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Elliptic CY3folds and non-perturbative modular transformation

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Abstract We study the refined topological string partition function of a class of toric elliptically fibered Calabi–Yau threefolds. These Calabi–Yau threefolds give rise to five dimensional quiver gauge theories and are dual to configurations of M5-M2-branes. We determine the Gopakumar–Vafa invariants for these threefolds and show that the genus g free energy is given by the weight 2 g Eisenstein series. We also show that although the free energy at all genera are modular invariant, the full partition function satisfies the non-perturbative modular transformation property discussed by Lockhart and Vafa in arXiv:1210.5909 and therefore the modularity of free energy is up to non-perturbative corrections.

1 Introduction

In this paper we study the refined topological string partition function of a class of toric elliptically fibered Calabi–Yau threefolds which are dual to a set of parallel M5-branes with a transverse direction compactified to a circle. The size of the circle is related to the Kähler parameter of the elliptic curve class on the Calabi–Yau threefolds. We determine the Gopakumar–Vafa invariants for all curve classes and also show that for these Calabi–Yau threefolds the genus g free energy takes a particularly simple for and is given by the Eisenstein series. We also study the modular properties of the refined partition function and show that it satisfies the non-perturbative modular transformation discovered in [1]. As was shown in [1] for the case of $N = 1$, N being the number of M5-branes, the partition function is not modular invariant but satisfies a more involved transformation which maps the g_s to $\frac{1}{g_s}$ where g_s is the topological string coupling constant—hence the name non-perturbative modular transformation.

This note is organized as follows. In Sect. 2 we discuss the Calabi–Yau geometry and the dual brane configuration. In Sect. 3 we use the refined topological vertex formalism to express the partition function as a trace of an operator acting on the fermionic Fock space. In Sect. 4 we determine the Gopakumar–Vafa invariants for all curve classes and the genus g fee energy. In Sect. 5 we express the partition function in terms of double elliptic gamma functions and show that it satisfies the non-perturbative modular transformation property.

2 Elliptic CY3fold and dual brane configuration

The class elliptic Calabi–Yau threefolds we are interested in are dual to certain brane configurations which arose in the study of M-strings [2]. These Calabi–Yau threefolds are birationally equivalent $\widehat{A}_{N-1} \times_f \mathbb{C}$ where $\widehat{A}_{N-1}$ is affine A_{N-1} space blown up at N points and it is fibered over $\mathbb{C}$ to obtain the Calabi–Yau threefold. We will denote these by X_N.

The affine A_{N-1} space has N $\mathbb{P}^1$'s corresponding to the simple roots of affine $SU(N)$. We denote these curve classes by C_a with $a = 1, 2, \ldots, N$. The blow-up introduces N new curve classes, which we denote by M_a with $a = 1, 2, \ldots, N$. The class of the elliptic curve E is given by

$$E = C_1 + C_2 + \cdots + C_N. \tag{2.1}$$

The complexified Kähler parameters associated with these curves are given by the Kähler form ω:

$$t_a = \int_{C_a} \omega, \quad \rho = \int_E \omega, \quad m = \int_{M_a} \omega, \quad a = 1, 2, \ldots, N. \tag{2.2}$$

Because of Eq. (2.1) ρ is given by

$$\rho = t_1 + t_2 + \cdots + t_N. \tag{2.3}$$

The (p, q) 5-brane web dual to X_N and the Newton polygon are given by Fig. 1a, b, respectively.

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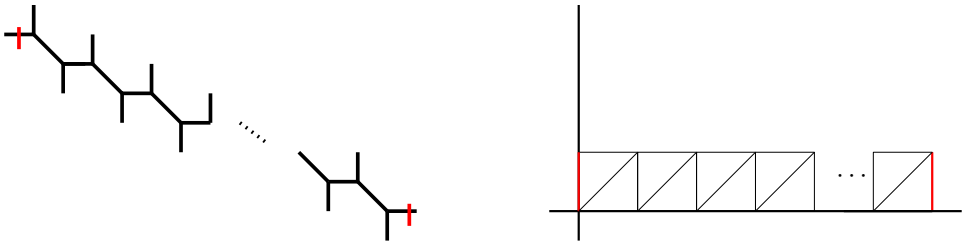

(a) **(b)**

Fig. 1 **a** The (p, q) brane configuration dual to X_N. The web lives on $\mathbb{R} \times S^1$ and the *horizontal lines* are glued together. **b** The Newton polygon of X_N

