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Degeneration of groups of type E_7 and minimal coupling in supergravity

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Abstract: We study properties of $D = 4$ $\mathcal{N} \geqslant 2$ extended supergravities (and related compactifications of superstring theory) and their consistent truncation to the phenomenologically interesting models of $\mathcal{N} = 1$ supergravity. This involves a detailed classification of the "degenerations" of the duality groups of type E_7, when the corresponding *quartic* invariant polynomial built from the symplectic irreducible representation of G_4 "degenerates" into a *perfect square*. With regard to cosmological applications, *minimal coupling* of vectors in consistent truncation to $\mathcal{N} = 1$ from higher-dimensional or higher-$\mathcal{N}$ theory is non-generic. On the other hand, *non-minimal coupling* involving vectors coupled to scalars and axions is generic. These features of supergravity, following from the electric-magnetic duality, may be useful in other applications, like stabilization of moduli, and in studies of non-perturbative black-hole solutions of supergravity/string theory.

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Contents

1 Introduction

In the present investigation, we relate a physical property of supergravity couplings to a mathematical property of the underlying electric-magnetic duality symmetries[1] of $\mathcal{N}\geqslant 2$ extended supergravity in $D=4$ space-time dimensions.

In the textbook [4], the coupling of $\mathcal{N}=1$ vector and chiral multiplets to supergravity is presented in its *minimal* form, i.e. it is assumed that the vector kinetic term

$$-\frac{1}{4}\delta_{\alpha\beta}F^{\alpha}_{\mu\nu}F^{\beta|\mu\nu} \tag{1.1}$$

is scalar independent. However, supersymmetry allows for the replacement of the constant kinetic vector matrix $\delta_{\alpha\beta}$ by an holomorphic function of the scalar fields z, $\delta_{\alpha\beta}\to f_{\alpha\beta}(z)$, such that kinetic vector term reads

$$-\frac{1}{4}\Big(\mathrm{Re}\,f_{\alpha\beta}(z)\Big)F^{\alpha}_{\mu\nu}F^{\beta\mu\nu}+\frac{i}{4}\Big(\mathrm{Im}\,f_{\alpha\beta}(z)\Big)F^{\alpha}_{\mu\nu}\tilde{F}^{\beta\mu\nu}\ . \tag{1.2}$$

Here the function $f_{\alpha\beta}(z)$ is holomorphic, so that a *non-minimal* coupling is introduced. For example, for one vector, in the simplest case, $f(z)=\phi+ia$ and we have a vector-vector-scalar, ϕF^2, and a vector-vector-axion, $aF\tilde{F}$, couplings.

In theories with global supersymmetry the choice of the minimal coupling is often preferred since only for constant, scalar independent $f_{\alpha\beta}$ the theory is renormalizable. It is the same consideration which suggested that a preferred Kähler potential is canonical. In the context of supergravity, however, the requirement of renormalizability is less relevant, the issue we address here is: what kind of vector coupling is preferred in the models originating from higher supersymmetries/higher dimensions.

Non-minimal vector scalar couplings may play an important rule in inflationary cosmology, because a direct coupling of the inflaton scalar field to matter vector fields (as heavy vector bosons, or photons) may provide the only way to complete the creation of matter in the early Universe. This problem was recently addressed in [5], where it was pointed out that in $\mathcal{N}=1$ supergravity obtained by reduction from higher-dimensional and/or higher-supersymmetric theories the *non-minimal* vector scalar couplings (1.2) are generic.

The present paper is intended to generalize the results of [5], because we believe that the issue of *minimal coupling* in $\mathcal{N}\geqslant 2$ extended supergravities deserves some attention.

[1] Further below, we use the term U-duality, meaning the "continuous" symmetries of [1, 2]. Their discrete versions are the U-duality non-perturbative string theory symmetries [3].

Indeed, such theories never[2] exhibit a constant $f_{\alpha\beta}$, and in [5] this fact was pointed out to be *a consequence of electric-magnetic-duality, which requires a special coupling of the non-linear sigma model of scalars to the vector sector* [6, 7]. The kinetic vector matrix $\mathcal{N}_{\Lambda\Sigma}$ which occurs in $\mathcal{N} \geqslant 2$, $D = 4$ extended supergravities is not holomorphic,

$$\mathrm{Im}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda}_{\mu\nu}F^{\mu\nu\Sigma} + i\mathrm{Re}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda}_{\mu\nu}\tilde{F}^{\Sigma\mu\nu}\,. \tag{1.3}$$

Here the kinetic term for vectors $\mathcal{N}_{\Lambda\Sigma}$ in general depends on scalars. The matrix $\mathrm{Im}\mathcal{N}_{\Lambda\Sigma}$ is a metric in the vector moduli space. Comparing the Maxwell term, $\mathcal{N}_{\Lambda\Sigma}$ should reduce to $-\frac{i}{4}\overline{f}_{\alpha\beta}(\overline{z})$ in the $\mathcal{N} = 1$ theory [9]. Consistent truncations of $\mathcal{N} \geqslant 2$ extended supergravities to $\mathcal{N} = 1$ have been studied in [10, 11], where it was shown how the non-holomorphic $\mathcal{N}_{\Lambda\Sigma}$ reduces to an anti-holomorphic $f_{\alpha\beta}$ in the corresponding truncated theories.

Let us remind that in $\mathcal{N} = 2$ special Kähler geometry, in a symplectic frame in which an holomorphic prepotential function $F(X)$ exists (such that $X^{\Lambda}\partial_{\Lambda}F = 2F$), the kinetic vector matrix is given by (see e.g. [8], and refs. therein):

$$\mathcal{N}_{\Lambda\Sigma} = \overline{F}_{\Lambda\Sigma} - 2i\overline{T}_{\Lambda}\overline{T}_{\Sigma}(L^{\Xi}\mathrm{Im}F_{\Xi\Omega}L^{\Omega}), \tag{1.4}$$

where $F_{\Lambda\Sigma} = \partial_{\Lambda}\partial_{\Sigma}F$, $L^{\Lambda} = e^{K/2}X^{\Lambda}$ is the covariantly holomorphic contravariant symplectic section, and

$$T_{\Lambda} = 2i\mathrm{Im}\mathcal{N}_{\Lambda\Sigma}L^{\Sigma} \tag{1.5}$$

is the projector on the graviphoton ($T^{-}_{\mu\nu} = T_{\Lambda}F^{\Lambda|-}_{\mu\nu}$), whose "flux" define the $\mathcal{N} = 2$ central charge Z (see e.g. [12, 13] and refs. therein). Note that $\mathcal{N}_{\Lambda\Sigma}$ is not anti-holomorphic because of the presence of the second term in the r.h.s. of (1.4). In order to have a consistent $\mathcal{N} = 1$ reduction, one needs to impose $T_{\Lambda} = 0$, i.e. that the graviphoton projection vanishes (when Λ is restricted to the index running on $\mathcal{N} = 1$ vector multiplets). One then obtains that *minimal coupling* demands $F(X)$ to be *quadratic* in the truncated scalars of the corresponding would-be $\mathcal{N} = 1$ vector multiplets.

It is here worth observing that, while *minimal coupling* seems natural in $\mathcal{N} = 1$ supergravity [4], its relaxation is actually natural if one considers $\mathcal{N} = 1$ theories coming from supergravity theory [14] or from higher dimensions [15]. In the systematic approach of the present paper, we will provide a detailed list of examples in which *minimal coupling* is impossible in the higher-dimensional or higher-$\mathcal{N}$ theory, but it can be achieved by a further suitable consistent truncation to $\mathcal{N} = 1$.

This is related to the mathematical property of the U-duality group G_4 of type E_7 [16]. Simple, *non-degenerate* groups G_4 are related to Freudenthal triple systems $\mathfrak{M}(J_3)$ on simple rank-3 Jordan algebras J_3. In general, $G_4 \equiv$Conf(J_3) =Aut$(\mathfrak{M}(J_3))$ (see e.g. [34–36] for a recent introduction, and a list of refs.). When considering a consistent reduction to a subgroup, G_4 groups of type E_7 may admit a *"degeneration"* in which the rank-4 invariant symmetric structure $\mathbf{q}$ is *reducible*, namely it is the product of two symmetric invariant tensors. As a consequence, the corresponding *quartic* invariant polynomial built from

[2]With exception of "pure" $\mathcal{N} = 2$ and $\mathcal{N} = 3$ supergravity theories, which have no scalars, with U(1) and U(3) U-duality group, respectively, consistent with the analysis of [6, 7].

the symplectic irrep. $\mathbf{R}$ of G_4 "degenerates" into a *perfect square*.[3] Here $\mathbf{R}$ denotes the symplectic representation of the U-duality group G_4 formed by a the chiral (or anti-chiral) vector field strengths $F^{\Lambda|\pm}$ and their duals $G_\Lambda^{\mp} \equiv \mp\frac{i}{2}\delta\mathcal{L}/\delta F^{\Lambda|\mp}$:

$$\mathbf{R} = F^{\Lambda|\pm}, G_\Lambda^{\pm}), \tag{1.6}$$

such that "fluxes" of suitably defined projections defines the central charge (matrix) and matter charges (*if any*; see (see e.g. [19, 20] and refs. therein). Sometimes, in order to simplify the analysis, in the treatment below we will switch to the basis of the fluxes of the corresponding field strengths, defining the dyonic vector of magnetic and electric charges ([6, 7]; see e.g. the treatment of [8]):

$$\mathbf{R} = (p^\Lambda, q_\Lambda) \equiv \mathcal{Q}, \tag{1.7}$$

even if our analysis does not only restrict to charged states, such as black holes. By truncation of the charged fluxes $\mathcal{Q}$ we here mean the reduction of the group G_4 and its irrep. $\mathbf{R}(G_4)$ to some proper subgroup G_4' and its irrep. $\mathbf{R}(G_4') \equiv \mathbf{R}'$.

Since $\mathcal{N} > 2$ theories are related to scalar manifolds which are symmetric spaces, we will consider $\mathcal{N} = 2$ theories with symmetric cosets. Therefore, $\mathcal{N} = 1$ truncations are simpler to investigate, because the $\mathcal{N} = 2$ theory leading to $\mathcal{N} = 1$ *minimal coupling* are the so-called $\mathcal{N} = 2$ *minimally coupled* Maxwell-Einstein supergravities [22], whose scalar manifold is a (non-compact) $\mathbb{CP}^n$ space. In a scalar-dressed symplectic frame of $\mathcal{N} = 2$ special Kähler geometry, the *"degeneration"* of the quartic polynomial invariant to a quadratic one corresponds to setting the C-tensor to zero ($C_{ijk} = 0$). Also for $\mathcal{N} > 2$, we will then consider those cases in which the reduction to $\mathcal{N} = 2$ gives rise to a $\mathbb{CP}^n$ special Kähler geometry ($C_{ijk} = 0$), in which the U-duality group $G_4 = \mathrm{U}(1, n)$ is a *degenerate*[4] group of type E_7 [21], with the rank-4 completely symmetric invariant $\mathbf{q}$-structure r*educible*, as pointed out above.

As recalled in Example 1.2 of [21] and proved in [16, 23], all degenerate Freudenthal triple systems are isomorphic to the degenerate triple system in which the resulting quartic invariant polynomial $\mathcal{I}_4$ is the square of a *quadratic* invariant polynomial $\mathcal{I}_2$ which, as pointed out above, also corresponds to the case relevant for $D = 4$ supergravity with symmetric scalar manifold (see the treatment of section 2, as well). The *degeneration* of a U-duality group G_4 of type E_7 is also confirmed by the fact that the fundamental identity characterizing simple, *non-degenerate* groups of type E_7 (proved in section 2 of [21] for E_7, and generalized in formula (2.19) further below *at least* for all groups listed in table 1) does *not* hold in these cases; see section 2. The cases of U-duality groups as *semi-simple*, *non-degenerate* groups of type E_7 relevant to $D = 4$ supergravity theories with symmetric (vector multiplets') scalar manifolds are also analyzed in subsection 2.4.

Simple, *degenerate* groups of type E_7 relevant to $D = 4$ supergravity (namely, $\mathrm{U}(1, n)$ or $\mathrm{U}(3, n)$) share the property that the dyonic charge vector $\mathcal{Q}$ (1.7) (element

[3]An analysis at the level of quartic invariant polynomial, and dependent on charge configurations, has been considered in [17].

[4]In [5] these groups were called *"not of type E_7"*.

of the Freudenthal triple system) fits into the sum of the fundamental and anti-fundamental irrep.

$$\mathcal{Q} \in \mathbf{R} \equiv \mathbf{Fund} + \overline{\mathbf{Fund}}, \tag{1.8}$$

thus naturally admitting a *complex* representation, endowed with an invariant Hermitian quadratic structure (see e.g. [24, 25]), whose real part gives rise to the aforementioned quadratic invariant polynomial $\mathcal{I}_2$; see the discussion in section 2.

It should be stressed that the conditions on truncations of fluxes and embeddings of scalar manifolds, under consideration in the treatment below, are generally only *necessary*, but *not sufficient* for minimal coupling. An analysis of the consistency of the truncations at the level of supersymmetry transformations, along the lines exploited in [10] and [11] (this latter on the further truncation $\mathcal{N} = 2 \to 1$) is required to determine also a sufficient condition.

The plan of the paper is as follows.

After axiomatically introducing groups of type E_7 in section 2, we analyze various truncations to *minimal coupling* models in subsequent sections. It is here worth pointing out that by truncation of a theory we here mean a sub-theory obtained from the original one by reducing the amount of supersymmetry. For *"pure"* ($\mathcal{N} \geqslant 5$) supergravities, this means to consistently truncate away the extra gravitino multiplet(s); these cases are considered in sections 3 and 4. On the other hand, for matter-coupled ($2 \leqslant \mathcal{N} \leqslant 4$) theories the truncation also requires to consistently truncate the matter multiplets' sector; such cases are analyzed in sections 5, 6 and 7. In presence of matter coupling, there is another way of obtaining sub-theories, namely to consistently reduce the matter sector but not the gravitino multiplet(s); section 8 deals with such cases. The list of examples produced by the systematic approach of the present investigation is much larger than the ones given in [5, 10, 11], and it is of some interest also because some truncations correspond to orbifolds and orientifolds of string theories with larger supersymmetry, as discussed in section 9, in which the further truncation $\mathcal{N} = 2 \to \mathcal{N} = 1$ is considered. Comments on the *"degeneration"* of the so-called Freudenthal duality are then given in section 10. Section 11 contain some remarks on fermions and minimal coupling. Conclusive remarks and an outlook are given in section 12. Appendix A, containing some details on the structure of Pauli terms, concludes the paper.

