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Local bifurcation of limit cycles and integrability of a class of nilpotent systems

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Abstract

In this article, center conditions and bifurcation of limit cycles at the nilpotent critical point in a class of seventh-degree systems are investigated. With the help of computer algebra system MATHEMATICA, the first 13 quasi-Lyapunov constants are deduced. As a result, sufficient and necessary conditions in order to have a center are obtained. The result that there exist 13 small amplitude limit cycles created from the three-order nilpotent critical point is also proved. Henceforth, we give a lower bound of cyclicity of three-order nilpotent critical point for seventh-degree nilpotent systems.

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Keywords: three-order nilpotent critical point, center-focus problem, bifurcation of limit cycles, quasi-Lyapunov constant

1 Introduction

Computation of Lyapunov quantities is a hot topic with large number of articles per year, but it is very difficult to obtain general results. The methods of computation of Lyapunov quantities when the critical points are non-degenerate have been greatly developed by many mathematicians. The method in [1,2] is based on the sequential construction of Lyapunov functions. Furthermore, computing Lyapunov quantities using the reduction of system to normal form could be seen in [3-5]. Another approach to numerical computation of Lyapunov quantities which uses the passage to the polar coordinates and the procedure of sequential construction of solution approximations is related with the obtaining of approximations of system solution, see [2], they also could be seen in [6,7]. But computations of Lyapunov quantities become difficult when the critical points are degenerate because the method of the Poincaré formal series cannot be used in order to compute Lyapunov constants in a neighborhood of the critical point.

The nilpotent center problem was investigated by Moussu [8] and Stróżyna and Żołądek [9]. In [10], Takens proved that Lyapunov system can be formally transformed into a generalized Liénard system. Furthermore, in [11], Álvarez and Gasull proved that the generalized Lienard system could be simplified even more by a reparametrization of the time. At the same time, Giacomini et al. [12,13] proved that the analytic nilpotent systems with a center can be expressed as limit of non-degenerate systems with a center.

As far as we know, there are essentially three differential ways of obtaining Lyapunov constant for nilpotent critical points in theory: by using normal form theory [14], by computing the Poincaré return map [15] or by using Lyapunov functions [16]. Álvarez investigated the momodromy and stability for nilpotent critical points with the method of computing the Poincaré return map, see for instance [17]; Chavarriga et al. investigated the local analytic integrability for nilpotent centers by using Lyapunov functions, see for instance [18]; Moussu investigated the center-focus problem of nilpotent critical points with the method of normal form theory, see for instance [8].

Nevertheless, these methods are so complicated that it is hard to use for an analytic system with a monodromic point, even in the case of a concrete polynomial systems. So there are very few results known for concrete differential systems with monodromic nilpotent critical points. Gasull and Torregrosa [19] have generalized the scheme of computation of Lyapunov constants for systems of the form

$$\begin{aligned}\dot{x} &= y + \sum_{k\geq n+1} F_k(x, y),\\ \dot{y} &= -x^{2n-1} + \sum_{k\geq 2n} G_k(x, y),\end{aligned} \tag{1.1}$$

where F_k and G_k are (1, n)-quasi-homogeneous functions of degree k. Chavarriga et al. investigated the integrability of centers perturbed by (p, q)-quasi-homogeneous polynomials in [20]. Fortunately, Yirong Liu and Jibin Li [21] found that there always exists a formal inverse integrating factor for three-order nilpotent critical points in 2009, and they gave a new definition of the focal values under the generalized triangle polar coordinates and the method of commuting Lyapunov constants using the inverse integral factors for the three-order nilpotent critical point.

For a given family of polynomial differential equations, the number of Lyapunov constants needed to solve the center-focus problem is also related with the so-called cyclicity of the point, i.e., the number of limit cycles that appear from it by small perturbations of the coefficients of the given differential equation inside the family considered (see, [22] for cases where this relation does not exist for the case of nondegenerate centers). Let $N(n)$ be the maximum possible number of limit cycles bifurcating from nilpotent critical points for analytic vector fields of degree n. It was found that $N(3) \geq 2$, $N(5) \geq 5$, $N(7) \geq 9$ in [23], $N(3) \geq 3$, $N(5) \geq 5$ in [17], and for Kukles system with six parameters $N(3) \geq 3$ in [11]. Recently, Yirong Liu and Jibin Li proved that $N(3) \geq 8$ in [24]. In this article, by employing the inverse integral factor method introduced in [21], we consider a planar septic ordinary differential equation having a three-order nilpotent critical point with the form

$$\begin{aligned}\frac{dx}{dt} &= \mu y + \mu x^3 - \mu x^2 y + a_{12} x y^2 + a_{03} y^3 + \left(1 - \frac{71275\mu}{378}\right) x^4 y + a_{32} x^3 y^2\\ &\quad + a_{23} x^2 y^3 + a_{14} x y^4 + a_{05} y^5 - \mu y (x^2 + y^2)^3,\\ \frac{dy}{dt} &= -2\mu x^3 + b_{21} x^2 y + \mu x y^2 + b_{03} y^3 - 2\left(1 - \frac{71275\mu}{378}\right) x^3 y^2 + b_{23} x^2 y^3\\ &\quad + b_{14} x y^4 + b_{05} y^5 + \mu x (x^2 + y^2)^3.\end{aligned} \tag{1.2}$$

