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# Barnes-type Daehee polynomials

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available at the end of the article**Abstract**

In this paper, we consider Barnes-type Daehee polynomials of the first kind and of the second kind. From the properties of the Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.

**MSC:** 05A15; 05A40; 11B68; 11B75; 65Q05**1 Introduction**

In this paper, we consider the polynomials  $D_n(x|a_1, \dots, a_r)$  and  $\widehat{D}_n(x|a_1, \dots, a_r)$  called the Barnes-type Daehee polynomials of the first kind and of the second kind, whose generating functions are given by

$$\prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) (1+t)^x = \sum_{n=0}^{\infty} D_n(x|a_1, \dots, a_r) \frac{t^n}{n!}, \quad (1)$$

$$\prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) (1+t)^x = \sum_{n=0}^{\infty} \widehat{D}_n(x|a_1, \dots, a_r) \frac{t^n}{n!}, \quad (2)$$

respectively, where  $a_1, \dots, a_r \neq 0$ . When  $x = 0$ ,  $D_n(a_1, \dots, a_r) = D_n(0|a_1, \dots, a_r)$  and  $\widehat{D}_n(a_1, \dots, a_r) = \widehat{D}_n(0|a_1, \dots, a_r)$  are called the Barnes-type Daehee numbers of the first kind and of the second kind, respectively.

Recall that the Daehee polynomials of the first kind and of the second kind of order  $r$ , denoted by  $D_n^{(r)}(x)$  and  $\widehat{D}_n^{(r)}(x)$ , respectively, are given by the generating functions to be

$$\left( \frac{\ln(1+t)}{t} \right)^r (1+t)^x = \sum_{n=0}^{\infty} D_n^{(r)}(x) \frac{t^n}{n!},$$
$$\left( \frac{(1+t) \ln(1+t)}{t} \right)^r (1+t)^x = \sum_{n=0}^{\infty} \widehat{D}_n^{(r)}(x) \frac{t^n}{n!},$$

respectively. If  $a_1 = \dots = a_r = 1$ , then  $D_n^{(r)}(x) = D_n(x|\underbrace{1, \dots, 1}_r)$  and  $\widehat{D}_n^{(r)}(x) = \widehat{D}_n(x|\underbrace{1, \dots, 1}_r)$ .

Daehee polynomials were defined by the second author [1] and have been investigated in [2–4].

In this paper, we consider Barnes-type Daehee polynomials of the first kind and of the second kind. From the properties of the Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.

## 2 Umbral calculus

Let  $\mathbb{C}$  be the complex number field and let  $\mathcal{F}$  be the set of all formal power series in the variable  $t$ :

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \mid a_k \in \mathbb{C} \right\}. \tag{3}$$

Let  $\mathbb{P} = \mathbb{C}[x]$  and let  $\mathbb{P}^*$  be the vector space of all linear functionals on  $\mathbb{P}$ .  $\langle L|p(x) \rangle$  is the action of the linear functional  $L$  on the polynomial  $p(x)$ , and we recall that the vector space operations on  $\mathbb{P}^*$  are defined by  $\langle L + M|p(x) \rangle = \langle L|p(x) \rangle + \langle M|p(x) \rangle$ ,  $\langle cL|p(x) \rangle = c\langle L|p(x) \rangle$ , where  $c$  is a complex constant in  $\mathbb{C}$ . For  $f(t) \in \mathcal{F}$ , let us define the linear functional on  $\mathbb{P}$  by setting

$$\langle f(t)|x^n \rangle = a_n \quad (n \geq 0). \tag{4}$$

In particular,

$$\langle t^k|x^n \rangle = n! \delta_{n,k} \quad (n, k \geq 0), \tag{5}$$

where  $\delta_{n,k}$  is the Kronecker symbol.

For  $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k \rangle}{k!} t^k$ , we have  $\langle f_L(t)|x^n \rangle = \langle L|x^n \rangle$ . That is,  $L = f_L(t)$ . The map  $L \mapsto f_L(t)$  is a vector space isomorphism from  $\mathbb{P}^*$  onto  $\mathcal{F}$ . Henceforth,  $\mathcal{F}$  denotes both the algebra of formal power series in  $t$  and the vector space of all linear functionals on  $\mathbb{P}$ , and therefore an element  $f(t)$  of  $\mathcal{F}$  will be thought of as both a formal power series and a linear functional. We call  $\mathcal{F}$  the *umbral algebra* and the *umbral calculus* is the study of umbral algebra. The order  $O(f(t))$  of a power series  $f(t) (\neq 0)$  is the smallest integer  $k$  for which the coefficient of  $t^k$  does not vanish. If  $O(f(t)) = 1$ , then  $f(t)$  is called a *delta series*; if  $O(f(t)) = 0$ , then  $f(t)$  is called an *invertible series*. For  $f(t), g(t) \in \mathcal{F}$  with  $O(f(t)) = 1$  and  $O(g(t)) = 0$ , there exists a unique sequence  $s_n(x)$  ( $\deg s_n(x) = n$ ) such that  $\langle g(t)f(t)^k|s_n(x) \rangle = n! \delta_{n,k}$ , for  $n, k \geq 0$ . Such a sequence  $s_n(x)$  is called the *Sheffer sequence* for  $(g(t), f(t))$ , which is denoted by  $s_n(x) \sim (g(t), f(t))$ .

For  $f(t), g(t) \in \mathcal{F}$  and  $p(x) \in \mathbb{P}$ , we have

$$\langle f(t)g(t)|p(x) \rangle = \langle f(t)|g(t)p(x) \rangle = \langle g(t)|f(t)p(x) \rangle \tag{6}$$

and

$$f(t) = \sum_{k=0}^{\infty} \langle f(t)|x^k \rangle \frac{t^k}{k!}, \quad p(x) = \sum_{k=0}^{\infty} \langle t^k|p(x) \rangle \frac{x^k}{k!} \tag{7}$$

[5, Theorem 2.2.5]. Thus, by (7), we get

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k} \quad \text{and} \quad e^{yt} p(x) = p(x + y). \tag{8}$$

Sheffer sequences are characterized by the generating function [5, Theorem 2.3.4].

**Lemma 1** *The sequence  $s_n(x)$  is Sheffer for  $(g(t), f(t))$  if and only if*

$$\frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} \frac{s_k(y)}{k!} t^k \quad (y \in \mathbb{C}),$$

where  $\bar{f}(t)$  is the compositional inverse of  $f(t)$ .

For  $s_n(x) \sim (g(t), f(t))$ , we have the following equations [5, Theorem 2.3.7, Theorem 2.3.5, Theorem 2.3.9]:

$$f(t)s_n(x) = ns_{n-1}(x) \quad (n \geq 0), \tag{9}$$

$$s_n(x) = \sum_{j=0}^n \frac{1}{j!} \langle g(\bar{f}(t))^{-1} \bar{f}(t) | x^n \rangle x^j, \tag{10}$$

$$s_n(x+y) = \sum_{j=0}^n \binom{n}{j} s_j(x) p_{n-j}(y), \tag{11}$$

where  $p_n(x) = g(t)s_n(x)$ .

Assume that  $p_n(x) \sim (1, f(t))$  and  $q_n(x) \sim (1, g(t))$ . Then the transfer formula [5, Corollary 3.8.2] is given by

$$q_n(x) = x \left( \frac{f(t)}{g(t)} \right)^n x^{-1} p_n(x) \quad (n \geq 1).$$

For  $s_n(x) \sim (g(t), f(t))$  and  $r_n(x) \sim (h(t), l(t))$ , assume that

$$s_n(x) = \sum_{m=0}^n C_{n,m} r_m(x) \quad (n \geq 0).$$

Then we have [5, p.132]

$$C_{n,m} = \frac{1}{m!} \left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} l(\bar{f}(t))^m | x^n \right\rangle. \tag{12}$$

### 3 Main results

We now note that  $D_n(x|a_1, \dots, a_r)$  is the Sheffer sequence for

$$g(t) = \prod_{j=1}^r \left( \frac{e^{a_j t} - 1}{t} \right) \quad \text{and} \quad f(t) = e^t - 1.$$

Therefore,

$$D_n(x|a_1, \dots, a_r) \sim \left( \prod_{j=1}^r \left( \frac{e^{a_j t} - 1}{t} \right), e^t - 1 \right). \tag{13}$$

$\widehat{D}_n(x|a_1, \dots, a_r)$  is the Sheffer sequence for

$$g(t) = \prod_{j=1}^r \left( \frac{e^{a_j t} - 1}{te^{a_j t}} \right) \quad \text{and} \quad f(t) = e^t - 1.$$

So,

$$\widehat{D}_n(x|a_1, \dots, a_r) \sim \left( \prod_{j=1}^r \left( \frac{e^{a_j t} - 1}{t e^{a_j t}} \right), e^t - 1 \right). \tag{14}$$