2 On groups of type E_7

2.1 Axiomatic characterization

The first axiomatic characterization of groups *"of type E_7"* through a module (irreducible representation) was given in 1967 by Brown [16].

A group G of type E_7 is a Lie group endowed with a representation $\mathbf{R}$ such that:

1. $\mathbf{R}$ is *symplectic*, i.e. (the subscripts "s" and "a" stand for symmetric and skew-symmetric throughout):

$$\exists ! \mathbb{C}_{[MN]} \equiv \mathbf{1} \in \mathbf{R} \times_a \mathbf{R}; \tag{2.1}$$

$\mathbb{C}_{[MN]}$ defines a non-degenerate skew-symmetric bilinear form (*symplectic product*); given two different charge vectors $\mathcal{Q}_x$ and $\mathcal{Q}_y$ in $\mathbf{R}$, such a bilinear form is defined as

$$\langle \mathcal{Q}_x, \mathcal{Q}_y \rangle \equiv \mathcal{Q}_x^M \mathcal{Q}_y^N \mathbb{C}_{MN} = -\langle \mathcal{Q}_y, \mathcal{Q}_x \rangle . \tag{2.2}$$

2. $\mathbf{R}$ admits a unique rank-4 completely symmetric primitive G-invariant structure, usually named K-tensor

$$\exists ! \mathbb{K}_{(MNPQ)} \equiv \mathbf{1} \in [\mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R}]_s ; \tag{2.3}$$

thus, by contracting the K-tensor with the same charge vector $\mathcal{Q}$ in $\mathbf{R}$, one can construct a rank-4 homogeneous G-invariant polynomial (whose ς is the normalization constant):

$$\mathbf{q}(\mathcal{Q}) \equiv \varsigma \mathbb{K}_{MNPQ} \mathcal{Q}^M \mathcal{Q}^N \mathcal{Q}^P \mathcal{Q}^Q , \tag{2.4}$$

which corresponds to the evaluation of the rank-4 symmetric invariant $\mathbf{q}$-structure induced by the K-tensor on four identical modules $\mathbf{R}$:

$$\begin{aligned} \mathbf{q}(Q) &\equiv \mathbf{q}(\mathcal{Q}_x, \mathcal{Q}_y, \mathcal{Q}_z, \mathcal{Q}_w)\big|_{\mathcal{Q}_x=\mathcal{Q}_y=\mathcal{Q}_z=\mathcal{Q}_w\equiv\mathcal{Q}} \\ &\equiv \varsigma \left[\mathbb{K}_{MNPQ} \mathcal{Q}_x^M \mathcal{Q}_y^N \mathcal{Q}_z^P \mathcal{Q}_w^Q\right]_{\mathcal{Q}_x=\mathcal{Q}_y=\mathcal{Q}_z=\mathcal{Q}_w\equiv\mathcal{Q}} . \end{aligned} \tag{2.5}$$

A famous example of *quartic* invariant in $G = E_7$ is the *Cartan-Cremmer-Julia* invariant ([27], p. 274), constructed out of the fundamental representation $\mathbf{R} = \mathbf{56}$.

3. if a trilinear map $T \colon \mathbf{R} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is defined such that

$$\langle T(\mathcal{Q}_x, \mathcal{Q}_y, \mathcal{Q}_z), \mathcal{Q}_w \rangle = \mathbf{q}(\mathcal{Q}_x, \mathcal{Q}_y, \mathcal{Q}_z, \mathcal{Q}_w) , \tag{2.6}$$

then it holds that

$$\langle T(\mathcal{Q}_x, \mathcal{Q}_x, \mathcal{Q}_y), T(\mathcal{Q}_y, \mathcal{Q}_y, \mathcal{Q}_y)\rangle = -2 \langle \mathcal{Q}_x, \mathcal{Q}_y \rangle \, \mathbf{q}(\mathcal{Q}_x, \mathcal{Q}_y, \mathcal{Q}_y, \mathcal{Q}_y). \tag{2.7}$$

This last property makes the group of type E_7 amenable to a treatment in terms of (rank-3) Jordan algebras and related Freudenthal triple systems.

Remarkably, groups of type E_7, appearing in $D = 4$ supergravity as U-duality groups, admit a $D = 5$ uplift to groups of type E_6, as well as a $D = 3$ downlift to groups of type E_8. It should also be recalled that split form of exceptional E — Lie groups appear in the exceptional Cremmer-Julia [1, 2] sequence $E_{11-D(11-D)}$ of U-duality groups of M-theory compactified on a D-dimensional torus, in $D = 3, 4, 5$. Other sequences, composed by non-split, non-compact real forms of exceptional groups, are also relevant to non-maximal supergravity in various dimensions (see e.g. the treatment in [26], also for a list of related refs.).

The connection of groups of type E_7 to supergravity can be summarized by stating that all $2 \leqslant N \leqslant 8$-extended supergravities in $D = 4$ with symmetric scalar manifolds $\frac{G_4}{H_4}$ have G_4 of type E_7 [17, 73]. It is intriguing to notice that the first paper on groups of type E_7

was written about a decade before the discovery of of extended ($\mathcal{N}=2$) supergravity [28], in which electromagnetic duality symmetry was observed [29].

An example of Lie group which is not of type E_7 is the exceptional Lie group E_6 in its fundamental representation[5] $\mathbf{27}$; this is relevant to both maximal ($\mathcal{N}=8$) and exceptional ($\mathcal{N}=2$) supergravity theories in $D=5$. The representation $\mathbf{27}$ is *not* symplectic, but rather it is conjugated to its contra-gradient counterpart ($a=1,\ldots,27$):

$$\exists!\delta^a_b \equiv \mathbf{1} \in \mathbf{27}\times\overline{\mathbf{27}}. \tag{2.8}$$

Furthermore, $\mathbf{27}$ admits a unique rank-3 completely symmetric primitive E_6-invariant structure, usually named d-tensor

$$\exists! d_{abc} \equiv \mathbf{1} \in \left[\mathbf{27}\times\mathbf{27}\times\mathbf{27}\right]_s ; \tag{2.9}$$

thus, by contracting the d-tensor with the same charge vector Q in $\mathbf{27}$, one can construct a rank-3 homogeneous E_6-invariant polynomial (whose ϑ is the normalization constant):

$$\mathbf{d}(Q) \equiv \vartheta d_{abc} Q^a Q^b Q^c, \tag{2.10}$$

which corresponds to the evaluation of the rank-3 symmetric invariant $\mathbf{d}$-structure induced by the d-tensor on four identical modules $\mathbf{27}$:

$$\mathbf{d}(Q) \equiv \mathbf{d}(Q_x,Q_y,Q_z)\big|_{Q_x=Q_y=Q_z\equiv\mathcal{Q}} \equiv \varsigma\left[\vartheta d_{abc} Q^a_x Q^b_y Q^c_z\right]_{\mathcal{Q}_x=\mathcal{Q}_y=\mathcal{Q}_z\equiv\mathcal{Q}}. \tag{2.11}$$

Focussing on the relevance to supergravity theories in $D=4$, in the remaining part of this section we will characterize various classes of groups of type E_7 in terms of (tensor and) scalar identities, along the lines of [21] and exploiting results of previous investigations, such as [26] and [30].

2.2 Simple, non-degenerate

In *simple*, *non-degenerate* groups G_4 of type E_7 [16] relevant to $D=4$ (super)gravity with symmetric scalar manifolds (listed in table 1[6]), the following identity holds (cfr. (5.18) of [26]):

$$\mathbb{K}_{MNPQ}\mathbb{K}_{RSTU}\mathbb{C}^{PT}\mathbb{C}^{QU} = \xi\left[(2\tau-1)\mathbb{K}_{MNRS} + \xi\tau(\tau-1)\mathbb{C}_{M(R}\mathbb{C}_{S)N}\right]. \tag{2.12}$$

$\mathbb{C}_{MN}$ is the symplectic metric, and $\mathbb{K}_{MNPQ}$ denotes the completely symmetric, rank-4 invariant "K-tensor" in the relevant symplectic irrep. $\mathbf{R}(G_4)$ (M is an index in $\mathbf{R}$):

$$\mathbb{C} \equiv \exists!\mathbf{1} \in \left[\mathbf{R}\times\mathbf{R}\right]_a ; \tag{2.13}$$

$$\mathbb{K} \equiv \exists!\mathbf{1} \in \left[\mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}\right]_s , \tag{2.14}$$

[5] Strictly speaking, the pair $(G,\mathbf{R})=(E_6,\mathbf{27})$ is the prototype of the so-called groups *"of type E_6"*.

[6] We only consider rank-3 Jordan algebras related to locally supersymmetric theories of gravity.

where the subscript "s" ("a") denotes the (anti)symmetric part of the tensor product. Moreover, the G_4-dependent parameters are defined as [26, 33]

$$\tau \equiv \frac{2d}{f(f+1)}; \tag{2.15}$$

$$\xi \equiv -\frac{1}{3\tau}, \tag{2.16}$$

where

$$f \equiv \dim_{\mathbb{R}}(\mathbf{R}(G_4)); \tag{2.17}$$

$$d \equiv \dim_{\mathbb{R}}(\mathbf{Adj}(G_4)). \tag{2.18}$$

By using (2.12), one can show that the following identity holds:

$$tr(p(x \otimes x)p(y \otimes y)) = \beta \left[\mathbf{q}(x,x,y,y) - 2b(y,x)^2\right], \tag{2.19}$$

where (recall definition (2.2))

$$b(x,y) \equiv -\mathbb{C}_{MN}\mathcal{Q}_x^M \mathcal{Q}_y^N = -\langle \mathcal{Q}_x, \mathcal{Q}_y \rangle . \tag{2.20}$$

$$\mathbf{q}(x,y,z,w) \equiv -6\mathbb{K}_{MNPQ}\mathcal{Q}_x^M \mathcal{Q}_y^N \mathcal{Q}_z^P \mathcal{Q}_w^Q; \tag{2.21}$$

$$\beta \equiv \frac{2}{\tau}, \tag{2.22}$$

and p denotes the following vector space map (cfr. section 2 of [21] for further detail)

$$p(x \otimes y)z \equiv t(x,y,z) - b(z,x)y - b(z,y)x, \tag{2.23}$$

where $t(x,y,z)$ is the trilinear product related to $\mathbf{q}(x,y,z,w)$ as

$$\mathbf{q}(x,y,z,w) \equiv b(x, t(y,z,w)). \tag{2.24}$$

The scalar identity (2.19) holds *at least* for all simple, *non-degenerate* groups G_4 of type E_7 listed in table 1 (and for all their other non-compact forms, as well as for the corresponding compact Lie group $G_{4,c}$), and it is a consequence of the tensor identity (2.12), which in turn follows from the identity for the K-tensor given by (5.17) of [26]. In the particular case of E_7 (see tables 1 and 2), it holds $\tau = 1/12 \Rightarrow \beta = 24$, and the identity proved in Theorem 2.3 of [21] is retrieved.

It is worth remarking that, by defining the parameter q as specified in table 2, the values of f (2.17), d (2.18), τ (2.16), ξ (2.15) and β (2.22) can be easily q-parametrized as follows ((2.26) was noticed in [26]):

$$f = 2(3q+4); \tag{2.25}$$

$$d = \frac{3(3q+4)(2q+3)}{q+4}; \tag{2.26}$$

$$\tau = \frac{1}{q+4}; \tag{2.27}$$

$$\xi = -\frac{(q+4)}{3}; \tag{2.28}$$

$$\beta = 2(q+4). \tag{2.29}$$

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J_3	G_4	$\mathbf{R}$	$\mathcal{N}$
$J_3^{\mathbb{O}}$	$E_{7(-25)}$	$\mathbf{56}$	2
$J_3^{\mathbb{O}_s}$	$E_{7(7)}$	$\mathbf{56}$	8
$J_3^{\mathbb{H}}$	$\mathrm{SO}^*(12)$	$\mathbf{32}$	2, 6
$J_3^{\mathbb{C}}$	$\mathrm{SU}(3,3)$	$\mathbf{20}$	2
$M_{1,2}(\mathbb{O})$	$\mathrm{SU}(1,5)$	$\mathbf{20}$	5
$J_3^{\mathbb{R}}$	$\mathrm{Sp}(6,\mathbb{R})$	$\mathbf{14}'$	2
$\mathbb{R}$ (T^3 model)	$\mathrm{SL}(2,\mathbb{R})$	$\mathbf{4}$	2

Table 1. Simple, *non-degenerate* groups G_4 related to Freudenthal triple systems $\mathfrak{M}(J_3)$ on simple rank-3 Jordan algebras J_3. The relevant symplectic irrep. $\mathbf{R}$ of G_4 is also reported. $\mathbb{O}$, $\mathbb{H}$, $\mathbb{C}$ and $\mathbb{R}$ respectively denote the four division algebras of octonions, quaternions, complex and real numbers, and $\mathbb{O}_s$, $\mathbb{H}_s$, $\mathbb{C}_s$ are the corresponding split forms. Note that the G_4 related to split forms $\mathbb{O}_s$, $\mathbb{H}_s$, $\mathbb{C}_s$ is the *maximally non-compact* (*split*) real form of the corresponding compact Lie group. The corresponding scalar manifolds are the *symmetric* cosets $\frac{G_4}{H_4}$, where H_4 is the maximal compact subgroup (with symmetric embedding) of G_4. The number of supercharges of the resulting supergravity theory in $D = 4$ is also listed. $M_{1,2}(\mathbb{O})$ is the Jordan triple system generated by 2×1 vectors over $\mathbb{O}$ [46–49]. The $D = 5$ uplift of the T^3 model based on $J_3 = \mathbb{R}$ is the *pure* $\mathcal{N} = 2$, $D = 5$ supergravity. $J_3^{\mathbb{H}}$ is related to both 8 and 24 supersymmetries, because the corresponding supergravity theories are *"twin"*, namely they share the very same bosonic sector [39, 41, 42, 46–49].