We will prove $N(7) \geq 13$. To the best of authors' knowledge, their result on the lower bounds of cyclicity of three-order nilpotent critical points for septic systems is new. It is helpful to Hilbert's 16th problem.

The rest of the article is organized as follows. In Section 2, some preliminary knowledge given in [21] which is useful throughout the article are introduced. In Section 3, using the linear recursive formulae in [21] to do direct computation, the first 13 quasi-Lyapunov constants and the sufficient and necessary conditions of center are obtained. This article is ended with Section 4 in which the 13-order weak focus conditions and the result that there exist 13 limit cycles in the neighborhood of the three-order nilpotent critical point is proved.

2 Preliminary knowledge

The idea of this section comes from [21,24], where the center-focus problem of three-order nilpotent critical points of the planar dynamical systems is studied. For more details, please refer to [21,24]. We will recall the related notions and results. The origin of system

$$\begin{aligned} \frac{dx}{dt} &= y + \sum_{i+j=2}^{\infty} a_{ij}x^i y^j = X(x, y), \\ \frac{dy}{dt} &= \sum_{i+j=2}^{\infty} b_{ij}x^i y^j = Y(x, y). \end{aligned} \tag{2.1}$$

is a three-order monodromic critical point if and only if the system could be written as follows:

$$\begin{aligned} \frac{dx}{dt} &= y + \mu x^2 + \sum_{i+2j=3}^{\infty} a_{ij}x^i y^j = X(x, y), \\ \frac{dy}{dt} &= -2x^3 + 2\mu xy + \sum_{i+2j=4}^{\infty} b_{ij}x^i y^j = Y(x, y). \end{aligned} \tag{2.2}$$

Under the transformation of generalized polar coordinates

$$x = r\cos\theta, y = r^2 \sin\theta, \tag{2.3}$$

system (2.2) can be changed into

$$\frac{dr}{d\theta} = \frac{-\cos\theta[\sin\theta(1 - 2\cos^2\theta) + \mu(\cos^2\theta + 2\sin^2\theta)]}{2(\cos^4\theta + \sin^2\theta)} r + o(r). \tag{2.4}$$

In a small neighborhood, we can define the successor function of system (2.2) as follows:

$$\Delta(h) = \tilde{r}(-2\pi, h) - h = \sum_{k=2}^{\infty} v_k(-2\pi)h^k. \tag{2.5}$$

We have the following result.

Lemma 2.1. *For any positive integer m,* $v_{2m+1}(-2\pi)$ *has the form*

$$v_{2m+1}(-2\pi) = \sum_{k=1}^{m} \varsigma_k^{(m)} v_{2k}(-2\pi), \tag{2.6}$$

where $\varsigma_k^{(m)}$ *is a polynomial of* $v_j(\pi)$, $v_j(2\pi)$, $v_j(-2\pi)$, $(j = 2,3,\ldots, 2m)$ *with rational coefficients.*

It is different from the center-focus problem for the elementary critical points, we know from Lemma 2.1 that when $k > 1$ for the first non-zero $v_k(-2\pi)$, k is an even integer.

Definition 2.1. *1. For any positive integer m,* $v_{2m}(-2\pi)$ *is called the m-order focal value of system* (2.2) *in the origin.*

2. If $v_2(-2\pi) \neq 0$, *the origin of system* (2.2) *is called 1-order weak focus. If there is an integer* $m > 1$, *such that* $v_2(-2\pi) = v_4(-2\pi) = \ldots = v_{2m-2}(-2\pi) = 0$, $v_{2m}(-2\pi) \neq 0$, *then, the origin of system* (2.2) *is called m-order weak focus.*