### 3.1 Explicit expressions

Recall that Barnes' multiple Bernoulli polynomials  $B_n(x|a_1, \dots, a_r)$  are defined by the generating function as

$$\frac{t^r}{\prod_{j=1}^r (e^{a_j t} - 1)} e^{xt} = \sum_{n=0}^{\infty} B_n(x|a_1, \dots, a_r) \frac{t^n}{n!}, \tag{15}$$

where  $a_1, \dots, a_r \neq 0$  [6–8]. Let  $(n)_j = n(n-1) \cdots (n-j+1)$  ( $j \geq 1$ ) with  $(n)_0 = 1$ . The (signed) Stirling numbers of the first kind  $S_1(n, m)$  are defined by

$$(x)_n = \sum_{m=0}^n S_1(n, m) x^m.$$

#### Theorem 1

$$D_n(x|a_1, \dots, a_r) = \sum_{m=0}^n S_1(n, m) B_m(x|a_1, \dots, a_r) \tag{16}$$

$$= \sum_{j=0}^n \left( \sum_{l=j}^n \binom{n}{l} S_1(l, j) D_{n-l}(a_1, \dots, a_r) \right) x^j \tag{17}$$

$$= \sum_{m=0}^n \binom{n}{m} D_{n-m}(a_1, \dots, a_r) (x)_m, \tag{18}$$

$$\widehat{D}_n(x|a_1, \dots, a_r) = \sum_{m=0}^n S_1(n, m) B_m(x + a_1 + \dots + a_r | a_1, \dots, a_r) \tag{19}$$

$$= \sum_{j=0}^n \left( \sum_{l=j}^n \binom{n}{l} S_1(l, j) \widehat{D}_{n-l}(a_1, \dots, a_r) \right) x^j \tag{20}$$

$$= \sum_{m=0}^n \binom{n}{m} \widehat{D}_{n-m}(a_1, \dots, a_r) (x)_m. \tag{21}$$

*Proof* Since

$$\prod_{j=1}^r \left( \frac{e^{a_j t} - 1}{t} \right) D_n(x|a_1, \dots, a_r) \sim (1, e^t - 1) \tag{22}$$

and

$$(x)_n \sim (1, e^t - 1), \tag{23}$$

we have

$$\begin{aligned}
 D_n(x|a_1, \dots, a_r) &= \prod_{j=1}^r \left( \frac{t}{e^{a_j t} - 1} \right) (x)_n \\
 &= \sum_{m=0}^n S_1(n, m) \prod_{j=1}^r \left( \frac{t}{e^{a_j t} - 1} \right) x^m \\
 &= \sum_{m=0}^n S_1(n, m) B_m(x|a_1, \dots, a_r).
 \end{aligned}$$

So, we get (16).

Similarly, by

$$\prod_{j=1}^r \left( \frac{e^{a_j t} - 1}{te^{a_j t}} \right) \widehat{D}_n(x|a_1, \dots, a_r) \sim (1, e^t - 1) \tag{24}$$

and (23), we have

$$\begin{aligned}
 \widehat{D}_n(x|a_1, \dots, a_r) &= \prod_{j=1}^r \left( \frac{te^{a_j t}}{e^{a_j t} - 1} \right) (x)_n \\
 &= \sum_{m=0}^n S_1(n, m) \prod_{j=1}^r \left( \frac{te^{a_j t}}{e^{a_j t} - 1} \right) x^m \\
 &= \sum_{m=0}^n S_1(n, m) e^{(a_1 + \dots + a_r)t} \prod_{j=1}^r \left( \frac{t}{e^{a_j t} - 1} \right) x^m \\
 &= \sum_{m=0}^n S_1(n, m) B_m(x + a_1 + \dots + a_r | a_1, \dots, a_r).
 \end{aligned}$$

Therefore, we get (19).

By (10) with (13), we get

$$D_n(x|a_1, \dots, a_r) = \sum_{j=0}^n \frac{1}{j!} \left\langle \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) (\ln(1+t))^j \middle| x^n \right\rangle x^j.$$

Since

$$\begin{aligned}
 &\left\langle \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) (\ln(1+t))^j \middle| x^n \right\rangle \\
 &= \left\langle \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \middle| (\ln(1+t))^j x^n \right\rangle \\
 &= \left\langle \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \middle| j! \sum_{l=j}^{\infty} S_1(l, j) \frac{t^l}{l!} x^n \right\rangle \\
 &= j! \sum_{l=j}^n \binom{n}{l} S_1(l, j) \left\langle \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \middle| x^{n-l} \right\rangle
 \end{aligned}$$

$$\begin{aligned}
 &= j! \sum_{l=j}^n \binom{n}{l} S_1(l, j) \left\langle \sum_{i=0}^{\infty} D_i(a_1, \dots, a_r) \frac{t^i}{i!} \middle| x^{n-l} \right\rangle \\
 &= j! \sum_{l=j}^n \binom{n}{l} S_1(l, j) D_{n-l}(a_1, \dots, a_r),
 \end{aligned}$$

we obtain (17).

Similarly, by (10) with (14), we get

$$\widehat{D}_n(x|a_1, \dots, a_r) = \sum_{j=0}^n \frac{1}{j!} \left\langle \prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) (\ln(1+t))^j \middle| x^n \right\rangle x^j.$$

Since

$$\begin{aligned}
 &\left\langle \prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) (\ln(1+t))^j \middle| x^n \right\rangle \\
 &= \left\langle \prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) \middle| (\ln(1+t))^j x^n \right\rangle \\
 &= j! \sum_{l=j}^n \binom{n}{l} S_1(l, j) \left\langle \prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) \middle| x^{n-l} \right\rangle \\
 &= j! \sum_{l=j}^n \binom{n}{l} S_1(l, j) \left\langle \sum_{i=0}^{\infty} \widehat{D}_i(a_1, \dots, a_r) \frac{t^i}{i!} \middle| x^{n-l} \right\rangle \\
 &= j! \sum_{l=j}^n \binom{n}{l} S_1(l, j) \widehat{D}_{n-l}(a_1, \dots, a_r),
 \end{aligned}$$

we obtain (20).

Next, we obtain

$$\begin{aligned}
 D_n(y|a_1, \dots, a_r) &= \left\langle \sum_{i=0}^{\infty} D_i(y|a_1, \dots, a_r) \frac{t^i}{i!} \middle| x^n \right\rangle \\
 &= \left\langle \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) (1+t)^y \middle| x^n \right\rangle \\
 &= \left\langle \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \middle| \sum_{m=0}^{\infty} (y)_m \frac{t^m}{m!} x^n \right\rangle \\
 &= \sum_{m=0}^n (y)_m \binom{n}{m} \left\langle \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \middle| x^{n-m} \right\rangle \\
 &= \sum_{m=0}^n \binom{n}{m} D_{n-m}(a_1, \dots, a_r) (y)_m.
 \end{aligned}$$

Thus, we get the identity (18).

Similarly,

$$\begin{aligned} \widehat{D}_n(y|a_1, \dots, a_r) &= \left\langle \sum_{i=0}^{\infty} \widehat{D}_i(y|a_1, \dots, a_r) \frac{t^i}{i!} \middle| x^n \right\rangle \\ &= \left\langle \prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) (1+t)^y \middle| x^n \right\rangle \\ &= \left\langle \prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) \middle| \sum_{m=0}^{\infty} (y)_m \frac{t^m}{m!} x^n \right\rangle \\ &= \sum_{m=0}^n (y)_m \binom{n}{m} \left\langle \prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) \middle| x^{n-m} \right\rangle \\ &= \sum_{m=0}^n \binom{n}{m} \widehat{D}_{n-m}(a_1, \dots, a_r)(y)_m. \end{aligned}$$

Thus, we get the identity (21). □

### 3.2 Sheffer identity

#### Theorem 2

$$D_n(x+y|a_1, \dots, a_r) = \sum_{j=0}^n \binom{n}{j} D_j(x|a_1, \dots, a_r)(y)_{n-j}, \tag{25}$$

$$\widehat{D}_n(x+y|a_1, \dots, a_r) = \sum_{j=0}^n \binom{n}{j} \widehat{D}_j(x|a_1, \dots, a_r)(y)_{n-j}. \tag{26}$$

*Proof* By (13) with

$$\begin{aligned} p_n(x) &= \prod_{j=1}^r \left( \frac{e^{a_j t} - 1}{t} \right) D_n(x|a_1, \dots, a_r) \\ &= (x)_n \sim (1, e^t - 1), \end{aligned}$$

using (11), we have (25).

By (14) with

$$\begin{aligned} p_n(x) &= \prod_{j=1}^r \left( \frac{e^{a_j t} - 1}{te^{a_j t}} \right) \widehat{D}_n(x|a_1, \dots, a_r) \\ &= (x)_n \sim (1, e^t - 1), \end{aligned}$$

using (11), we have (26). □

### 3.3 Difference relations

#### Theorem 3

$$D_n(x+1|a_1, \dots, a_r) - D_n(x|a_1, \dots, a_r) = nD_{n-1}(x|a_1, \dots, a_r), \tag{27}$$

$$\widehat{D}_n(x+1|a_1, \dots, a_r) - \widehat{D}_n(x|a_1, \dots, a_r) = n\widehat{D}_{n-1}(x|a_1, \dots, a_r). \tag{28}$$

*Proof* By (9) with (13), we get

$$(e^t - 1)D_n(x|a_1, \dots, a_r) = nD_{n-1}(x|a_1, \dots, a_r).$$

By (8), we have (27).

