the integer string winding number and β is the real deformation parameter, satisfying $|n_2 - \beta J| \leq 1/2$ [65]. For more applications in this direction, one could consult the review article [82]. Finally, finite-size effects may be studied for higher dimensional analogues of GMs and single spike strings, e.g. for M2-branes on $\text{AdS}_4 \times S^7$ [83]. Our results are also directly applicable to stringy membranes that rotate inside $\text{AdS}_{4/7} \times S^{7/4}$ [84, 85].

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A Lambert’s W-function

In this appendix we briefly review Lambert’s W-function. It is defined by the following implicit relation:

$$W(z) e^{W(z)} = z \Leftrightarrow W(z e^z) = z. \quad (\text{A.1})$$

¹⁸Aka real Lunin-Maldacena backgrounds.

$W(x)$ has two real branches, $W_0(x)$ in $[-e^{-1}, \infty)$ and $W_{-1}(x)$ in $[-e^{-1}, 0]$, plotted in figure 8. The branch point is $(-e^{-1}, -1)$. The Taylor series at $x = 0$ for each of the two branches are [86]:

$$W_0(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)^n}{(n+1)!} \cdot x^{n+1} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^{n-1}}{n!} \cdot x^n, \quad |x| \leq e^{-1} \quad (\text{A.2})$$

$$W_{-1}(x) = \ln|x| - \ln \ln|x| + \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^n}{m!} \begin{bmatrix} n+m \\ n+1 \end{bmatrix} (\ln|x|)^{-n-m} (\ln \ln|x|)^m, \quad (\text{A.3})$$

where the unsigned Stirling numbers of the first kind, $\begin{bmatrix} n+m \\ n+1 \end{bmatrix}$ can be defined recursively as [87]:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1) \begin{bmatrix} n-1 \\ k \end{bmatrix} \quad \& \quad \begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ k \end{bmatrix} = 0, \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1, \quad n, k \geq 1. \quad (\text{A.4})$$

In our paper we have used the following identities of unsigned Stirling numbers:

$$\begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)!, \quad \begin{bmatrix} n \\ 2 \end{bmatrix} = (n-1)! H_{n-1}, \quad \begin{bmatrix} n \\ 3 \end{bmatrix} = \frac{1}{2} (n-1)! [H_{n-1}^2 - H_{n-1}^{(2)}]. \quad (\text{A.5})$$

The W-function also provides a series for the expression $x^{x^{\dots}}$. The result is:

$$x^{x^{\dots}} = \infty(x^z) = \frac{W(-\ln x)}{-\ln x}. \quad (\text{A.6})$$

Due to its defining property (A.1), the derivatives and antiderivatives of Lambert's function get significantly simplified. Below, we give a list of some useful expressions involving the W_0 function:

$$W'(x) = \frac{W(x)}{x(1+W(x))} \quad (\text{A.7})$$

$$x W'(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^n}{n!} \cdot x^n = \frac{W(x)}{1+W(x)} \quad (\text{A.8})$$

$$x (x W'(x))' = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^{n+1}}{n!} \cdot x^n = \frac{W(x)}{(1+W(x))^3} \quad (\text{A.9})$$

$$\int W(x) dx = x \left(W(x) - 1 + \frac{1}{W(x)} \right) \quad (\text{A.10})$$

$$\int \frac{W(x)}{x} dx = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^{n-2}}{n!} \cdot x^n = W(x) + \frac{W^2(x)}{2} \quad (\text{A.11})$$

$$\int \frac{1}{x} \int \frac{W(x)}{x} dx^2 = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^{n-3}}{n!} \cdot x^n = W(x) + \frac{3W^2(x)}{4} + \frac{W^3(x)}{6}. \quad (\text{A.12})$$

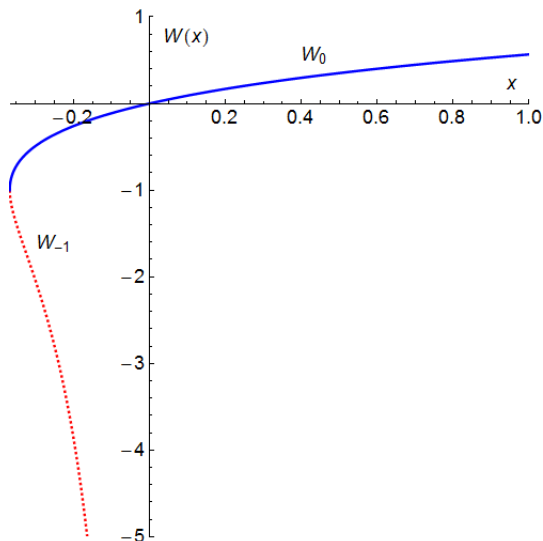


Figure 8. Lambert’s W-function.

B Elliptic integrals and Jacobian elliptic functions

This appendix is a brief reminder of the definitions and some basic properties of elliptic integrals and Jacobian elliptic functions that we have used in our paper. Our conventions are similar to those of Abramowitz-Stegun [69].