3. If for all positive integer m, we have $v_{2m}(-2\pi) = 0$, *then, the origin of system* (2.2) *is called a center.*

Consider the system

$$\begin{aligned} \frac{dx}{dt} &= \delta x + y + \sum_{k+j=2}^{\infty} a_{kj}(\gamma) x^k y^j, \\ \frac{dy}{dt} &= 2\delta y + \sum_{k+j=2}^{\infty} b_{kj}(\gamma) x^k y^j, \end{aligned} \tag{2.7}$$

where $\gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_{m-1}$ is $(m\text{-}1)$-dimensional parameter vector. Let $\gamma_0 = \{\gamma_1^{(0)}, \gamma_2^{(0)}, \ldots, \gamma_{m-1}^{(0)}\}$ be a point at the parameter space. Suppose that for $||\gamma - \gamma_0|| \ll 1$, the functions of the right hand of system (2.7) are power series of x, y with a non-zero convergence radius and have continuous partial derivatives with respect to γ. In addition,

$$a_{20}(\gamma) \equiv \mu, b_{20}(\gamma) \equiv 0, b_{11}(\gamma) \equiv 2\mu, b_{30}(\gamma) \equiv -2. \tag{2.8}$$

For an integer k, letting $v_{2k}(-2\pi, \gamma)$ be the k-order focal value of the origin of system $(2.7)_{\delta = 0}$.

Theorem 2.1. *If for* $\gamma = \gamma_0$, *the origin of system* $(2.7)_{\delta = 0}$ *is a m-order weak focus, and the Jacobin*

$$\frac{\partial(v_2, v_4, \ldots, v_{2m-2})}{\partial(\gamma_1, \gamma_2, \ldots, \gamma_{m-1})} |_{\gamma=\gamma_0} \neq 0, \tag{2.9}$$

then, there exist two positive numbers δ^* *and* γ^*, *such that for* $0 < |\delta| < \delta^*$, $0 < ||\gamma - \gamma_0|| < \gamma^*$, *in a neighborhood of the origin, system* (2.7) *has at most m limit cycles which enclose the origin (an elementary node)* $O(0,0)$. *In addition, under the above conditions, there exist* $\tilde{\gamma}, \tilde{\delta}$, *such that when* $\gamma = \tilde{\gamma}, \delta = \tilde{\delta}$, *there exist exact m limit cycles of* (2.7) *in a small neighborhood of the origin.*

The following key results which define the quasi-Lyapunov constants and provide a way of computing them were also given by Liu and Li [24].

Theorem 2.2. *If the origin of system* (2.2) *is a s-class or ∞-class, one can construct successively the terms of the formal power series* $M(x, y) = x^4 + y^2 + o(r^4)$, *such that*

$$\frac{\partial}{\partial x}\left(\frac{X}{M^{s+1}}\right) + \frac{\partial}{\partial y}\left(\frac{Y}{M^{s+1}}\right) = \frac{1}{M^{s+2}}\sum_{m=1}^{\infty}(2m - 4s - 1)\lambda_m[x^{2m+4} + o(r^{2m+4})], \quad (2.10)$$

Theorem 2.3. *For system* (2.2), *if there exists a natural number s and a formal series* $M(x,y) = x^4 + y^2 + o(r^4)$, *such that* (2.10) *holds, then*

$$\{v_{2m}(-2\pi)\} \sim [\sigma_m(s, \mu)\lambda_m\}, \quad (2.11)$$

where

$$\sigma_m(s, \mu) = \frac{1}{2}\int_0^{2\pi}\frac{(1 + \sin^2\theta)\cos^{2m+4}\theta}{(\cos^4\theta + \sin^2\theta)^{s+2}}v_1^{2m-4s-1}(\theta)d\theta. \quad (2.12)$$

Definition 2.2. *If there exists a natural number s and a formal series* $M(x, y) = x^4 + y^2 + o(r^4)$, *such that* (2.10) *holds, then,* λ_m *is called the m-th quasi-Lyapunov constants of the origin of system* (2.2).

Theorem 2.4. *For any positive integer s and a given number sequence*

$$\{c_{0\beta}\}, \beta \geq 3, \quad (2.13)$$

one can construct successively the terms with the coefficients $c_{\alpha\beta}$ *satisfying* $\alpha \neq 0$ *of the formal series*

$$M(x, y) = y^2 + \sum_{\alpha+\beta=3}^{\infty} c_{\alpha\beta}x^{\alpha}y^{\beta} = \sum_{k=2}^{\infty} M_k(x, y), \quad (2.14)$$

such that

$$\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right)M - (s + 1)\left(\frac{\partial M}{\partial x}X + \frac{\partial M}{\partial y}Y\right) = \sum_{m=3}^{\infty}\omega_m(s, \mu)x^m. \quad (2.15)$$

where for all k, $M_k(x, y)$ *is a k-homogeneous polynomial of x, y and* $s\mu = 0$.

It is easy to see that (2.15) is linear with respect to the function M, so that we can easily find the following recursive formulae for the calculation of $c_{\alpha\beta}$ and $\omega_m(s, \mu)$.