Similarly, by (9) with (14), we get

$$(e^t - 1)\widehat{D}_n(x|a_1, \dots, a_r) = n\widehat{D}_{n-1}(x|a_1, \dots, a_r).$$

By (8), we have (28). □

### 3.4 Recurrence

#### Theorem 4

$$\begin{aligned}
 D_{n+1}(x|a_1, \dots, a_r) &= xD_n(x-1|a_1, \dots, a_r) \\
 &\quad - \sum_{m=0}^n \left( \sum_{i=m}^n \sum_{l=i}^n \sum_{j=1}^r \frac{1}{i+1} \binom{n}{l} \binom{i+1}{m} S_1(l, i) \right. \\
 &\quad \left. \times B_{i+1-m}(-a_j)^{i+1-m} D_{n-i}(a_1, \dots, a_r) \right) (x-1)^m, \tag{29}
 \end{aligned}$$

$$\begin{aligned}
 \widehat{D}_{n+1}(x|a_1, \dots, a_r) &= \left( x + \sum_{j=1}^r a_j \right) \widehat{D}_n(x-1|a_1, \dots, a_r) \\
 &\quad - \sum_{m=0}^n \left( \sum_{i=m}^n \sum_{l=i}^n \sum_{j=1}^r \frac{1}{i+1} \binom{n}{l} \binom{i+1}{m} S_1(l, i) \right. \\
 &\quad \left. \times B_{i+1-m}(-a_j)^{i+1-m} \widehat{D}_{n-i}(a_1, \dots, a_r) \right) (x-1)^m, \tag{30}
 \end{aligned}$$

where  $B_n$  is the  $n$ th ordinary Bernoulli number.

*Proof* By applying

$$s_{n+1}(x) = \left( x - \frac{g'(t)}{g(t)} \right) \frac{1}{f'(t)} s_n(x) \tag{31}$$

[5, Corollary 3.7.2] with (13), we get

$$D_{n+1}(x|a_1, \dots, a_r) = xD_n(x-1|a_1, \dots, a_r) - e^{-t} \frac{g'(t)}{g(t)} D_n(x|a_1, \dots, a_r).$$

Now,

$$\begin{aligned}
 \frac{g'(t)}{g(t)} &= (\ln g(t))' \\
 &= \left( \sum_{j=1}^r \ln(e^{a_j t} - 1) - r \ln t \right)'
 \end{aligned}$$



$$\begin{aligned}
 &= \sum_{j=1}^r \frac{a_j e^{a_j t}}{e^{a_j t} - 1} - \frac{r}{t} \\
 &= \frac{\sum_{j=1}^r \prod_{i \neq j} (e^{a_i t} - 1)(a_j t e^{a_j t} - e^{a_j t} + 1)}{t \prod_{j=1}^r (e^{a_j t} - 1)}.
 \end{aligned}$$

Since

$$\begin{aligned}
 \sum_{j=1}^r \frac{a_j t e^{a_j t}}{e^{a_j t} - 1} - r &= \frac{\sum_{j=1}^r \prod_{i \neq j} (e^{a_i t} - 1)(a_j t e^{a_j t} - e^{a_j t} + 1)}{\prod_{j=1}^r (e^{a_j t} - 1)} \\
 &= \frac{\frac{1}{2}(\sum_{j=1}^r a_1 \cdots a_{j-1} a_j^2 a_{j+1} \cdots a_r) t^{r+1} + \cdots}{(a_1 \cdots a_r) t^r + \cdots} \\
 &= \frac{1}{2} \left( \sum_{j=1}^r a_j \right) t + \cdots
 \end{aligned}$$

is a series with order  $\geq 1$ , by (17) we have

$$\begin{aligned}
 &D_{n+1}(x|a_1, \dots, a_r) \\
 &= xD_n(x-1|a_1, \dots, a_r) - e^{-t} \frac{\sum_{j=1}^r \frac{a_j t e^{a_j t}}{e^{a_j t} - 1} - r}{t} D_n(x|a_1, \dots, a_r) \\
 &= xD_n(x-1|a_1, \dots, a_r) - e^{-t} \frac{\sum_{j=1}^r \frac{a_j t e^{a_j t}}{e^{a_j t} - 1} - r}{t} \left( \sum_{i=0}^n \sum_{l=i}^n \binom{n}{l} S_1(l, i) D_{n-l}(a_1, \dots, a_r) x^i \right) \\
 &= xD_n(x-1|a_1, \dots, a_r) - \sum_{i=0}^n \sum_{l=i}^n \binom{n}{l} S_1(l, i) D_{n-l}(a_1, \dots, a_r) e^{-t} \left( \sum_{j=1}^r \frac{a_j t e^{a_j t}}{e^{a_j t} - 1} - r \right) \frac{x^{i+1}}{i+1}.
 \end{aligned}$$

Since

$$\begin{aligned}
 e^{-t} \left( \sum_{j=1}^r \frac{a_j t e^{a_j t}}{e^{a_j t} - 1} - r \right) x^{i+1} &= e^{-t} \left( \sum_{j=1}^r \sum_{m=0}^{\infty} \frac{(-1)^m B_m a_j^m}{m!} t^m - r \right) x^{i+1} \\
 &= e^{-t} \left( \sum_{j=1}^r \sum_{m=0}^{i+1} \binom{i+1}{m} B_m (-a_j)^m x^{i+1-m} - r x^{i+1} \right) \\
 &= \sum_{j=1}^r \sum_{m=1}^{i+1} \binom{i+1}{m} B_m (-a_j)^m (x-1)^{i+1-m} \\
 &= \sum_{j=1}^r \sum_{m=0}^i \binom{i+1}{m} B_{i+1-m} (-a_j)^{i+1-m} (x-1)^m, \tag{32}
 \end{aligned}$$

we have

$$\begin{aligned}
 D_{n+1}(x|a_1, \dots, a_r) &= xD_n(x-1|a_1, \dots, a_r) \\
 &\quad - \sum_{i=0}^n \sum_{l=i}^n \sum_{j=1}^r \sum_{m=0}^i \frac{1}{i+1} \binom{n}{l} \binom{i+1}{m} S_1(l, i)
 \end{aligned}$$

$$\begin{aligned} & \times B_{i+1-m}(-a_j)^{i+1-m} D_{n-l}(a_1, \dots, a_r)(x-1)^m \\ & = xD_n(x-1|a_1, \dots, a_r) \\ & - \sum_{m=0}^n \left( \sum_{i=m}^n \sum_{l=i}^n \sum_{j=1}^r \frac{1}{i+1} \binom{n}{l} \binom{i+1}{m} S_1(l, i) \right. \\ & \left. \times B_{i+1-m}(-a_j)^{i+1-m} D_{n-l}(a_1, \dots, a_r) \right) (x-1)^m, \end{aligned}$$

which is the identity (29).

Next, by applying (31) with (14), we get

$$\widehat{D}_{n+1}(x|a_1, \dots, a_r) = x\widehat{D}_n(x-1|a_1, \dots, a_r) - e^{-t} \frac{g'(t)}{g(t)} \widehat{D}_n(x|a_1, \dots, a_r).$$

Now,

$$\begin{aligned} \frac{g'(t)}{g(t)} & = (\ln g(t))' \\ & = \left( \sum_{j=1}^r \ln(e^{a_j t} - 1) - r \ln t - \left( \sum_{j=1}^r a_j \right) t \right)' \\ & = \sum_{j=1}^r \frac{a_j e^{a_j t}}{e^{a_j t} - 1} - \frac{r}{t} - \sum_{j=1}^r a_j. \end{aligned}$$

By (20) we have

$$\begin{aligned} & \widehat{D}_{n+1}(x|a_1, \dots, a_r) \\ & = \left( x + \sum_{j=1}^r a_j \right) \widehat{D}_n(x-1|a_1, \dots, a_r) \\ & - e^{-t} \frac{\sum_{j=1}^r \frac{a_j t e^{a_j t}}{e^{a_j t} - 1} - r}{t} \widehat{D}_n(x|a_1, \dots, a_r) \\ & = \left( x + \sum_{j=1}^r a_j \right) \widehat{D}_n(x-1|a_1, \dots, a_r) \\ & - e^{-t} \frac{\sum_{j=1}^r \frac{a_j t e^{a_j t}}{e^{a_j t} - 1} - r}{t} \left( \sum_{i=0}^n \sum_{l=i}^n \binom{n}{l} S_1(l, i) \widehat{D}_{n-l}(a_1, \dots, a_r) x^i \right) \\ & = \left( x + \sum_{j=1}^r a_j \right) \widehat{D}_n(x-1|a_1, \dots, a_r) \\ & - \sum_{i=0}^n \sum_{l=i}^n \binom{n}{l} S_1(l, i) \widehat{D}_{n-l}(a_1, \dots, a_r) e^{-t} \left( \sum_{j=1}^r \frac{a_j t e^{a_j t}}{e^{a_j t} - 1} - r \right) \frac{x^{i+1}}{i+1}. \end{aligned}$$

By (32), we have the identity (30). □

### 3.5 Differentiation

#### Theorem 5

$$\frac{d}{dx}D_n(x|a_1, \dots, a_r) = n! \sum_{l=0}^{n-1} \frac{(-1)^{n-l-1}}{l!(n-l)} D_l(x|a_1, \dots, a_r), \tag{33}$$

$$\frac{d}{dx}\widehat{D}_n(x|a_1, \dots, a_r) = n! \sum_{l=0}^{n-1} \frac{(-1)^{n-l-1}}{l!(n-l)} \widehat{D}_l(x|a_1, \dots, a_r). \tag{34}$$