Jacobian elliptic functions.

$$u \equiv \int_0^\varphi \frac{d\theta}{(1 - m \sin^2 \theta)^{1/2}}, \quad \varphi \equiv am(u|m), \quad \Delta(\varphi) \equiv (1 - \sin^2 \theta)^{1/2} \equiv dn(u|m)$$

$$x = \sin \varphi \equiv sn(u|m), \quad \cos \varphi \equiv cn(u|m).$$

Elliptic integral of the first kind.

$$\mathbb{F}(\varphi|m) \equiv \int_0^\varphi (1 - m \sin^2 \theta)^{-1/2} d\theta = \int_0^x [(1 - t^2)(1 - mt^2)]^{-1/2} dt = u \tag{B.1}$$

$$\mathbb{K}(m) \equiv \mathbb{F}\left(\frac{\pi}{2}|m\right) = \frac{\pi}{2} \cdot {}_2\mathcal{F}_1\left[\frac{1}{2}, \frac{1}{2}; 1; m\right] \quad (\text{complete}) \tag{B.2}$$

$$\begin{aligned} \mathbb{K}(m) &= \frac{\pi}{2} \cdot \sum_{n=0}^{\infty} \left(\frac{(2n-1)!!}{(2n)!!}\right)^2 m^n = \\ &= \frac{\pi}{2} \cdot \left[1 + \left(\frac{1}{2}\right)^2 m + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 m^2 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 m^3 + \dots\right], \quad |m| < 1 \end{aligned} \tag{B.3}$$

$$\begin{aligned}
\mathbb{K}(m) &= \frac{1}{2\pi} \cdot \sum_{n=0}^{\infty} \left(\frac{\Gamma(n+1/2)}{n!} \right)^2 \left[2\psi(n+1) - 2\psi\left(n+\frac{1}{2}\right) - \ln(1-m) \right] (1-m)^n = \\
&= \sum_{n=0}^{\infty} \left(\frac{(2n-1)!!}{(2n)!!} \right)^2 \left[\psi(n+1) - \psi\left(n+\frac{1}{2}\right) - \frac{1}{2} \ln(1-m) \right] (1-m)^n, \\
&|1-m| < 1,
\end{aligned} \tag{B.4}$$

where $\psi(z) \equiv \Gamma'(z)/\Gamma(z)$ is the psi (digamma) function.

Elliptic integral of the second kind,

$$\mathbb{E}(\varphi|m) \equiv \int_0^\varphi (1-m \sin^2 \theta)^{1/2} d\theta = \int_0^x (1-t^2)^{-1/2} (1-mt^2)^{1/2} dt \tag{B.5}$$

$$\mathbb{E}(m) \equiv \mathbb{E}\left(\frac{\pi}{2}|m\right) = \frac{\pi}{2} \cdot {}_2F_1\left[-\frac{1}{2}, \frac{1}{2}; 1; m\right] \quad (\text{complete}) \tag{B.6}$$

$$\begin{aligned}
\mathbb{E}(m) &= -\frac{\pi}{2} \cdot \sum_{n=0}^{\infty} \left(\frac{(2n-1)!!}{(2n)!!} \right)^2 \frac{m^n}{2n-1} = \\
&= \frac{\pi}{2} \cdot \left[1 - \left(\frac{1}{2}\right)^2 \frac{m}{1} - \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \frac{m^2}{3} - \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 \frac{m^3}{5} + \dots \right], \quad |m| < 1 \tag{B.7}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}(m) &= 1 - \frac{1}{2\pi} \cdot \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)\Gamma(n+3/2)}{n!(n+1)!} \left[\ln(1-m) + \psi(n+1/2) + \psi(n+3/2) - \right. \\
&\quad \left. - \psi(n+1) - \psi(n+2) \right] (1-m)^{n+1} = \\
&= 1 + \frac{1}{2\pi} \cdot \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)\Gamma(n+3/2)}{n!(n+1)!} \left[2\psi(n+1) - 2\psi(n+1/2) - \ln(1-m) - \right. \\
&\quad \left. - \frac{1}{(n+1)(2n+1)} \right] (1-m)^{n+1}, \quad |1-m| < 1. \tag{B.8}
\end{aligned}$$

C Short and slow strings

In this appendix we summarize our computations on short and slow GKP strings. We have the following cases.

- $\mathbb{R} \times S^2$ folded strings ($\omega > 1$): for short folded strings on S^2 ($\omega \rightarrow \infty$) the expansions of energy (3.14) and spin (3.15) in terms of the angular frequency ω assume the following forms (cf. appendix B):

$$E = \sqrt{\lambda} \cdot \sum_{n=0}^{\infty} \left(\frac{(2n-1)!!}{(2n)!!} \right)^2 \frac{1}{\omega^{2n+1}} \tag{C.1}$$

$$J = \sqrt{\lambda} \cdot \sum_{n=1}^{\infty} \left(\frac{(2n-1)!!}{(2n)!!} \right)^2 \frac{2n}{2n-1} \frac{1}{\omega^{2n}}. \tag{C.2}$$

Series (C.2) can be inverted with a symbolic computations program, then the inverse spin function $x = x(\mathcal{J})$ may be inserted into the expression for energy (C.1), leading to $\mathcal{E} = \mathcal{E}(\mathcal{J})$. The results are:

$$x = 1 - \frac{4\mathcal{J}}{\pi} + \frac{6\mathcal{J}^2}{\pi^2} - \frac{3\mathcal{J}^3}{\pi^3} - \frac{5\mathcal{J}^4}{4\pi^4} + \frac{9\mathcal{J}^5}{16\pi^5} + \frac{21\mathcal{J}^6}{16\pi^6} + \frac{35\mathcal{J}^7}{64\pi^7} - \frac{459\mathcal{J}^8}{512\pi^8} - \frac{5.835\mathcal{J}^9}{4.096\pi^9} - \dots \quad (\text{C.3})$$

$$\mathcal{E} = \pi^{1/2}\mathcal{J}^{1/2} + \frac{\mathcal{J}^{3/2}}{4\pi^{1/2}} + \frac{3\mathcal{J}^{5/2}}{32\pi^{3/2}} + \frac{\mathcal{J}^{7/2}}{128\pi^{5/2}} - \frac{61\mathcal{J}^{9/2}}{2.048\pi^{7/2}} - \frac{201\mathcal{J}^{11/2}}{8.192\pi^{9/2}} + \frac{199\mathcal{J}^{13/2}}{65.536\pi^{11/2}} + \dots \quad (\text{C.4})$$