The Calabi–Yau threefolds X_N are dual to certain M5-brane configurations. Consider N M5-brane with worldvolume coordinates X^A, $A = 0, 1, 2, 3, 4, 5$. The transverse space is $\mathbb{R}^5$ with coordinates X^I, $I = 6, 7, 8, 9, 10$. We consider the X^6 direction and separate the N M5-branes along this direction and then compactify this direction to a circle so that M5-branes are sprinkled on the X^6 circle. The transverse space to the M5-branes is $\mathbb{R}^4_\perp \times S^1$ and one can introduce a deformation by fibering this transverse space over the X^6 circle so that as one goes around the X^6 circle there is an $SO(4)$ action on $\mathbb{R}^4_\perp$ given by [2],

$$(w_1, w_2) \mapsto (e^{2\pi i m} w_1, e^{-2\pi i m} w_2). \tag{2.4}$$

The deformation m breaks the maximal supersymmetry of the brane configuration. In the type IIB picture the theory living on the (p, q) brane web is five dimensional quiver gauge theory with eight supercharges. The partition function of this gauge theory can be calculated using Nekrasov's instanton calculus or by applying the refined topological vertex formalism to the (p, q) brane web of Fig. 1a.

3 The partition function from topological vertex

The partition function of the (p, q) brane web shown in Fig. 1 can be calculated by applying refined topological vertex formalism to the web. The refined vertex requires a choice of preferred direction [3] which determines the form of the partition function as a sum over Young diagrams but the partition function itself is invariant under the change in the preferred direction. In this case the simplest choice for the preferred direction is vertical and, as we will see below, in this case the sum can be carried out exactly and partition function becomes an infinite product. We denote the topological string partition function of this Calabi–Yau threefold by Z_N and it is given by [4]

$$Z_N := \sum_{\vec{\lambda}} \prod_{a=0}^{N-1} \left[(-Q_a)^{|\lambda_{(a)}|} \times \left(\sum_{\mu} (-Q_m)^{|\mu|} C_{\lambda^t_{(a)}\mu\emptyset}(t, q)\, C_{\lambda_{(a+1)}\mu^t\emptyset}(q, t) \right) \right], \tag{3.1}$$

where

$$C_{\lambda\mu\emptyset}(t, q) = \left(\frac{q}{t}\right)^{\frac{|\lambda|-|\mu|}{2}} \times \sum_{\eta} \left(\frac{q}{t}\right)^{\frac{|\eta|}{2}} s_{\lambda^t/\eta}(t^{-\rho})\, s_{\mu/\eta}(q^{-\rho}) \tag{3.2}$$

is the refined topological vertex. The length of the slanted lines in Fig. 1 are all equal to m and we have defined $Q_m = e^{-m}$, similarly the length of the horizontal lines is T_a and we have defined $Q_a = e^{-T_a}$ such that $t_a = T_a + m$ is the distance between the two vertical lines. λ, μ and η are Young diagrams and $s_{\lambda/\mu}(x_1, x_2, \ldots)$ is the skew-Schur polynomial in the variables $x_1, x_2, \ldots$. $t^{-\rho}$ denotes the following set of variables: $t^{-\rho} = \{t^{1/2}, t^{3/2}, \ldots\}$.

Using repeatedly the identity

$$\sum_{\lambda} s_{\lambda^t/\eta}(x) s_{\lambda/\sigma}(y) = \prod_{i,j}(1 + x_i y_j) \sum_{\tau} s_{\sigma^t/\tau}(x)\, s_{\eta^t/\tau^t}(y) \tag{3.3}$$

and the following properties of the skew-Schur functions:

$$s_{\lambda^t/\sigma^t}(q^{-\rho}) = s_{\lambda/\sigma}(-q^{\rho}), \quad \sum_{\eta} s_{\lambda/\eta}(x)\, s_{\eta/\sigma}(y) = s_{\lambda/\sigma}(x, y), \tag{3.4}$$

the partition function in Eq. (3.1) can be written as

$$Z_N = \Pi(Q_m)^N \sum_{\boldsymbol{\lambda}, \boldsymbol{\tau}} Q_\rho^{|\lambda_{(0)}|} \times \prod_{a=0}^{N-1} s_{\lambda_{(a)}/\tau_{(a+1)}}(\mathbf{x}_{a+1})\, s_{\lambda_{(a+1)}/\tau_{(a+1)}}(\mathbf{y}_{a+1}), \tag{3.5}$$