*Proof* We shall use

$$\frac{d}{dx}s_n(x) = \sum_{l=0}^{n-1} \binom{n}{l} (\bar{f}(t)|x^{n-l})s_l(x)$$

(cf. [5, Theorem 2.3.12]). Since

$$\begin{aligned} (\bar{f}(t)|x^{n-l}) &= (\ln(1+t)|x^{n-l}) = \left\langle \sum_{m=1}^{\infty} \frac{(-1)^{m-1}t^m}{m} \middle| x^{n-l} \right\rangle \\ &= \sum_{m=1}^{n-l} \frac{(-1)^{m-1}}{m} (t^m|x^{n-l}) \\ &= \sum_{m=1}^{n-l} \frac{(-1)^{m-1}}{m} (n-l)! \delta_{m,n-l} \\ &= (-1)^{n-l-1} (n-l-1)!, \end{aligned}$$

with (13), we have

$$\begin{aligned} \frac{d}{dx}D_n(x|a_1, \dots, a_r) &= \sum_{l=0}^{n-1} \binom{n}{l} (-1)^{n-l-1} (n-l-1)! D_l(x|a_1, \dots, a_r) \\ &= n! \sum_{l=0}^{n-1} \frac{(-1)^{n-l-1}}{l!(n-l)} D_l(x|a_1, \dots, a_r), \end{aligned}$$

which is the identity (33). Similarly, with (14), we have the identity (34). □

### 3.6 More relations

The classical Cauchy numbers  $c_n$  are defined by

$$\frac{t}{\ln(1+t)} = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!}$$

(see e.g. [9, 10]).

#### Theorem 6

$$\begin{aligned} D_n(x|a_1, \dots, a_r) &= xD_{n-1}(x-1|a_1, \dots, a_r) \\ &\quad + \frac{r}{n} \sum_{l=0}^n \binom{n}{l} c_l D_{n-l}(x-1|a_1, \dots, a_r) \end{aligned}$$

$$-\frac{1}{n} \sum_{j=1}^r \sum_{l=0}^n \binom{n}{l} a_j c_l D_{n-l}(x + a_j - 1 | a_1, \dots, a_r, a_j), \tag{35}$$

$$\begin{aligned} \widehat{D}_n(x | a_1, \dots, a_r) &= \left( x + \sum_{j=1}^r a_j \right) \widehat{D}_{n-1}(x - 1 | a_1, \dots, a_r) \\ &\quad + \frac{r}{n} \sum_{l=0}^n \binom{n}{l} c_l \widehat{D}_{n-l}(x - 1 | a_1, \dots, a_r) \\ &\quad - \frac{1}{n} \sum_{j=1}^r \sum_{l=0}^n \binom{n}{l} a_j c_l \widehat{D}_{n-l}(x - 1 | a_1, \dots, a_r, a_j). \end{aligned} \tag{36}$$

*Proof* For  $n \geq 1$ , we have

$$\begin{aligned} D_n(y | a_1, \dots, a_r) &= \left\langle \sum_{l=0}^{\infty} D_l(y | a_1, \dots, a_r) \frac{t^l}{l!} \middle| x^n \right\rangle \\ &= \left\langle \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) (1+t)^y \middle| x^n \right\rangle \\ &= \left\langle \partial_t \left( \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) (1+t)^y \right) \middle| x^{n-1} \right\rangle \\ &= \left\langle \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) (\partial_t (1+t)^y) \middle| x^{n-1} \right\rangle \\ &\quad + \left\langle \left( \partial_t \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \right) (1+t)^y \middle| x^{n-1} \right\rangle \\ &= y D_{n-1}(y - 1 | a_1, \dots, a_r) \\ &\quad + \left\langle \left( \partial_t \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \right) (1+t)^y \middle| x^{n-1} \right\rangle. \end{aligned}$$

Observe that

$$\begin{aligned} \partial_t \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) &= \sum_{j=1}^r \prod_{i \neq j} \left( \frac{\ln(1+t)}{(1+t)^{a_i} - 1} \right) \frac{\frac{1}{1+t}((1+t)^{a_j} - 1) - \ln(1+t)(a_j(1+t)^{a_j-1})}{((1+t)^{a_j} - 1)^2} \\ &= \frac{1}{1+t} \prod_{i=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_i} - 1} \right) \sum_{j=1}^r \left( \frac{1}{\ln(1+t)} - \frac{a_j(1+t)^{a_j}}{(1+t)^{a_j} - 1} \right) \\ &= \frac{1}{1+t} \prod_{i=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_i} - 1} \right) \frac{\sum_{j=1}^r \left( \frac{t}{\ln(1+t)} - \frac{a_j t(1+t)^{a_j}}{(1+t)^{a_j} - 1} \right)}{t}. \end{aligned}$$

Since

$$\sum_{j=1}^r \left( \frac{t}{\ln(1+t)} - \frac{a_j t(1+t)^{a_j}}{(1+t)^{a_j} - 1} \right) = -\frac{1}{2} \left( \sum_{j=1}^r a_j \right) t + \dots$$

is a series with order  $\geq 1$ , we have

$$\begin{aligned}
 & \left\langle \left( \partial_t \prod_{i=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_i} - 1} \right) \right) (1+t)^y \middle| x^{n-1} \right\rangle \\
 &= \left\langle \prod_{i=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_i} - 1} \right) (1+t)^{y-1} \middle| \frac{\sum_{j=1}^r \left( \frac{t}{\ln(1+t)} - \frac{a_j t (1+t)^{a_j}}{(1+t)^{a_j} - 1} \right)}{t} x^{n-1} \right\rangle \\
 &= \frac{1}{n} \sum_{j=1}^r \left\langle \prod_{i=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_i} - 1} \right) (1+t)^{y-1} \middle| \left( \frac{t}{\ln(1+t)} - \frac{a_j t (1+t)^{a_j}}{(1+t)^{a_j} - 1} \right) x^n \right\rangle \\
 &= \frac{r}{n} \left\langle \prod_{i=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_i} - 1} \right) (1+t)^{y-1} \middle| \frac{t}{\ln(1+t)} x^n \right\rangle \\
 &\quad - \frac{1}{n} \sum_{j=1}^r a_j \left\langle \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \prod_{i=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_i} - 1} \right) (1+t)^{y+a_j-1} \middle| \frac{t}{\ln(1+t)} x^n \right\rangle \\
 &= \frac{r}{n} \left\langle \prod_{i=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_i} - 1} \right) (1+t)^{y-1} \middle| \sum_{l=0}^{\infty} c_l \frac{t^l}{l!} x^n \right\rangle \\
 &\quad - \frac{1}{n} \sum_{j=1}^r a_j \left\langle \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \prod_{i=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_i} - 1} \right) (1+t)^{y+a_j-1} \middle| \sum_{l=0}^{\infty} c_l \frac{t^l}{l!} x^n \right\rangle \\
 &= \frac{r}{n} \sum_{l=0}^n c_l \binom{n}{l} \left\langle \prod_{i=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_i} - 1} \right) (1+t)^{y-1} \middle| x^{n-l} \right\rangle \\
 &\quad - \frac{1}{n} \sum_{j=1}^r a_j \sum_{l=0}^n c_l \binom{n}{l} \left\langle \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \prod_{i=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_i} - 1} \right) (1+t)^{y+a_j-1} \middle| x^{n-l} \right\rangle \\
 &= \frac{r}{n} \sum_{l=0}^n \binom{n}{l} c_l D_{n-l}(y-1|a_1, \dots, a_r) - \frac{1}{n} \sum_{j=1}^r \sum_{l=0}^n \binom{n}{l} a_j c_l D_{n-l}(y+a_j-1|a_1, \dots, a_r, a_j).
 \end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
 D_n(x|a_1, \dots, a_r) &= x D_{n-1}(x-1|a_1, \dots, a_r) \\
 &\quad + \frac{r}{n} \sum_{l=0}^n \binom{n}{l} c_l D_{n-l}(x-1|a_1, \dots, a_r) \\
 &\quad - \frac{1}{n} \sum_{j=1}^r \sum_{l=0}^n \binom{n}{l} a_j c_l D_{n-l}(x+a_j-1|a_1, \dots, a_r, a_j),
 \end{aligned}$$

which is the identity (35).