The latter may also be written as follows:

$$E = \left(2\sqrt{\lambda}J\right)^{1/2} \cdot \left[1 + \frac{J}{8\sqrt{\lambda}} + \frac{3J^2}{128\lambda} + \frac{J^3}{1024\lambda^{3/2}} - \frac{61J^4}{32768\lambda^2} - \frac{201J^5}{262144\lambda^{5/2}} + O\left(\frac{J^6}{\lambda^3}\right)\right]. \quad (\text{C.5})$$

- $\mathbb{R} \times \mathbb{S}^2$ circular strings ($\omega < 1$): slow circular strings on \mathbb{S}^2 ($\omega \rightarrow 0^+$) have the following series for the energy (3.11) and spin (3.12), in terms of the angular frequency ω :

$$E = \sqrt{\lambda} \cdot \sum_{n=0}^{\infty} \left(\frac{(2n-1)!!}{(2n)!!}\right)^2 \omega^{2n} \quad (\text{C.6})$$

$$J = \sqrt{\lambda} \cdot \sum_{n=1}^{\infty} \left(\frac{(2n-1)!!}{(2n)!!}\right)^2 \frac{2n}{2n-1} \cdot \omega^{2n-1}. \quad (\text{C.7})$$

Again, series (C.7) is reverted for $\omega = \omega(J)$ and then the inverse spin function $\tilde{x} \equiv 1 - \omega^2 = \tilde{x}(\mathcal{J})$ is plugged into the expression for energy (C.6), leading to:

$$\tilde{x} = 1 - \frac{16\mathcal{J}^2}{\pi^2} + \frac{192\mathcal{J}^4}{\pi^4} - \frac{2112\mathcal{J}^6}{\pi^6} + \frac{22.400\mathcal{J}^8}{\pi^8} - \frac{233.088\mathcal{J}^{10}}{\pi^{10}} + \frac{2.397.696\mathcal{J}^{12}}{\pi^{12}} - \dots \quad (\text{C.8})$$

$$\mathcal{E} = \frac{\pi}{2} + \frac{2\mathcal{J}^2}{\pi} - \frac{6\mathcal{J}^4}{\pi^3} + \frac{32\mathcal{J}^6}{\pi^5} - \frac{206\mathcal{J}^8}{\pi^7} + \frac{1464\mathcal{J}^{10}}{\pi^9} - \frac{11.064\mathcal{J}^{12}}{\pi^{11}} + \frac{87.200\mathcal{J}^{14}}{\pi^{13}} - \dots \quad (\text{C.9})$$

(C.9) can also be written as follows:

$$E = \sqrt{\lambda} \cdot \left[1 + \frac{J^2}{\lambda} - \frac{3J^4}{4\lambda^2} + \frac{J^6}{\lambda^3} - \frac{103J^8}{64\lambda^4} + \frac{183J^{10}}{64\lambda^5} - \frac{1383J^{12}}{256\lambda^6} + \frac{2725J^{14}}{256\lambda^7} - O\left(\frac{J^{16}}{\lambda^8}\right)\right]. \quad (\text{C.10})$$

- AdS_3 folded strings ($\omega > 1$): for the short-string limit of AdS_3 strings, we may expand expressions (5.10)–(5.11) around $\omega \rightarrow \infty$ and obtain the corresponding short-string series (cf. appendix B):

$$E = \sqrt{\lambda} \cdot \sum_{n=0}^{\infty} \left(\frac{(2n-1)!!}{(2n)!!}\right)^2 \frac{2n+1}{\omega^{2n+1}} \quad (\text{C.11})$$

$$S = \sqrt{\lambda} \cdot \sum_{n=1}^{\infty} \left(\frac{(2n-1)!!}{(2n)!!}\right)^2 \frac{2n}{\omega^{2n}}, \quad (\text{C.12})$$

where, in order to obtain series (C.11), the identity

$$(2n + 1) \left(\frac{(2n - 1)!!}{(2n)!!} \right)^2 + \sum_{k=0}^n \frac{1}{2k - 1} \left(\frac{(2k - 1)!!}{(2k)!!} \right)^2 = 0 \quad (\text{C.13})$$

was used. Observe that the short S^2 -string coefficients of the energy series (C.1) differ from the corresponding AdS_3 ones (C.11) by a factor of $2n + 1$, $n = 0, 1, \dots$, while the coefficients of the angular momentum J in (C.2) differ from those of the spin S in (C.12) by $1/2n - 1$. This is related to the fact the the corresponding functions (5.10), (5.11) can appropriately be taken by differentiation/integration of (3.14), (3.15). Using series reversion in Mathematica, we may also obtain expressions for the inverse spin function $x = x(S)$ and energy $\mathcal{E} = \mathcal{E}(S)$ as functions of the spin S :

$$x = 1 - \frac{4S}{\pi} + \frac{18S^2}{\pi^2} - \frac{87S^3}{\pi^3} + \frac{1.765S^4}{4\pi^4} - \frac{37.071S^5}{16\pi^5} + \frac{199.815S^6}{16\pi^6} - \frac{4.397.017S^7}{64\pi^7} + \dots \quad (\text{C.14})$$

$$\mathcal{E} = \pi^{1/2} S^{1/2} + \frac{3S^{3/2}}{4\pi^{1/2}} - \frac{21S^{5/2}}{32\pi^{3/2}} + \frac{187S^{7/2}}{128\pi^{5/2}} - \frac{9.261S^{9/2}}{2.048\pi^{7/2}} + \frac{136.245S^{11/2}}{8.192\pi^{9/2}} - \dots \quad (\text{C.15})$$

The 't Hooft coupling dependence of the last expression may be restored as follows:

$$E = \left(2\sqrt{\lambda} S \right)^{1/2} \left[1 + \frac{3S}{8\sqrt{\lambda}} - \frac{21S^2}{128\lambda} + \frac{187S^3}{1024\lambda^{3/2}} - \frac{9261S^4}{32768\lambda^2} + \frac{136245S^5}{262144\lambda^{5/2}} - O\left(\frac{S^6}{\lambda^3}\right) \right]. \quad (\text{C.16})$$

D Long and fast strings

This appendix contains some of our symbolic computations on long and fast GKP strings. Just as for short/slow strings in the preceding appendix, the inverse spin functions $x(\mathcal{J})$ and $x(S)$ of long/fast strings on $\mathbb{R} \times S^2$ and AdS_3 have been obtained by inverting the corresponding series for spins (3.20), (4.41) and (5.16) with Mathematica. Then the results are plugged into the series of the anomalous dimensions $\mathcal{E} = \mathcal{E}(x)$, leading to the expressions for $\mathcal{E}(\mathcal{J})$ and $\mathcal{E}(S)$. Only the first few terms of each series are presented here. All of these results agree with the analytic formulas and series coefficients that were derived in our paper and we summarized in section 2. We have addressed the following cases.