where

$$Q_\rho = \prod_{a=0}^{N-1} (Q_a Q_m), \quad Q_{1,a+1} = (Q_1 Q_2 \ldots Q_a) Q_m^a$$

$$\mathbf{x}_{a+1} = Q_m^{-1/2} Q_{1,a+1} \left\{ Q_m \sqrt{\frac{q}{t}} t^{\rho}, t^{-\rho} \right\},$$

$$\mathbf{y}_{a+1} = Q_m^{-1/2} Q_{1,a+1}^{-1} \left\{ Q_m \sqrt{\frac{t}{q}} q^{-\rho}, q^{\rho} \right\},$$

and

$$\Pi(x) = \prod_{i,j=1}^{\infty} \left(1 - x\, q^{-\rho_i}\, t^{-\rho_j}\right). \tag{3.6}$$

In the above equations $-log(Q_\rho)$ is the Kähler parameter associated to the elliptic curve which is dual to the circle transverse to the M5-branes. The form of the partition function given in Eq. (3.5) was obtained in [4] where it was related to the periodic Schur process of period N.

3.1 Product representation of partition function using free fermions

In this section we express the partition function given in Eq. (3.5) as an infinite product. To do this, following [5] we introduce the free fermion Fock space spanned by creation and annihilation operators (ψ_a, ψ_a^*) satisfying the relations

$$\{\psi_a, \psi_b\} = \{\psi_a^*, \psi_b^*\} = 0, \quad \{\psi_a, \psi_b^*\} = \delta_{ab},\ a, b \in \mathbb{Z} + \tfrac{1}{2}. \tag{3.7}$$

With these we can construct the operators

$$\Gamma_\pm(z) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n J_n}{n}\right), \quad \text{where}$$
$$J_n = \sum_{k \in \mathbb{Z}+\frac{1}{2}} \psi_{k+n}\psi_k^*, \quad n = \pm 1, \pm 2, \ldots. \tag{3.8}$$

These operators satisfy the following commutation relation:

$$\Gamma_+(z)\Gamma_-(w) = (1 - zw)\Gamma_-(w)\Gamma_+(z). \tag{3.9}$$

The fermionic Fock space is spanned by states which are in a one-to-one correspondence with partitions. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ we have the state

$$|\lambda\rangle = \psi_{\lambda_1 - \frac{1}{2}} \psi_{\lambda_2 - \frac{3}{2}} \psi_{\lambda_3 - \frac{5}{2}} \ldots |0\rangle. \tag{3.10}$$

The ground state $|0\rangle$ is annihilated by ψ_k for $k < 0$. The operators defined in Eq. (3.8) are useful when working with symmetric polynomials due to the following:

$$\prod_i \Gamma_+(x_i)|\lambda\rangle = \sum_\mu s_{\mu/\lambda}(x_1, x_2, \ldots)|\mu\rangle$$
$$\prod_i \Gamma_-(x_i)|\lambda\rangle = \sum_\mu s_{\lambda/\mu}(x_1, x_2, \ldots)|\mu\rangle. \tag{3.11}$$

To simplify the equations we will use the following notation: $\prod_i \Gamma_\pm(x_i) = \Gamma_\pm(\mathbf{x})$. Using Eqs. (3.5) and (3.11) we can write $\widehat{Z}_N := Z_N / \Pi(Q_m)^N$ as a trace, over the free fermionic Fock space, of an infinite product of operators:

$$\widehat{Z}_N = \mathrm{Tr}\Big((Q_1 Q_m)^{L_0} \mathcal{O} (Q_2 Q_m)^{L_0} \mathcal{O} \ldots (Q_N Q_m)^{L_0} \mathcal{O}\Big), \tag{3.12}$$

with the operator insertions $\mathcal{O}$ being built from $\Gamma_\pm$,

$$\mathcal{O} = \prod_{i=1}^{\infty} \left(\Gamma_+\left(Q_m^{-1}\, t^{-i+\frac{1}{2}}\right)\Gamma_+\left(\sqrt{\frac{q}{t}}\, t^{i-\frac{1}{2}}\right)\right.$$
$$\left. \times \prod_{j=1}^{\infty} \left(\Gamma_-\left(q^{j-\frac{1}{2}}\right)\Gamma_-\left(Q_m \sqrt{\frac{t}{q}}\, q^{-j+\frac{1}{2}}\right)\right). \tag{3.13}$$