$G_{4,c}$	q	f	d	τ	ξ	β
E_7	8	56	133	1/12	−4	24
SO(12)	4	32	66	1/8	−8/3	16
SU(6)	2	20	35	1/6	−2	12
USp(6)	1	14	21	1/5	−5/3	10
SU(2)	−2/3	4	3	3/10	−10/9	20/3

Table 2. The parameter q and the related q-parametrized quantites f (2.25), d (2.26), τ (2.27), ξ (2.28) and β (2.29). The corresponding compact form $G_{4,c}$ of G_4 is listed.

The specific values for the groups listed in table 1 are reported in table 2. Note that, speaking in terms of compact form $G_{4,c}$ of G_4, for $G_{4,c} = E_7$, SO(12), SU(6) and USp(6), q can be defined as

$$q \equiv \dim_{\mathbb{R}} \mathbb{A}, \tag{2.30}$$

where $\mathbb{A}$ denotes the division algebra on which the corresponding rank-3 simple Jordan algebra $J_3^{\mathbb{A}}$ is constructed ($q = 8,\ 4,\ 2,\ 1$ for $\mathbb{A} = \mathbb{O}, \mathbb{H}, \mathbb{C}, \mathbb{R}$, respectively). Note that the *triality symmetric* so-called $\mathcal{N} = 2$ *STU* model [37, 38], based on $J_3 = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$, can be obtained by setting $q = 0$; however, since the corresponding G_4 is *semi-simple*, it will be considered further below.

Also, note that the dimensions f and d of G_4's listed in table 1 satisfy the relation [26]

$$d = \frac{3f(f+1)}{f+16}. \tag{2.31}$$

2.3 Simple, degenerate

As pointed out in section 2 of [21], the story changes for *degenerate* groups of type E_7.

Confining ourselves to the ones relevant in $D = 4$ supergravity with symmetric scalar manifold, they are nothing but $G_4 = \mathrm{U}(r, s)$ with $r = 1$ ($\mathcal{N} = 2$ *minimally coupled* to s vector multiplets [22]) or $r = 3$ ($\mathcal{N} = 3$ coupled to s vector multiplets [43]), and the relevant (complex) symplectic representation is $\mathbf{R}(G_4) = \mathbf{r} + \mathbf{s}$. In these cases, it can be computed that

$$\mathbb{K}_{MNPQ} = \frac{\zeta^2}{3}\mathbb{S}_{M(N}\mathbb{S}_{PQ)}, \tag{2.32}$$

where ζ is a real constant, and the rank-2 symmetric invariant symplectic tensor $\mathbb{S}$ ($\mathbb{S}^T = \mathbb{S}$, $\mathbb{S}\mathbb{C}\mathbb{S} = \mathbb{C}$) is defined by the following formula:

$$\mathbf{Q}^i_x \overline{\mathbf{Q}}^{\bar{j}}_y \eta_{i\bar{j}} = \mathbb{S}_{MN}\mathcal{Q}^M_x\mathcal{Q}^N_y + i\mathbb{C}_{MN}\mathcal{Q}^M_x\mathcal{Q}^N_y, \tag{2.33}$$

where $\eta_{i\bar{j}}$ is the invariant metric of the fundamental irrep. $\mathbf{r}+\mathbf{s}$ of $\mathrm{U}(r,s)$, and $\mathbf{Q}^i_x$ and $\mathbf{Q}^i_x$ are the charge vectors in the *complex* (manifestly $\mathrm{U}(r,s)$-covariant) symplectic frame. By introducing

$$\mathcal{I}_2(x,y) \equiv \zeta\mathbb{S}_{MN}\mathcal{Q}^M_x\mathcal{Q}^N_y, \tag{2.34}$$

it is immediate to check the *degenerate* nature of the quartic invariant $\mathbf{q}$-structure (2.21):

$$\begin{aligned}
\mathbf{q}(x,y,z,w) &\equiv -6\mathbb{K}_{MNPQ}\mathcal{Q}^M_x\mathcal{Q}^N_y\mathcal{Q}^P_z\mathcal{Q}^Q_w \\
&= -2\left[\mathcal{I}_2(x,y)\mathcal{I}_2(z,w) + \mathcal{I}_2(x,z)\mathcal{I}_2(y,w) + \mathcal{I}_2(x,w)\mathcal{I}_2(y,z)\right]; \qquad (2.35)\\
&\Downarrow \\
\mathbf{q}(x,x,y,y) &= -2\left[2\mathcal{I}_2(x,y)^2 + \mathcal{I}_2(x,x)\mathcal{I}_2(y,y)\right]; \qquad (2.36)\\
&\Downarrow \\
-\frac{1}{6}\mathbf{q}(x,x,x,x) &= \mathcal{I}_2(x,x)^2. \qquad (2.37)
\end{aligned}$$

The analogue of identity (2.19) for such *degenerate* groups of type E_7 enjoys a very simple form ($\mathbb{C}_{MN}\mathbb{C}^{MN} = 2(r+s)$):

$$\mathbb{K}_{QPNR}\mathbb{K}_{SMTU}\mathbb{C}^{RS} = \zeta^4\mathbb{S}_{(QP}\mathbb{C}_{N)(M}\mathbb{S}_{TU)}; \tag{2.38}$$

$$\Downarrow$$

$$\mathbb{K}_{QPNR}\mathbb{K}_{SMTU}\mathbb{C}^{NM}\mathbb{C}^{RS} = \frac{\zeta^4}{9}\left[(2(r+s)+4)\mathbb{S}_{PQ}\mathbb{S}_{TU} + 2\mathbb{C}_{PT}\mathbb{C}_{QU} + 2\mathbb{C}_{PU}\mathbb{C}_{QT}\right]. \tag{2.39}$$

By exploiting (2.39), one can thus compute:

$$
\begin{aligned}
tr(p(x\otimes x)p(y\otimes y)) = {} & 4\left[\mathbf{q}(x,x,y,y)-(4\zeta^4+1)b(y,x)^2\right] \\
& -4\zeta^2\left[2(r+s)+4\right]\mathcal{I}_2(x,x)\mathcal{I}_2(y,y),
\end{aligned} \tag{2.40}
$$

which can be considered the analogue of (2.19) for the *degenerate* groups of type E_7 under consideration. The validity of the postulate (2.7) implies $\zeta^2 = 1/2$.

It should be remarked that, according to the discussion in Example 1.2 of [21] (and to the whole treatment therein), the invariant $\mathbf{q}$-structure of *any* degenerate Freudenthal triple system enjoys the form (2.37), up to isomorphisms. Therefore, the simple, *degenerate* groups of type E_7 mentioned above (relevant to $\mathcal{N}=2$ *minimally coupled* and $\mathcal{N}=3$ supergravity in $D=4$; see also the treatment below) can be regarded as "prototypes" (up to isomorphisms) of (simple) *degenerate* groups of type E_7.

2.4 Semi-simple, non-degenerate

Let us now consider *semi-simple, non-degenerate* groups of type E_7.

Confining ourselves to the ones relevant in $D=4$ supergravity with symmetric scalar manifold, they are nothing but $G_4 = \mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(m,n)$ with $m=2$ ($\mathcal{N}=2$ coupled to $n+1$ vector multiplets) or $m=6$ ($\mathcal{N}=4$ coupled to n vector multiplets), and the relevant symplectic representation is the bi-fundamental $\mathbf{R}(G_4)=(\mathbf{2},\mathbf{m}+\mathbf{n})$. They are respectively related to *semi-simple* rank-3 Jordan algebras $\mathbb{R}\oplus\mathbf{\Gamma}_{m-1,n-1}$, where $\mathbf{\Gamma}_{m-1,n-1}$ is a Jordan algebra with a quadratic form of pseudo-Euclidean $(m-1,n-1)$ signature, i.e. the Clifford algebra of $O(m-1,n-1)$ [61].The aforementioned $\mathcal{N}=2$ *STU* model [37, 38], based on $J_3=\mathbb{R}\oplus\mathbf{\Gamma}_{1,1}\sim\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$, is recovered by setting $m=n=2$.

In these cases, electro-magnetic splitting of the symplectic representation $\mathbf{R}$ can be implemented in a *manifestly* G_4*-covariant* fashion. Namely, $\mathcal{Q}$ is an electro-magnetic doublet $\mathbf{2}$ of the $\mathrm{SL}(2,\mathbb{R})$ factor of G_4 itself. The symplectic index M thus splits as follows (cfr. eq. (3.7) of [31])

$$
\left.\begin{array}{l} M=\alpha\Lambda, \\ \alpha=1,2,\ \ \Lambda=1,\ldots,m+n-2. \end{array}\right\}\Rightarrow \mathcal{Q}^M\equiv\mathcal{Q}^{\Lambda}_{\alpha}, \tag{2.41}
$$

and it should be pointed out that in the $\mathcal{N}=2$ case usually $\Lambda=0,1,\ldots,n-1$, with "0" pertaining to the $D=4$ graviphoton vector. The manifestly G_4-covariant symplectic frame (2.41) is usually dubbed *Calabi-Vesentini* frame [32], and it was firstly introduced in supergravity in [12].

The symplectic metric $\mathbb{C}_{MN}=\mathbb{C}^{\alpha\beta}_{\Lambda\Sigma}$ and rank-4 completely symmetric $\mathbb{K}$-tensor $\mathbb{K}_{MNPQ}=\mathbb{K}^{\alpha\beta\gamma\delta}_{\Lambda\Sigma\Xi\Omega}$ enjoy the following expression in term of the invariant structures $\epsilon^{\alpha\beta}$ and $\eta_{\Lambda\Xi}$ of $\mathrm{SL}_v(2,\mathbb{R})$ and of $\mathrm{SO}(m,n-2)$, respectively [30]:

$$
\mathbb{C}^{\alpha\beta}_{\Lambda\Sigma}=\eta_{\Lambda\Sigma}\epsilon^{\alpha\beta}; \tag{2.42}
$$

$$
\begin{aligned}
\mathbb{K}^{\alpha\beta\gamma\delta}_{\Lambda\Sigma\Xi\Omega} = {} & \frac{1}{12}\Big[\Big(\epsilon^{\alpha\beta}\epsilon^{\gamma\delta}+\epsilon^{\alpha\delta}\epsilon^{\beta\gamma}\Big)\eta_{\Lambda\Xi}\eta_{\Sigma\Omega} \\
& +\Big(\epsilon^{\alpha\beta}\epsilon^{\delta\gamma}+\epsilon^{\alpha\gamma}\epsilon^{\delta\beta}\Big)\eta_{\Lambda\Omega}\eta_{\Sigma\Xi}+\Big(\epsilon^{\alpha\gamma}\epsilon^{\beta\delta}+\epsilon^{\alpha\delta}\epsilon^{\beta\gamma}\Big)\eta_{\Lambda\Sigma}\eta_{\Xi\Omega}\Big].
\end{aligned} \tag{2.43}
$$

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From this, one can compute the analogue of identities (2.12) and (2.39) for the *semi-simple*, non-degenerate groups of type E_7 under consideration ($\epsilon_{\alpha\beta}\epsilon^{\alpha\beta} = 2$, $\eta_{\Lambda\Sigma}\eta^{\Lambda\Sigma} = m + n$):

$$\begin{aligned}\mathbb{K}_{MNPQ}\mathbb{K}_{RSTU}\mathbb{C}^{PT}\mathbb{C}^{QU} &= \mathbb{K}^{\alpha\beta\gamma\delta}_{\Lambda\Sigma\Xi\Omega}\mathbb{K}^{\eta\xi\lambda\rho}_{\Delta\Theta\Phi\Psi}\mathbb{C}^{\Omega\Delta}_{\delta\eta}\mathbb{C}^{\Xi\Theta}_{\gamma\xi} \\ &= \frac{1}{6}\mathbb{K}^{\alpha\beta\lambda\rho}_{\Lambda\Sigma\Phi\Psi} - \frac{1}{36}\mathbb{C}^{\alpha\lambda}_{\Lambda(\Phi}\mathbb{C}^{\rho\beta}_{\Psi)\Sigma} \\ &\quad -\frac{1}{72}\left[\begin{array}{l}\eta_{\Lambda\Psi}\eta_{\Sigma\Phi}\left(\epsilon^{\alpha\beta}\epsilon^{\lambda\rho}+\epsilon^{\alpha\lambda}\epsilon^{\rho\beta}\right)\\ +\eta_{\Lambda\Phi}\eta_{\Sigma\Psi}\left(\epsilon^{\alpha\beta}\epsilon^{\rho\lambda}+2\epsilon^{\alpha\lambda}\epsilon^{\beta\rho}+\epsilon^{\alpha\rho}\epsilon^{\beta\lambda}\right)\\ +(m+n-1)\eta_{\Lambda\Sigma}\eta_{\Phi\Psi}\left(\epsilon^{\alpha\rho}\epsilon^{\lambda\beta}+\epsilon^{\alpha\lambda}\epsilon^{\rho\beta}\right)\end{array}\right].\end{aligned} \tag{2.44}$$

By exploiting (2.44), one can thus compute:

$$\begin{aligned}tr(p(x\otimes x)p(y\otimes y)) &= 5\mathbf{q}(x,x,y,y) - 4b(y,x)^2 \\ &\quad -\left[\epsilon^{\alpha\beta}\epsilon^{\rho\lambda}\eta_{\Lambda\Psi}\eta_{\Sigma\Phi} + (m+n-2)\epsilon^{\alpha\rho}\epsilon^{\beta\lambda}\eta_{\Lambda\Sigma}\eta_{\Phi\Psi}\right]\mathcal{Q}^{\Lambda}_{\alpha|x}\mathcal{Q}^{\Sigma}_{\beta|x}\mathcal{Q}^{\Phi}_{\lambda|y}\mathcal{Q}^{\Psi}_{\rho|y},\end{aligned} \tag{2.45}$$

where we recall that the quartic invariant form $\mathbf{q}$ is defined by (2.21). The identity (2.45) can be considered the analogue of (2.19) and (2.40) for the *semi-simple*, *non-degenerate* groups of type E_7 under consideration, and it is different from them both.