- Folded string on $\mathbb{R} \times S^2$ ($\omega > 1$).¹⁹

$$\begin{aligned} x &= 16 e^{-2\mathcal{J}-2} - 128 (\mathcal{J} + 1) e^{-4\mathcal{J}-4} + 64 (24\mathcal{J}^2 + 34\mathcal{J} + 15) e^{-6\mathcal{J}-6} \\ &\quad - \frac{512}{3} (128\mathcal{J}^3 + 216\mathcal{J}^2 + 153\mathcal{J} + 42) e^{-8\mathcal{J}-8} \\ &\quad + \frac{32}{3} (32.000\mathcal{J}^4 + 60.800\mathcal{J}^3 + 54.960\mathcal{J}^2 + 25.452\mathcal{J} + 4989) e^{-10\mathcal{J}-10} \\ &\quad - \frac{512}{5} (55.296\mathcal{J}^5 + 115.200\mathcal{J}^4 + 122.400\mathcal{J}^3 + 74.460\mathcal{J}^2 + 25.480\mathcal{J} + 3855) e^{-12\mathcal{J}-12} + \dots \end{aligned} \quad (\text{D.2})$$

¹⁹By making the transformation

$$S \equiv \frac{1}{16} e^{2\mathcal{J}+2} \Leftrightarrow \mathcal{J} = \frac{1}{2} (\ln S + 4 \ln 2 - 2) \quad (\text{D.1})$$

the inverse spin function and the anomalous dimensions of strings spinning in $\mathbb{R} \times S^2$, assume a form that resembles the corresponding formula for strings in AdS_3 and permits comparisons between the two. Interestingly, and in contradistinction with the AdS case, all $\ln^n S/S^n$ terms are absent.

$$\begin{aligned}
\mathcal{E} - \mathcal{J} = & 1 - 4e^{-2\mathcal{J}-2} + 4(4\mathcal{J} - 1) e^{-4\mathcal{J}-4} - 32(4\mathcal{J}^2 - \mathcal{J} + 1) e^{-6\mathcal{J}-6} \\
& + \frac{4}{3}(1024\mathcal{J}^3 - 192\mathcal{J}^2 + 456\mathcal{J} - 63) e^{-8\mathcal{J}-8} \\
& - \frac{8}{3}(6400\mathcal{J}^4 - 640\mathcal{J}^3 + 3888\mathcal{J}^2 - 660\mathcal{J} + 279) e^{-10\mathcal{J}-10} \\
& + \frac{32}{5}(36.864\mathcal{J}^5 + 27.840\mathcal{J}^3 - 4040\mathcal{J}^2 + 4160\mathcal{J} - 405) e^{-12\mathcal{J}-12} - \dots
\end{aligned} \tag{D.3}$$

- Circular string on $\mathbb{R} \times \mathbb{S}^2$ ($\omega < 1$).

$$\begin{aligned}
\tilde{x} = & 16e^{-2\mathcal{J}-2} + 128(\mathcal{J} - 1) e^{-4\mathcal{J}-4} + 192(8\mathcal{J}^2 - 10\mathcal{J} + 5) e^{-6\mathcal{J}-6} \\
& + \frac{512}{3}(128\mathcal{J}^3 - 168\mathcal{J}^2 + 129\mathcal{J} - 42) e^{-8\mathcal{J}-8} \\
& + \frac{32}{3}(32.000\mathcal{J}^4 - 41.600\mathcal{J}^3 + 39.600\mathcal{J}^2 - 20.628\mathcal{J} + 4989) e^{-10\mathcal{J}-10} \\
& + \frac{1536}{5}(18.432\mathcal{J}^5 - 23.040\mathcal{J}^4 + 25.440\mathcal{J}^3 - 16.460\mathcal{J}^2 + 6.720\mathcal{J} - 1285) e^{-12\mathcal{J}-12} + \dots
\end{aligned} \tag{D.4}$$

$$\begin{aligned}
\mathcal{E} - \mathcal{J} = & 1 + 4e^{-2\mathcal{J}-2} + 4(4\mathcal{J} - 1) e^{-4\mathcal{J}-4} + 32(4\mathcal{J}^2 - \mathcal{J} + 1) e^{-6\mathcal{J}-6} \\
& + \frac{4}{3}(1024\mathcal{J}^3 - 192\mathcal{J}^2 + 456\mathcal{J} - 63) e^{-8\mathcal{J}-8} \\
& + \frac{8}{3}(6400\mathcal{J}^4 - 640\mathcal{J}^3 + 3888\mathcal{J}^2 - 660\mathcal{J} + 279) e^{-10\mathcal{J}-10} \\
& + \frac{32}{5}(36.864\mathcal{J}^5 + 27.840\mathcal{J}^3 - 4040\mathcal{J}^2 + 4160\mathcal{J} - 405) e^{-12\mathcal{J}-12} + \dots
\end{aligned} \tag{D.5}$$

- Folded string on AdS_3 ($\omega > 1$).

