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Research Article

Asymptotic Behavior of a Discrete Nonlinear Oscillator with Damping Dynamical System

Hadi Khatibzadeh^{1,2}

¹ Department of Mathematics, Zanjan University, P.O. Box 45195-313, Zanjan, Iran

² School of Mathematics, Institute for Research in Fundamental Sciences (IPM),
 P.O. Box 19395-5746, Tehran, Iran

Correspondence should be addressed to Hadi Khatibzadeh, hkhatibzadeh@znu.ac.ir

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We propose a new discrete version of nonlinear oscillator with damping dynamical system governed by a general maximal monotone operator. We show the weak convergence of solutions and their weighted averages to a zero of a maximal monotone operator A . We also prove some strong convergence theorems with additional assumptions on A . This iterative scheme gives also an extension of the proximal point algorithm for the approximation of a zero of a maximal monotone operator. These results extend previous results by Brézis and Lions (1978), Lions (1978) as well as Djafari Rouhani and H. Khatibzadeh (2008).

1. Introduction

Let H be a real Hilbert space with inner product (\cdot, \cdot) and norm $|\cdot|$. We denote weak convergence in H by \rightharpoonup and strong convergence by \rightarrow . Let A be a nonempty subset of $H \times H$ which we will refer to as a (nonlinear) possibly multivalued operator in H . A is called monotone (resp. strongly monotone) if $(y_2 - y_1, x_2 - x_1) \geq 0$ (resp. $(y_2 - y_1, x_2 - x_1) \geq \alpha|x_1 - x_2|^2$ for some $\alpha > 0$) for all $[x_i, y_i] \in A, i = 1, 2$. A is maximal monotone if A is monotone and $I + A$ is surjective, where I is the identity operator on H .

Nonlinear oscillator with damping dynamical system,

$$\begin{aligned} u''(t) + \gamma u'(t) + Au(t) &\ni 0, \\ u(0) = u_0, \quad u'(0) &= u_1, \end{aligned} \tag{1.1}$$

where A is a maximal monotone operator and $\gamma > 0$, has been investigated by many authors specially for asymptotic behavior. We refer the reader to [1–6] and references in there.

Following discrete version of (1.1),

$$u_{n+1} = (I + \lambda_n A)^{-1}(u_n + \alpha_n(u_n - u_{n-1})) \quad (1.2)$$

is called inertial proximal method and has been studied in [3]. This iterative algorithm gives a method for approximation of a zero of a maximal monotone operator. In this paper, we propose another discrete version of (1.1) and study asymptotic behavior of its solutions. By using approximations

$$\begin{aligned} u'(t) &= \frac{u(t+h) - u(t-h)}{2h} + o(h), \\ u''(t) &= \frac{u(t+h) - 2u(t) + u(t-h)}{h^2} + o(h), \end{aligned} \quad (1.3)$$

for (1.1), we get

$$\frac{u_{n+1} - 2u_n + u_{n-1}}{h_n^2} + \gamma \frac{u_{n+1} - u_{n-1}}{2h_n} + Au_{n+1} \ni 0. \quad (1.4)$$

By letting $\beta = \gamma/2$, $\lambda_{n+1} = h_n^2/(1 + \beta h_n)$ and $\alpha_n = (\beta h_n - 1)/(\beta h_n + 1)$, we get

$$\begin{aligned} u_{n+1} &= J_{\lambda_{n+1}}((1 - \alpha_n)u_n + \alpha_n u_{n-1}), \quad n \geq 0, \\ u_{-1} &= 0, \quad u_0 = x \in H, \end{aligned} \quad (1.5)$$

where α_n (resp. λ_n) is nonnegative (resp. positive) sequence and $J_\lambda = (I + \lambda A)^{-1}$. This discrete version gives also an algorithm for approximation of a zero of maximal monotone operator A . This algorithm extends proximal point algorithm which was introduced by Martinet in [7] with $\lambda_n = \lambda$ and $\alpha_n = 0$ and then generalized by Rockafellar [8]. We investigate asymptotic behavior of solutions of (1.5) as discrete version of (1.1) which also extend previous results of [9–11] on proximal point algorithm.