Theorem 2.5. *For* $\alpha \geq 1$, $\alpha + \beta \geq 3$ *in* (2.14) *and* (2.15), $c_{\alpha\beta}$ *can be uniquely determined by the recursive formula*

$$c_{\alpha\beta} = \frac{1}{(s + 1)\alpha}(A_{\alpha-1,\beta+1} + B_{\alpha-1,\beta+1}). \quad (2.16)$$

For $m \geq 1$, $\omega_m(s, \mu)$ *can be uniquely determined by the recursive formula*

$$\omega_m(s, \mu) = A_{m,0} + B_{m,0}, \quad (2.17)$$

where

$$A_{\alpha\beta} = \sum_{k+j=2}^{\alpha+\beta-1}[k - (s + 1)(\alpha - k + 1)]a_{kj}c_{\alpha-k+1,\beta-j},$$
$$B_{\alpha\beta} = \sum_{k+j=2}^{\alpha+\beta-1}[j - (s + 1)(\beta - j + 1)]b_{kj}c_{\alpha-k,\beta-j+1}. \quad (2.18)$$

Notice that in (2.18), *we set*

$$\begin{aligned} &c_{00} = c_{10} = c_{01} = 0, \\ &c_{20} = c_{11} = 0, c_{02} = 1, \\ &c_{\alpha\beta} = 0, \text{ if } \alpha < 0 \text{ or } \beta < 0. \end{aligned} \tag{2.19}$$

We see from Theorem 2.2 that if the origin of system (2.2) is s-class or ∞-class, then, by choosing $\{c_{\alpha\beta}\}$, such that

$$\omega_{2k+1}(s, \mu) = 0, k = 1, 2, \ldots, \tag{2.20}$$

we can obtain a solution group of $\{c_{\alpha\beta}\}$ of (2.20), thus, we have

$$\lambda_m = \frac{\omega_{2m+4}(s, \mu)}{2m - 4s - 1}. \tag{2.21}$$

Clearly, the recursive formulae by Theorem 2.5 is linear with respect to all $c_{\alpha\beta}$. Therefore, it is convenient to perform the computations by using computer algebraic system like MATHEMATICA.

3 Quasi-Lyapunov constants and center conditions

Theorem 2.4 implies that we can find a positive integer s and a formal series $M(x,y) = x^4 + y^2 + o(r^4)$ for system (1.2) such that (2.15) holds. Meanwhile, with the help of MATH-EMATICA, by applying the recursive formulae presented in Theorem 2.5 to carry out calculations, we have

$$\begin{aligned} &\omega_3 = \omega_4 = \omega_5 = 0, \\ &\omega_6 = -\frac{1}{3\mu}(b_{21} + 3\mu)(-1 + 4s), \\ &\omega_7 = 3(s + 1)c_{03}, \\ &\omega_8 = -\frac{2}{5\mu}(a_{12} + 3b_{03})(-3 + 4s). \end{aligned} \tag{3.1}$$

(2.21) and (3.1) implies that the first two quasi-Lyapunov constants of system (1.2):

$$\begin{aligned} &\lambda_1 = \frac{\omega_6}{1 - 4s} = -\frac{1}{3\mu}(b_{21} + 3\mu), \\ &\lambda_2 = \frac{\omega_8}{3 - 4s} = \frac{2}{5\mu}(a_{12} + 3b_{03}). \end{aligned} \tag{3.2}$$

we see from $\omega_7 = 0$ that

$$c_{03} = 0. \tag{3.3}$$

Furthermore, the following conclusion holds.

Proposition 3.1. *For system* (1.2), *one can determine successively the terms of the formal series* $M(x, y) = x^4 + y^2 + o(r^4)$, *such that*

$$\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) M - 2\left(\frac{\partial M}{\partial x} X + \frac{\partial M}{\partial y} Y\right) = \sum_{m=1}^{13} \lambda_m [(2m - 5)x^{2m+4} + o(r^{30})], \tag{3.4}$$

where λ_m *is the mth quasi-Lyapunov constant at the origin of system* (1.2), $m = 1, 2, \ldots, 13$.

Theorem 3.1. *For system* (1.2), *the first 13 quasi-Lyapunov constants at the origin are given by*

