Math Sci (2014) 8:65–69
DOI 10.1007/s40096-014-0127-4

ORIGINAL RESEARCH

A Pata-type fixed point theorem

Sriram Balasubramanian

Received: 28 May 2014 / Accepted: 22 July 2014 / Published online: 8 November 2014

Abstract We prove a fixed point theorem for a Pata-type map defined on a complete (normal) cone metric space. Our results generalize the recent work of M. Chakraborty and S. K. Samanta. An example demonstrating this fact is also presented.

Keywords Fixed point · Kannan · Pata · Contraction · Cone metric spaces

Mathematics Subject Classification 47H10 · 54H25 (Primary) · 37C25 (Secondary)

Introduction

The classical Banach fixed point theorem states that if (X, d) is a complete metric space and $T : X \to X$ is a contraction map, i.e., T satisfies

$$d(Tx, Ty) \leq \alpha d(x, y), \tag{1}$$

for all $x, y \in X$ and some $\alpha \in [0, 1)$, then T has a unique fixed point. i.e., there exists a unique $a \in X$ such that $Ta = a$.

In [4], Pata considered a map $T : X \to X$ on the complete metric space (X, d) that satisfied the condition: for all $x, y \in X$,

$$d(Tx, Ty) \leq (1 - \epsilon)d(x, y) + \Lambda\epsilon^{\alpha}\psi(\epsilon)[1 + \|x\| + \|y\|]^{\beta}, \tag{2}$$

for every $\epsilon \in [0, 1]$, fixed constants $\Lambda \geq 0$, $\alpha \geq 1$ and $\beta \in [0, \alpha]$, a fixed element $x_0 \in X$, $\|z\| = d(z, x_0)$ and an increasing function $\psi : [0, 1] \to [0, \infty)$ which vanishes at and is continuous at 0. He proved that the map T satisfying (2) has a unique fixed point. Moreover, he also demonstrated that if $T : X \to X$ is a contraction map, then T satisfies condition (2), thereby obtaining a generalization of the Banach fixed point theorem.

Another fixed point theorem that is widely popular, is due to Kannan which states that if (X, d) is a complete metric space and the map $T : X \to X$ is a Kannan contraction, i.e., T satisfies

$$d(Tx, Ty) \leq \frac{\gamma}{2}\{d(x, Tx) + d(y, Ty)\} \tag{3}$$

for all $x, y \in X$ and some $\gamma \in [0, 1)$, then T has a unique fixed point.

There are several generalizations of the Kannan fixed point theorem, but the one of particular interest to us is due to Chakraborty and Samanta in [1]. The authors, in [1], consider a map $T : X \to X$ defined on the complete metric space (X, d) that satisfies the condition: for all $x, y \in X$,

$$\begin{aligned} d(Tx, Ty) \leq \frac{1 - \epsilon}{2}\{d(x, Tx) + d(y, Ty)\} + \\ \Lambda\epsilon^{\alpha}\psi(\epsilon)[1 + \|x\| + \|y\| + \|Tx\| + \|Ty\|]^{\beta}, \end{aligned} \tag{4}$$

for every $\epsilon \in [0, 1]$, fixed constants $\Lambda \geq 0$, $\alpha \geq 1$ and $\beta \in [0, \infty)$, a fixed element $x_0 \in X$, $\|z\| = d(z, x_0)$ and a function $\psi : [0, 1] \to [0, \infty)$ which vanishes at and is continuous at 0. The authors prove that the map T has a unique fixed point. Moreover, they also demonstrate that if $T : X \to X$ is a Kannan contraction map, then T satisfies condition (4).

This article contains a generalization of the main result in [1]. The setting considered is that of complete cone

S. Balasubramanian (✉)
Department of Mathematics and Statistics, Indian Institute of Science Education and Research, Kolkata, India
e-mail: bsriram@iiserkol.ac.in

metric spaces, where the underlying cone is normal (see [3]). It is shown that for a map $T : X \to X$ defined on the complete (normal) cone metric space (X, d), which satisfies a Pata-type condition, that is an improved version of (4), there exists a unique fixed point. An example to illustrate the main result is also provided.

Preliminaries and the main result

Let E be a real Banach space. A non-empty closed subset P of E is said to be a *cone* if

(a) $\alpha P + \beta P \subset P$ for all $\alpha, \beta \in [0, \infty)$.
(b) $P \cap (-P) = \{\theta\}$, where $\theta \in E$ is the zero vector.