Let $w_n := (\sum_{k=1}^n \lambda_k)^{-1}(\sum_{k=1}^n \lambda_k u_k)$. Under suitable assumptions, we investigate weak and strong convergence of w_n and u_n to an element of $A^{-1}(0)$ if and only if $\{u_n\}$ is bounded. Therefore, $A^{-1}(0) \neq \emptyset$ if and only if $\{u_n\}$ is bounded provided $\sum_{n=1}^{+\infty} \lambda_n = +\infty$. Our results extend previous results in [2, 3, 5].

Throughout the paper, we denote $Au_{n+1} = ((1 - \alpha_n)u_n + \alpha_n u_{n-1} - u_{n+1})/\lambda_{n+1}$, and we assume the following assumptions on the sequence $\{\alpha_n\}$:

$$0 \leq \alpha_n \leq 1, \quad \{\alpha_n\} \text{ is nonincreasing and } \alpha_n \rightarrow 0 \text{ as } n \rightarrow +\infty. \quad (1.6)$$

2. Main Results

In this section, we establish convergence of the sequence $\{u_n\}$ or its weighted average to an element of $A^{-1}(0)$. First we recall the following elementary lemma without proof.

Lemma 2.1. *Suppose that $\{\alpha_n\}$ is a nonnegative sequence and $\{\lambda_n\}$ is a positive sequence such that $\sum_{n=1}^{+\infty} \lambda_n = +\infty$. If $\alpha_n/\lambda_n \rightarrow 0$ as $n \rightarrow +\infty$, then $\sum_{k=1}^n \alpha_k / \sum_{k=1}^n \lambda_k \rightarrow 0$ as $n \rightarrow +\infty$.*

We start with a weak ergodic theorem which extends a theorem of Lions [11] (see also [12] page 139 Theorem 3.1 as well as [10] Theorem 2.1).

Theorem 2.2. *Assume that u_n is a solution to (1.5) and $\{\alpha_n\}$ satisfies (1.6). If $\sum_{k=1}^{+\infty} \lambda_k = +\infty$ and $\alpha_n/\lambda_n \rightarrow 0$, then $w_n \rightarrow p \in A^{-1}(0)$ as $n \rightarrow \infty$ if and only if u_n is bounded.*

Proof. Suppose that $w_n \rightarrow p \in A^{-1}(0)$ by (1.5); we get

$$|u_{n+1} - p| \leq |J_{\lambda_{n+1}}((1 - \alpha_n)u_n + \alpha_n u_{n-1}) - p| \leq (1 - \alpha_n)|u_n - p| + \alpha_n|u_{n-1} - p|. \quad (2.1)$$

This implies that

$$|u_{n+1} - p| \leq \max\{|u_1 - p|, |u_0 - p|\}. \quad (2.2)$$

Then $\{u_n\}$ is bounded and this proves necessity. Now, we prove sufficiency. By monotonicity of A , we have

$$(Au_{n+1}, u_{m+1}) + (Au_{m+1}, u_{n+1}) \leq (Au_{m+1}, u_{m+1}) + (Au_{n+1}, u_{n+1}) \quad (2.3)$$

for all $m, n \geq 0$. Multiplying both sides of the above inequality by $\lambda_{m+1}\lambda_{n+1}$ and using (1.5), we deduce

$$\begin{aligned} & (1 - \alpha_n)(u_n - u_{n+1}, \lambda_{m+1}u_{m+1}) + \alpha_n(u_{n-1} - u_{n+1}, \lambda_{m+1}u_{m+1}) \\ & \quad + (1 - \alpha_m)(u_m - u_{m+1}, \lambda_{n+1}u_{n+1}) + \alpha_m(u_{m-1} - u_{m+1}, \lambda_{n+1}u_{n+1}) \\ & \leq \lambda_{m+1}(1 - \alpha_n)(u_n - u_{n+1}, u_{n+1}) + \lambda_{m+1}\alpha_n(u_{n-1} - u_{n+1}, u_{n+1}) \\ & \quad + \lambda_{n+1}(1 - \alpha_m)(u_m - u_{m+1}, u_{m+1}) + \lambda_{n+1}\alpha_m(u_{m-1} - u_{m+1}, u_{m+1}). \end{aligned} \quad (2.4)$$