$$
\begin{aligned}
\lambda_1 &= -\frac{1}{3\mu}(b_{21}+3\mu),\\
\lambda_2 &= \frac{2}{5\mu}(a_{12}+3b_{03}),\\
\lambda_3 &= \frac{2}{7\mu}(a_{32}+b_{23}),\\
\lambda_4 &= \frac{4}{15\mu}(a_{14}-a_{23}+5b_{05}-2b_{14}),\\
\lambda_5 &= -\frac{40}{77\mu^2}(a_{23}+2b_{14})(-b_{03}+\mu),\\
\lambda_6 &= -\frac{2}{7371\mu^2}(a_{23}+2b_{14})(-756+252a_{32}+142739\mu),\\
\lambda_7 &= -\frac{8}{6237\mu^2}(a_{23}+2b_{14})(189-378b_{05}+189b_{14}-35516\mu),\\
\lambda_8 &= -\frac{8}{125307\mu^2}(a_{23}+2b_{14})(-945+2646a_{03}+231391\mu),\\
\lambda_9 &= -\frac{1}{358435\mu^2}(a_{23}+2b_{14})(-34020+4116a_{23}+1372b_{14}+5024095\mu),\\
\lambda_{10} &= \frac{1}{386822709\mu^3}(a_{23}+2b_{14})-714420-594712314\mu-14002632a_{05}\mu+90016920a_{23}\mu\\
&\quad+102674821303\mu^2),\\
\lambda_{11} &= -\frac{17}{6544091169\mu^3}(a_{23}+2b_{14})(-359281818+42007896a_{23}+36356476968\mu+2196158788177\mu^2),\\
\lambda_{12} &= -\frac{1}{35823651080490\mu^4}(a_{23}+2b_{14})(-26843045544+15035266598814\mu-2015039855231046\mu^2\\
&\quad-154659205390636303\mu^3+30287803437510297551\mu^4),\\
\lambda_{13} &= -\frac{1}{1549838616695238870\mu^4}(a_{23}+2b_{14})(-153852012319592664+349339887516834993324\mu\\
&\quad+10292695776995494843134\mu^2-3251791005302797375353453\mu^3\\
&\quad+21854333536925240101931559 1\mu^4).
\end{aligned}
\tag{3.5}
$$

In the above expressions of λ_k, we have already, let $\lambda_1 = \lambda_2 = \cdots = \lambda_{k-1} = 0$, $k = 2,\ldots,$ 13.

It follows from Theorem 3.1 that

Proposition 3.2. *The first 13 quasi-Lyapunov constants at the origin of system* (1.2) *are zero if and only if the following condition is satisfied:*

$$b_{21} = -3\mu,\ a_{12} = -3b_{03},\ b_{23} = -a_{32},\ a_{14} = -5b_{05},\ a_{23} = -2b_{14}.$$

When the condition of Proposition 3.2 holds, system (1.2) can be brought to

$$
\begin{aligned}
\frac{dx}{dt} &= \mu y + \mu x^3 - \mu x^2 y - 3b_{03}xy^2 + a_{03}y^3 + \left(1 - \frac{71275\mu}{378}\right)x^4 y + a_{32}x^3y^2\\
&\quad - 2b_{14}x^2y^3 - 5b_{05}xy^4 + a_{05}y^5 - \mu y(x^2+y^2)^3,\\
\frac{dy}{dt} &= -2\mu x^3 - 3\mu x^2 y + \mu xy^2 + b_{03}y^3 - 2\left(1 - \frac{71275\mu}{378}\right)x^3y^2 - a_{32}x^2y^3\\
&\quad + b_{14}xy^4 + b_{05}y^5 + \mu x(x^2+y^2)^3.
\end{aligned}
\tag{3.6}
$$

system (3.6) has an analytic first integral

$$
\begin{aligned}
H(x,y) &= \frac{\mu}{2}y^2 + \frac{\mu}{2}x^4 + \frac{a_{03}}{4}y^4 + \frac{a_{05}}{5}y^5 - b_{05}xy^5 - b_{03}xy^3 - \frac{\mu}{2}x^2y^2 + \mu x^3 y\\
&\quad + \frac{1}{2}\left(1 - \frac{71275}{378}\mu\right)x^4y^2 - \frac{b_{14}}{2}x^2y^4 + \frac{a_{32}}{3}x^3y^3 - \frac{\mu}{8}(x^2+y^2)^4
\end{aligned}
\tag{3.7}
$$

Proposition 3.2 implies that

Proposition 3.3. *The origin of systems* (3.6) *is a center.*

From Propositions 3.2 and 3.3, we further have

Theorem 3.2. *The origin of system* (1.2) *is a center if and only if the first 13 quasi-Lyapunov constants are zero, that is, the condition in Proposition 3.2 is satisfied.*

4 Multiple bifurcation of limit cycles

It is very interesting to study the number of limit cycles which could be bifurcated from critical point $O(0,0)$, because it is closely related with 16th problem of 23 problems of Hilbert. In this section, we will prove that the perturbed system of (1.2) can generate 13 limit cycles enclosing an elementary node at the origin when the three-order nilpotent critical point $O(0,0)$ is a 13-order weak focus.