In the above equations L_0 is such that $L_0|\lambda\rangle = |\lambda|\,|\lambda\rangle$. Using the commutation relations of $\Gamma_\pm(\mathbf{x})$, we can write Eq. (3.12) as

$$\widehat{Z}_N = \mathrm{Tr}\left(Q_\rho^{L_0} \prod_{a=1}^{k} \Gamma_+\left(Q_1 Q_2 \ldots Q_a Q_m^a \mathbf{x}\right)\right.$$
$$\left. \times\ \Gamma_-\left(Q_1^{-1} Q_2^{-1} \ldots Q_a^{-1} Q_m^{-a} \mathbf{y}\right)\right), \tag{3.14}$$

where

$$\mathbf{x} = \left\{Q_m^{-1} t^{-i+\frac{1}{2}}, \sqrt{\frac{q}{t}}\, t^{i-\frac{1}{2}} \,|\, i = 1, 2, \ldots\right\},$$
$$\mathbf{y} = \left\{q^{i-\frac{1}{2}}, Q_m \sqrt{\frac{t}{q}}\, q^{-i+\frac{1}{2}} \,|\, i = 1, 2, \ldots\right\}. \tag{3.15}$$

Using the commutation relation of $\Gamma_\pm(x)$ repeatedly we get

$$\widehat{Z}_N = \left(\prod_{1 \le a < b \le k} F_{ab}\right) \mathrm{Tr}\left(Q_\rho^{L_0} \Gamma_+(\mathbf{X})\Gamma_-(\mathbf{Y})\right), \tag{3.16}$$

where we have denoted

$$\mathbf{X} = \left\{Q_1 Q_m \mathbf{x}, Q_1 Q_2 Q_m^2 \mathbf{x}, \ldots, Q_1 Q_2 \ldots Q_N Q_m^N \mathbf{x}\right\}, \tag{3.17}$$
$$\mathbf{Y} = \left\{Q_1^{-1} Q_m^{-1} \mathbf{y}, Q_1^{-1} Q_2^{-1} Q_m^{-2} \mathbf{y}, \ldots, Q_1^{-1} Q_2^{-1} \ldots Q_N^{-1} Q_m^{-N} \mathbf{y}\right\}, \tag{3.18}$$
$$F_{ab} = \prod_{i,j=1}^{\infty} \frac{\left(1 - Q_{ab} Q_m^{-1} t^{i-\frac{1}{2}} q^{j-\frac{1}{2}}\right)\left(1 - Q_{ab} Q_m t^{i-\frac{1}{2}} q^{j-\frac{1}{2}}\right)}{(1 - Q_{ab} t^i q^{j-1})(1 - Q_{ab} t^{i-1} q^j)}, \tag{3.19}$$
$$Q_{ab} = Q_{a+1} \ldots Q_b Q_m^{b-a}. \tag{3.20}$$

The trace appearing in Eq. (3.16) can be written in the form of a product [6],

$$\mathrm{Tr}\left(Q_\rho^{L_0} \Gamma_+(\mathbf{x})\Gamma_-(\mathbf{y})\right) = \prod_{n=1}^{\infty} \left(1 - Q_\rho^n\right)^{-1} \prod_{i,j} \left(1 - Q_\rho^n x_i y_j\right)^{-1}. \tag{3.21}$$

such that the partition function is given by

$$\widehat{Z}_N = \left(\prod_{n=1}^{\infty}\left(1-Q_\rho^n\right)^{-1}\right)\left(\prod_{1\le a<b\le N} F_{ab}\right)\left(\prod_{a,b=1}^{N} H_{ab}\right), \tag{3.22}$$

where

$$H_{ab} = \prod_{n,i,j=1}^{\infty} \frac{\left(1-Q_\rho^n \widetilde{Q}_{ab} Q_m^{-1} t^{i-\frac{1}{2}} q^{j-\frac{1}{2}}\right)\left(1-Q_\rho^n \widetilde{Q}_{ab} Q_m\, t^{i-\frac{1}{2}} q^{j-\frac{1}{2}}\right)}{\left(1-Q_\rho^n \widetilde{Q}_{ab} t^i q^{j-1}\right)\left(1-Q_\rho^n \widetilde{Q}_{ab} t^{i-1} q^j\right)},$$
$$\widetilde{Q}_{ab} = Q_1 Q_2 \dots Q_a Q_1^{-1} \dots Q_b^{-1} Q_m^{a-b}. \tag{3.23}$$