The cone P is said to be *solid* if the interior of P, which we will denote by $int\,P$, is non-empty.

Examples of solid cones

(1) Let $E = \mathbb{R}$ and $P = [0, \infty)$.
(2) Let $E = \mathbb{R}^2$ and $P = \{(x, y) : x, y \geq 0\}$.
(3) Let $E = \ell^2$ and $P = \{(x_n)_{n \geq 1} : x_n \geq 0\}$.

The norms on E in the examples above are the usual norms.

A cone P in a real Banach space E, induces the following partial order $\preceq$ on E. For $x, y \in E$,

$$x \preceq y \Leftrightarrow y - x \in P.$$

In the case of a solid cone P, we will use the notation $x \ll y$ to denote $y - x \in int\,P$.

A cone P is said to be *normal* if for all $x, y \in P$ such that $x \preceq y$, there exists a constant $\kappa \geq 1$ such that $\|x\| \leq \kappa\|y\|$. The examples (1), (2) and (3) above are normal cones with $\kappa = 1$.

Let X be a nonempty set, E be a real Banach space and $P \subset E$ be a solid normal cone with normal constant $\kappa \geq 1$. A map $d : X \times X \to E$ is said to be a *cone metric* if for all $x, y, z \in E$,

(a) $d(x, y) \succeq \theta$, i.e., $d(x, y) \in P$.
(b) $d(x, y) = \theta$ if and only if $y = x$.
(c) $d(x, y) \preceq d(x, z) + d(z, y)$.

The pair (X, d) is called a *cone metric space*. It is indeed the case that every metric space is a cone metric space.

Examples of cone metric spaces

(1) ([3]) Let E and P be as in example (2) above, $d : \mathbb{R} \times \mathbb{R} \to E$ be the map $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. The pair $(\mathbb{R}, d)$ is a cone metric space.
(2) ([2]) Let (X, ρ) be a metric space, E and P be as in example (3) above, $d : X \times X \to E$ be the map $d(x, y) = \left(\sqrt{2^{-n}\rho(x, y)}\right)_{n \geq 1}$. It can be verified that (X, d) is a cone metric space.

Let (X, d) be a cone metric space. A sequence (x_n) of points in X is said to be *Cauchy* if for any given $c \gg 0$, there exists $N \in \mathbb{N}$ such that $d(x_m, x_n) \ll c$, for all $m, n \geq N$, or equivalently, there exists $M \in \mathbb{N}$ which is independent of p such that $d(x_n, x_{n+p}) \ll c$ for all $n \geq M$.

The sequence (x_n) is said to be *convergent*, if there exists $x \in X$ such that for any given $c \gg 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n \geq N$.

The cone metric space (X, d) is said to be *complete* if every Cauchy sequence in X converges.

The proof of the following lemma can be found in [3].

Lemma 1 Let (X, d) be a cone metric space and (x_n) be a sequence in X.

(1) (x_n) is Cauchy if and only if $\|d(x_m, x_n)\| \to 0$ as $m, n \to \infty$. i.e., Given $\eta > 0$ and $p \in \mathbb{N}$, there exists $N \in \mathbb{N}$ which is independent of p such that $\|d(x_n, x_{n+p})\| < \eta$, for all $n \geq N$.
(2) (x_n) is convergent to x if and only if $\|d(x_n, x)\| \to 0$ as $n \to \infty$.
(3) If (x_n) converges to $x, y \in X$, then $x = y$.

The following is our main result which generalizes Theorem 2.2 in [1].

Theorem 1 Let (X, d) be a complete cone metric space with normal constant $\kappa \geq 1$, $x_0 \in X$, $\Lambda \geq 0$, $\alpha \geq 1$ and $\beta \in [0, \infty)$ be fixed constants and $\psi : [0, 1] \to [0, \infty)$ be such that $\lim_{\epsilon \to 0^+} \psi(\epsilon) = 0$. If for every $x, y \in X$, the map $T : X \to X$ satisfies

$$\|d(Tx, Ty)\| \leq \frac{(1 - \epsilon)}{\kappa} M(x, y) + \Lambda\epsilon^{\alpha}\psi(\epsilon)[1 + \|x\| + \|y\| + \|Tx\| + \|Ty\|]^{\beta}, \tag{5}$$

for every $\epsilon \in [0, 1]$, where $M(x, y) = \max\{\|d(x, Tx)\|, \|d(y, Ty)\|, \frac{1}{2\kappa}\|d(x, y)\|\}$ and $\|z\|$ denotes $\|d(z, x_0)\|$, then T has a unique fixed point.