According to the relations

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = \lambda_9 = \lambda_{10} = \lambda_{11} = \lambda_{12} = 0, \lambda_{13} \neq 0,$$

we have that

Theorem 4.1. *The three-order nilpotent critical point* $O(0,0)$ *of system* (1.2) *is a 13-order weak focus if and only if*

$$\begin{aligned}
&b_{21} = -3\mu,\ a_{12} = -3b_{03},\ b_{23} = -a_{32,}\\
&a_{14} = a_{23} - 5b_{05} + 2b_{14},\ b_{03} = \mu,\\
&a_{32} = \frac{756 - 142739\mu}{252},\\
&b_{05} = \frac{189 + 189b_{14} - 35516\mu}{378},\\
&a_{03} = \frac{945 - 231391\mu}{2646},\\
&b_{14} = \frac{34020 - 4116a_{23} - 5024095\mu}{1372},\\
&a_{05} = \frac{-714420 - 594712314\mu + 90016920a_{23}\mu + 102674821303\mu]^2}{14002632\mu}\\
&a_{23} = \frac{359281818 - 36356476968\mu - 2196158788177\mu^2}{42007896},\\
&\quad \mu \approx \pm 0.00917916.
\end{aligned} \tag{4.1}$$

Proof. Let

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = \lambda_9 = \lambda_{10} = \lambda_{11} = 0,$$

we obtain relations of

$$b_{21},\ a_{12},\ b_{23},\ a_{14},\ b_{03}\, a_{32},\ b_{05},\ a_{03},\ b_{14},\ a_{05},\ a_{23},$$

Solving the equation $\lambda_{12} = 0$ about μ, we could get four solutions

$$A_1 \approx 0.00358818 - 0.000840459i, A_2 \approx 0.00358818 + 0.000840459i,$$
$$A_3 \approx -0.00917916, A_4 \approx 0.00917916.$$

$\mu \in R$, so $\mu \approx$ ±- 0.00917916, and

$$\begin{aligned}
\text{Resultant}[\lambda_{12}, \lambda_{13}, \mu] = {} & 708862143887297940928622843069693973671321637260998996555\\
& 670402478173244030656968763426347748941357929168220619\\
& 37088068805885358452543012305600000.
\end{aligned} \tag{4.2}$$

So $\lambda_{13} \neq 0$, the origin of system (1.2) is a 13-order weak focus.

The perturbed system of (1.2) could be written as follows:

$$\begin{aligned} \frac{dx}{dt} &= \delta x + \mu y + \mu x^3 - \mu x^2 y + a_{12}xy^2 + a_{03}y^3 + \left(1 - \frac{71275\mu}{378}\right)x^4 y + a_{32}x^3y^2 \\ &\quad + a_{23}x^2y^3 + a_{14}xy^4 + a_{05}y^5 - \mu y(x^2+y^2)^3, \\ \frac{dy}{dt} &= \delta y - 2\mu x^3 + b_{21}x^2y + \mu xy^2 + b_{03}y^3 - 2\left(1 - \frac{71275\mu}{378}\right)x^3y^2 + b_{23}x^2y^3 \\ &\quad + b_{14}xy^4 + b_{05}y^5 + \mu x(x^2+y^2)^3.. \end{aligned} \tag{4.3}$$

When conditions of (4.1) hold, by the relationships $\lambda 1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = \lambda_9 = \lambda_{10} = \lambda_{11} = \lambda 12 = 0$, the values of b_{21}, a_{12}, b_{23}, a_{14}, b_{03} a_{32}, b_{05}, a_{03}, b_{14}, a_{05}, a_{23}, μ could be determined. Notice that $\mu = \pm$ -0.00917916 are the simple zeros of $\lambda_{12} = 0$. Hence, when conditions in (4.1) hold, we have

when $\mu \approx -0.00917916$

$$\frac{\partial(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7,\lambda_8,\lambda_9,\lambda_{10},\lambda_{11},\lambda_{12})}{\partial(b_{21},a_{12},b_{23},a_{14},b_{03},a_{32},b_{05},a_{03},b_{14},a_{05},a_{23},\mu)} \approx 1.83793 \times 10^{51},$$

when $\mu \approx 0.00917916$

$$\frac{\partial(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7,\lambda_8,\lambda_9,\lambda_{10},\lambda_{11},\lambda_{12})}{\partial(b_{21},a_{12},b_{23},a_{14},b_{03},a_{32},b_{05},a_{03},b_{14},a_{05},a_{23},\mu)} \approx 4.05781 \times 10^{37}.$$

In fact,

$$\begin{aligned} J &= \frac{\partial(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7,\lambda_8,\lambda_9,\lambda_{10},\lambda_{11},\lambda_{12})}{\partial(b_{21},a_{12},b_{23},a_{14},b_{03},a_{32},b_{05},a_{03},b_{14},a_{05},a_{23})} \\ &= \frac{3125(57367926 - 25174619316\mu + 2196158788177\mu^2)^8}{3553644238836727429387691170820702439358739754227457606557653317975629911417669374\mu^{24}} \\ &\quad \times (15338883168 - 6443685685206\mu + 575725672923156\mu^2 + 22094172198662329\mu^3,) \end{aligned} \tag{4.4}$$

and

$$\text{Resultant}[J, \lambda_{12}, \mu] \not= 0.$$

So when $\mu \approx \pm 0.00917916$, $J \neq 0$.