$$\begin{aligned}
x = & \frac{1}{\mathcal{S}} - \left[\frac{1}{4} \ln \mathcal{S} + \left(\ln 2 + \frac{1}{4} \right) \right] \frac{1}{\mathcal{S}^2} + \left[\frac{1}{16} \ln^2 \mathcal{S} + \left(\frac{\ln 2}{2} + \frac{1}{32} \right) \ln \mathcal{S} + \left(\ln^2 2 + \frac{\ln 2}{8} + \frac{3}{64} \right) \right] \frac{1}{\mathcal{S}^3} \\
& - \left[\frac{1}{64} \ln^3 \mathcal{S} + \left(\frac{3 \ln 2}{16} - \frac{1}{64} \right) \ln^2 \mathcal{S} + \left(\frac{3 \ln^2 2}{4} - \frac{\ln 2}{8} + \frac{3}{128} \right) \ln \mathcal{S} \right. \\
& \left. + \left(\ln^3 2 - \frac{\ln^2 2}{4} + \frac{3 \ln 2}{32} \right) \right] \frac{1}{\mathcal{S}^4} + \dots
\end{aligned} \tag{D.6}$$

$$\begin{aligned}
\gamma = & \frac{1}{2} \ln \mathcal{S} + \left[2 \ln 2 - \frac{1}{2} \right] + \left[\frac{1}{8} \ln \mathcal{S} + \left(\frac{\ln 2}{2} - \frac{1}{8} \right) \right] \frac{1}{\mathcal{S}} - \left[\frac{1}{64} \ln^2 \mathcal{S} + \left(\frac{\ln 2}{8} - \frac{9}{128} \right) \ln \mathcal{S} \right. \\
& \left. + \left(\frac{\ln^2 2}{4} - \frac{9 \ln 2}{32} + \frac{5}{128} \right) \right] \frac{1}{\mathcal{S}^2} + \left[\frac{1}{384} \ln^3 \mathcal{S} + \left(\frac{\ln 2}{32} - \frac{3}{128} \right) \ln^2 \mathcal{S} \right. \\
& \left. + \left(\frac{\ln^2 2}{8} - \frac{3 \ln 2}{16} + \frac{11}{256} \right) \ln \mathcal{S} + \left(\frac{\ln^3 2}{6} - \frac{3 \ln^2 2}{8} + \frac{11 \ln 2}{64} - \frac{7}{384} \right) \right] \frac{1}{\mathcal{S}^3} \\
& - \left[\frac{1}{2048} \ln^4 \mathcal{S} + \left(\frac{\ln 2}{128} - \frac{43}{6144} \right) \ln^3 \mathcal{S} + \left(\frac{3 \ln^2 2}{64} - \frac{43 \ln 2}{512} + \frac{51}{2048} \right) \ln^2 \mathcal{S} \right. \\
& \left. + \left(\frac{\ln^3 2}{8} - \frac{43 \ln^2 2}{128} + \frac{51 \ln 2}{256} - \frac{937}{32.768} \right) \ln \mathcal{S} \right. \\
& \left. + \left(\frac{\ln^4 2}{8} - \frac{43 \ln^3 2}{96} + \frac{51 \ln^2 2}{128} - \frac{937 \ln 2}{8192} + \frac{1919}{196.608} \right) \right] \cdot \frac{1}{\mathcal{S}^4} + \dots
\end{aligned} \tag{D.7}$$

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References

- [1] J.M. Maldacena, *The Large- N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [[hep-th/9711200](#)] [[INSPIRE](#)].
- [2] S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys. Lett. B* **428** (1998) 105 [[hep-th/9802109](#)] [[INSPIRE](#)].
- [3] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [[hep-th/9802150](#)] [[INSPIRE](#)].
- [4] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz, *Large- N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183 [[hep-th/9905111](#)] [[INSPIRE](#)].
- [5] E. D'Hoker and D.Z. Freedman, *Supersymmetric gauge theories and the AdS /CFT correspondence*, [hep-th/0201253](#) [[INSPIRE](#)].
- [6] C. Kristjansen, M. Staudacher and A. Tseytlin, *Gauge-string duality and integrability: Progress and outlook*, *J. Phys. A* **42** (2009) 250301 [[INSPIRE](#)].
- [7] N. Beisert, C. Ahn, L.F. Alday, Z. Bajnok, J.M. Drummond et al., *Review of AdS/CFT Integrability: An Overview*, *Lett. Math. Phys.* **99** (2012) 3 [[arXiv:1012.3982](#)] [[INSPIRE](#)].
- [8] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *A Semiclassical limit of the gauge/string correspondence*, *Nucl. Phys. B* **636** (2002) 99 [[hep-th/0204051](#)] [[INSPIRE](#)].
- [9] A.A. Tseytlin, *Review of AdS/CFT Integrability, Chapter II.1: Classical AdS $_5$ xS 5 string solutions*, *Lett. Math. Phys.* **99** (2012) 103 [[arXiv:1012.3986](#)] [[INSPIRE](#)].
- [10] M. Axenides, E. Floratos and A. Kehagias, *Scaling violations in Yang-Mills theories and strings in AdS $_5$* , *Nucl. Phys. B* **662** (2003) 170 [[hep-th/0210091](#)] [[INSPIRE](#)].
- [11] H. Georgi and H.D. Politzer, *Electroproduction scaling in an asymptotically free theory of strong interactions*, *Phys. Rev. D* **9** (1974) 416 [[INSPIRE](#)].
- [12] D.J. Gross and F. Wilczek, *Asymptotically free gauge theories. 2.*, *Phys. Rev. D* **9** (1974) 980 [[INSPIRE](#)].
- [13] E.G. Floratos, D.A. Ross and C.T. Sachrajda, *Higher Order Effects in Asymptotically Free Gauge Theories: The Anomalous Dimensions of Wilson Operators*, *Nucl. Phys. B* **129** (1977) 66 [*Erratum ibid.* **B 139** (1978) 545-546] [[INSPIRE](#)].
- [14] E.G. Floratos, D.A. Ross and C.T. Sachrajda, *Higher Order Effects in Asymptotically Free Gauge Theories. 2. Flavor Singlet Wilson Operators and Coefficient Functions*, *Nucl. Phys. B* **152** (1979) 493 [[INSPIRE](#)].
- [15] G. Curci, W. Furmanski and R. Petronzio, *Evolution of Parton Densities Beyond Leading Order: The Nonsinglet Case*, *Nucl. Phys. B* **175** (1980) 27 [[INSPIRE](#)].
- [16] E.G. Floratos, C. Kounnas and R. Lacaze, *Higher Order QCD Effects in Inclusive Annihilation and Deep Inelastic Scattering*, *Nucl. Phys. B* **192** (1981) 417 [[INSPIRE](#)].
- [17] S. Moch, J.A.M. Vermaseren and A. Vogt, *The Three loop splitting functions in QCD: The Nonsinglet case*, *Nucl. Phys. B* **688** (2004) 101 [[hep-ph/0403192](#)] [[INSPIRE](#)].
- [18] A. Vogt, S. Moch and J.A.M. Vermaseren, *The Three-loop splitting functions in QCD: The Singlet case*, *Nucl. Phys. B* **691** (2004) 129 [[hep-ph/0404111](#)] [[INSPIRE](#)].