Next, for  $n \geq 1$  we have

$$\begin{aligned}
 \widehat{D}_n(y|a_1, \dots, a_r) &= \left\langle \sum_{l=0}^{\infty} \widehat{D}_l(y|a_1, \dots, a_r) \frac{t^l}{l!} \middle| x^n \right\rangle \\
 &= \left\langle \prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) (1+t)^y \middle| x^n \right\rangle
 \end{aligned}$$

$$\begin{aligned}
 &= \left\langle \partial_t \left( \prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) (1+t)^y \right) \middle| x^{n-1} \right\rangle \\
 &= \left\langle \prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) (\partial_t (1+t)^y) \middle| x^{n-1} \right\rangle \\
 &\quad + \left\langle \left( \partial_t \prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) \right) (1+t)^y \middle| x^{n-1} \right\rangle \\
 &= y \widehat{D}_{n-1}(y-1|a_1, \dots, a_r) \\
 &\quad + \left\langle \left( \partial_t \prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) \right) (1+t)^y \middle| x^{n-1} \right\rangle.
 \end{aligned}$$

Observe that

$$\begin{aligned}
 &\partial_t \prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) \\
 &= \partial_t \left( \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \prod_{j=1}^r (1+t)^{a_j} \right) \\
 &= \left( \partial_t \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \right) \prod_{j=1}^r (1+t)^{a_j} \\
 &\quad + \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \left( \partial_t \prod_{j=1}^r (1+t)^{a_j} \right) \\
 &= \frac{1}{1+t} \prod_{i=1}^r \left( \frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i} - 1} \right) \frac{\sum_{j=1}^r \left( \frac{t}{\ln(1+t)} - \frac{a_j t (1+t)^{a_j}}{(1+t)^{a_j} - 1} \right)}{t} \\
 &\quad + \frac{1}{1+t} \prod_{i=1}^r \left( \frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i} - 1} \right) \sum_{j=1}^r a_j.
 \end{aligned}$$

Thus, we have

$$\begin{aligned}
 &\left\langle \left( \partial_t \prod_{i=1}^r \left( \frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i} - 1} \right) \right) (1+t)^y \middle| x^{n-1} \right\rangle \\
 &= \left\langle \prod_{i=1}^r \left( \frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i} - 1} \right) (1+t)^{y-1} \middle| \frac{\sum_{j=1}^r \left( \frac{t}{\ln(1+t)} - \frac{a_j t (1+t)^{a_j}}{(1+t)^{a_j} - 1} \right)}{t} x^{n-1} \right\rangle \\
 &\quad + \left( \sum_{j=1}^r a_j \right) \left\langle \prod_{i=1}^r \left( \frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i} - 1} \right) (1+t)^{y-1} \middle| x^{n-1} \right\rangle \\
 &= \left( \sum_{j=1}^r a_j \right) \widehat{D}_{n-1}(y-1|a_1, \dots, a_r) \\
 &\quad + \frac{1}{n} \left\langle \prod_{i=1}^r \left( \frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i} - 1} \right) (1+t)^{y-1} \middle| \sum_{j=1}^r \left( \frac{t}{\ln(1+t)} - \frac{a_j t (1+t)^{a_j}}{(1+t)^{a_j} - 1} \right) x^n \right\rangle
 \end{aligned}$$

$$\begin{aligned}
 &= \left( \sum_{j=1}^r a_j \right) \widehat{D}_{n-1}(y-1|a_1, \dots, a_r) + \frac{r}{n} \left\langle \prod_{i=1}^r \left( \frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i} - 1} \right) (1+t)^{y-1} \left| \frac{t}{\ln(1+t)} x^n \right. \right\rangle \\
 &\quad - \frac{1}{n} \sum_{j=1}^r a_j \left\langle \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \prod_{i=1}^r \left( \frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i} - 1} \right) (1+t)^{y-1} \left| \frac{t}{\ln(1+t)} x^n \right. \right\rangle \\
 &= \left( \sum_{j=1}^r a_j \right) \widehat{D}_{n-1}(y-1|a_1, \dots, a_r) + \frac{r}{n} \left\langle \prod_{i=1}^r \left( \frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i} - 1} \right) (1+t)^{y-1} \left| \sum_{l=0}^{\infty} c_l \frac{t^l}{l!} x^n \right. \right\rangle \\
 &\quad - \frac{1}{n} \sum_{j=1}^r a_j \left\langle \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \prod_{i=1}^r \left( \frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i} - 1} \right) (1+t)^{y-1} \left| \sum_{l=0}^{\infty} c_l \frac{t^l}{l!} x^n \right. \right\rangle \\
 &= \left( \sum_{j=1}^r a_j \right) \widehat{D}_{n-1}(y-1|a_1, \dots, a_r) \\
 &\quad + \frac{r}{n} \sum_{l=0}^n c_l \binom{n}{l} \left\langle \prod_{i=1}^r \left( \frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i} - 1} \right) (1+t)^{y-1} \left| x^{n-l} \right. \right\rangle \\
 &\quad - \frac{1}{n} \sum_{j=1}^r a_j \sum_{l=0}^n c_l \binom{n}{l} \left\langle \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \prod_{i=1}^r \left( \frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i} - 1} \right) (1+t)^{y-1} \left| x^{n-l} \right. \right\rangle \\
 &= \left( \sum_{j=1}^r a_j \right) \widehat{D}_{n-1}(y-1|a_1, \dots, a_r) + \frac{r}{n} \sum_{l=0}^n \binom{n}{l} c_l \widehat{D}_{n-l}(y-1|a_1, \dots, a_r) \\
 &\quad - \frac{1}{n} \sum_{j=1}^r \sum_{l=0}^n \binom{n}{l} a_j c_l \widehat{D}_{n-l}(y-1|a_1, \dots, a_r, a_j).
 \end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
 \widehat{D}_n(x|a_1, \dots, a_r) &= \left( x + \sum_{j=1}^r a_j \right) \widehat{D}_{n-1}(x-1|a_1, \dots, a_r) \\
 &\quad + \frac{r}{n} \sum_{l=0}^n \binom{n}{l} c_l \widehat{D}_{n-l}(x-1|a_1, \dots, a_r) \\
 &\quad - \frac{1}{n} \sum_{j=1}^r \sum_{l=0}^n \binom{n}{l} a_j c_l \widehat{D}_{n-l}(x-1|a_1, \dots, a_r, a_j),
 \end{aligned}$$

which is the identity (36). □

### 3.7 Relations including the Stirling numbers of the first kind

**Theorem 7** For  $n-1 \geq m \geq 1$ , we have

$$\begin{aligned}
 &\sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l, m) D_l(a_1, \dots, a_r) \\
 &= \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1, m-1) D_l(-1|a_1, \dots, a_r) \\
 &\quad + \frac{1}{n} \sum_{l=0}^{n-m-1} \binom{n}{l+1} S_1(n-l-1, m)
 \end{aligned}$$

$$\begin{aligned}
 & \times \left( r \sum_{i=0}^{l+1} \binom{l+1}{i} c_i D_{l+1-i}(-1|a_1, \dots, a_r) \right. \\
 & \left. - \sum_{j=1}^r \sum_{i=0}^{l+1} \binom{l+1}{i} a_j c_i D_{l+1-i}(a_j - 1|a_1, \dots, a_r, a_j) \right), \tag{37} \\
 & \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l, m) \widehat{D}_l(a_1, \dots, a_r) \\
 & = \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1, m-1) \widehat{D}_l(-1|a_1, \dots, a_r) \\
 & \quad + \frac{1}{n} \sum_{l=0}^{n-m-1} \binom{n}{l+1} S_1(n-l-1, m) \\
 & \quad \times \left( r \sum_{i=0}^{l+1} \binom{l+1}{i} c_i \widehat{D}_{l+1-i}(-1|a_1, \dots, a_r) \right. \\
 & \quad \left. - \sum_{j=1}^r \sum_{i=0}^{l+1} \binom{l+1}{i} a_j c_i \widehat{D}_{l+1-i}(-1|a_1, \dots, a_r, a_j) \right) \\
 & \quad + \sum_{l=0}^{n-m-1} \binom{n-1}{l} S_1(n-l-1, m) \sum_{j=1}^r a_j \widehat{D}_l(-1|a_1, \dots, a_r). \tag{38}
 \end{aligned}$$

*Proof* We shall compute

$$\left\langle \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) (\ln(1+t))^m \middle| x^n \right\rangle$$

in two different ways. On the one hand,

$$\begin{aligned}
 & \left\langle \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) (\ln(1+t))^m \middle| x^n \right\rangle \\
 & = \left\langle \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \middle| (\ln(1+t))^m x^n \right\rangle \\
 & = \left\langle \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \middle| \sum_{l=0}^{\infty} \frac{m!}{(l+m)!} S_1(l+m, m) t^{l+m} x^n \right\rangle \\
 & = \sum_{l=0}^{n-m} \frac{m!}{(l+m)!} S_1(l+m, m) (n)_{l+m} \left\langle \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \middle| x^{n-l-m} \right\rangle \\
 & = \sum_{l=0}^{n-m} m! \binom{n}{l+m} S_1(l+m, m) D_{n-l-m}(a_1, \dots, a_r) \\
 & = \sum_{l=0}^{n-m} m! \binom{n}{l} S_1(n-l, m) D_l(a_1, \dots, a_r).
 \end{aligned}$$



On the other hand,

$$\begin{aligned}
 & \left\langle \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) (\ln(1+t))^m \middle| x^n \right\rangle \\
 &= \left\langle \partial_t \left( \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) (\ln(1+t))^m \right) \middle| x^{n-1} \right\rangle \\
 &= \left\langle \left( \partial_t \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \right) (\ln(1+t))^m \middle| x^{n-1} \right\rangle \\
 & \quad + \left\langle \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \partial_t ((\ln(1+t))^m) \middle| x^{n-1} \right\rangle. \tag{39}
 \end{aligned}$$