The proof of the above Theorem is given in Sect. 3. We point out that there is no loss of generality in choosing any such x_0 in (5), simply because a change in x_0 can essentially be absorbed by assigning a different value to Λ, thanks to the triangle inequality of the cone metric d and the sub-additivity of the norm.

Proof of The main result

This section contains a proof of our main result. Although the proof follows a similar pattern as that of Theorem 2.2 in [1], the arguments provided here are different and sometimes simpler.

Proof of Theorem 1 First we prove uniqueness of the fixed point. Suppose that $x, y \in X$ are such that $x \neq y$, $Tx = x$ and $Ty = y$. Letting $\epsilon = 0$ in inequality (5) yields, $\|d(x,y)\| \leq \frac{1}{2\kappa}\|d(x,y)\|$, a contradiction to the fact that $x \neq y$. The uniqueness follows.

To show the existence of a fixed point, consider the sequence $(T^n x_0)$. Without loss of generality, we assume that $T^n x_0 \neq T^{n+1}x_0$, for all $n = 0, 1, 2, \ldots$. Since for each $n \in \mathbb{N}$, and $\epsilon \in [0, 1]$,

$$\begin{aligned}\|d(T^{n+1}x_0, T^n x_0)\| &\leq (1-\epsilon)M(T^n x_0, T^{n-1}x_0) \\ &\quad + \Lambda\epsilon^\alpha\psi(\epsilon)[1 + 2\|T^n x_0\| + \|T^{n-1}x_0\| \\ &\quad + \|T^{n+1}x_0\|]^\beta, \qquad (6)\end{aligned}$$

it follows that there exists no $n \in \mathbb{N}$, for which $M(T^n x_0, T^{n-1}x_0) = \|d(T^{n+1}x_0, T^n x_0)\|$. For otherwise, it would mean that there exists some $m \in \mathbb{N}$ such that for every $\epsilon \in (0, 1]$,

$$\begin{aligned}&\|d(T^{m+1}x_0, T^m x_0)\| \leq \Lambda\epsilon^{\alpha-1}\psi(\epsilon)[1 + 2\|T^m x_0\| \\ &\quad + \|T^{m-1}x_0\| + \|T^{m+1}x_0\|]^\beta.\end{aligned}$$

Letting $\epsilon \to 0^+$ yields, $\|d(T^{m+1}x_0, T^m x_0)\| = 0$, i.e., $T^{m+1}x_0 = T^m x_0$, a contradiction. Thus, for each $n \in \mathbb{N}$, letting $\epsilon = 0$ in inequality (6) yields,

$$\|d(T^{n+1}x_0, T^n x_0)\| \leq \|d(T^n x_0, T^{n-1}x_0)\|. \qquad (7)$$

Iterating we obtain

$$\|d(T^{n+1}x_0, T^n x_0)\| \leq \|d(Tx_0, x_0)\|, \qquad (8)$$

for all $n = 0, 1, 2, \ldots$

Next, we show that the sequence $(\|d(T^n x_0, x_0)\|)$ is bounded above by $c = 2\kappa\|d(x_0, Tx_0)\|$. This is certainly the case when $n = 1$. Assume that $\|d(T^{m-1}x_0, x_0)\| \leq c$. The claim follows from induction, if we show that $\|d(T^m x_0, x_0)\| \leq c$. Using (8), it follows that

$$\begin{aligned}\|d(T^m x_0, x_0)\| &\leq \kappa\{\|d(T^m x_0, Tx_0)\| + \|d(Tx_0, x_0)\|\} \\ &\leq \kappa M(T^{m-1}x_0, x_0) + \frac{c}{2} \\ &= \kappa \max\Big\{\|d(T^{m-1}x_0, T^m x_0)\|, \|d(x_0, Tx_0)\|, \\ &\quad \times \frac{1}{2\kappa}\|d(T^{m-1}x_0, x_0)\|\Big\} + \frac{c}{2} \\ &\leq \kappa\max\left\{\|d(x_0, Tx_0)\|, \frac{1}{2\kappa}\|d(T^{m-1}x_0, x_0)\|\right\} \\ &\quad + \frac{c}{2} \leq \kappa\left\{\frac{c}{2\kappa}\right\} + \frac{c}{2} = c.\end{aligned}$$