4 Free energy and Gopakumar–Vafa invariants

Using the refined topological string partition function we can calculate the free energy and the Gopakumar–Vafa invariants [7,8] for the different curve classes. The free energy is given by

$$\begin{aligned} F_N(\rho, t_1, \dots, t_{N-1}, m) &= \ln Z_N \\ &= N \ln \Pi(Q_m) + \ln \widehat{Z}_N, \end{aligned} \tag{4.1}$$

where $\Pi(Q_m)$ is given by Eq. (3.6) and $\widehat{Z}_N$ is given by Eq. (3.22). After some simplification F_N can be written as

$$\begin{aligned} &F_N(\rho, t_1, \dots, t_{N-1}, m) \\ &= -N \sum_{k=1}^{\infty} \frac{Q_m^k}{k}\left[\frac{1}{\left(q^{\frac{k}{2}}-q^{-\frac{k}{2}}\right)\left(t^{\frac{k}{2}}-t^{-\frac{k}{2}}\right)}\right] + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{Q_\rho^{n\,k}}{k} \\ &+ \sum_{1\le a<b\le N}\sum_{k=1}^{\infty}\frac{Q_{ab}^k}{k}\left[\frac{(t/q)^{\frac{k}{2}}+(q/t)^{\frac{k}{2}}-\left(Q_m^k+Q_m^{-k}\right)}{\left(q^{\frac{k}{2}}-q^{-\frac{k}{2}}\right)\left(t^{\frac{k}{2}}-t^{-\frac{k}{2}}\right)}\right] \\ &+ \sum_{n=1}^{\infty}\sum_{a,b=1}^{N}\sum_{k=1}^{\infty}\frac{Q_\rho^{n\,k}\widetilde{Q}_{ab}^k}{k}\left[\frac{(t/q)^{\frac{k}{2}}+(q/t)^{\frac{k}{2}}-\left(Q_m^k+Q_m^{-k}\right)}{\left(q^{\frac{k}{2}}-q^{-\frac{k}{2}}\right)\left(t^{\frac{k}{2}}-t^{-\frac{k}{2}}\right)}\right]. \end{aligned} \tag{4.2}$$

From Eq. (4.2) we can see that the $SU(2)_L \times SU(2)_R$ spin content of various curve classes is given by

Curve	$\sum_{j_L,j_R} N_C^{j_L,j_R}(j_L, j_R)$
$n\,E + C_{ab}$, $n \ge 0$, $1 \le a < b \le N$, $c = a-1, b+1$	$(0, \frac{1}{2})$
$n\,E + C_{ab} + M_c$, $n \ge 0$, $1 \le a < b \le N$, $c = a-1, b+1$	$(0, 0)$
$n\,E + C_{ab} - M_c$, $n \ge 0$, $1 \le a < b \le N$, $c = a+1, b-1$	$(0, 0)$
$n\,E - C_{ab}$, $n \ge 1$, $1 \le a < b \le N$	$(0, \frac{1}{2})$
$n\,E - C_{ab} + M_c$, $n \ge 1$, $1 \le a < b \le N$, $c = a-1, b+1$	$(0, 0)$
$n\,E - C_{ab} - M_c$, $n \ge 1$, $1 \le a < b \le N$, $c = a+1, b-1$	$(0, 0)$
$n\,E$, $n \ge 1$	$(\frac{1}{2}, 0) \oplus (N-1)(0, \frac{1}{2})$

Here

$$C_{ab} = C_a + C_{a+1} + \dots + C_b.$$

From Eq. (4.2) we can isolate the contribution of the elliptic curve E to F_N, which we will denote by F_N^E, by considering only those terms which depend only on Q_ρ and the other Kähler parameter,

$$F_N^E(\rho, q, t) = \sum_{n,k=1}^{\infty}\frac{Q_\rho^{n\,k}}{k}\left[1 + N\left(\frac{t^k+q^k}{(1-q^k)(1-t^k)}\right)\right]. \tag{4.3}$$