Summing both sides of this inequality from $m = 0$ to $m = k - 1$, we get

$$\begin{aligned}
& (1 - \alpha_n) \left(u_n - u_{n+1}, \sum_{m=0}^{k-1} \lambda_{m+1} u_{m+1} \right) + \alpha_n \left(u_{n-1} - u_{n+1}, \sum_{m=0}^{k-1} \lambda_{m+1} u_{m+1} \right) \\
& \leq \lambda_{n+1} |u_{n+1}| \sum_{m=0}^{k-1} \alpha_m |u_{m-1} - u_m| + \sum_{m=0}^{k-1} (u_{m+1} - u_m, \lambda_{n+1} u_{n+1}) \\
& \quad + \left(\sum_{m=0}^{k-1} \lambda_{m+1} \right) (1 - \alpha_n) (u_n - u_{n+1}, u_{n+1}) + \left(\sum_{m=0}^{k-1} \lambda_{m+1} \right) \alpha_n (u_{n-1} - u_{n+1}, u_{n+1}) \\
& \quad + \lambda_{n+1} \sum_{m=0}^{k-1} \left(\frac{(1 - \alpha_m)}{2} |u_m|^2 - \frac{(1 - \alpha_m)}{2} |u_{m+1}|^2 \right) + \lambda_{n+1} \sum_{m=0}^{k-1} \left(\frac{\alpha_m}{2} |u_{m-1}|^2 - \frac{\alpha_m}{2} |u_{m+1}|^2 \right) \quad (2.5) \\
& = \lambda_{n+1} |u_{n+1}| \sum_{m=0}^{k-1} \alpha_m |u_{m-1} - u_m| + (u_k - u_0, \lambda_{n+1} u_{n+1}) \\
& \quad + \left(\sum_{m=0}^{k-1} \lambda_{m+1} \right) (1 - \alpha_n) (u_n - u_{n+1}, u_{n+1}) + \left(\sum_{m=0}^{k-1} \lambda_{m+1} \right) \alpha_n (u_{n-1} - u_{n+1}, u_{n+1}) \\
& \quad + \lambda_{n+1} \sum_{m=0}^{k-1} \left(\frac{1}{2} |u_m|^2 - \frac{1}{2} |u_{m+1}|^2 \right) + \lambda_{n+1} \sum_{m=0}^{k-1} \left(\frac{\alpha_m}{2} |u_{m-1}|^2 - \frac{\alpha_m}{2} |u_m|^2 \right).
\end{aligned}$$

Divide both sides of the above inequality by $\sum_{m=0}^{k-1} \lambda_{m+1}$ and suppose that $k = n_j$ and $w_{n_j} \rightarrow p$ as $j \rightarrow +\infty$. By assumptions on $\{\alpha_n\}$, $\{\lambda_n\}$ and Lemma 2.1, we have

$$(1 - \alpha_n) (u_n - u_{n+1}, p) + \alpha_n (u_{n-1} - u_{n+1}, p) \leq (1 - \alpha_n) (u_n - u_{n+1}, u_{n+1}) + \alpha_n (u_{n-1} - u_{n+1}, u_{n+1}). \quad (2.6)$$

This implies that

$$((1 - \alpha_n) u_n + \alpha_n u_{n-1} - u_{n+1}, u_{n+1} - p) \geq 0. \quad (2.7)$$

From (1.6), we get

$$|u_{n+1} - p| + \alpha_n |u_n - p| \leq |u_n - p| + \alpha_{n-1} |u_{n-1} - p|. \quad (2.8)$$

By (1.6) and boundedness of $\{u_n\}$, we get $\lim_{n \rightarrow +\infty} |u_n - p|$ exists. If $w_{n_k} \rightarrow q$, we obtain again $\lim_{n \rightarrow +\infty} |u_n - q|$ exists. Therefore, $\lim_{n \rightarrow +\infty} (1/2)(|u_n - p|^2 - |u_n - q|^2)$, and hence $\lim_{n \rightarrow +\infty} (u_n, p - q)$ exists. This follows that $\lim_{n \rightarrow +\infty} (w_n, p - q)$ exists. It implies that