2.5 The unified limit

The different structure exhibited by the scalar identities (2.19) (holding for simple, *non-degenerate* groups of type E_7), (2.40) (holding for simple, *degenerate* groups of type E_7) and (2.45) (holding for semi-simple, *non-degenerate* groups of type E_7) is manifest: the structure of (2.19) is the same as the structure of the first line of (2.40) and of (2.45), but the second line of (2.40) and of (2.45) is not compatible with such a structure.

Therefore, along the lines of [21], the scalar identities (2.19), (2.40) and (2.45) (or the corresponding tensor identities) can be considered as defining identities for *simple non-degenerate*, *simple degenerate*, and *semi-simple non-degenerate* groups of type E_7, respectively.

However, it should be also noted that (2.19), (2.40) and (2.45) share the very same $x \equiv y$ limit:

$$tr\left(p\left(x\otimes x\right)p\left(x\otimes x\right)\right) = \beta\mathbf{q}(x,x,x,x), \tag{2.46}$$

modulo the renamings

$$\beta \equiv 4\left[1+\frac{\zeta^2}{3}(r+s+2)\right] \equiv \left[5+\frac{1}{3}(m+n)\right]. \tag{2.47}$$

Before proceeding to the analysis of various truncation patterns to *minimal coupling* models, it is worth stressing a peculiar feature of the $\mathcal{N}=2$ theory among $D=4$ extended supergravity theories.

$\mathcal{N}=2$ supergravity is the unique extended supergravity which admits two different types of matter multiplets, namely vector and hyper multiplets. Thus, out of the three classes (simple non-degenerate, simple degenerate and semi-simple non-degenerate, respectively treated in subsections 2.2, 2.3 and 2.4) of groups G_4 of type E_7 treated above, one can always construct a semi-simple group of type E_7 with the following structure:

$$\begin{aligned}&G_4 \times \mathcal{G}_4;\\&(\mathbf{R}(G_4),\ \mathcal{R}(\mathcal{G}_4) = \mathbf{1})\,.\end{aligned} \tag{2.48}$$

As pointed out above, G_4 is the U-duality group of the $\mathcal{N} = 2$ theory (which is also the global isometry group of the special Kähler vector multiplets' scalar manifold), whereas $\mathcal{G}_4$ is the global isometry group of the quaternonic Kähler hypermultiplets' scalar manifold. The various truncations analyzed in subsequent sections provide a number of examples of truncations of simple non-degenerate groups of type E_7 down to $\mathcal{N} = 2$ semi-simple degenerate (see e.g. section 6) or semi-simple non-degenerate (see e.g. the models with $n_V, n_H \neq 0$ in table 5 below) groups of type E_7 of type (2.48).

3 Maximal truncations from $\mathcal{N} = 8$ ($J_3^{\mathbb{O}_s}$)

One can perform the kinematical reduction of $\mathcal{N}$-extended supergravity multiplets down to $\mathcal{N}' < \mathcal{N}$ multiplets (massless multiplets in $\mathcal{N}$-extended $D = 4$ supergravity are reported in tables 3 and 4). The reduction is subjected to the following dynamical conditions: the inclusion of U-duality groups: $G_4 \supset G'_4$, as well as of the stabilizers of the scalar manifold: $H_4 \supset H'_4$, such that the scalar manifold of the truncated theory is a proper sub-manifold of the scalar manifold of the starting theory: $G_4/H_4 \supset G'_4/H'_4$. At the level of electric and magnetic fluxes, the branching $\mathbf{R}(G_4) = \mathbf{R}'(G'_4) + \ldots$ has to hold, where $\mathbf{R}'(G'_4)$ is the relevant symplectic representation of G'_4 itself.

If $\mathcal{N}, \mathcal{N}' \geqslant 4$, the kinematical multiplet truncation actually coincides with the dynamical truncation, because there is a unique choice of matter multiplets in these cases. On the other hand, already for $\mathcal{N}' = 3$ this is no longer true for $\mathcal{N} \geqslant 6$, and for $\mathcal{N} = 8 \to \mathcal{N}' = 2$ many possibilities exist; the *maximal* truncations (in the sense of $G_4 \supset G'_4$ specified above) are listed in table 5. Kinematical truncations $\mathcal{N} = 6 \to 5$, $\mathcal{N} = 6 \to 4$ and $\mathcal{N} = 5 \to 4$ actually coincide with the corresponding dynamical reduction. The two latter cases yield 2 and no matter multiplets, respectively. Further truncation of these theories down to $\mathcal{N} = 1$ reduces to some of the general examples we consider further below.

Before proceeding with the analysis of the various truncations $\mathcal{N} = 8 \to \mathcal{N}' < 8$, we would like here to add a brief discussion of the general consistency conditions yielded by the supersymmetry transformations of the $\mathcal{N} = 8$ fermionic fields, along the lines of the treatment in section 5 of [10] (a similar discussion related to the Attractor Mechanism has been given also in section 5 of [18]). Neglecting three fermion terms, the transformations of the gravitinos ψ_A and of spin $\frac{1}{2}$ fermions χ_{ABC} read as follows ($A = 1, \ldots, 8$; cfr. e.g. (5.2)–(5.3) of [10]):

$$\delta\psi_{A\mu} = \nabla_\mu \epsilon_A + T^-_{AB|\nu\rho}\gamma_\mu{}^\nu\gamma^\rho\epsilon^B; \tag{3.1}$$

$$\delta\chi_{ABC} = P_{ABCD,\alpha}\partial_\mu\phi^\alpha\gamma^\mu\epsilon^D + T^-_{[AB|\mu\nu}\gamma^{\mu\nu}\epsilon_{|C]}, \tag{3.2}$$

$\mathcal{N}$	massless $\lambda_{\rm MAX}=2$ multiplet	massless $\lambda_{\rm MAX}=3/2$ multiplet
8	$\left[(2),8(\frac{3}{2}),28(1),56(\frac{1}{2}),70(0)\right]$	none
6	$\left[(2),6(\frac{3}{2}),16(1),26(\frac{1}{2}),30(0)\right]$	$\left[(\frac{3}{2}),6(1),15(\frac{1}{2}),20(0)\right]$
5	$\left[(2),5(\frac{3}{2}),10(1),11(\frac{1}{2}),10(0)\right]$	$\left[(\frac{3}{2}),6(1),15(\frac{1}{2}),20(0)\right]$
4	$\left[(2),4(\frac{3}{2}),6(1),4(\frac{1}{2}),2(0)\right]$	$\left[(\frac{3}{2}),4(1),7(\frac{1}{2}),8(0)\right]$
3	$\left[(2),3(\frac{3}{2}),3(1),(\frac{1}{2})\right]$	$\left[(\frac{3}{2}),3(1),3(\frac{1}{2}),2(0)\right]$
2	$\left[(2),2(\frac{3}{2}),(1)\right]$	$\left[(\frac{3}{2}),2(1),(\frac{1}{2})\right]$
1	$\left[(2),(\frac{3}{2})\right]$	$\left[(\frac{3}{2}),(1)\right]$

Table 3. Massless multiplets with maximal helicity $\lambda_{\rm MAX}=2,\ 3/2$ [40].

$\mathcal{N}$	massless $\lambda_{\rm MAX}=1$ multiplet	massless $\lambda_{\rm MAX}=1/2$ multiplet
8,6,5	none	none
4	$\left[(1),4(\frac{1}{2}),6(0)\right]$	none
3	$\left[(1),4(\frac{1}{2}),6(0)\right]$	none
2	$\left[(1),2(\frac{1}{2}),2(0)\right]$	$\left[2(\frac{1}{2}),4(0)\right]$
1	$\left[(1),(\frac{1}{2})\right]$	$\left[(\frac{1}{2}),2(0)\right]$

Table 4. Massless multiplets with maximal helicity $\lambda_{\rm MAX}=1,\ 1/2$ [40].

where $\nabla_\mu\epsilon_A \equiv \mathcal{D}_\mu\epsilon_A + \omega_A{}^B\epsilon_B$, T^-_{AB} is the (dressed) graviphotonic field strengths' 2-form, P_{ABCD} is the *Vielbein* 1-form, and ϕ^α are the 70 real scalars of the rank-7 symmetric $\mathcal{N}=8$ scalar manifold $E_{7(7)}/\mathrm{SU}(8)/\mathbb{Z}_2$.

When considering a truncation $\mathcal{N}=8\longrightarrow\mathcal{N}'<8$, it holds that

$$\mathrm{SU}(8)\supset\mathrm{SU}(\mathcal{N}')\times\mathrm{SU}(8-\mathcal{N}')\times\mathrm{U}(1); \tag{3.3}$$

$$\mathbf{8}=\left(\mathcal{N}',\mathbf{1}\right)_{\mathcal{N}'-8}+\left(\mathbf{1},\mathbf{8}\text{–}\mathcal{N}'\right)_{\mathcal{N}'}. \tag{3.4}$$

Correspondingly, the supersymmetry parameters, the gravitinos and the spin 1/2 fermions branch as ($a=1,\ldots,\mathcal{N}'$, $i=1,\ldots,8-\mathcal{N}'$):

$$\epsilon_A=\epsilon_a,\quad \epsilon_i; \tag{3.5}$$

$$\psi_A=\psi_a,\quad \psi_i; \tag{3.6}$$

$$\chi_{ABC}=\chi_{abc},\ \chi_{abi},\ \chi_{aij},\ \chi_{ijk}. \tag{3.7}$$

The conditions of consistent truncation read

$$\left\{\begin{array}{l}\epsilon_i=0;\\ \psi_i=0;\\ \chi_{abi}=\chi_{aij}=\chi_{ijk}=0,\end{array}\right.\qquad\text{such that}\quad\left\{\begin{array}{l}\delta\psi_i=0;\\ \\ \delta\chi_{abi}=\delta\chi_{aij}=\delta\chi_{ijk}=0,\end{array}\right. \tag{3.8}$$

with the exception of the case $\mathcal{N}' = 6$, for which χ_{aij}, as well as its corresponding supersymmetry variation, does not vanish.

By exploiting (3.8) and (3.1)–(3.2), one obtains the following general consistency conditions:

$$\begin{cases} i) : \omega_i{}^a = 0; \\ ii) : T^-_{ai} = 0; \\ iii) : P_{abci} = P_{aijk} = 0; \\ iv) : P_{abij} = 0; \\ v) : T^-_{ij} = 0, \end{cases} \tag{3.9}$$

where conditions $iv)$ and $v)$ do not hold for $\mathcal{N}\prime = 6$. Condition $i)$ (on the spin connection ω) confirms the consystency condition on the reduction of the holonomy ($\mathcal{R}$-symmetry) group, as discussed in sections 3 and 4 of [10]. Conditions $iii)$ and $iv)$ (on the *Vielbein* P) confirm the consistency conditions from the embedding of the scalar manifold of the $\mathcal{N}'$-extended supergravity sub-theory into the scalar manifold of $\mathcal{N} = 8$ theory, as discussed in section 4 of [10]. Furthermore, conditions $ii)$ and $v)$ (on the graviphotonic field strengths T), which are the (necessary but noot necessary) conditions which we discuss in the present investigation, are needed in order for T^-_{ab} to consistently parametrize the (dressed) graviphotons which survive the truncation under consideration (in the case $\mathcal{N}' = 6$, one should consider also T^-_{ij} non-vanishing).

3.1 $\to \mathcal{N} = 6$

In this subsection, we discuss, at the level of the consistency conditions yielded by supersymmetry, the case $\mathcal{N} = 8 \longrightarrow 6$ (for the "*twin*" case $\mathcal{N} = 2$ $(n_V, n_H) = (15, 0)$, and further decomposition, see point 1 of subsubsection 3.4.1):

$$J_3^{\mathbb{O}_s} \ : \ \mathcal{N} = 8 \longrightarrow J_3^{\mathbb{H}} : \mathcal{N} = 6$$
$$E_{7(7)} \supset SO^*(12) \times SU(2) \supset \mathrm{SU}(6) \times \mathrm{SU}(2) \times \mathrm{U}(1) : \tag{3.10}$$
$$\begin{cases} \mathbf{56} = (\mathbf{32}, \mathbf{1}) + (\mathbf{12}, \mathbf{2}) \\ = (\mathbf{1}, \mathbf{1})_3 + (\mathbf{1}, \mathbf{1})_{-3} + (\mathbf{15}, \mathbf{1})_{-1} + \left(\overline{\mathbf{15}}, \mathbf{1}\right)_1 + (\mathbf{6}, \mathbf{2})_1 + \left(\overline{\mathbf{6}}, \mathbf{2}\right)_{-1}; \end{cases} \tag{3.11}$$

$$\mathrm{SU}(8) \supset \mathrm{SU}(6) \times \mathrm{SU}(2) \times \mathrm{U}(1) : \begin{cases} \mathbf{8} = (\mathbf{6}, \mathbf{1})_{-2} + (\mathbf{1}, \mathbf{2})_6 ; \\ \mathbf{28} = (\mathbf{15}, \mathbf{1})_{-4} + (\mathbf{6}, \mathbf{2})_4 + (\mathbf{1}, \mathbf{1})_{12} ; \end{cases} \tag{3.12}$$

$$\frac{E_{7(7)}}{SU(8)} \supset \frac{SO^*(12)}{SU(6) \times U(1)} \supset \frac{SL(2, \mathbb{R})}{U(1)} \times \frac{SO(6,2)}{SO(6) \times SO(2)}. \tag{3.13}$$

In particular, the decomposition of the $\mathbf{28}$ of $SU(8)$ yields the following branching of the $\mathcal{N} = 8$ dressed graviphotonic field strengths ($a = 1, \ldots, 6$, $i = 1, 2$):

$$\underset{\mathbf{28}}{T^-_{AB}} = \underset{(\mathbf{15},\mathbf{1})}{T^-_{ab}} , \ \underset{(\mathbf{6},\mathbf{2})}{T^-_{ai}} , \ \underset{(\mathbf{1},\mathbf{1})}{T^-_{ij}} . \tag{3.14}$$

The $SU(2)$ commuting factor in (3.10) and (3.12) is the $\mathcal{R}$-symmetry truncated away in the supersymmetry reduction $\mathcal{N} = 8 \to 6$ (a further truncation $\mathcal{N} = 6 \to \mathcal{N} = 3$ is considered in subsection 4.1). The truncation condition on the two-form Abelian field strengths' fluxes reads

$$(\mathbf{12},\mathbf{2}) = (\mathbf{6},\mathbf{2}) + \left(\overline{\mathbf{6}},\mathbf{2}\right) = 0 \Leftrightarrow T^-_{ai} = 0. \tag{3.15}$$

We anticipate that this truncation condition on fluxes is complementary to the condition (3.46) considered in subsection 3.3; indeed, the embedding (3.42)–(3.45) is a different non-compact, real form of the embedding (3.12).