Thus, for all $n \in \mathbb{N}$,

$$\|d(T^n x_0, x_0)\| \leq c. \qquad (9)$$

Consider the monotonically decreasing sequence $(\|d(T^n x_0, T^{n+1}x_0)\|)$ [see (7)]. Since it is bounded below by 0, it is convergent. Let $\ell = \lim_{n\to\infty}\|d(T^n x_0, T^{n+1}x_0)\|$. Clearly $\ell \geq 0$. We will in fact, prove that $\ell = 0$. For each $\epsilon \in (0, 1]$, it follows from (6) and (9) that

$$\|d(T^n x_0, T^{n+1}x_0)\| \leq (1-\epsilon)\|d(T^{n-1}x_0, T^n x_0)\| + K\Lambda\epsilon^\alpha\psi(\epsilon),$$

where $K = (1+4c)^\beta$. Rearranging the above inequality and using the fact that the sequence $(\|d(T^n x_0, T^{n+1}x_0)\|)$ is monotonically decreasing yields,

$$\|d(T^n x_0, T^{n+1}x_0)\| \leq \frac{\|d(T^{n-1}x_0, T^n x_0)\|}{(1+\epsilon)} + \frac{K\Lambda\epsilon^\alpha\psi(\epsilon)}{(1+\epsilon)}, \qquad (10)$$

for each $\epsilon \in (0, 1]$. Fixing such an ϵ and letting $n \to \infty$ in (10), we obtain

$$\ell \leq \frac{\ell}{(1+\epsilon)} + \frac{K\Lambda\epsilon^\alpha\psi(\epsilon)}{(1+\epsilon)}$$

Multiplying both sides by $(1+\epsilon)$ and simplifying yields, for each $\epsilon \in (0, 1]$,

$$\ell \leq K\Lambda\epsilon^{\alpha-1}\psi(\epsilon).$$

Letting $\epsilon \to 0^+$ yields, $\ell \leq 0$. Thus, in fact,

$$\ell = 0. \qquad (11)$$

Next, we show that the sequence $(T^n x_0)$ is Cauchy. In view of Lemma 1 (i), it suffices to show that, given $\eta > 0$ and $p \in \mathbb{N}$, there exists $M \in \mathbb{N}$ which is independent of p, such that $\|d(T^n x_0, T^{n+p}x_0)\| < \eta$, for all $n \geq M$.

By (11), choose $N \in \mathbb{N}$ such that

$$\|d(T^n x_0, T^{n+1}x_0)\| < \frac{\eta}{2} \qquad (12)$$

for all $n \geq N$.

Consider $\|d(T^n x_0, T^{n+p}x_0)\|$ for all $n \geq N+1$. Letting $\epsilon = 0$ in (5) yields,

$$\|d(T^n x_0, T^{n+p}x_0)\| \leq M(T^{n-1}x_0, T^{n+p-1}x_0).$$

If $M(T^{n-1}x_0, T^{n+p-1}x_0) = \|d(T^{n-1}x_0, T^n x_0)\|$ or $\|d(T^{n+p-1}x_0, T^{n+p}x_0)\|$, then it follows from (12) that

$$\|d(T^n x_0, T^{n+p}x_0)\| < \frac{\eta}{2}. \qquad (13)$$

If $M(T^{n-1}x_0, T^{n+p-1}x_0) = \frac{1}{2\kappa}\|d(T^{n-1}x_0, T^{n+p-1}x_0)\|$, then the normality of the underlying cone and an application of the triangle inequality yield,

$$\begin{aligned}\|d(T^n x_0, T^{n+p}x_0)\| &\leq \frac{1}{2\kappa}\|d(T^{n-1}x_0, T^{n+p-1}x_0)\| \\ &\leq \frac{1}{2}\{\|d(T^{n-1}x_0, T^n x_0)\| + \|d(T^n x_0, T^{n+p}x_0)\| \\ &\quad + \|d(T^{n+p}x_0, T^{n+p-1}x_0)\|\}.\end{aligned}$$

From a rearrangement of terms, it follows from (12) that

$$\|d(T^n x_0, T^{n+p}x_0)\| \leq \|d(T^{n-1}x_0, T^n x_0)\| + \|d(T^{n+p}x_0, T^{n+p-1}x_0)\| < \eta. \quad (14)$$

Thus, it follows from (13) and (14) that given $\eta > 0$ and $p \in \mathbb{N}$, by choosing $M = N + 1$, which is independent of p, one has that

$$\|d(T^n x_0, T^{n+p}x_0)\| < \eta,$$

for all $n \geq M$. i.e., the sequence $(T^n x_0)$ is Cauchy in the cone metric space (X, d). By completeness, there exists a unique $u \in X$ such that $T^n x_0 \to u$ as $n \to \infty$.