- [19] A.V. Kotikov and L.N. Lipatov, *DGLAP and BFKL evolution equations in the $N = 4$ supersymmetric gauge theory*, [hep-ph/0112346](#) [INSPIRE].
- [20] A.V. Kotikov, L.N. Lipatov and V.N. Velizhanin, *Anomalous dimensions of Wilson operators in $N = 4$ SYM theory*, *Phys. Lett. B* **557** (2003) 114 [[hep-ph/0301021](#)] [INSPIRE].
- [21] A.V. Kotikov, L.N. Lipatov, A.I. Onishchenko and V.N. Velizhanin, *Three loop universal anomalous dimension of the Wilson operators in $N = 4$ SUSY Yang-Mills model*, *Phys. Lett. B* **595** (2004) 521 [*Erratum ibid.* **B 632** (2006) 754-756] [[hep-th/0404092](#)] [INSPIRE].
- [22] B. Eden and M. Staudacher, *Integrability and transcendentality*, *J. Stat. Mech.* **0611** (2006) P11014 [[hep-th/0603157](#)] [INSPIRE].
- [23] N. Beisert, B. Eden and M. Staudacher, *Transcendentality and Crossing*, *J. Stat. Mech.* **0701** (2007) P01021 [[hep-th/0610251](#)] [INSPIRE].
- [24] B. Basso, G.P. Korchemsky and J. Kotiański, *Cusp anomalous dimension in maximally supersymmetric Yang-Mills theory at strong coupling*, *Phys. Rev. Lett.* **100** (2008) 091601 [[arXiv:0708.3933](#)] [INSPIRE].
- [25] I. Kostov, D. Serban and D. Volin, *Functional BES equation*, *JHEP* **08** (2008) 101 [[arXiv:0801.2542](#)] [INSPIRE].
- [26] S. Frolov and A.A. Tseytlin, *Semiclassical quantization of rotating superstring in $AdS_5 \times S^5$* , *JHEP* **06** (2002) 007 [[hep-th/0204226](#)] [INSPIRE].
- [27] R. Roiban, A. Tirziu and A.A. Tseytlin, *Two-loop world-sheet corrections in $AdS_5 \times S^5$ superstring*, *JHEP* **07** (2007) 056 [[arXiv:0704.3638](#)] [INSPIRE].
- [28] R. Roiban and A.A. Tseytlin, *Strong-coupling expansion of cusp anomaly from quantum superstring*, *JHEP* **11** (2007) 016 [[arXiv:0709.0681](#)] [INSPIRE].
- [29] A.V. Kotikov, A. Rej and S. Zieme, *Analytic three-loop Solutions for $\mathcal{N} = 4$ SYM Twist Operators*, *Nucl. Phys. B* **813** (2009) 460 [[arXiv:0810.0691](#)] [INSPIRE].
- [30] M. Beccaria, A.V. Belitsky, A.V. Kotikov and S. Zieme, *Analytic solution of the multiloop Baxter equation*, *Nucl. Phys. B* **827** (2010) 565 [[arXiv:0908.0520](#)] [INSPIRE].
- [31] Z. Bajnok, R.A. Janik and T. Lukowski, *Four loop twist two, BFKL, wrapping and strings*, *Nucl. Phys. B* **816** (2009) 376 [[arXiv:0811.4448](#)] [INSPIRE].
- [32] T. Lukowski, A. Rej and V.N. Velizhanin, *Five-Loop Anomalous Dimension of Twist-Two Operators*, *Nucl. Phys. B* **831** (2010) 105 [[arXiv:0912.1624](#)] [INSPIRE].
- [33] G. Georgiou, *Two and three-point correlators of operators dual to folded string solutions at strong coupling*, *JHEP* **02** (2011) 046 [[arXiv:1011.5181](#)] [INSPIRE].
- [34] G. Georgiou, *$SL(2)$ sector: weak/strong coupling agreement of three-point correlators*, *JHEP* **09** (2011) 132 [[arXiv:1107.1850](#)] [INSPIRE].
- [35] D. Bombardelli, D. Fioravanti and R. Tateo, *Thermodynamic Bethe Ansatz for planar AdS/CFT : A Proposal*, *J. Phys. A* **42** (2009) 375401 [[arXiv:0902.3930](#)] [INSPIRE].
- [36] M. Beccaria, V. Forini, A. Tirziu and A.A. Tseytlin, *Structure of large spin expansion of anomalous dimensions at strong coupling*, *Nucl. Phys. B* **812** (2009) 144 [[arXiv:0809.5234](#)] [INSPIRE].
- [37] M. Beccaria, G.V. Dunne, V. Forini, M. Pawellek and A.A. Tseytlin, *Exact computation of one-loop correction to energy of spinning folded string in $AdS_5 \times S^5$* , *J. Phys. A* **43** (2010) 165402 [[arXiv:1001.4018](#)] [INSPIRE].