The second term of (39) is equal to

$$\begin{aligned}
 & \left\langle \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \partial_t ((\ln(1+t))^m) \middle| x^{n-1} \right\rangle \\
 &= m \left\langle \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) (1+t)^{-1} (\ln(1+t))^{m-1} x^{n-1} \right\rangle \\
 &= m \left\langle \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) (1+t)^{-1} \sum_{l=0}^{n-m} \frac{(m-1)!}{(l+m-1)!} S_1(l+m-1, m-1) t^{l+m-1} x^{n-1} \right\rangle \\
 &= m \sum_{l=0}^{n-m} \frac{(m-1)!}{(l+m-1)!} S_1(l+m-1, m-1) (n-1)_{l+m-1} \\
 & \quad \times \left\langle \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) (1+t)^{-1} \middle| x^{n-l-m} \right\rangle \\
 &= m! \sum_{l=0}^{n-m} \binom{n-1}{l+m-1} S_1(l+m-1, m-1) D_{n-l-m}(-1|a_1, \dots, a_r) \\
 &= m! \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1, m-1) D_l(-1|a_1, \dots, a_r).
 \end{aligned}$$

The first term of (39) is equal to

$$\begin{aligned}
 & \left\langle \left( \partial_t \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \right) (\ln(1+t))^m \middle| x^{n-1} \right\rangle \\
 &= \left\langle \partial_t \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) (\ln(1+t))^m x^{n-1} \right\rangle \\
 &= \left\langle \partial_t \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \sum_{l=0}^{n-m-1} \frac{m!}{(l+m)!} S_1(l+m, m) t^{l+m} x^{n-1} \right\rangle \\
 &= \sum_{l=0}^{n-m-1} \frac{m!}{(l+m)!} S_1(l+m, m) (n-1)_{l+m} \left\langle \partial_t \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \middle| x^{n-l-m-1} \right\rangle
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{l=0}^{n-m-1} m! \binom{n-1}{l+m} S_1(l+m, m) \\
 &\quad \times \left\langle \prod_{i=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_i} - 1} \right) (1+t)^{-1} \left| \frac{\sum_{j=1}^r \left( \frac{t}{\ln(1+t)} - \frac{a_j t(1+t)^{a_j}}{(1+t)^{a_j} - 1} \right)}{t} x^{n-l-m-1} \right. \right\rangle \\
 &= m! \sum_{l=0}^{n-m-1} \frac{1}{n-l-m} \binom{n-1}{l+m} S_1(l+m, m) \\
 &\quad \times \left\langle \prod_{i=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_i} - 1} \right) (1+t)^{-1} \left| \sum_{j=1}^r \left( \frac{t}{\ln(1+t)} - \frac{a_j t(1+t)^{a_j}}{(1+t)^{a_j} - 1} \right) x^{n-l-m} \right. \right\rangle \\
 &= \frac{m!}{n} \sum_{l=0}^{n-m-1} \binom{n}{l+1} S_1(n-1-l, m) \left( r \left\langle \prod_{i=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_i} - 1} \right) (1+t)^{-1} \left| \frac{t}{\ln(1+t)} x^{l+1} \right. \right\rangle \right. \\
 &\quad \left. - \left( \sum_{j=1}^r a_j \right) \left\langle \frac{\ln(1+t)}{(1+t)^{a_j} - 1} (1+t)^{a_j-1} \prod_{i=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_i} - 1} \right) \left| \frac{t}{\ln(1+t)} x^{l+1} \right. \right\rangle \right) \\
 &= \frac{m!}{n} \sum_{l=0}^{n-m-1} \binom{n}{l+1} S_1(n-l-1, m) \left( r \left\langle \prod_{i=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_i} - 1} \right) (1+t)^{-1} \left| \sum_{i=0}^{l+1} c_i \frac{t^i}{i!} x^{l+1} \right. \right\rangle \right. \\
 &\quad \left. - \left( \sum_{j=1}^r a_j \right) \left\langle \frac{\ln(1+t)}{(1+t)^{a_j} - 1} (1+t)^{a_j-1} \prod_{i=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_i} - 1} \right) \left| \sum_{i=0}^{l+1} c_i \frac{t^i}{i!} x^{l+1} \right. \right\rangle \right) \\
 &= \frac{m!}{n} \sum_{l=0}^{n-m-1} \binom{n}{l+1} S_1(n-l-1, m) \left( r \sum_{i=0}^{l+1} \binom{l+1}{i} c_i D_{l+1-i}(-1|a_1, \dots, a_r) \right. \\
 &\quad \left. - \sum_{j=1}^r a_j \sum_{i=0}^{l+1} \binom{l+1}{i} c_i D_{l+1-i}(a_j - 1|a_1, \dots, a_r, a_j) \right).
 \end{aligned}$$

Therefore, we have, for  $n-1 \geq m \geq 1$ ,

$$\begin{aligned}
 &m! \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l, m) D_l(a_1, \dots, a_r) \\
 &= m! \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1, m-1) D_l(-1|a_1, \dots, a_r) \\
 &\quad + \frac{m!}{n} \sum_{l=0}^{n-m-1} \binom{n}{l+1} S_1(n-l-1, m) \\
 &\quad \times \left( r \sum_{i=0}^{l+1} \binom{l+1}{i} c_{l+1-i} D_i(-1|a_1, \dots, a_r) \right. \\
 &\quad \left. - \sum_{j=1}^r \sum_{i=0}^{l+1} a_j \binom{l+1}{i} c_{l+1-i} D_i(a_j - 1|a_1, \dots, a_r, a_j) \right).
 \end{aligned}$$

Thus, we get (37).

Next, we shall compute

$$\left\langle \prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) (\ln(1+t))^m \left| x^n \right. \right\rangle$$

in two different ways. On the one hand,

$$\begin{aligned}
 & \left\langle \prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) (\ln(1+t))^m \middle| x^n \right\rangle \\
 &= \left\langle \prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) \middle| (\ln(1+t))^m x^n \right\rangle \\
 &= \left\langle \prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) \middle| \sum_{l=0}^{\infty} \frac{m!}{(l+m)!} S_1(l+m, m) t^{l+m} x^n \right\rangle \\
 &= \sum_{l=0}^{n-m} \frac{m!}{(l+m)!} S_1(l+m, m) (n)_{l+m} \left\langle \prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) \middle| x^{n-l-m} \right\rangle \\
 &= \sum_{l=0}^{n-m} m! \binom{n}{l+m} S_1(l+m, m) \widehat{D}_{n-l-m}(a_1, \dots, a_r) \\
 &= \sum_{l=0}^{n-m} m! \binom{n}{l} S_1(n-l, m) \widehat{D}_l(a_1, \dots, a_r).
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 & \left\langle \prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) (\ln(1+t))^m \middle| x^n \right\rangle \\
 &= \left\langle \partial_t \left( \prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) (\ln(1+t))^m \right) \middle| x^{n-1} \right\rangle \\
 &= \left\langle \left( \partial_t \prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) \right) (\ln(1+t))^m \middle| x^{n-1} \right\rangle \\
 &\quad + \left\langle \prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) \partial_t ((\ln(1+t))^m) \middle| x^{n-1} \right\rangle. \tag{40}
 \end{aligned}$$

The second term of (40) is equal to

$$\begin{aligned}
 & \left\langle \prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) \partial_t ((\ln(1+t))^m) \middle| x^{n-1} \right\rangle \\
 &= m \left\langle \prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) (1+t)^{-1} (\ln(1+t))^{m-1} x^{n-1} \right\rangle \\
 &= m \left\langle \prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) (1+t)^{-1} \sum_{l=0}^{n-m} \frac{(m-1)!}{(l+m-1)!} S_1(l+m-1, m-1) t^{l+m-1} x^{n-1} \right\rangle \\
 &= m \sum_{l=0}^{n-m} \frac{(m-1)!}{(l+m-1)!} S_1(l+m-1, m-1) (n-1)_{l+m-1} \\
 &\quad \times \left\langle \prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) (1+t)^{-1} \middle| x^{n-l-m} \right\rangle
 \end{aligned}$$

$$\begin{aligned}
 &= m! \sum_{l=0}^{n-m} \binom{n-1}{l+m-1} S_1(l+m-1, m-1) \widehat{D}_{n-l-m}(-1|a_1, \dots, a_r) \\
 &= m! \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1, m-1) \widehat{D}_l(-1|a_1, \dots, a_r).
 \end{aligned}$$

The first term of (40) is equal to

$$\begin{aligned}
 &\left\langle \left( \partial_t \prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) \right) (\ln(1+t))^m \Big| x^{n-1} \right\rangle \\
 &= \left\langle \partial_t \prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) \Big| (\ln(1+t))^m x^{n-1} \right\rangle \\
 &= \left\langle \partial_t \prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) \Big| \sum_{l=0}^{n-m-1} \frac{m!}{(l+m)!} S_1(l+m, m) t^{l+m} x^{n-1} \right\rangle \\
 &= \sum_{l=0}^{n-m-1} \frac{m!}{(l+m)!} S_1(l+m, m) (n-1)_{l+m} \left\langle \partial_t \prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) \Big| x^{n-l-m-1} \right\rangle.
 \end{aligned}$$

From the proof of (36), we recall

$$\begin{aligned}
 &\partial_t \prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) \\
 &= \frac{1}{1+t} \prod_{i=1}^r \left( \frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i} - 1} \right) \frac{\sum_{j=1}^r \left( \frac{t}{\ln(1+t)} - \frac{a_j t (1+t)^{a_j}}{(1+t)^{a_j} - 1} \right)}{t} \\
 &\quad + \frac{1}{1+t} \prod_{i=1}^r \left( \frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i} - 1} \right) \sum_{j=1}^r a_j.
 \end{aligned}$$