We will restrict ourselves to the unrefined case so that $q = t$ and in terms of the topological string coupling constant g_s,

$$q = t = e^{i\,g_s}. \tag{4.4}$$

Since

$$\frac{2q}{(1-q)^2} = \frac{2}{g_s^2} - \frac{1}{6} - 2\sum_{g\ge 2} g_s^{2g-2}\frac{B_{2g}}{(2g)(2g-2)!}, \tag{4.5}$$

where B_k are the Bernoulli numbers, we have

$$F_N^E(\rho, q) = \sum_{g=0}^{\infty} g_s^{2g-2} F_{N,g}^E(\rho) \tag{4.6}$$

with

$$\begin{aligned} F_{N,g\ge 2}^E &= N\frac{B_{2g}B_{2g-2}}{2(2g)\,(2g-2)!} \\ &\quad - 2\,N\frac{B_{2g}}{(2g)\,(2g-2)!}\,\frac{B_{2g-2}}{4}\,E_{2g-2}(\rho), \\ F_1^E &= -\frac{1}{12}\ln\left[\prod_{n=1}^{\infty}\left(1-Q_\rho^n\right)^{12-2N}\right], \\ \frac{\partial^2 F_0^E}{\partial\rho^2} &= 2N\sum_{n=1}^{\infty} n^2 \ln\left(1-Q_\rho^n\right)^{-1} \end{aligned} \tag{4.7}$$

where $E_{2g}(\rho)$ is the Eisenstein series defined by

$$E_{2g}(\rho) = 1 - \frac{4}{B_{2g}}\sum_{n\ge 1} n^{2g-1}\left(\frac{Q_\rho^n}{1-Q_\rho^n}\right). \tag{4.8}$$

5 Non-perturbative modular transformation

For an elliptic Calabi–Yau threefold with modular parameter ρ one naively would expect the partition function to be invariant under the modular transformation,

$$Z_N\left(-\frac{1}{\tau}, \frac{\epsilon_1}{\tau}, \frac{\epsilon_2}{\tau}\right) = Z_N\left(\tau, \epsilon_1, \epsilon_2\right) \tag{5.1}$$

where $q = e^{i\epsilon_1}$ and $t = e^{-i\epsilon_2}$. However, quite surprisingly, in [1] it was shown that the refined topological partition function

of X_1, which is dual to a single M5-brane wrapped on a circle, satisfies a non-perturbative modular transformation,

$$Z_1\left(-\frac{1}{\tau},\frac{\epsilon_1}{\tau},\frac{\epsilon_2}{\tau}\right)=\frac{Z_1(\tau,\epsilon_1,\epsilon_2)}{Z_1\left(\frac{\tau}{\epsilon_1},-\frac{1}{\epsilon_1},\frac{\epsilon_2}{\epsilon_1}\right)Z_1\left(\frac{\tau}{\epsilon_2},\frac{\epsilon_1}{\epsilon_2},-\frac{1}{\epsilon_2}\right)}. \tag{5.2}$$

Since ϵ_1,ϵ_2 are related to the topological string coupling constant g_s by

$$\epsilon_1\epsilon_2=-g_s^2, \tag{5.3}$$

Equation (5.2) implies that

$$Z_1\left(-\frac{1}{\tau},\frac{\epsilon_1}{\tau},\frac{\epsilon_2}{\tau}\right)=Z_1(\tau,\epsilon_1,\epsilon_2)+O\left(e^{-\frac{1}{gs}}\right), \tag{5.4}$$

i.e., modularity holds only up to non-perturbative corrections. For this reason the transformation in Eq. (5.2) was called a non-perturbative modular transformation and it was argued that this gives the non-perturbative completion of the topological string partition function for the case of elliptic Calabi–Yau threefolds.

To show that Z_N also satisfies an equation similar to Eq. (5.2) we write it in terms of double elliptic gamma functions. Recall that the double elliptic Gamma function is defined by

$$G_2(x;\tau,\epsilon_1,\epsilon_2)=\prod_{k,i,j=1}^{\infty}\left(1-Q_\tau^{k-1}q^{i-1}t^{-j+1}x\right)\times\left(1-Q_\tau^{k}q^{i}t^{-j}x^{-1}\right) \tag{5.5}$$

and satisfies the following modular transformation:

$$\begin{aligned}&G_2(z;\rho,\epsilon_1,\epsilon_2)\\&=G_2\left(\frac{z}{\rho};-\frac{1}{\rho},\frac{\epsilon_1}{\rho},\frac{\epsilon_2}{\rho}\right)G_2\left(\frac{z}{\epsilon_1};\frac{\rho}{\epsilon_1},-\frac{1}{\epsilon_1},\frac{\epsilon_2}{\epsilon_1}\right)\\&\quad\times G_2\left(\frac{z}{\epsilon_2};\frac{\rho}{\epsilon_2},\frac{\epsilon_1}{\epsilon_2},-\frac{1}{\epsilon_2}\right)\exp\left(\frac{i\pi}{12}B_{44}\right),\end{aligned} \tag{5.6}$$