As a consequence of the decomposition (3.12) of the $\mathbf{8}$ of SU(8), the supersymmetry parameters, the gravitinos and the spin 1/2 fermions respectively branch as:

$$\underset{\mathbf{8}}{\epsilon_A} = \underset{(\mathbf{6},\mathbf{1})}{\epsilon_a}\,, \quad \underset{(\mathbf{1},\mathbf{2})}{\epsilon_i}\,; \tag{3.16}$$

$$\underset{\mathbf{8}}{\psi_A} = \underset{(\mathbf{6},\mathbf{1})}{\psi_a}\,, \quad \underset{(\mathbf{1},\mathbf{2})}{\psi_i}\,; \tag{3.17}$$

$$\underset{\mathbf{56}}{\chi_{ABC}} = \underset{\mathbf{20}}{\chi_{abc}}, \quad \underset{(\mathbf{15},\mathbf{2})}{\chi_{abi}}\,, \quad \underset{(\mathbf{6},\mathbf{1})}{\chi_{aij}}. \tag{3.18}$$

Correspondingly, the conditions of consistent truncation read

$$\left\{\begin{array}{l} \epsilon_i = 0; \\ \psi_i = 0; \\ \chi_{abi} = 0, \end{array}\right. \quad \text{such that} \quad \left\{\begin{array}{l} \delta\psi_i = 0; \\ \\ \delta\chi_{abi} = 0. \end{array}\right. \tag{3.19}$$

By evaluating the supersymmetry transformations (3.1)–(3.2) on the truncation conditions (3.19), one obtains

$$\delta\psi_{a\mu} = \nabla_\mu\epsilon_a + T^-_{ab|\nu\rho}\gamma_\mu{}^\nu\gamma^\rho\epsilon^b; \tag{3.20}$$

$$\delta\chi_{abc} = P_{abcd,\alpha}\partial_\mu\phi^\alpha\gamma^\mu\epsilon^d + \frac{1}{3}\left(T^-_{ab|\mu\nu}\gamma^{\mu\nu}\epsilon_c + T^-_{ca|\mu\nu}\gamma^{\mu\nu}\epsilon_b + T^-_{bc|\mu\nu}\gamma^{\mu\nu}\epsilon_a\right); \tag{3.21}$$

$$\delta\chi_{aij} = P_{abij,\alpha}\partial_\mu\phi^\alpha\gamma^\mu\epsilon^b + \frac{1}{3}T^-_{ij|\mu\nu}\gamma^{\mu\nu}\epsilon_a \tag{3.22}$$

in the untruncated sector, and

$$\delta\psi_{i\mu} = \omega_{i|\mu}^{\ a}\epsilon_a + T^-_{ia|\nu\rho}\gamma_\mu{}^\nu\gamma^\rho\epsilon^a; \tag{3.23}$$

$$\delta\chi_{abi} = -P_{abci,\alpha}\partial_\mu\phi^\alpha\gamma^\mu\epsilon^c + \frac{2}{3}T^-_{i[a||\mu\nu}\gamma^{\mu\nu}\epsilon_{|b]} \tag{3.24}$$

in the truncated sector.

By then imposing (3.19) on (3.23) and (3.24), one obtains

$$\omega_i{}^a = 0 = T^-_{ia} = P_{abci}. \tag{3.25}$$

In particular, $T^-_{ia} = 0$ is nothing but the condition (3.15).

Thus, one can conclude that the truncation (3.25) is fully consistent.

A similar analysis at the level of supersymmetry can be performed in all cases. We observe that whenever the truncation $\mathcal{N} = 8 \longrightarrow \mathcal{N}' < 8$ is consistent with supersymmetry, and thus it actually exists, there occurs an $\mathrm{SU}(8-\mathcal{N}')$ factor commuting with the the $\mathcal{R}$-symmetry $\mathrm{U}(\mathcal{N}\prime)$ of the truncated sub-theory inside the $\mathcal{N} = 8$ $\mathcal{R}$-symmetry SU(8).

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3.2 $\rightarrow \mathcal{N}=5 \rightarrow \mathcal{N}=3,2$

Next, we consider the maximal *non-symmetric* embedding:

$$J_3^{\mathbb{O}_s} \;:\; \mathcal{N}=8 \longrightarrow M_{1,2}(\mathbb{O}) : \mathcal{N}=5; \tag{3.26}$$

$$E_{7(7)} \supset \mathrm{SU}(1,5)\times\mathrm{SU}(3); \tag{3.27}$$

$$\mathbf{56} = (\mathbf{6},\mathbf{3}) + \left(\overline{\mathbf{6}},\overline{\mathbf{3}}\right) + (\mathbf{20},\mathbf{1})\,; \tag{3.28}$$

$$\frac{E_{7(7)}}{\mathrm{SU}(8)} \supset \frac{\mathrm{SU}(1,5)}{\mathrm{U}(5)}. \tag{3.29}$$

$M_{1,2}(\mathbb{O})$ is the Jordan triple system (not upliftable to $D=5$) generated by 2×1 matrices over $\mathbb{O}$ [46–49]. The $\mathbf{20}$ is the rank-3 antisymmetric self-real irrep. of $\mathrm{SU}(1,5)$. The commuting $\mathrm{SU}(3)$ factor can be interpreted as the part of the $\mathcal{R}$-symmetry truncated away in the supersymmetry reduction $\mathcal{N}=8\rightarrow\mathcal{N}=5$. On the two-form Abelian field strengths' fluxes, the truncation condition reads

$$(\mathbf{6},\mathbf{3}) = 0. \tag{3.30}$$

As discussed in section 8 of [41], the *quartic* invariant of the $\mathbf{R}=\mathbf{20}$ of $\mathrm{SU}(1,5)$, after skew-diagonalization in the scalar-dressed $\mathcal{R}$-symmetry $\mathrm{U}(5)$-basis and use of the Hua-Bloch-Messiah-Zumino Theorem [50–52], is a *perfect square*. On this respect the couples $(\mathrm{SU}(1,5),\ \mathbf{R}=\mathbf{20})$ and $(\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(6),\ \mathbf{R}=(\mathbf{2},\mathbf{6}))$ (this latter pertaining to $\mathcal{N}=4$ *"pure"* supergravity) stand on a particular footing among simple and respectively semisimple groups *"of type E_7"* [16]. Thus, this embedding does not concern a proper *"degeneration"* of a group of type E_7, but it is however noteworthy.

In turn, the *"pure"* $\mathcal{N}=5$ theory admits two maximal *"degenerative"* truncations, which precisely match the kinematical decomposition of the $\mathcal{N}=5$ gravity multiplet into matter $\mathcal{N}=2$ multiplets.

1. The first reads:

$$\mathcal{N}=5 \longrightarrow \mathcal{N}=3,\ n_V=1 \overset{\text{"twin"}}{\Leftrightarrow} \mathcal{N}=2\ \mathbb{CP}^3; \tag{3.31}$$

$$\mathrm{SU}(1,5) \supset \mathrm{SU}(1,3)\times\mathrm{SU}(2)\times\mathrm{U}(1); \tag{3.32}$$

$$\mathbf{20} = (\mathbf{4},\mathbf{1})_{+3} + \left(\overline{\mathbf{4}},\mathbf{1}\right)_{-3} + (\mathbf{6},\mathbf{2})_0\,; \tag{3.33}$$

$$\frac{\mathrm{SU}(1,5)}{\mathrm{U}(5)} \supset \frac{\mathrm{SU}(1,3)}{\mathrm{U}(3)}, \tag{3.34}$$

   and it admits two possible interpretations, due to the fact that $\mathcal{N}=3$ supergravity coupled to 1 vector multiplet and $\mathcal{N}=2$ supergravity *minimally coupled* to 3 vector multiplets share the very same bosonic sector (namely, they are *"twin"* theories; see the discussion in section 9 of [41]). In the $\mathcal{N}=3$ interpretation, one gets a theory with 1 vector multiplets, and the $\mathrm{SU}(2)$ commuting factor can be interpreted as the part of the $\mathcal{R}$-symmetry truncated away in the supersymmetry reduction $\mathcal{N}=5\rightarrow\mathcal{N}=3$. On the other hand, in the $\mathcal{N}=2$ interpretation, one gets a theory with 3 *minimally coupled* vector multiplets without hypermultiplets, and the $\mathrm{SU}(2)$ commuting factor is the global $\mathcal{N}=2$ hyper $\mathcal{R}$-symmetry. In both cases, on the two-form Abelian field strengths' fluxes the truncation condition reads

$$(\mathbf{6},\mathbf{2})_0 = 0. \tag{3.35}$$

One can also prove that the *quartic* invariant of the $\mathbf{R}=\mathbf{20}$ of $SU(1,5)$, under the truncation (3.35) becomes the *square* of the *quadratic* invariant of the $\mathbf{R}=\mathbf{4}$ of $SU(1,3)$.

2. The second maximal *"degenerative"* truncation of the *"pure"* $\mathcal{N}=5$ theory reads

$$\mathcal{N}=5 \longrightarrow \mathcal{N}=2,\ n_V=0,\ n_H=1; \tag{3.36}$$

$$SU(1,5) \supset SU(1,2)\times SU(3)\times U(1); \tag{3.37}$$

$$\mathbf{20} = (\mathbf{1},\mathbf{1})_{+3} + (\mathbf{1},\mathbf{1})_{-3} + \left(\mathbf{3},\overline{\mathbf{3}}\right)_{-1} + \left(\overline{\mathbf{3}},\mathbf{3}\right)_{+1}; \tag{3.38}$$

$$\frac{SU(1,5)}{U(5)} \supset \frac{SU(1,2)}{U(2)}. \tag{3.39}$$

The $\mathcal{N}=2$ theory is coupled to the *universal* hypermultiplet, in absence of vector multiplets. The $SU(3)$ commuting factor can be interpreted as the part of the $\mathcal{R}$-symmetry truncated away in the supersymmetry reduction $\mathcal{N}=5\to\mathcal{N}=2$, whereas the commuting $U(1)$ factor is the global $\mathcal{N}=2$ vector $\mathcal{R}$-symmetry. On the two-form Abelian field strengths' fluxes, the truncation condition reads

$$(\mathbf{3},\overline{\mathbf{3}})_{-1} = 0, \tag{3.40}$$

such that only the graviphoton charges $(\mathbf{1},\mathbf{1})_{+3}+(\mathbf{1},\mathbf{1})_{-3}$ survive the truncation. One can prove that the *quartic* invariant of the $\mathbf{R}=\mathbf{20}$ of $SU(1,5)$, under the truncation (3.40) becomes nothing but the *square* of the Reissner-Nördstrom entropy

$$\frac{S_{RN}}{\pi} = \frac{1}{2}\left[(p^0)^2 + q_0^2\right]. \tag{3.41}$$

It is here worth pointing out that a consistent truncation to an hypermultiplet(s)-coupled $\mathcal{N}=2$ theory with no vector multiplets should necessarily contain two real singlets (namely the electric and magnetic charge of the graviphoton) in the branching of the original flux representation, as it holds e.g. for (3.38) and (3.72) respectively pertaining to truncations (3.36) and (3.70). However, such truncations are not interesting for our investigation, because they yield no vectors when further reduced down to $\mathcal{N}=1$ models (the $\mathcal{N}=2$ graviphoton is contained in the $\mathcal{N}=1$ gravitino multiplet, which is truncated away).