We complete the proof by showing that this u is a fixed point of T. Let $\eta > 0$ be arbitrary, $B = \Lambda\kappa(1 + \|u\| + \|Tu\| + 2c)^{\beta}$. Choose $\epsilon_0 \in (0, 1]$ such that

$$B\epsilon_0^{\alpha-1}\psi(\epsilon_0) < \frac{\eta}{2}. \quad (15)$$

Choose $L \in \mathbb{N}$ such that

$$\|d(T^L x_0, u)\| + \|d(T^L x_0, T^{L+1}x_0)\| < \frac{\epsilon_0\eta}{4\kappa}. \quad (16)$$

Consider $\|d(Tu, u)\|$. From (5), it follows that

$$\begin{aligned}\|d(Tu, u)\| &\leq \kappa\{\|d(Tu, T^{L+1}x_0)\| + \|d(T^{L+1}x_0, T^L x_0)\| + \|d(T^L x_0, u)\|\}\\ &\leq (1 - \epsilon_0)M(u, T^L x_0) + \kappa\Lambda\epsilon_0^{\alpha}\psi(\epsilon_0)(1 + \|u\| + \|T^L x_0\| + \|Tu\| + \|T^{L+1}x_0\|)^{\beta} + \kappa\{\|d(T^{L+1}x_0, T^L x_0)\| + \|d(T^L x_0, u)\|\}\\ &\leq (1 - \epsilon_0)\{\|d(u, Tu)\| + \|d(T^{L+1}x_0, T^L x_0)\| + \|d(u, T^L x_0)\|\} + B\epsilon_0^{\alpha}\psi(\epsilon_0) + \kappa\{\|d(T^{L+1}x_0, T^L x_0)\| + \|d(T^L x_0, u)\|\}\\ &\leq (1 - \epsilon_0)\|d(u, Tu)\| + 2\kappa\{\|d(T^{L+1}x_0, T^L x_0)\| + \|d(u, T^L x_0)\|\} + B\epsilon_0^{\alpha}\psi(\epsilon_0).\end{aligned}$$

Rearranging the terms in the above inequality yields,

$$\|d(u, Tu)\| \leq \frac{2\kappa}{\epsilon_0}\{\|d(T^{L+1}x_0, T^L x_0)\| + \|d(u, T^L x_0)\|\} + B\epsilon_0^{\alpha-1}\psi(\epsilon_0).$$

It follows from (15) and (16) that

$$\|d(u, Tu)\| < \eta.$$

Since $\eta > 0$ is arbitrary, the proof is complete. □

Applications and examples

This section contains applications of Theorem 1, presented as corollaries. Recall the definition of $M(x, y)$ and the notation $\|\cdot\|$ from Theorem 1.

Corollary 1 Let (X, d) be a complete cone metric space with normal constant $\kappa \geq 1$ and $\delta \in \left(0, \frac{1}{\kappa}\right)$. If the map $T : X \to X$ satisfies

$$\|d(Tx, Ty)\| \leq \delta M(x, y)$$

for all $x, y \in X$, then T has a unique fixed point.

Proof It suffices to show that T satisfies condition (5). The result then follows from Theorem 1. Fix $x_0 \in X$. Observe that

$$\begin{aligned}M(x, y) &= \max\left\{\|d(x, Tx)\|, \|d(y, Ty)\|, \frac{1}{2\kappa}\|d(x, y)\|\right\}\\ &\leq 1 + \|d(x, x_0)\| + \|d(x_0, Tx)\| + \|d(y, x_0)\| + \|d(x_0, Ty)\|\\ &= 1 + \|x\| + \|Tx\| + \|y\| + \|Ty\|.\end{aligned} \quad (17)$$

It follows from (17) and a Bernoulli inequality argument similar to Sect. 3 in [1], that for all $\epsilon \in [0, 1]$, $x, y \in X$,