- [38] G. Georgiou and G. Savvidy, *Large spin behavior of anomalous dimensions and short-long strings duality*, *J. Phys. A* **44** (2011) 305402 [[arXiv:1012.5580](#)] [[INSPIRE](#)].
- [39] G. Savvidy, *Non-Abelian Tensor Gauge Fields*, *Proc. Steklov Inst. Math.* **272** (2011) 201 [[arXiv:1004.4456](#)] [[INSPIRE](#)].
- [40] Z. Chong, H. Lü and C. Pope, *Rotating strings in massive type IIA supergravity*, [hep-th/0402202](#) [[INSPIRE](#)].
- [41] D.M. Hofman and J.M. Maldacena, *Giant Magnons*, *J. Phys. A* **39** (2006) 13095 [[hep-th/0604135](#)] [[INSPIRE](#)].
- [42] N. Beisert, *The SU(2|2) dynamic S-matrix*, *Adv. Theor. Math. Phys.* **12** (2008) 945 [[hep-th/0511082](#)] [[INSPIRE](#)].
- [43] J.A. Minahan and K. Zarembo, *The Bethe ansatz for $N = 4$ super Yang-Mills*, *JHEP* **03** (2003) 013 [[hep-th/0212208](#)] [[INSPIRE](#)].
- [44] A.V. Kotikov, L.N. Lipatov, A. Rej, M. Staudacher and V.N. Velizhanin, *Dressing and wrapping*, *J. Stat. Mech.* **0710** (2007) P10003 [[arXiv:0704.3586](#)] [[INSPIRE](#)].
- [45] S. Schäfer-Nameki, M. Zamaklar and K. Zarembo, *Quantum corrections to spinning strings in $AdS_5 \times S^5$ and Bethe ansatz: A Comparative study*, *JHEP* **09** (2005) 051 [[hep-th/0507189](#)] [[INSPIRE](#)].
- [46] S. Schäfer-Nameki and M. Zamaklar, *Stringy sums and corrections to the quantum string Bethe ansatz*, *JHEP* **10** (2005) 044 [[hep-th/0509096](#)] [[INSPIRE](#)].
- [47] S. Schäfer-Nameki, *Exact expressions for quantum corrections to spinning strings*, *Phys. Lett. B* **639** (2006) 571 [[hep-th/0602214](#)] [[INSPIRE](#)].
- [48] S. Schäfer-Nameki, M. Zamaklar and K. Zarembo, *How Accurate is the Quantum String Bethe Ansatz?*, *JHEP* **12** (2006) 020 [[hep-th/0610250](#)] [[INSPIRE](#)].
- [49] J. Ambjørn, R.A. Janik and C. Kristjansen, *Wrapping interactions and a new source of corrections to the spin-chain/string duality*, *Nucl. Phys. B* **736** (2006) 288 [[hep-th/0510171](#)] [[INSPIRE](#)].
- [50] A.B. Zamolodchikov, *Thermodynamic Bethe Ansatz in Relativistic Models. Scaling Three State Potts and Lee-Yang Models*, *Nucl. Phys. B* **342** (1990) 695 [[INSPIRE](#)].
- [51] Z. Bajnok, *Review of AdS/CFT Integrability, Chapter III.6: Thermodynamic Bethe Ansatz*, *Lett. Math. Phys.* **99** (2012) 299 [[arXiv:1012.3995](#)] [[INSPIRE](#)].
- [52] N. Gromov, V. Kazakov and P. Vieira, *Exact Spectrum of Anomalous Dimensions of Planar $N = 4$ Supersymmetric Yang-Mills Theory*, *Phys. Rev. Lett.* **103** (2009) 131601 [[arXiv:0901.3753](#)] [[INSPIRE](#)].
- [53] G. Arutyunov, S. Frolov and M. Zamaklar, *Finite-size Effects from Giant Magnons*, *Nucl. Phys. B* **778** (2007) 1 [[hep-th/0606126](#)] [[INSPIRE](#)].
- [54] D. Astolfi, V. Forini, G. Grignani and G.W. Semenoff, *Gauge invariant finite size spectrum of the giant magnon*, *Phys. Lett. B* **651** (2007) 329 [[hep-th/0702043](#)] [[INSPIRE](#)].
- [55] J.A. Minahan and O. Ohlsson Sax, *Finite size effects for giant magnons on physical strings*, *Nucl. Phys. B* **801** (2008) 97 [[arXiv:0801.2064](#)] [[INSPIRE](#)].
- [56] M. Lüscher, *Volume Dependence of the Energy Spectrum in Massive Quantum Field Theories. 1. Stable Particle States*, *Commun. Math. Phys.* **104** (1986) 177 [[INSPIRE](#)].