3.3 $\rightarrow \mathcal{N}=4$ $\mathbb{R}\oplus\mathbf{\Gamma}_{5,5}$

Let's consider now the embedding:

$$J_3^{\mathbb{O}_s}\ :\ \mathcal{N}=8 \longrightarrow \mathbb{R}\oplus\mathbf{\Gamma}_{5,5} : \mathcal{N}=4,\ n_V=6; \tag{3.42}$$

$$E_{7(7)} \supset SL(2,\mathbb{R})\times SO(6,6); \tag{3.43}$$

$$\mathbf{56} = (\mathbf{2},\mathbf{12}) + (\mathbf{1},\mathbf{32}); \tag{3.44}$$

$$\frac{E_{7(7)}}{SU(8)} \supset \frac{SL(2,\mathbb{R})}{U(1)}\times\frac{SO(6,6)}{SO(6)\times SO(6)}. \tag{3.45}$$

	G_V	G_H	H_V	H_H	$\frac{G_V}{H_V} \times \frac{G_H}{H_H}$	(n_V, n_H)
$J_3^{\mathbb{H}}$	$\mathrm{SO}^*(12)$	$\mathrm{SU}(2)$	$\mathrm{SU}(6) \times \mathrm{U}(1)$	–	$\frac{\mathrm{SO}^*(12)}{\mathrm{SU}(6)\otimes\mathrm{U}(1)}$	$(15, 0)$
$J_3^{\mathbb{C}}$	$\mathrm{SU}(3,3)$	$\mathrm{SU}(2,1)$	$\mathrm{SU}(3) \times \mathrm{SU}(3) \times \mathrm{U}(1)$	$\mathrm{SU}(2) \times \mathrm{U}(1)$	$\frac{\mathrm{SU}(3,3)}{S(\mathrm{U}(3)\times\mathrm{U}(3))} \times \frac{\mathrm{SU}(2,1)}{\mathrm{SU}(2)\times\mathrm{U}(1)}$	$(9, 1)$
$J_3^{\mathbb{R}}$	$\mathrm{Sp}(6, \mathbb{R})$	$G_{2(2)}$	$\mathrm{SU}(3) \times \mathrm{U}(1)$	$\mathrm{SU}(2) \times \mathrm{SU}(2)$	$\frac{\mathrm{Sp}(6,\mathbb{R})}{\mathrm{SU}(3)\times\mathrm{U}(1)} \times \frac{G_{2(2)}}{\mathrm{SO}(4)}$	$(6, 2)$
STU	$\mathrm{SU}(1,1) \times \mathrm{SO}(2,2)$	$\mathrm{SO}(4,4)$	$\mathrm{U}(1) \times \mathrm{SO}(2) \times \mathrm{SO}(2)$	$\mathrm{SO}(4) \times \mathrm{SO}(4)$	$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(2,2)}{\mathrm{SO}(2)\times\mathrm{SO}(2)} \times \frac{\mathrm{SO}(4,4)}{\mathrm{SO}(4)\otimes\mathrm{SO}(4)}$	$(3, 4)$
$J_{3,M}^{\mathbb{R}}$	$\mathrm{SU}(1,1)$	$F_{4(4)}$	$\mathrm{U}(1)$	$\mathrm{USp}(6) \times \mathrm{SU}(2)$	$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{F_{4(4)}}{\mathrm{USp}(6)\otimes\mathrm{SU}(2)}$	$(1, 7)$
$J_{3,M}^{\mathbb{C}}$	$\mathrm{U}(1)$	$E_{6(2)}$	–	$\mathrm{SU}(6) \times \mathrm{SU}(2)$	$\frac{E_{6(2)}}{\mathrm{SU}(6)\times\mathrm{SU}(2)}$	$(0, 10)$

Table 5. $\mathcal{N} = 2$ supergravities obtained as consistent *maximal* truncation of $\mathcal{N} = 8$ supergravity.

The $\mathcal{N} = 4$ theory is coupled to 6 vector multiplets and, on the two-form Abelian field strengths' fluxes, the truncation condition reads

$$(\mathbf{1}, \mathbf{32}) = 0. \tag{3.46}$$

It still exhibits a *quartic* U-invariant $\mathcal{I}_4$, but it can be further truncated to a theory with U-duality group $\mathrm{U}(3,3)$ with *quadratic* invariant through the procedure considered in section 5, to which we address the reader for further elucidation (also the treatment given in section 8.3 can be considered).

3.4 $\to \mathcal{N} = 2$

We now consider the reduction of $\mathcal{N} = 8$ supergravity to an $N = 2$ theory with n_V vector and n_H hypermultiplets:

$$(n_V, n_H) \equiv \left(\dim_{\mathbb{C}}\left(\frac{G_V}{H_V}\right), \ \dim_{\mathbb{H}}\left(\frac{G_H}{H_H}\right)\right), \ n_V \leqslant 15, \ n_H \leqslant 20, \tag{3.47}$$

where $\frac{G_V}{H_V}$ and $\frac{G_H}{H_H}$ respectively stand for the special Kähler and quaternionic Kähler scalar manifolds, where $H_V = mcs(G_V)$ and $H_H = mcs(G_H)$. H_V always contains a factorized commuting $\mathrm{U}(1)$ subgroup, which is promoted to global symmetry when $n_V = 0$; on the other hand, H_H always contains a factorized commuting $\mathrm{SU}(2)$ subgroup, which is promoted to global symmetry when $n_H = 0$ [53].

We consider only $\mathcal{N} = 2$ *maximal* supergravities, i.e. $\mathcal{N} = 2$ theories (obtained by consistent truncations of $\mathcal{N} = 8$ supergravity) which cannot be obtained by a further reduction from some other $\mathcal{N} = 2$ theory, which are also *magic*. They are called *magic*, since their symmetry groups are the groups of the famous *Magic Square* of Freudenthal, Rozenfeld and Tits associated with some remarkable geometries [54–56]. From the analysis performed in [10], only six $\mathcal{N} = 2$, $d = 4$ *maximal magic* supergravities[7] exist which can be obtained by consistently truncating $\mathcal{N} = 8$, $d = 4$ supergravity; they are given by table 5. After [57], we also include the case of STU model [58–60] with $n_H = 4$ hypermultiplets; see below.

The models have been denoted by referring to their special geometry. $J_3^{\mathbb{H}}$, $J_3^{\mathbb{C}}$ and $J_3^{\mathbb{R}}$ stand for three of the four $\mathcal{N} = 2$, $d = 4$ magic supergravities which, as their 5-dim. versions, are respectively defined by the three simple Jordan algebras $J_3^{\mathbb{H}}$, $J_3^{\mathbb{C}}$ and $J_3^{\mathbb{R}}$ of degree 3 with irreducible norm forms, namely by the Jordan algebras of Hermitian 3×3 matrices over the division algebras of quaternions $\mathbb{H}$, complex numbers $\mathbb{C}$ and real numbers $\mathbb{R}$ [46–49, 61–64].

In table 5, the subscript "M" denotes the model obtained by performing a $D = 4$ *mirror map* (i.e. the composition of two c-maps [65] in $D = 4$) from the original manifold; such an operation maps a model with content (n_V, n_H) to a model with content $(n_H - 1, n_V + 1)$, and thus the mirror $J_{3,M}^{\mathbb{H}}$ of $J_3^{\mathbb{H}}$, with $(n_V, n_H) = (-1, 16)$ and quaternionic manifold $\frac{E_{7(-5)}}{\mathrm{SO}(12)\otimes\mathrm{SU}(2)}$ does not exist, *at least* in $D = 4$. The STU model is *self-mirror*: $STU = STU_M$.

3.4.1 Further truncation to *minimal coupling*

Then, we consider further truncations to $\mathcal{N} = 2$ theories exhibiting scalar-vector *minimal coupling*; since hyperscalars are always minimally coupled, we study only truncations of the vector multiplets' scalar sector.

Out of the cases reported in table 5, some deserve immediate comments:

- The case pertaining to the *self-mirror* STU_M model is included in the treatment of section 6 starting from $\mathcal{N} = 4$ theory coupled to $n = 6$ vector multiplets (which in turn is *maximally* embedded into $\mathcal{N} = 8$ theory), and considering the splitting $(n_1, n - n_1) = (2, 4)$.

- The case pertaining to the *mirror* model $J_{3,M}^{\mathbb{R}}$ is not interesting in our investigation: indeed, in the vector multiplets sector, $J_{3,M}^{\mathbb{R}}$ is nothing but the so-called $\mathcal{N} = 2$ T^3 model, in which the complex scalar field T is *not* minimally coupled to vectors, and no further truncation to minimally coupled $\mathcal{N} = 2$ or $\mathcal{N} = 1$ models is possible.

Let's now list the various relevant possibilities from the models reported in table 5:

[7] By $E_{7(p)}$ we denote a non-compact form of E_7, where $p \equiv$ (# non-compact − # compact) generators of the group [24, 25]. In such a notation, the compact form of E_7 is $E_{7(-133)}$ ($dim_{\mathbb{R}} E_7 = 133$).

1.

$$
\begin{aligned}
J_3^{\mathbb{O}_s} \quad &: \quad \mathcal{N}=8 \longrightarrow J_3^{\mathbb{H}} : \begin{cases} \mathcal{N}=2 \ (n_V,n_H)=(15,0) \\ \quad\quad \Updownarrow \text{ “}twin\text{”} \\ \quad\quad \mathcal{N}=6 \end{cases} \\
&\longrightarrow \mathbb{R}\oplus\mathbf{\Gamma}_{1,5} \ : \begin{cases} \mathcal{N}=2 \ (n_V,n_H)=(7,0) \\ \quad\quad \Updownarrow \text{ “}twin\text{”} \\ \quad \mathcal{N}=4 \ n_V=2 \end{cases}
\end{aligned} \tag{3.48}
$$

$$
\begin{aligned}
E_{7(7)} &\supset \mathrm{SO}^*(12)\times\mathrm{SU}(2) \\
&\supset \mathrm{SO}^*(8)\times\mathrm{SO}^*(4)\times\mathrm{SU}(2)\sim\mathrm{SO}(6,2)\times\mathrm{SL}(2,\mathbb{R})\times\mathrm{SU}(2)\times\mathrm{SU}(2);
\end{aligned} \tag{3.49}
$$

$$
\begin{aligned}
\mathbf{56} &= (\mathbf{32},\mathbf{1})+(\mathbf{12},\mathbf{2}) \\
&= (\mathbf{8}_s,\mathbf{2},\mathbf{1},\mathbf{1})+(\mathbf{8}_c,\mathbf{1},\mathbf{2},\mathbf{1})+(\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{2})+(\mathbf{8}_v,\mathbf{1},\mathbf{1},\mathbf{2})\,;
\end{aligned} \tag{3.50}
$$

$$
\frac{E_{7(7)}}{\mathrm{SU}(8)} \supset \frac{\mathrm{SO}^*(12)}{\mathrm{SU}(6)\times\mathrm{U}(1)} \supset \frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)}\times\frac{\mathrm{SO}(6,2)}{\mathrm{SO}(6)\times\mathrm{SO}(2)}. \tag{3.51}
$$

The $J_3^{\mathbb{H}}$-based theory can either be interpreted as $\mathcal{N}=2$ or as its “twin” $\mathcal{N}=6$ [39, 41, 69, 70]; in the former case, the $\mathrm{SU}(2)$ commuting factor is the *global* hyper $\mathcal{R}$-symmetry, whereas in the latter case it is the $\mathcal{R}$-symmetry truncated away in the supersymmetry reduction $\mathcal{N}=8\to\mathcal{N}=6$ (a further truncation $\mathcal{N}=6\to\mathcal{N}=3$ is considered in subsection 4.1). In both cases, the truncation condition on the two-form Abelian is given in eq. (3.31). Thence, one can proceed by truncating to the $(\mathbb{R}\oplus\mathbf{\Gamma}_{1,5})$-based theory still enjoys a *“twin”* interpretation [41, 70], either $\mathcal{N}=2$ or $\mathcal{N}=4$ supergravity; in the former case, the second $\mathrm{SU}(2)$ commuting factor also be interpreted as the *global* hyper $\mathcal{R}$-symmetry, whereas in the latter case it is the $\mathcal{R}$-symmetry truncated away in the supersymmetry reduction $\mathcal{N}=6\to\mathcal{N}=4$. In both cases, the truncation condition is

$$
(\mathbf{8}_c,\mathbf{1},\mathbf{2},\mathbf{1})=0 \ \ or \ \ (\mathbf{8}_s,\mathbf{2},\mathbf{1},\mathbf{1})=0. \tag{3.52}
$$

The resulting theory still exhibits a *quartic* U-invariant $\mathcal{I}_4$, but it can be further truncated to a theory with U-duality group $\mathrm{U}(1,3)$ with *quadratic* invariant. It is here worth remarking that such a theory still admits a *“twin”* interpretation [41], namely either as $\mathcal{N}=3$ with $n_V=1$ vector multiplet or as $\mathcal{N}=2$ *minimally coupled* to $n_V=3$ vector multiplets (and no hypermultiplets).

2.
$$J_3^{\mathbb{O}_s}\ :\ \mathcal{N}=8 \longrightarrow J_3^{\mathbb{C}} : \mathcal{N}=2\ (n_V, n_H) = (9,1) \longrightarrow \mathcal{N}=2\ \mathbb{CP}^3\ (n_V, n_H) = (4,1); \tag{3.53}$$

$$E_{7(7)} \supset \mathrm{SU}(3,3)\times \mathrm{SU}(2,1) \supset \mathrm{SU}(1,3)\times \mathrm{SU}(2)\times \mathrm{SU}(2,1)\times \mathrm{U}(1); \tag{3.54}$$

$$\begin{aligned}\mathbf{56} &= (\mathbf{6},\mathbf{3}) + \left(\overline{\mathbf{6}},\overline{\mathbf{3}}\right) + (\mathbf{20},\mathbf{1}) \\ &= (\mathbf{1},\mathbf{2},\mathbf{3})_2 + (\mathbf{4},\mathbf{1},\mathbf{3})_{-1} + \left(\mathbf{1},\mathbf{2},\overline{\mathbf{3}}\right)_{-2} + \left(\overline{\mathbf{4}},\mathbf{1},\overline{\mathbf{3}}\right)_1 + (\mathbf{4},\mathbf{1},\mathbf{0})_{+3} + \left(\overline{\mathbf{4}},\mathbf{1},\mathbf{0}\right)_{-3} + (\mathbf{6},\mathbf{2})_0\,;\end{aligned} \tag{3.55}$$

$$\frac{E_{7(7)}}{\mathrm{SU}(8)} \supset \frac{\mathrm{SU}(3,3)}{S\left(\mathrm{U}(3)\times\mathrm{U}(3)\right)}\times\frac{\mathrm{SU}(2,1)}{\mathrm{U}(2)} \supset \frac{\mathrm{SU}(1,3)}{\mathrm{U}(3)}\times\frac{\mathrm{SU}(2,1)}{\mathrm{U}(2)}. \tag{3.56}$$

The $J_3^{\mathbb{C}}$-based theory is *magic* $\mathcal{N}=2$ with 9 vector multiplets and 1 universal hypermultiplet. The truncation condition reads

$$(\mathbf{6},\mathbf{3}) = 0. \tag{3.57}$$

A different realization of this truncation has been studied in subsection 3.2. Thence, one can proceed by truncating to $\mathcal{N}=2$ *minimally coupled* to 3 vector multiplets (hyper sector untouched); the further truncation condition is

$$(\mathbf{6},\mathbf{2})_0 = 0. \tag{3.58}$$

Through this chain of truncation one can prove that the *quartic* invariant of the $\mathbf{R}=\mathbf{20}$ of $\mathrm{SU}(3,3)$ becomes the *square* of the *quadratic* invariant of the $\mathbf{R}=\mathbf{4}$ of $\mathrm{SU}(1,3)$.