From the statement mentioned above, Theorem 2.1 follows that

Theorem 4.2. *If the three-order nilpotent critical point* $O(0,0)$ *of system* (1.2) *is a 13-order weak focus, for* $0 < \delta \ll 1$, *making a small perturbation to the coefficients of system* (1.2), *then, for system* (4.3), *in a small neighborhood of the origin, there exist exactly 13 small amplitude limit cycles enclosing the origin* $O(0,0)$, *which is an elementary node.*

Appendix A

Detailed recursive MATHEMATICA code to compute the quasi-Lyapunov constants at the origin of system (1.2):

$$c[0,0] = 0, c[1,0] = 0, c[0,1] = 0, c[2,0] = 0, c[1,1] = 0, c[0,2] = 1;$$

when $k < 0$, or $j < 0$, $c[k, j] = 0$;

else

$$
\begin{aligned}
c[k,j] = {} & \frac{1}{378k(1+s)\mu}(-756\mu c[-8+k,2+j] - 378j\mu c[-8+k,2+j] - 756s\mu c[-8+k,2+j] \\
& - 378js\mu c[-8+k,2+j] - 2268\mu c[-6+k,j] - 1134j\mu c[-6+k,j] + 378k\mu c[-6+k,j] \\
& - 2268s\mu c[-6+k,j] - 1134js\mu c[-6+k,j] + 378ks\mu c[-6+k,j] - 2268\mu c[-4+k,-2+j] \\
& - 1134j\mu c[-4+k,-2+j] + 1134k\mu c[-4+k,-2+j] - 2268s\mu c[-4+k,-2+j] \\
& - 1134js\mu c[-4+k,-2+j] + 1134ks\mu c[-4+k,-2+j] + 1512c[-4+k,j] \\
& - 378kc[-4+k,j] + 1512sc[-4+k,j] + 756jsc[-4+k,j] - 378ksc[-4+k,j] \\
& - 142550j\mu c[-4+k,j] + 71275k\mu c[-4+k,j] - 285100s\mu c[-4+k,j] \\
& - 142550js\mu c[-4+k,j] + 71275ks\mu c[-4+k,j] + 1512\mu c[-4+k,2+j] \\
& + 756j\mu c[-4+k,2+j] + 1512s\mu c[-4+k,2+j] + 756js\mu c[-4+k,2+j] \\
& + 2268a_{32}c[-3+k,-1+j] + 1512b_{23}c[-3+k,-1+j] - 378b_{23}jc[-3+k,-1+j] \\
& - 378a_{32}kc[-3+k,-1+j] + 1134a_{32}sc[-3+k,-1+j] + 378b_{23}sc[-3+k,-1+j] \\
& - 378b_{23}jsc[-3+k,-1+j] - 378a_{32}ksc[-3+k,-1+j] - 378b_{21}jc[-3+k,1+j] \\
& - 378b_{21}sc[-3+k,1+j] - 378b_{21}jsc[-3+k,1+j] + 2268\mu c[-3+k,1+j] \\
& - 378k\mu c[-3+k,1+j] + 1134s\mu c[-3+k,1+j] - 378ks\mu c[-3+k,1+j] \\
& - 756\mu c[-2+k,-4+j] - 378j\mu c[-2+k,-4+j] + 1134k\mu c[-2+k,-4+j] \\
& - 756s\mu c[-2+k,-4+j] - 378js\mu c[-2+k,-4+j] + 1134ks\mu c[-2+k,-4+j] \\
& + 1512a_{23}c[-2+k,-2+j] + 2268b_{14}c[-2+k,-2+j] - 378b_{14}jc[-2+k,-2+j] \\
& - 378a_{23}kc[-2+k,-2+j] + 756a_{23}sc[-2+k,-2+j] + 756b_{14}sc[-2+k,-2+j] \\
& - 378b_{14}jsc[-2+k,-2+j] - 378a_{23}ksc[-2+k,-2+j] - 756\mu c[-2+k,j] \\
& - 378j\mu c[-2+k,j] + 378k\mu c[-2+k,j] - 756s\mu c[-2+k,j] - 378js\mu c[-2+k,j] \\
& + 378ks\mu c[-2+k,j] + 756a_{14}c[-1+k,-3+j] + 3024b_{05}c[-1+k,-3+j] \\
& - 378b_{05}jc[-1+k,-3+j] - 378a_{14}kc[-1+k,-3+j] + 378a_{14}sc[-1+k,-3+j] \\
& + 1134b_{05}sc[-1+k,-3+j] - 378b_{05}jsc[-1+k,-3+j] - 378a_{14}ksc[-1+k,-3+j] \\
& + 756a12c[-1+k,-1+k,-1+j] + 1512b_{03}c[-1+k,-1+j] - 378b_{03}jc[-1+k,-1+j] \\
& - 378a12kc[-1+k,-1+j] + 378a12sc[-1+k,-1+j] + 378b_{03}sc[-1+k,-1+j] \\
& - 378b_{03}jsc[-1+k,-1+j] - 378a12ksc[-1+k,-1+j] + 378k\mu c[k,-6+j] \\