$$\begin{aligned}\|d(Tx, Ty)\| &\leq \delta M(x, y)\\ &= \frac{(1-\epsilon)}{\kappa}M(x, y) + \left(\delta + \frac{(\epsilon - 1)}{\kappa}\right)M(x, y)\\ &= \frac{(1-\epsilon)}{\kappa}M(x, y) + \delta\left(1 + \frac{(\epsilon - 1)}{\kappa\delta}\right)M(x, y)\\ &\leq \frac{(1-\epsilon)}{\kappa}M(x, y) + \delta(1 + (\epsilon - 1))^{\frac{1}{\kappa\delta}}M(x, y)\\ &\leq \frac{(1-\epsilon)}{\kappa}M(x, y) + \delta\epsilon^{\frac{1}{\kappa\delta}}[1 + \|x\| + \|Tx\| + \|y\| + \|Ty\|].\\ &= \frac{(1-\epsilon)}{\kappa}M(x, y) + \delta\epsilon^{1+\eta}[1 + \|x\| + \|Tx\| + \|y\| + \|Ty\|],\end{aligned} \quad (18)$$

where $\eta = \left(\frac{1}{\kappa\delta} - 1\right) > 0$. Comparing (18) with (5), one sees that T satisfies (5) with $\Lambda = \delta$, $\beta = \alpha = 1$, $\psi(\epsilon) = \epsilon^{\eta}$. □

Corollary 2 (The Kannan Fixed Point Theorem) Let (X, d) be a complete metric space and $\delta \in (0, 1)$. If the map $T : X \to X$ satisfies

$$d(Tx, Ty) \leq \frac{\delta}{2}(d(x, Tx) + d(y, Ty))$$

for all $x, y \in X$, then T has a unique fixed point.

Proof Observe that $\kappa = 1$ and T satisfies the hypothesis of Corollary 1. The result follows. □

We end with an example which demonstrates that Theorem 1 indeed generalizes the main result in [1]. Observe that it suffices to produce a complete metric space (X, d) and a map $T : X \to X$ which satisfies

(A) $d(Tx, Ty) \le \delta \max\{d(x, Tx), d(y, Ty), \frac{1}{2}d(x, y)\}$ for some $\delta \in (0, 1)$ and $x, y \in X$,

(B) $d(Ta, Tb) > \frac{1}{2}\{d(a, Ta) + d(b, Tb)\}$ for some $a, b \in X$.

Example Let $X = \left[0, \frac{1}{2}\right] \cup \{1, 2\}$ with d being the usual metric. It is clear that (X, d) is a complete cone metric space with $\kappa = 1$. Define the map $T : X \to X$ by $Tx = 2$ if $x \ne 1, 2$ and $T1 = T2 = 1$. Observe that for $x \ne 1, 2$, $d(Tx, T1) = 1$ and

$$\max\left\{d(x, Tx), d(1, T1), \frac{1}{2}d(x, 1)\right\} = \max\left\{2 - x, 0, \frac{1}{2}(1 - x)\right\} = 2 - x.$$

Similarly, for $x \ne 1, 2$, $d(Tx, T2) = 1$ and

$$\max\left\{d(x, Tx), d(2, T2), \frac{1}{2}d(x, 2)\right\} = \max\left\{2 - x, 1, \frac{1}{2}(2 - x)\right\} = 2 - x.$$

Since, by choice, $x \in \left[0, \frac{1}{2}\right]$, it follows that $1 \le \frac{2}{3}(2 - x)$. Moreover, $d(T1, T2) = d(Tx, Ty) = 0$ for all $x, y \in \left[0, \frac{1}{2}\right]$. Putting all this together implies that T satisfies condition (A) above with $\delta = \frac{2}{3}$. Since $\frac{1}{2}\left\{d\left(\frac{1}{4}, T\frac{1}{4}\right) + d(1, T1)\right\} = \frac{\left(2 - \frac{1}{4}\right)}{2} < 1 = d\left(T\frac{1}{4}, T1\right)$, it follows that T also satisfies condition (B) above with $a = \frac{1}{4}$ and $b = 1$. From the proof of Corollary 1, it follows that T satisfies (5). However, T does not satisfy (4), as it satisfies condition (B) above. This can be seen by setting $\epsilon = 0$ in (4). It is immediate that T has a unique fixed point.

References

1. Chakraborty, M., Samanta, S.K.: A fixed point theorem for kannan-type maps in metric spaces, pre-print (2012), arXiv:1211.7331v2 [math.GN], Nov 2012
2. Haghi, R.H., Rezapour, Sh.: Fixed points of multifunctions on regular cone metric spaces. Expositiones Mathematicae **28**(1), 71–77 (2010)
3. Huang, L.-G., Zhang, X.: Cone metric spaces and fixed point theorems of contractive mappings. J. Math. Anal. Appl. **332**, 1468–1476 (2007)
4. Pata, V.: A fixed point theorem in metric spaces. J. Fixed Point Theory Appl. **10**, 299–305 (2011)