$(q, p - q) = (p, p - q)$ and hence $p = q$ and $w_n \rightarrow p \in H$ as $n \rightarrow +\infty$. Now we prove $p \in A^{-1}(0)$. Suppose that $[x, y] \in A$. By monotonicity of A and Assumption (1.6), we get

$$\begin{aligned}
 & \left(x - \left(\sum_{i=0}^{n-1} \lambda_{i+1} \right)^{-1} \sum_{i=0}^{n-1} \lambda_{i+1} u_{i+1}, y \right) \\
 &= \left(\sum_{i=0}^{n-1} \lambda_{i+1} \right)^{-1} \sum_{i=0}^{n-1} \lambda_{i+1} (x - u_{i+1}, y) \\
 &\geq \left(\sum_{i=0}^{n-1} \lambda_{i+1} \right)^{-1} \sum_{i=0}^{n-1} \lambda_{i+1} (x - u_{i+1}, Au_{i+1}) \\
 &= \left(\sum_{i=0}^{n-1} \lambda_{i+1} \right)^{-1} \sum_{i=0}^{n-1} (x - u_{i+1}, (1 - \alpha_i)u_i + \alpha_i u_{i-1} - u_{i+1}) \\
 &= \left(\sum_{i=0}^{n-1} \lambda_{i+1} \right)^{-1} \sum_{i=0}^{n-1} \left(-(1 - \alpha_i)(u_{i+1} - x, u_i - x) - \alpha_i(u_{i+1} - x, u_{i-1} - x) + |u_{i+1} - x|^2 \right) \\
 &\geq \left(\sum_{i=0}^{n-1} \lambda_{i+1} \right)^{-1} \sum_{i=0}^{n-1} \left(\frac{1}{2} (|u_{i+1} - x|^2 - |u_i - x|^2) + \frac{1}{2} (\alpha_i |u_i - x|^2 - \alpha_{i-1} |u_{i-1} - x|^2) \right).
 \end{aligned} \tag{2.9}$$

Letting $n \rightarrow +\infty$, we get: $(x - p, y) \geq 0$. By maximality of A , we get $p \in A^{-1}(0)$. □

Remark 2.3. Since range of J_{λ_n} is $D(A)$ (the domain of A), as a trivial consequence of Theorem 2.2, we have that If $D(A)$ is bounded then $A^{-1}(0) \neq \emptyset$.

In the following, we prove a weak convergence theorem. Since the necessity is obvious, we omit the proof of necessity in the next theorems.

Theorem 2.4. *Let u_n be a solution to (1.5) and $\lambda_n \geq \lambda_0 > 0$. If $\{\alpha_n\}$ satisfies (1.6), then $u_n \rightarrow p \in A^{-1}(0)$ as $n \rightarrow +\infty$ if and only if $\{u_n\}$ is bounded.*

Proof. Since assumption on $\{\lambda_n\}$ implies that $\sum_{n=1}^{+\infty} \lambda_n = +\infty$, from (1.5) and (2.7), we get

$$\begin{aligned}
 \lambda_{n+1}^2 |Au_{n+1}|^2 &= |u_{n+1} - p + \lambda_{n+1} Au_{n+1}|^2 - |u_{n+1} - p|^2 - 2\lambda_{n+1} (Au_{n+1}, u_{n+1} - p) \\
 &\leq |(1 - \alpha_n)(u_n - p) + \alpha_n(u_{n-1} - p)|^2 - |u_{n+1} - p|^2 \\
 &\leq (1 - \alpha_n) |u_n - p|^2 + \alpha_n |u_{n-1} - p|^2 - |u_{n+1} - p|^2 \\
 &\leq \alpha_{n-1} |u_{n-1} - p|^2 - \alpha_n |u_n - p|^2 + |u_n - p|^2 - |u_{n+1} - p|^2.
 \end{aligned} \tag{2.10}$$

(The last inequality follows from Assumption (1.6)). Summing both sides of this inequality from $n = 1$ to m and letting $m \rightarrow +\infty$, since $\{\alpha_n\}$ satisfies (1.6), we have

$$\sum_{n=1}^{+\infty} \lambda_{n+1}^2 |Au_{n+1}|^2 < +\infty. \tag{2.11}$$

By assumption on $\{\lambda_n\}$, we have $|Au_n| \rightarrow 0$ as $n \rightarrow +\infty$. Assume $u_{n_j} \rightarrow q$ as $j \rightarrow +\infty$, by the monotonicity of A , we have $(Au_m - Au_{n_j}, u_m - u_{n_j}) \geq 0$. Letting $j \rightarrow +\infty$, we get $(Au_m, u_m - q) \geq 0$. Similar to the proof of Theorem 2.2, $\lim_{m \rightarrow +\infty} |u_m - q|$ exists. This implies that $u_n \rightarrow q = p \in A^{-1}(0)$ as $n \rightarrow +\infty$. \square

In two following, theorems we show strong convergence of $\{u_n\}$ under suitable assumptions on operator A and the sequence $\{\lambda_n\}$.