where $B_{4,4}$ is given by

$$B_{4,4}(z;\rho,\epsilon_1,\epsilon_2)=\frac{\mathrm{d}^4}{\mathrm{d}x^4}\frac{x^4\,e^{z\,x}}{(e^{\rho\,x}-1)(e^{\epsilon_1\,x}-1)(e^{\epsilon_2\,x}-1)}|_{x=0}. \tag{5.7}$$

The triple infinite product in Eq. (3.23) can be written in terms of double elliptic gamma functions so that the full partition function Z_N is given by

$$\begin{aligned}Z_N&=Z_1^N\left(\prod_{n=1}^{\infty}(1-Q_\rho^n)\right)^{N-1}\\&\quad\times\prod_{1\le a<b\le N}\frac{G_2\left(Q_{ab}Q_m\sqrt{t\,q};\rho,\epsilon_1,-\epsilon_2\right)G_2\left(Q_{ab}\,Q_m^{-1}\sqrt{t\,q};\rho,\epsilon_1,-\epsilon_2\right)}{G_2\left(Q_{ab}\,t;\rho,\epsilon_1,-\epsilon_2\right)G_2\left(Q_{ab}\,q;\rho,\epsilon_1,-\epsilon_2\right)}\\&=Z_1^N\left(\prod_{n=1}^{\infty}(1-Q_\rho^n)\right)^{N-1}\\&\quad\times\prod_{1\le a<b\le N}\frac{G_2\left(Q_{ab};\rho,\epsilon_1,\epsilon_2\right)G_2\left(Q_\rho Q_{ab}^{-1};\tau,\epsilon_1,\epsilon_2\right)}{G_2\left(Q_{ab}Q_m\sqrt{t\,q};\rho,\epsilon_1,\epsilon_2\right)G_2\left(Q_{ab}\,Q_m^{-1}\sqrt{t\,q};\rho,\epsilon_1,\epsilon_2\right)}\end{aligned} \tag{5.8}$$

with the explicit expression

$$\begin{aligned}Z_1&=\left(\prod_{n=1}^{\infty}(1-Q_\rho^n)^{-1}\right)\\&\quad\times\prod_{i,j=1}^{\infty}\frac{\left(1-Q_\rho^{n-1}\,Q_m\,q^{i-\frac{1}{2}}\,t^{j-\frac{1}{2}}\right)\left(1-Q_\rho^n\,Q_m^{-1}\,q^{i-\frac{1}{2}}t^{j-\frac{1}{2}}\right)}{\left(1-Q_\rho^n q^i t^{j-1}\right)\left(1-Q_\rho^n q^{i-1}t^j\right)}.\end{aligned} \tag{5.9}$$

Using Eq. (5.8) and the modular transformation satisfied by $G_2(z;\rho,\epsilon_1,\epsilon_2)$ we see that

$$\frac{Z_N(\tau,\epsilon_1,\epsilon_2)}{Z_N\left(\frac{\tau}{\epsilon_1},-\frac{1}{\epsilon_1},\frac{\epsilon_2}{\epsilon_1}\right)Z_N\left(\frac{\tau}{\epsilon_2},\frac{\epsilon_1}{\epsilon_2},-\frac{1}{\epsilon_2}\right)}=Z_N\left(-\frac{1}{\tau},\frac{\epsilon_1}{\tau},\frac{\epsilon_2}{\tau}\right). \tag{5.10}$$

Thus this class of elliptic Calabi–Yau threefolds do indeed satisfy the non-perturbative modular transformation of Lockhart and Vafa [1].

6 Discussion

In this paper we have shown that there is a simple class of elliptic Calabi–Yau threefolds for which the product representation of the refined topological string partition function can be determined using the relation between Schur functions and free fermionic Fock space. The product representation allows us to study the non-perturbative modular transformations of [1]. The class of elliptic Calabi–Yau threefolds discussed in this paper were simplest in the sense that they had no compact 4-cycles. It would be interesting to see if the non-perturbative modular transformation holds for an elliptic Calabi–Yau threefold with compact 4-cycles. We hope to report on this in the future [9].

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