Hence, the first term of (39) is equal to

$$\begin{aligned}
 &\sum_{l=0}^{n-m-1} m! \binom{n-1}{l+m} S_1(l+m, m) \\
 &\quad \times \left( \left\langle \prod_{i=1}^r \left( \frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i} - 1} \right) (1+t)^{-1} \Big| \frac{\sum_{j=1}^r \left( \frac{t}{\ln(1+t)} - \frac{a_j t (1+t)^{a_j}}{(1+t)^{a_j} - 1} \right)}{t} x^{n-l-m-1} \right\rangle \right. \\
 &\quad \left. + \left( \sum_{j=1}^r a_j \right) \left\langle \prod_{i=1}^r \left( \frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i} - 1} \right) (1+t)^{-1} \Big| x^{n-l-m-1} \right\rangle \right) \\
 &= m! \sum_{l=0}^{n-m-1} \binom{n-1}{l} S_1(n-l-1, m) \\
 &\quad \times \left( \left\langle \prod_{i=1}^r \left( \frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i} - 1} \right) (1+t)^{-1} \Big| \frac{\sum_{j=1}^r \left( \frac{t}{\ln(1+t)} - \frac{a_j t (1+t)^{a_j}}{(1+t)^{a_j} - 1} \right)}{t} x^l \right\rangle \right)
 \end{aligned}$$

$$\begin{aligned}
 & + \left( \sum_{j=1}^r a_j \right) \left\langle \prod_{i=1}^r \left( \frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i} - 1} \right) (1+t)^{-1} \middle| x^l \right\rangle \\
 = & m! \sum_{l=0}^{n-m-1} \binom{n-1}{l} S_1(n-l-1, m) \\
 & \times \left( \frac{1}{l+1} \left\langle \prod_{i=1}^r \left( \frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i} - 1} \right) (1+t)^{-1} \middle| \sum_{j=1}^r \left( \frac{t}{\ln(1+t)} - \frac{a_j t (1+t)^{a_j}}{(1+t)^{a_j} - 1} \right) x^{l+1} \right\rangle \right. \\
 & + \left. \left( \sum_{j=1}^r a_j \right) \left\langle \prod_{i=1}^r \left( \frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i} - 1} \right) (1+t)^{-1} \middle| x^l \right\rangle \right) \\
 = & m! \sum_{l=0}^{n-m-1} \binom{n-1}{l} S_1(n-l-1, m) \\
 & \times \left( \frac{r}{l+1} \left\langle \prod_{i=1}^r \left( \frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i} - 1} \right) (1+t)^{-1} \middle| \frac{t}{\ln(1+t)} x^{l+1} \right\rangle \right. \\
 & - \frac{1}{l+1} \left( \sum_{j=1}^r a_j \right) \left\langle \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \prod_{i=1}^r \left( \frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i} - 1} \right) (1+t)^{-1} \middle| \frac{t}{\ln(1+t)} x^{l+1} \right\rangle \\
 & + \left. \left( \sum_{j=1}^r a_j \right) \left\langle \prod_{i=1}^r \left( \frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i} - 1} \right) (1+t)^{-1} \middle| x^l \right\rangle \right) \\
 = & m! \sum_{l=0}^{n-m-1} \binom{n-1}{l} S_1(n-l-1, m) \\
 & \times \left( \frac{r}{l+1} \left\langle \prod_{i=1}^r \left( \frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i} - 1} \right) (1+t)^{-1} \middle| \sum_{i=0}^{l+1} c_i \frac{t^i}{i!} x^{l+1} \right\rangle \right. \\
 & - \frac{1}{l+1} \left( \sum_{j=1}^r a_j \right) \left\langle \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \prod_{i=1}^r \left( \frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i} - 1} \right) (1+t)^{-1} \middle| \sum_{i=0}^{l+1} c_i \frac{t^i}{i!} x^{l+1} \right\rangle \\
 & + \left. \left( \sum_{j=1}^r a_j \right) \left\langle \prod_{i=1}^r \left( \frac{(1+t)^{a_i} \ln(1+t)}{(1+t)^{a_i} - 1} \right) (1+t)^{-1} \middle| x^l \right\rangle \right) \\
 = & m! \sum_{l=0}^{n-m-1} \binom{n-1}{l} S_1(n-l-1, m) \\
 & \times \left( \frac{r}{l+1} \sum_{i=0}^{l+1} \binom{l+1}{i} c_i \widehat{D}_{l+1-i}(-1|a_1, \dots, a_r) \right. \\
 & - \frac{1}{l+1} \sum_{j=1}^r a_j \sum_{i=0}^{l+1} \binom{l+1}{i} c_i \widehat{D}_{l+1-i}(-1|a_1, \dots, a_r, a_j) + \sum_{j=1}^r a_j \widehat{D}_l(-1|a_1, \dots, a_r) \left. \right) \\
 = & \frac{m!}{n} \sum_{l=0}^{n-m-1} \binom{n}{l+1} S_1(n-l-1, m) \\
 & \times \left( r \sum_{i=0}^{l+1} \binom{l+1}{i} c_i \widehat{D}_{l+1-i}(-1|a_1, \dots, a_r) \right)
 \end{aligned}$$

$$\begin{aligned}
 & - \sum_{j=1}^r \sum_{i=0}^{l+1} \binom{l+1}{i} a_j c_i \widehat{D}_{l+1-i}(-1|a_1, \dots, a_r, a_j) \\
 & + m! \sum_{l=0}^{n-m-1} \binom{n-1}{l} S_1(n-l-1, m) \sum_{j=1}^r a_j \widehat{D}_l(-1|a_1, \dots, a_r).
 \end{aligned}$$

Therefore, we get (38). □

### 3.8 Relations with the falling factorials

#### Theorem 8

$$D_n(x|a_1, \dots, a_r) = \sum_{m=0}^n \binom{n}{m} D_{n-m}(a_1, \dots, a_r)(x)_m, \tag{41}$$

$$\widehat{D}_n(x|a_1, \dots, a_r) = \sum_{m=0}^n \binom{n}{m} \widehat{D}_{n-m}(a_1, \dots, a_r)(x)_m. \tag{42}$$

*Proof* For (13) and (23), assume that  $D_n(x|a_1, \dots, a_r) = \sum_{m=0}^n C_{n,m}(x)_m$ . By (12), we have

$$\begin{aligned}
 C_{n,m} &= \frac{1}{m!} \left\langle \frac{1}{\prod_{j=1}^r \left( \frac{e^{a_j \ln(1+t)} - 1}{\ln(1+t)} \right)} t^m \middle| x^n \right\rangle \\
 &= \frac{1}{m!} \left\langle \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \middle| t^m x^n \right\rangle \\
 &= \binom{n}{m} \left\langle \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) \middle| x^{n-m} \right\rangle \\
 &= \binom{n}{m} D_{n-m}(a_1, \dots, a_r).
 \end{aligned}$$

Thus, we get the identity (41).

Similarly, for (13) and (23), assume that  $\widehat{D}_n(x|a_1, \dots, a_r) = \sum_{m=0}^n C_{n,m}(x)_m$ . By (12), we have

$$\begin{aligned}
 C_{n,m} &= \frac{1}{m!} \left\langle \frac{1}{\prod_{j=1}^r \left( \frac{e^{a_j \ln(1+t)} - 1}{e^{a_j \ln(1+t)} \ln(1+t)} \right)} t^m \middle| x^n \right\rangle \\
 &= \frac{1}{m!} \left\langle \prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) \middle| t^m x^n \right\rangle \\
 &= \binom{n}{m} \left\langle \prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) \middle| x^{n-m} \right\rangle \\
 &= \binom{n}{m} \widehat{D}_{n-m}(a_1, \dots, a_r).
 \end{aligned}$$

Thus, we get the identity (42). □

### 3.9 Relations with higher-order Frobenius-Euler polynomials

For  $\lambda \in \mathbb{C}$  with  $\lambda \neq 1$ , the Frobenius-Euler polynomials of order  $r$ ,  $H_n^{(r)}(x|\lambda)$  are defined by the generating function

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(x|\lambda) \frac{t^n}{n!}$$

(see e.g. [11, 12]).