Theorem 2.5. *Assume that $(I + A)^{-1}$ is compact and $\sum_{n=1}^{+\infty} \lambda_n^2 = +\infty$. If α_n satisfies (1.6), then $u_n \rightarrow p \in A^{-1}(0)$ as $n \rightarrow +\infty$ if and only if $\{u_n\}$ is bounded.*

Proof. By (2.11) and assumption on $\{\lambda_n\}$, we get $\liminf_{n \rightarrow +\infty} |Au_n| = 0$ and $u_n \rightarrow p$ as $n \rightarrow +\infty$. Therefore, there exists a subsequence $\{Au_{n_j}\}$ of $\{Au_n\}$ such that $|Au_{n_j}| \rightarrow 0$ as $j \rightarrow +\infty$ and $\{u_{n_j} + Au_{n_j}\}$ is bounded. The compactity of $(I + A)^{-1}$ implies that $\{u_{n_j}\}$ has a strongly convergent subsequence (we denote again by $\{u_{n_j}\}$) to p . By the monotonicity of A , we have $(Au_n - Au_{n_j}, u_n - u_{n_j}) \geq 0$. Letting $j \rightarrow +\infty$, we obtain $(Au_n, u_n - p) \geq 0$. Now, the proof of Theorem 2.2 shows that $\lim_{n \rightarrow +\infty} |u_n - p|^2$ exists. This implies that $u_n \rightarrow p$ as $n \rightarrow +\infty$. \square

Theorem 2.6. *Assume that A is strongly monotone operator and $\sum_{n=1}^{+\infty} \lambda_n = +\infty$. If $\{\alpha_n\}$ satisfies (1.6), then $u_n \rightarrow p \in A^{-1}(0)$ as $n \rightarrow +\infty$ if and only if $\{u_n\}$ is bounded.*

Proof. By the proof of Theorem 2.2, $w_n \rightarrow p \in A^{-1}(0)$ as $n \rightarrow +\infty$, and $\lim_{n \rightarrow +\infty} |u_n - p|^2$ exists. Since A is strongly monotone, we have

$$(Au_{n+1}, u_{n+1} - p) \geq \alpha |u_{n+1} - p|^2. \quad (2.12)$$

Multiplying both sides of (2.12) by λ_{n+1} and summing from $n = 1$ to m , we have

$$\begin{aligned} \alpha \sum_{n=1}^m \lambda_{n+1} |u_{n+1} - p|^2 &\leq \sum_{n=1}^m ((1 - \alpha_n)u_n + \alpha_n u_{n-1} - u_{n+1}, u_{n+1} - p) \\ &= \sum_{n=1}^m \left[(1 - \alpha_n)(u_n - p, u_{n+1} - p) + \alpha_n (u_{n-1} - p, u_{n+1} - p) - |u_{n+1} - p|^2 \right] \\ &\leq \frac{1}{2} \sum_{n=1}^m \left[(1 - \alpha_n) |u_n - p|^2 + \alpha_n |u_{n-1} - p|^2 - |u_{n+1} - p|^2 \right] \\ &\leq \frac{1}{2} \sum_{n=1}^m \left[|u_n - p|^2 - |u_{n+1} - p|^2 + \alpha_{n-1} |u_{n-1} - p|^2 - \alpha_n |u_n - p|^2 \right]. \end{aligned} \quad (2.13)$$

(The last inequality follows from Assumption (1.6)). Letting $m \rightarrow +\infty$, we get:

$$\sum_{n=1}^{+\infty} \lambda_{n+1} |u_{n+1} - p|^2 < +\infty. \quad (2.14)$$

So, $\liminf_{n \rightarrow +\infty} |u_n - p|^2 = 0$. This implies that $u_n \rightarrow p$ as $n \rightarrow +\infty$. \square

In the following theorem, we assume that $A = \partial\varphi$, where φ is a proper, lower semicontinuous and convex function and $\text{Argmin } \varphi \neq \emptyset$.