3. From $\mathcal{N}=2$ $J_3^{\mathbb{C}}$ theory another truncation is possible, namely:
$$J_3^{\mathbb{O}_s}\ :\ \mathcal{N}=8 \longrightarrow J_3^{\mathbb{C}} : \mathcal{N}=2\ (n_V, n_H)=(9,1) \longrightarrow \mathcal{N}=2\,\mathbb{R}\oplus\mathbf{\Gamma}_{1,3}\,(n_V, n_H)=(5,1); \tag{3.59}$$

$$\begin{aligned}E_{7(7)} &\supset \mathrm{SU}(3,3)\times\mathrm{SU}(2,1) \\ &\supset \mathrm{SU}(1,1)\times\mathrm{SU}(2,2)\times\mathrm{SU}(2,1)\times\mathrm{U}(1) \sim \mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(2,4)\times\mathrm{SU}(2,1)\times\mathrm{U}(1);\end{aligned} \tag{3.60}$$

$$\begin{aligned}\mathbf{56} &= (\mathbf{6},\mathbf{3}) + \left(\overline{\mathbf{6}},\overline{\mathbf{3}}\right) + (\mathbf{20},\mathbf{1}) \\ &= (\mathbf{2},\mathbf{1},\mathbf{3})_2 + (\mathbf{1},\mathbf{4},\mathbf{3})_{-1} + \left(\mathbf{2},\mathbf{1},\overline{\mathbf{3}}\right)_{-2} + \left(\mathbf{1},\overline{\mathbf{4}},\overline{\mathbf{3}}\right)_1 + (\mathbf{1},\mathbf{4},\mathbf{1})_3 + \left(\mathbf{1},\overline{\mathbf{4}},\mathbf{1}\right)_{-3} + (\mathbf{2},\mathbf{6})_0\,;\end{aligned} \tag{3.61}$$

$$\frac{E_{7(7)}}{\mathrm{SU}(8)} \supset \frac{\mathrm{SU}(3,3)}{S\left(\mathrm{U}(3)\times\mathrm{U}(3)\right)}\times\frac{\mathrm{SU}(2,1)}{\mathrm{U}(2)} \supset \frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)}\times\frac{\mathrm{SO}(2,4)}{\mathrm{SO}(2)\times\mathrm{SO}(4)}\times\frac{\mathrm{SU}(2,1)}{\mathrm{U}(2)}. \tag{3.62}$$

As for the point 2 above, the first truncation condition is given by (3.57), but the second one is the very opposite of (3.58): only $(\mathbf{2},\mathbf{6})_0$ does not vanish, or equivalently:

$$(\mathbf{1},\mathbf{4},\mathbf{1})_3 = 0. \tag{3.63}$$

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The resulting theory still exhibits a *quartic* U-invariant $\mathcal{I}_4$, but it can be *non-maximally* further truncated to an $\mathcal{N}=2$ $\mathbb{CP}^2$ model with *quadratic* invariant through the procedure considered in section 8.3, to which we address the reader for further elucidation.

4.

$$J_3^{\mathbb{O}_s} \;:\; \mathcal{N}{=}8 \longrightarrow J_3^{\mathbb{R}} : \mathcal{N}{=}2\ (n_V,n_H){=}(6,2) \longrightarrow \mathcal{N}{=}2\,\mathbb{R}\oplus\mathbf{\Gamma}_{1,2}\,(n_V,n_H){=}(4,2); \tag{3.64}$$

$$E_{7(7)} \supset \mathrm{Sp}(6,\mathbb{R})\times G_{2(2)} \supset \mathrm{Sp}(2,\mathbb{R})\times\mathrm{Sp}(4,\mathbb{R})\times G_{2(2)} \sim \mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(2,3)\times G_{2(2)}; \tag{3.65}$$

$$\mathbf{56} = (\mathbf{14}',\mathbf{1})+(\mathbf{6},\mathbf{7})=(\mathbf{1},\mathbf{4},\mathbf{1})+(\mathbf{2},\mathbf{5},\mathbf{1})+(\mathbf{2},\mathbf{1},\mathbf{7})+(\mathbf{1},\mathbf{4},\mathbf{7})\,; \tag{3.66}$$

$$\frac{E_{7(7)}}{\mathrm{SU}(8)} \supset \frac{\mathrm{Sp}(6,\mathbb{R})}{\mathrm{U}(3)}\times\frac{G_{2(2)}}{\mathrm{SO}(4)} \supset \frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)}\times\frac{\mathrm{SO}(2,3)}{\mathrm{SO}(2)\times\mathrm{SO}(3)}\times\frac{G_{2(2)}}{\mathrm{SO}(4)}. \tag{3.67}$$

The $J_3^{\mathbb{R}}$-based theory is *magic* $\mathcal{N}=2$ with 6 vector multiplets and 2 hypermultiplets. The truncation condition reads

$$(\mathbf{6},\mathbf{7})=0. \tag{3.68}$$

Thence, one can proceed by truncating to $(\mathbb{R}\oplus\mathbf{\Gamma}_{1,2})$-based $\mathcal{N}=2$ theory (hyper sector untouched); the further truncation condition is

$$(\mathbf{1},\mathbf{4},\mathbf{1})=0. \tag{3.69}$$

The resulting theory still exhibits a *quartic* U-invariant $\mathcal{I}_4$, but it can be *non-maximally* further truncated to an $\mathcal{N}=2$ $\mathbb{CP}^1$ model with *quadratic* invariant through the procedure considered in section 8.3 (see also comment in subsection 8.3.1), to which we address the reader for further elucidation.

5.

$$J_3^{\mathbb{O}_s} \;:\; \mathcal{N}=8 \longrightarrow J_{3,M}^{\mathbb{C}} : \mathcal{N}=2\ (n_V,n_H)=(0,10); \tag{3.70}$$

$$E_{7(7)} \supset E_{6(2)}\times\mathrm{U}(1)\supset\mathrm{U}(1); \tag{3.71}$$

$$\mathbf{56} = \mathbf{27}_{+1}+\mathbf{27}'_{-1}+\mathbf{1}_{+3}+\mathbf{1}'_{-3}; \tag{3.72}$$

$$\frac{E_{7(7)}}{\mathrm{SU}(8)} \supset \frac{E_{6(2)}}{\mathrm{SU}(6)\times\mathrm{SU}(2)}. \tag{3.73}$$

The resulting $\mathcal{N}=2$ theory is coupled to 10 hypermultiplets, in absence of vector multiplets. The commuting U(1) factor is the global $\mathcal{N}=2$ vector $\mathcal{R}$-symmetry. On the two-form Abelian field strengths' fluxes, the truncation condition reads

$$\mathbf{27}_{+1}=0, \tag{3.74}$$

such that only the graviphoton charges $\mathbf{1}_{+3} + \mathbf{1}'_{-3}$ survive the truncation. One can prove that the *quartic* invariant of the $\mathbf{R} = \mathbf{56}$ of $E_{7(7)}$, under the truncation (3.74) becomes nothing but the *square* of the Reissner-Nördstrom entropy (3.41).

4 Maximal truncations from $\mathcal{N} = 6$ $(J_3^{\mathbb{H}})$

4.1 $\rightarrow \mathcal{N} = 3$

From $\mathcal{N} = 6$ *"pure"* theory, one can consider the following maximal *"degenerative"* truncation:

$$J_3^{\mathbb{H}} \ : \ \mathcal{N} = 6 \longrightarrow \mathcal{N} = 3, \ n_V = 3; \tag{4.1}$$

$$\mathrm{SO}^*(12) \supset \mathrm{SU}(3,3) \times \mathrm{U}(1); \tag{4.2}$$

$$\mathbf{32} = \mathbf{6}_{-2} + \overline{\mathbf{6}}_{+2} + \mathbf{20}_0; \tag{4.3}$$

$$\frac{\mathrm{SO}^*(12)}{\mathrm{SU}(6) \times \mathrm{U}(1)} \supset \frac{\mathrm{SU}(3,3)}{\mathrm{SU}(3) \times \mathrm{SU}(3) \times \mathrm{U}(1)}. \tag{4.4}$$

The $\mathcal{N} = 3$ theory is coupled to 3 vector multiplets and, on the two-form Abelian field strengths' fluxes, the truncation condition reads

$$\mathbf{20}_0 = 0. \tag{4.5}$$

One can prove that the *quartic* invariant of the $\mathbf{R} = \mathbf{32}$ of $\mathrm{SO}^*(12)$, under the truncation (4.5) becomes the *square* of the *quadratic* invariant of the $\mathbf{R} = \mathbf{6}$ of $\mathrm{SU}(3,3)$.

4.2 $\mathcal{N} = 5$

Note that, one might consider another truncation by setting

$$\mathbf{6}_{-2} = 0 \tag{4.6}$$

in (4.3); this corresponds to a truncation $\mathcal{N} = 6 \longrightarrow \mathcal{N} = 2$ based on $J_3^{\mathbb{C}}$ or, equivalently (due to the fact that $\mathcal{N} = 6$ and $\mathcal{N} = 2$ based on $J_3^{\mathbb{H}}$ are *"twin"*, i.e. they share the very same bosonic sector [39, 41, 69, 70]) to $\mathcal{N} = 2$ $J_3^{\mathbb{H}} \longrightarrow \mathcal{N} = 2$ $J_3^{\mathbb{C}}$. However, the resulting $\mathcal{N} = 2$ "magic" complex theory exhibits a generally *"non-degenerate" quartic* U-invariant $\mathcal{I}_4$.

On the other hand, if in (4.2) $\mathrm{SU}(3,3)$ is changed into $\mathrm{SU}(1,5)$, another, complementary, realization of the above truncation reads

$$\mathcal{N} = 6 \longrightarrow \mathcal{N} = 5; \tag{4.7}$$

$$\mathrm{SO}^*(12) \supset \mathrm{SU}(1,5) \times \mathrm{U}(1); \tag{4.8}$$

$$\mathbf{32} = \mathbf{6}_{-2} + \overline{\mathbf{6}}_{+2} + \mathbf{20}_0; \tag{4.9}$$

$$\frac{\mathrm{SO}^*(12)}{\mathrm{SU}(6) \times \mathrm{U}(1)} \supset \frac{\mathrm{SU}(1,5)}{\mathrm{U}(5)}. \tag{4.10}$$

The $\mathcal{N}=5$ theory is *"pure"* and the commuting $\mathrm{U}(1)$ factor corresponds to the part of the $\mathcal{R}$-symmetry truncated away in the supersymmetry reduction $\mathcal{N}=6 \to \mathcal{N}=5$. On the two-form Abelian field strengths' fluxes, the truncation condition is

$$\mathbf{6}_{-2}=0. \tag{4.11}$$

In turn, the *"pure"* $\mathcal{N}=5$ theory admits two maximal *"degenerative"* truncations, treated in section 3.2, which precisely match the kinematical decomposition of the $\mathcal{N}=5$ gravity multiplet into matter $\mathcal{N}=2$ multiplets.

5 $\mathcal{N}=4$ $\mathbb{R}\oplus\mathbf{\Gamma}_{5,2n-1} \longrightarrow \mathcal{N}=3$

We start with $\mathcal{N}=4$ supergravity coupled to $n_V=2n$ matter (vector) multiplets , which is based on the rank-3 Jordan algebra $\mathbb{R}\oplus\mathbf{\Gamma}_{1,2n-1}$, with data

$$\frac{G_4}{H_4} = \frac{\mathrm{SL}_v(2,\mathbb{R})}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(6,2n)}{\mathrm{SO}(6)\times\mathrm{SO}(n)}; \tag{5.1}$$

$$\mathbf{R} = (\mathbf{2},\mathbf{6}+\mathbf{n}). \tag{5.2}$$

The relevant products of electric and magnetic charges read

$$\begin{aligned} p^2 &\equiv p^\Lambda p^\Sigma \eta_{\Lambda\Sigma} = \sum_{a=1}^{6}(p^a)^2 - \sum_{I=1}^{2n}(p^I)^2; \\ q^2 &\equiv q_\Lambda q_\Sigma \eta^{\Lambda\Sigma} = \sum_{a=1}^{6} q_a^2 - \sum_{I=1}^{2n} q_I^2; \\ p\cdot q &\equiv p^\Lambda q_\Lambda, \end{aligned} \tag{5.3}$$

where η is the symmetric invariant structure of the vector (**Fund**) irrep. $\mathbf{6}+\mathbf{2n}$ of $\mathrm{SO}(6,2n)$, with $\Lambda=1,\ldots,2n+6$, where the indices $1,\ldots,6$ pertain to the 6 graviphotons.