- [57] T.R. Klassen and E. Melzer, *On the relation between scattering amplitudes and finite size mass corrections in QFT*, *Nucl. Phys. B* **362** (1991) 329 [INSPIRE].
- [58] R.A. Janik and T. Lukowski, *Wrapping interactions at strong coupling: The Giant magnon*, *Phys. Rev. D* **76** (2007) 126008 [arXiv:0708.2208] [INSPIRE].
- [59] M.P. Heller, R.A. Janik and T. Lukowski, *A New derivation of Lüscher F-term and fluctuations around the giant magnon*, *JHEP* **06** (2008) 036 [arXiv:0801.4463] [INSPIRE].
- [60] N. Gromov, S. Schäfer-Nameki and P. Vieira, *Quantum Wrapped Giant Magnon*, *Phys. Rev. D* **78** (2008) 026006 [arXiv:0801.3671] [INSPIRE].
- [61] K. Pohlmeyer, *Integrable Hamiltonian Systems and Interactions Through Quadratic Constraints*, *Commun. Math. Phys.* **46** (1976) 207 [INSPIRE].
- [62] T. Klose and T. McLoughlin, *Interacting finite-size magnons*, *J. Phys. A* **41** (2008) 285401 [arXiv:0803.2324] [INSPIRE].
- [63] C.-S. Chu, G. Georgiou and V.V. Khoze, *Magnons, classical strings and β -deformations*, *JHEP* **11** (2006) 093 [hep-th/0606220] [INSPIRE].
- [64] N. Bobev and R. Rashkov, *Multispin Giant Magnons*, *Phys. Rev. D* **74** (2006) 046011 [hep-th/0607018] [INSPIRE].
- [65] D.V. Bykov and S. Frolov, *Giant magnons in TsT-transformed $AdS_5 \times S^5$* , *JHEP* **07** (2008) 071 [arXiv:0805.1070] [INSPIRE].
- [66] C. Ahn and P. Bozhilov, *Finite-Size Dyon Giant Magnons in TsT-transformed $AdS_5 \times S^5$* , *JHEP* **07** (2010) 048 [arXiv:1005.2508] [INSPIRE].
- [67] D. Gaiotto, S. Giombi and X. Yin, *Spin Chains in $\mathcal{N} = 6$ Superconformal Chern-Simons-Matter Theory*, *JHEP* **04** (2009) 066 [arXiv:0806.4589] [INSPIRE].
- [68] G. Grignani, T. Harmark and M. Orselli, *The $SU(2) \times SU(2)$ sector in the string dual of $\mathcal{N} = 6$ superconformal Chern-Simons theory*, *Nucl. Phys. B* **810** (2009) 115 [arXiv:0806.4959] [INSPIRE].
- [69] M. Abramowitz and I. Stegun eds., *Handbook of Mathematical Functions*, Dover, New York (1972).
- [70] E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis*, Cambridge University Press, Cambridge (1958).
- [71] I.N. Galidakis, *On an Application of Lambert's W Function to Infinite Exponentials*, *Complex Var. Theory App.* **49** (2006) 759.
- [72] M. Beccaria, V. Forini and G. Macorini, *Generalized Gribov-Lipatov Reciprocity and AdS/CFT*, *Adv. High Energy Phys.* **2010** (2010) 753248 [arXiv:1002.2363] [INSPIRE].
- [73] T. Fukushima, *Numerical Computation of Inverse Complete Elliptic Integrals of First and Second Kinds*, *J. Comput. Appl. Math.* **249** (2013) 37.
- [74] N. Khuri and H. Ren, *Explicit Solutions for the Running Coupling Constant and the Separatrix of Quantum Field Theories*, *Annals Phys.* **189** (1989) 142 [INSPIRE].
- [75] T. Appelquist, A. Ratnaweera, J. Terning and L.C.R. Wijewardhana, *The Phase structure of an $SU(N)$ gauge theory with N_f flavors*, *Phys. Rev. D* **58** (1998) 105017 [hep-ph/9806472] [INSPIRE].

- [76] E. Gardi, G. Grunberg and M. Karliner, *Can the QCD running coupling have a causal analyticity structure?*, *JHEP* **07** (1998) 007 [[hep-ph/9806462](#)] [[INSPIRE](#)].
- [77] B.A. Magradze, *Analytic approach to perturbative QCD*, *Int. J. Mod. Phys. A* **15** (2000) 2715 [[hep-ph/9911456](#)] [[INSPIRE](#)].
- [78] C. Csáki and M. Reece, *Toward a systematic holographic QCD: A Braneless approach*, *JHEP* **05** (2007) 062 [[hep-ph/0608266](#)] [[INSPIRE](#)].
- [79] H. Sonoda, *Solving renormalization group equations with the Lambert W function*, *Phys. Rev. D* **87** (2013) 085023 [[arXiv:1302.6069](#)] [[INSPIRE](#)].
- [80] T.L. Curtright and C.K. Zachos, *Renormalization Group Functional Equations*, *Phys. Rev. D* **83** (2011) 065019 [[arXiv:1010.5174](#)] [[INSPIRE](#)].
- [81] M. Kruczenski, *Spiky strings and single trace operators in gauge theories*, *JHEP* **08** (2005) 014 [[hep-th/0410226](#)] [[INSPIRE](#)].
- [82] K. Zoubos, *Review of AdS/CFT Integrability, Chapter IV.2: Deformations, Orbifolds and Open Boundaries*, *Lett. Math. Phys.* **99** (2012) 375 [[arXiv:1012.3998](#)] [[INSPIRE](#)].
- [83] C. Ahn and P. Bozhilov, *Finite-size effects of Membranes on $AdS_4 \times S^7$* , *JHEP* **08** (2008) 054 [[arXiv:0807.0566](#)] [[INSPIRE](#)].
- [84] S.A. Hartnoll and C. Nuñez, *Rotating membranes on G_2 manifolds, logarithmic anomalous dimensions and $N = 1$ duality*, *JHEP* **02** (2003) 049 [[hep-th/0210218](#)] [[INSPIRE](#)].
- [85] M. Axenides, E. Floratos and G. Linares-Delgado, *Stringy Membranes in AdS/CFT*, *JHEP* **08** (2013) 089 [[arXiv:1306.0220](#)] [[INSPIRE](#)].
- [86] R. Corless, G.H. Gonnet, D.E.G. Hare, D.J. Jeffrey and D.E. Knuth, *On the Lambert W function*, *Adv. Comput. Math.* **5** (1996) 329 [[INSPIRE](#)].
- [87] L. Comtet, *Advanced Combinatorics*, Reidel (1974).