Theorem 2.7. *Let $A = \partial\varphi$, where φ is a proper, lower semicontinuous, and convex function. Assume that $A^{-1}(0)$ is nonempty (i.e., φ has at least one minimum point) and $\sum_{n=1}^{+\infty} \lambda_n = +\infty$. If $\{\alpha_n\}$ satisfies (1.6), then $u_n \rightarrow p \in A^{-1}(0)$ as $n \rightarrow +\infty$.*

Proof. Since A is subdifferential of φ and $p \in A^{-1}(0)$, by Assumption (1.6), we have

$$\begin{aligned} \varphi(u_{n+1}) - \varphi(p) &\leq \frac{1}{\lambda_{n+1}} ((1 - \alpha_n)u_n + \alpha_n u_{n-1} - u_{n+1}, u_{n+1} - p) \\ &\leq \frac{1}{\lambda_{n+1}} \left(\frac{(1 - \alpha_n)}{2} (|u_n - p|^2 - |u_{n+1} - p|^2) + \frac{\alpha_n}{2} (|u_{n-1} - p|^2 - |u_{n+1} - p|^2) \right) \\ &\leq \frac{1}{\lambda_{n+1}} \left(\frac{1}{2} (|u_n - p|^2 - |u_{n+1} - p|^2) + \frac{1}{2} (\alpha_{n-1} |u_{n-1} - p|^2 - \alpha_n |u_n - p|^2) \right). \end{aligned} \tag{2.15}$$

Multiplying both sides of the above inequality by λ_{n+1} and summing from $n = 1$ to m and letting $m \rightarrow +\infty$, we get

$$\sum_{n=1}^{+\infty} \lambda_{n+1} (\varphi(u_{n+1}) - \varphi(p)) < +\infty. \tag{2.16}$$

By assumption on $\{\lambda_n\}$, we deduce

$$\liminf_{n \rightarrow +\infty} \varphi(u_n) = \varphi(p). \tag{2.17}$$

By convexity of φ , we have

$$\begin{aligned} &\varphi(u_{n+1}) - (1 - \alpha_n)\varphi(u_n) - \alpha_n\varphi(u_{n-1}) \\ &\leq \varphi(u_{n+1}) - \varphi((1 - \alpha_n)u_n + \alpha_n(u_{n-1})) \\ &\leq \frac{1}{\lambda_{n+1}} ((1 - \alpha_n)u_n + \alpha_n u_{n-1} - u_{n+1}, u_{n+1} - (1 - \alpha_n)u_n - \alpha_n u_{n-1}) \\ &\leq 0. \end{aligned} \tag{2.18}$$

Therefore,

$$\varphi(u_{n+1}) \leq (1 - \alpha_n)\varphi(u_n) + \alpha_n\varphi(u_{n-1}). \tag{2.19}$$

From (2.19), by Assumption (1.6), we get

$$\varphi(u_{n+1}) + \alpha_n\varphi(u_n) \leq \varphi(u_n) + \alpha_{n-1}\varphi(u_{n-1}). \tag{2.20}$$

Again by (2.19), we get

$$\varphi(u_n) \leq \max\{\varphi(u_0), \varphi(u_1)\} \quad (2.21)$$

for all $n > 1$. By (2.20) and (2.21), we have that

$$\lim_{n \rightarrow +\infty} (\varphi(u_{n+1}) + \alpha_n \varphi(u_n)) \quad (2.22)$$

exists. From Assumptions (1.6), (2.17), and (2.21), we get

$$\lim_{n \rightarrow +\infty} \varphi(u_n) = \varphi(p). \quad (2.23)$$

If $u_{n_j} \rightarrow q$, then $\varphi(p) = \liminf_{j \rightarrow +\infty} \varphi(u_{n_j}) \geq \varphi(q)$. This implies that $q \in A^{-1}(0)$. On the other hand, for each $p \in A^{-1}(0)$ by (1.5), we get (2.7). The proof of Theorem 2.2 implies that there exists $\lim_{n \rightarrow +\infty} |u_n - p|$. Then the theorem is concluded by Opial's Lemma (see [13]). \square

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