#### Theorem 9

$$D_n(x|a_1, \dots, a_r) = \sum_{m=0}^n \left( \sum_{j=0}^{n-m} \sum_{l=0}^{n-m-j} \binom{s}{j} \binom{n-j}{l} (n)_j \right. \\ \left. \times (1-\lambda)^{-j} S_1(n-j-l, m) D_l(a_1, \dots, a_r) \right) H_m^{(s)}(x|\lambda), \quad (43)$$

$$\widehat{D}_n(x|a_1, \dots, a_r) = \sum_{m=0}^n \left( \sum_{j=0}^{n-m} \sum_{l=0}^{n-m-j} \binom{s}{j} \binom{n-j}{l} (n)_j \right. \\ \left. \times (1-\lambda)^{-j} S_1(n-j-l, m) \widehat{D}_l(a_1, \dots, a_r) \right) H_m^{(s)}(x|\lambda). \quad (44)$$

*Proof* For (13) and

$$H_n^{(s)}(x|\lambda) \sim \left( \left( \frac{e^t-\lambda}{1-\lambda} \right)^s, t \right), \quad (45)$$

assume that  $D_n(x|a_1, \dots, a_r) = \sum_{m=0}^n C_{n,m} H_m^{(s)}(x|\lambda)$ . By (12), similarly to the proof of (37), we have

$$C_{n,m} = \frac{1}{m!} \left\langle \frac{(e^{\ln(1+t)}-\lambda)^s}{1-\lambda} (\ln(1+t))^m \middle| x^n \right\rangle \\ = \frac{1}{m!(1-\lambda)^s} \left\langle \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) (\ln(1+t))^m (1-\lambda+t)^s \middle| x^n \right\rangle \\ = \frac{1}{m!(1-\lambda)^s} \left\langle \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) (\ln(1+t))^m \middle| \sum_{i=0}^{\min\{s,n\}} \binom{s}{i} (1-\lambda)^{s-i} t^i x^n \right\rangle \\ = \frac{1}{m!(1-\lambda)^s} \sum_{i=0}^{n-m} \binom{s}{i} (1-\lambda)^{s-i} (n)_i \left\langle \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j}-1} \right) \middle| (\ln(1+t))^m x^{n-i} \right\rangle \\ = \frac{1}{m!(1-\lambda)^s} \sum_{i=0}^{n-m} \binom{s}{i} (1-\lambda)^{s-i} (n)_i \sum_{l=0}^{n-m-i} m! \binom{n-i}{l} S_1(n-i-l, m) D_l(a_1, \dots, a_r) \\ = \sum_{i=0}^{n-m} \sum_{l=0}^{n-m-i} \binom{s}{i} \binom{n-i}{l} (n)_i (1-\lambda)^{-i} S_1(n-i-l, m) D_l(a_1, \dots, a_r).$$

Thus, we get the identity (43).

Next, for (14) and (45), assume that  $\widehat{D}_n(x|a_1, \dots, a_r) = \sum_{m=0}^n C_{n,m} H_m^{(s)}(x|\lambda)$ . By (12), similarly to the proof of (38), we have

$$\begin{aligned} C_{n,m} &= \frac{1}{m!} \left\langle \frac{\left(\frac{e^{\ln(1+t)} - \lambda}{1-\lambda}\right)^s}{\prod_{j=1}^r \left(\frac{e^{a_j \ln(1+t)} - 1}{e^{a_j \ln(1+t)} \ln(1+t)}\right)} (\ln(1+t))^m \middle| x^n \right\rangle \\ &= \frac{1}{m!(1-\lambda)^s} \left\langle \prod_{j=1}^r \left(\frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1}\right) (\ln(1+t))^m \middle| (1-\lambda+t)^s x^n \right\rangle \\ &= \frac{1}{m!(1-\lambda)^s} \left\langle \prod_{j=1}^r \left(\frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1}\right) (\ln(1+t))^m \middle| \sum_{i=0}^{\min\{s,n\}} \binom{s}{i} (1-\lambda)^{s-i} t^i x^n \right\rangle \\ &= \frac{1}{m!(1-\lambda)^s} \sum_{i=0}^{n-m} \binom{s}{i} (1-\lambda)^{s-i} (n)_i \left\langle \prod_{j=1}^r \left(\frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1}\right) \middle| (\ln(1+t))^m x^{n-i} \right\rangle \\ &= \frac{1}{m!(1-\lambda)^s} \sum_{i=0}^{n-m} \binom{s}{i} (1-\lambda)^{s-i} (n)_i \sum_{l=0}^{n-m-i} m! \binom{n-i}{l} S_1(n-i-l, m) \widehat{D}_l(a_1, \dots, a_r) \\ &= \sum_{i=0}^{n-m} \sum_{l=0}^{n-m-i} \binom{s}{i} \binom{n-i}{l} (n)_i (1-\lambda)^{-i} S_1(n-i-l, m) \widehat{D}_l(a_1, \dots, a_r). \end{aligned}$$

Thus, we get the identity (44). □

### 3.10 Relations with higher-order Bernoulli polynomials

Bernoulli polynomials  $\mathfrak{B}_n^{(r)}(x)$  of order  $r$  are defined by

$$\left(\frac{t}{e^t - 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} \frac{\mathfrak{B}_n^{(r)}(x)}{n!} t^n$$

(see e.g. [5, Section 2.2]). In addition, Cauchy numbers of the first kind  $\mathfrak{C}_n^{(r)}$  of order  $r$  are defined by

$$\left(\frac{t}{\ln(1+t)}\right)^r = \sum_{n=0}^{\infty} \frac{\mathfrak{C}_n^{(r)}}{n!} t^n$$

(see e.g. [13, (2.1)], [14, (6)]).

#### Theorem 10

$$\begin{aligned} D_n(x|a_1, \dots, a_r) &= \sum_{m=0}^n \left( \sum_{i=0}^{n-m} \sum_{l=0}^{n-m-i} \binom{n}{i} \binom{n-i}{l} \mathfrak{C}_i^{(s)} S_1(n-i-l, m) D_l(a_1, \dots, a_r) \right) \mathfrak{B}_m^{(s)}(x), \end{aligned} \tag{46}$$

$$\begin{aligned} \widehat{D}_n(x|a_1, \dots, a_r) &= \sum_{m=0}^n \left( \sum_{i=0}^{n-m} \sum_{l=0}^{n-m-i} \binom{n}{i} \binom{n-i}{l} \mathfrak{C}_i^{(s)} S_1(n-i-l, m) \widehat{D}_l(a_1, \dots, a_r) \right) \mathfrak{B}_m^{(s)}(x). \end{aligned} \tag{47}$$



*Proof* For (13) and

$$\mathfrak{B}_n^{(s)}(x) \sim \left( \left( \frac{e^t - 1}{t} \right)^s, t \right), \tag{48}$$

assume that  $D_n(x|a_1, \dots, a_r) = \sum_{m=0}^n C_{n,m} \mathfrak{B}_m^{(s)}(x)$ . By (12), similarly to the proof of (37), we have

$$\begin{aligned} C_{n,m} &= \frac{1}{m!} \left\langle \frac{(e^{\ln(1+t)} - 1)^s}{\prod_{j=1}^r (e^{a_j \ln(1+t)} - 1)} (\ln(1+t))^m \middle| x^n \right\rangle \\ &= \frac{1}{m!} \left\langle \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) (\ln(1+t))^m \middle| \left( \frac{t}{\ln(1+t)} \right)^s x^n \right\rangle \\ &= \frac{1}{m!} \left\langle \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) (\ln(1+t))^m \middle| \sum_{i=0}^{\infty} \mathfrak{C}_i^{(s)} \frac{t^i}{i!} x^n \right\rangle \\ &= \frac{1}{m!} \sum_{i=0}^{n-m} \mathfrak{C}_i^{(s)} \binom{n}{i} \left\langle \prod_{j=1}^r \left( \frac{\ln(1+t)}{(1+t)^{a_j} - 1} \right) (\ln(1+t))^m \middle| x^{n-i} \right\rangle \\ &= \frac{1}{m!} \sum_{i=0}^{n-m} \mathfrak{C}_i^{(s)} \binom{n}{i} \sum_{l=0}^{n-m-i} m! \binom{n-i}{l} S_1(n-i-l, m) D_l(a_1, \dots, a_r) \\ &= \sum_{i=0}^{n-m} \sum_{l=0}^{n-m-i} \binom{n}{i} \binom{n-i}{l} \mathfrak{C}_i^{(s)} S_1(n-i-l, m) D_l(a_1, \dots, a_r). \end{aligned}$$

Thus, we get the identity (46).

Next, for (13) and (48), assume that  $\widehat{D}_n(x|a_1, \dots, a_r) = \sum_{m=0}^n C_{n,m} \mathfrak{B}_m^{(s)}(x)$ . By (12), similarly to the proof of (38), we have

$$\begin{aligned} C_{n,m} &= \frac{1}{m!} \left\langle \frac{(e^{\ln(1+t)} - 1)^s}{\prod_{j=1}^r \left( \frac{e^{a_j \ln(1+t)} - 1}{e^{a_j \ln(1+t)}} \right)} (\ln(1+t))^m \middle| x^n \right\rangle \\ &= \frac{1}{m!} \left\langle \prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) (\ln(1+t))^m \middle| \left( \frac{t}{\ln(1+t)} \right)^s x^n \right\rangle \\ &= \frac{1}{m!} \left\langle \prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) (\ln(1+t))^m \middle| \sum_{i=0}^{\infty} \mathfrak{C}_i^{(s)} \frac{t^i}{i!} x^n \right\rangle \\ &= \frac{1}{m!} \sum_{i=0}^{n-m} \mathfrak{C}_i^{(s)} \binom{n}{i} \left\langle \prod_{j=1}^r \left( \frac{(1+t)^{a_j} \ln(1+t)}{(1+t)^{a_j} - 1} \right) (\ln(1+t))^m \middle| x^{n-i} \right\rangle \\ &= \frac{1}{m!} \sum_{i=0}^{n-m} \mathfrak{C}_i^{(s)} \binom{n}{i} \sum_{l=0}^{n-m-i} m! \binom{n-i}{l} S_1(n-i-l, m) \widehat{D}_l(a_1, \dots, a_r) \\ &= \sum_{i=0}^{n-m} \sum_{l=0}^{n-m-i} \binom{n}{i} \binom{n-i}{l} \mathfrak{C}_i^{(s)} S_1(n-i-l, m) \widehat{D}_l(a_1, \dots, a_r). \end{aligned}$$

Thus, we get the identity (47). □

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

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