& + 378ks\mu c[k,-6+j] - 378a_{05}kc[k,-4+j] - 378a_{05}ksc[k,-4+j] \\
& - 378a_{03}kc[k,-2+j] - 378a_{03}ksc[k,-2+j] + 756jc[-4+k,j] - 285100\mu c[-4+k,j]) \\
\omega_m = {} & -\frac{1}{378\mu}(378\mu c[-7+m,1] + 378s\mu c[-7+m,1] + 756\mu c[-5+m,-1] \\
& - 378m\mu c[-5+m,-1] + 756s\mu c[-5+m,-1] - 378ms\mu c[-5+m,-1] \\
& - 1134m\mu c[-3+m,-3] - 1134ms\mu c[-3+m,-3] - 378c[-3+m,-1] \\
& + 378mc[-3+m,-1] - 71275m\mu c[-3+m,-1] - 756s\mu c[-3+m,1] \\
& - 378sc[-3+m,-1] + 378msc[-3+m,-1] + 71275\mu c[-3+m,-1] \\
& + 71275s\mu c[-3+m,-1] - 71275ms\mu c[-3+m,-1] - 756\mu c[-3+m,1] \\
& - 1890a_{32}c[-2+m,-2] - 1890b_{23}c[-2+m,-2] + 378a_{32}mc[-2+m,-2] \\
& - 756a_{32}sc[-2+m,-2] - 756b_{23}sc[-2+m,-2] + 378a_{32}msc[-2+m,-2] \\
& - 378b_{21}c[-2+m,0] - 1890\mu c[-2+m,0] + 378m\mu c[-2+m,0] \\
& + 378ms\mu c[-2+m,0] - 756\mu c[-1+m,-5] - 1134m\mu c[-1+m,-5] \\
& - 1134ms\mu c[-1+m,-5] - 1134a_{23}c[-1+m,-3] - 2646b_{14}c[-1+m,-3] \\
& + 378a_{23}mc[-1+m,-3] - 378a_{23}sc[-1+m,-3] - 1134b_{14}sc[-1+m,-3] \\
& + 378a_{23}msc[-1+m,-3] - 378m\mu c[-1+m,-1] - 378ms\mu c[-1+m,-1] \\
& - 3402b_{05}c[m,-4] + 378a_{14}mc[m,-4] - 1512b_{05}sc[m,-4] + 378a_{14}msc[m,-4] \\
& - 378a12c[m,-2] - 1890b_{03}c[m,-2] + 378a12mc[m,-2] - 756b_{03}sc[m,-2] \\
& + 378a12msc[m,-2] - 378\mu c[1+m,-7] - 378m\mu c[1+m,-7] - 378s\mu c[1+m,-7] \\
& - 378ms\mu c[1+m,-7] + 378a_{05}c[1+m,-5] + 378a_{05}mc[1+m,-5] \\
& + 378a_{05}msc[1+m,-5] + 378a_{03}c[1+m,-3] + 378a_{03}mc[1+m,-3] \\
& + 378a_{03}msc[1+m,-3] + 378\mu c[1+m,-1] + 378m\mu c[1+m,-1] \\
& - 756s\mu c[-2+m,0] - 756s\mu c[-1+m,-5] - 378a_{14}c[m,-4] \\
& + 378s\mu c[1+m,-1] + 378ms\mu c[1+m,-1] + 378a_{05}sc[1+m,-5] + 378a_{03}sc[1+m,-3]) \\
\lambda_m = {} & \frac{\omega_{2m+4}}{2m-4s-1}.
\end{aligned}
$$

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Authors' contributions

We constructed a concrete system of seventh-degree and investigated the bifurcations of limit cycles from the nilpotent critical point. First of all, the first 13 quasi-Lyapunov constants were deduced. Furthermore, sufficient and necessary conditions in order to have a center were obtained. Moreover, we proved that 13 limit cycles could be bifurcated from its three-order nilpotent critical point. Although this system is very specific, comparing our result with $N(7) \geq 9$ in [23], we give a lower bound of cyclicity of three-order nilpotent critical point for seventh-degree nilpotent systems. The result of this article is helpful to the 16th Hilbert problem, it enriches the literatures as to the Hilbert's 16th problem. All authors read and approved the manuscript.

Competing interests

The authors declare that they have no competing interests.

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