We consider a complexification of the electric and magnetic charge vectors p^Λ and q_Λ as follows:

$$\begin{cases} P^1 \equiv p^1+ip^2; \\ P^2 \equiv p^3+ip^4; \\ P^3 \equiv p^5+ip^6; \\ P^4 \equiv p^7+ip^8; \\ \ldots. \\ P^{n+3} \equiv p^{2n+5}+ip^{2n+6}, \end{cases} \tag{5.4}$$

and analogously for the electric charges. Thus (5.3) can be rewritten as

$$p^2 = \sum_{\mathcal{A}=1}^{3}\left|P^{\mathcal{A}}\right|^2 - \sum_{A=4}^{n+3}\left|P^{A}\right|^2 = P^i\overline{P}^{\bar{j}}\eta_{i\bar{j}}; \tag{5.5}$$

$$q^2 = \sum_{\mathcal{A}=1}^{3}\left|Q_{\mathcal{A}}\right|^2 - \sum_{A=4}^{n+3}\left|Q_{A}\right|^2 = \eta^{i\bar{j}}Q_i\overline{Q}_{\bar{j}}; \tag{5.6}$$

$$p\cdot q = \sum_{i=1}^{n+3}\mathrm{Re}(P^i\overline{Q}_{\bar{i}}), \tag{5.7}$$

with η here denoting the invariant rank-2 structure in the product $(\mathbf{3}+\mathbf{n}) \times \left(\overline{\mathbf{3}+\mathbf{n}}\right)$ of $\mathrm{U}(3,n)$, with $i = 1,\ldots,n+3$ (in section 2, the complex charge vector (P^i, Q_i) has been indicated by $\mathbf{Q}$). Therefore:

$$\frac{1}{4}\mathcal{I}_{4,\mathbb{R}\oplus\mathbf{\Gamma}_{5,2n-1}} = p^2q^2 - (p\cdot q)^2 \tag{5.8}$$

$$= \eta_{i\bar{j}}\eta^{k\bar{l}}P^i\overline{P}^{\bar{j}}Q_k\overline{Q}_{\bar{l}} - \left(\sum_{i=1}^{n+3}\mathrm{Re}\left(P^i\overline{Q}_{\bar{i}}\right)\right)^2 \tag{5.9}$$

$$= \frac{1}{4}\left(S_1^2 - |S_2|^2\right), \tag{5.10}$$

where the following quantities have been introduced [39, 66]:

$$S_1 \equiv p^2 + q^2 = \left(P^i\overline{P}^{\bar{j}} + Q^i\overline{Q}^{\bar{j}}\right)\eta_{i\bar{j}}; \tag{5.11}$$

$$S_2 \equiv (p^2 - q^2) + 2ip\cdot q = \left(P^i\overline{P}^{\bar{j}} - Q^i\overline{Q}^{\bar{j}}\right)\eta_{i\bar{j}} + 2i\sum_{i=1}^{n+3}\mathrm{Re}\left(P^i\overline{Q}_{\bar{i}}\right). \tag{5.12}$$

The *"degeneration"* condition we exploit reads as follows:

$$S_2 = 0 \Leftrightarrow \begin{cases} \mathrm{Re}S_2 = 0 \Leftrightarrow \left(P^i\overline{P}^{\bar{j}} - Q^i\overline{Q}^{\bar{j}}\right)\eta_{i\bar{j}} = 0; \\ \\ \mathrm{Im}S_2 = 0 \Leftrightarrow \sum_{i=1}^{n+3}\mathrm{Re}(P^i\overline{Q}_{\bar{i}}) = 0, \end{cases} \tag{5.13}$$

whose a solution is

$$Q_j = \pm iP^j \ \forall j, \tag{5.14}$$

with j-dependent "$\pm$" branches. One thus obtains:

$$\mathcal{I}_{4,\mathbb{R}\oplus\mathbf{\Gamma}_{5,2n-1}}\big|_{S_2=0} = (S^1)^2 = 4\left(P^i\overline{P}^{\bar{j}}\eta_{i\bar{j}}\right)^2 = (\mathcal{I}_{2,\mathcal{N}=3})^2. \tag{5.15}$$

Namely, the *quartic* invariant $\mathcal{I}_{4,\mathbb{R}\oplus\mathbf{\Gamma}_{5,2n-1}}$ of the real irrep. $\mathbf{R} = (\mathbf{2}, \mathbf{6}+\mathbf{2n})$ of the semisimple group of type E_7 $G_4 = \mathrm{SL}_v(2,\mathbb{R}) \times \mathrm{SO}(6,2n) = Conf\left(\mathbb{R}\oplus\mathbf{\Gamma}_{5,2n-1}\right)$ *"degenerates"* into the square of the *quadratic* invariant $\mathcal{I}_{2,\mathcal{N}=3}$ of the complex irrep. $\mathbf{R}' = \mathbf{3}+\mathbf{n}$ of the "degenerate" group of type E_7 $G_4' = \mathrm{U}(3,n)$. This latter is the U-duality group of $\mathcal{N}=3$ supergravity coupled to n vector multiplets.

In a manifestly $\mathrm{U}(3,n)$-covariant symplectic basis, $\mathcal{I}_{2,\mathcal{N}=3}$ reads:

$$\mathcal{I}_{2,\mathcal{N}=3} = \sum_{\mathfrak{A}=1}^{3}\left[\left(\mathfrak{p}^{\mathfrak{A}}\right)^2 + \mathfrak{q}_{\mathfrak{A}}^2\right] - \sum_{\alpha=1}^{n}\left[(\mathfrak{p}^{\alpha})^2 + \mathfrak{q}_{\alpha}^2\right]. \tag{5.16}$$

In order to make (5.16) consistent with (5.15), the following *dyonic identification* of charges can be performed:

$$\begin{aligned} P^{\mathcal{A}} &\equiv \tfrac{1}{\sqrt{2}}\left(\mathfrak{p}^{\mathfrak{A}} + i\mathfrak{q}_{\mathfrak{A}}\right); \\ P^{A} &\equiv \tfrac{1}{\sqrt{2}}\left(\mathfrak{p}^{\alpha} + i\mathfrak{q}_{\alpha}\right). \end{aligned} \tag{5.17}$$

In group-theoretical terms, the *"degeneration"* procedure under consideration goes as follows:

$$
\begin{aligned}
\mathrm{SL}_v(2,\mathbb{R})\times\mathrm{SO}(6,2n) &\supset \mathrm{SL}_v(2,\mathbb{R})\times\mathrm{U}(3,n)\supset\mathrm{U}(3,n);\\
(\mathbf{2},\mathbf{6}+\mathbf{2n}) &= \left(\mathbf{2},(\mathbf{3}+\mathbf{n})_{+1}\right)+\left(\mathbf{2},\left(\overline{\mathbf{3}+\mathbf{n}}\right)_{-1}\right)\\
&= 2\cdot\left[(\mathbf{3}+\mathbf{n})_{+1}+\left(\overline{\mathbf{3}+\mathbf{n}}\right)_{-1}\right],
\end{aligned}
\tag{5.18}
$$

with the double-counting eventually removed by the *"degeneration"* truncating condition (5.13)–(5.14), which in this case sets to zero $n+3$ complex, i.e. $2n+6$ real, charge combinations. Notice that, also in this case, (5.14) breaks $\mathrm{SL}_v(2,\mathbb{R})$, and its various branches, generated by the various possibilities in the choice of "$\pm$" for each index i, are all interrelated by suitable $\mathrm{U}(3,n)$-transformations. At the level of the vector multiplets' scalar manifolds, it holds

$$
\underbrace{\frac{\mathrm{SL}_v(2,\mathbb{R})}{\mathrm{U}(1)}\times\frac{\mathrm{SO}(6,2n)}{\mathrm{SO}(2)\times\mathrm{SO}(2n)}}_{\mathcal{N}=4,\ \mathbb{R}\oplus\mathbf{\Gamma}_{5,2n-1}}\supset\underbrace{\frac{\mathrm{SU}(3,n)}{\mathrm{U}(3)\times\mathrm{SU}(n)}}_{\mathcal{N}=3}.
\tag{5.19}
$$

6 $\mathcal{N}=4\ \mathbb{R}\oplus\mathbf{\Gamma}_{5,n-1}\longrightarrow\mathcal{N}=2\ \mathbb{R}\oplus\mathbf{\Gamma}_{1,n-1}$ + hypermultiplets

$\mathcal{N}=2$ hypermultiplets can be added to the *"degenerative"* truncation procedures (starting from the $\mathcal{N}=2$ factorized sequence) treated above, by considering the following truncation:

$$
\mathcal{N}=4\ \mathbb{R}\oplus\mathbf{\Gamma}_{5,n-1}\longrightarrow\mathcal{N}=2\ \mathbb{R}\oplus\mathbf{\Gamma}_{1,n_1-1}+(n-n_1)\ \text{hypermults.}
\tag{6.1}
$$

$$
\mathrm{SL}_v(2,\mathbb{R})\times\mathrm{SO}(6,n)\supset\mathrm{SL}_v(2,\mathbb{R})\times\mathrm{SO}(2,n_1)\times\mathrm{SO}(4,n-n_1);
\tag{6.2}
$$

$$
(\mathbf{2},\mathbf{6}+\mathbf{n})=(\mathbf{2},\mathbf{2}+\mathbf{n}_1,\mathbf{1})+(\mathbf{2},\mathbf{1},\mathbf{4}+\mathbf{n}-\mathbf{n}_1),
\tag{6.3}
$$

where the hyperscalars fit into the quaternionic Kähler symmetric space

$$
\frac{\mathrm{SO}(4,n-n_1)}{\mathrm{SO}(4)\times\mathrm{SO}(n-n_1)}.
\tag{6.4}
$$

Thus, the $\mathcal{N}=2$ theory is obtained by setting

$$
(\mathbf{2},\mathbf{1},\mathbf{4}+\mathbf{n}-\mathbf{n}_1)=0.
\tag{6.5}
$$

At the level of the scalar manifolds, the truncation (6.1)–(6.5) corresponds to

$$
\frac{\mathrm{SL}_v(2,\mathbb{R})}{\mathrm{U}(1)}\times\frac{\mathrm{SO}(6,n)}{\mathrm{SO}(6)\times\mathrm{SO}(n)}\supset\frac{\mathrm{SL}_v(2,\mathbb{R})}{\mathrm{U}(1)}\times\frac{\mathrm{SO}(2,n_1)}{\mathrm{SO}(2)\times\mathrm{SO}(n_1)}\times\frac{\mathrm{SO}(4,n-n_1)}{\mathrm{SO}(4)\times\mathrm{SO}(n-n_1)}.
\tag{6.6}
$$

It is worth recalling that the case $n=0$ of the truncation (6.6) has been considered in section 5 of [68] (see also the considerations in subsection 8.3.1).

Starting from the $\mathcal{N}=2$ theory with n_1 vector multiplets and $n-n_1$ hypermultiplets, with scalar manifolds given by the direct product on the righthand side of (6.6), iff n_1 is *even* (i.e. iff $n_1=2m$) one can then consider the further *"degenerative"* truncation down to $\mathcal{N}=2$ *minimally coupled* supergravity with m vector multiplets and $n-n_1=n-2m$

hypermultiplets: in practice, the procedure outlined in subsection 8.3, with $n \to m$, and the hypermultiplets which are insensitive of the truncation:

$$\frac{\mathrm{SL}_v(2,\mathbb{R})}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(2,n_1)}{\mathrm{SO}(2)\times\mathrm{SO}(n_1)} \times \frac{\mathrm{SO}(4,n-n_1)}{\mathrm{SO}(4)\times\mathrm{SO}(n-n_1)} \overset{\text{iff } n_1=2m}{\supset} \frac{\mathrm{SU}(1,m)}{\mathrm{U}(m)} \times \frac{\mathrm{SO}(4,n-n_1)}{\mathrm{SO}(4)\times\mathrm{SO}(n-n_1)}. \tag{6.7}$$

Let's finally mention that the quaternionic manifolds (6.4) are maximal in the framework under consideration, but, iff $n-n_1$ is *even* (i.e. iff $n-n_1=2k$) the further following truncation in the hyper sector can be considered:

$$\frac{\mathrm{SO}(4,n-n_1)}{\mathrm{SO}(4)\times\mathrm{SO}(n-n_1)} \overset{\text{iff } n-n_1=2k}{\supset} \frac{\mathrm{SU}(2,k)}{\mathrm{SU}(2)\times\mathrm{SU}(k)\times\mathrm{U}(1)}. \tag{6.8}$$

Thus, by combining the two above observations, iff

$$\left.\begin{array}{l} n_1 = 2m; \\ \\ n-n_1 = 2k; \end{array}\right\} \Rightarrow n = 2(m+k)\,even, \tag{6.9}$$

one can consider, along the very same lines of subsection 8.3, the following further *non-maximal "degenerative"* truncation down to $\mathcal{N}=2$ *minimally coupled* supergravity with m vector multiplets and k hypermultiplets:

$$\frac{\mathrm{SL}_v(2,\mathbb{R})}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(2,n_1)}{\mathrm{SO}(2)\times\mathrm{SO}(n_1)} \times \frac{\mathrm{SO}(4,n-n_1)}{\mathrm{SO}(4)\times\mathrm{SO}(n-n_1)} \overset{\text{iff } n=2(m+k)}{\supset} \frac{\mathrm{SU}(1,m)}{\mathrm{U}(m)} \times \frac{\mathrm{SU}(2,k)}{\mathrm{SU}(2)\times\mathrm{SU}(k)\times\mathrm{U}(1)}. \tag{6.10}$$

7 $\mathcal{N}=3 \longrightarrow \mathcal{N}=2$ $\mathbb{CP}^n$ + hypermultiplets

Finally, let us consider the following truncation:

$$\mathcal{N}=3\ p \text{ vector mults.} \longrightarrow \mathcal{N}=2\ \mathbb{CP}^{s_1} + (p-s_1) \text{ hypermults.} \tag{7.1}$$

$$\mathrm{U}(3,p) \supset \mathrm{U}(1,s_1)\times\mathrm{SU}(2,p-s_1)\times\mathrm{U}(1); \tag{7.2}$$

$$(\mathbf{3}+\mathbf{n}) = (\mathbf{1}+\mathbf{s}_1)_{+1} + (\mathbf{2}+\mathbf{p}-\mathbf{s}_1)_{-\frac{(1+s_1)}{2+p-s_1}}, \tag{7.3}$$

which, at the level of scalar manifolds corresponds to the following maximal embedding:

$$\frac{\mathrm{SU}(3,p)}{\mathrm{SU}(3)\times\mathrm{SU}(p)\times\mathrm{U}(1)} \supset \frac{\mathrm{SU}(1,s_1)}{\mathrm{U}(s_1)} \times \frac{\mathrm{SU}(2,p-s_1)}{\mathrm{SU}(2)\times\mathrm{SU}(p-s_1)\times\mathrm{U}(1)}. \tag{7.4}$$

Thus, the $\mathcal{N}=2$ *minimally coupled* theory is obtained by setting

$$(\mathbf{2}+\mathbf{p}-\mathbf{s}_1) = 0. \tag{7.5}$$

Notice that the starting $\mathcal{N}=3$ theory can be seen to be obtained from $\mathcal{N}=4$ theory coupled to $2p$ matter (vector) multiplets through the *"degenerative"* truncation procedure outlined in subsection 5, with $n \to p$.

8 Maximal truncations within $\mathcal{N}=2$

8.1 $J_3^{\mathbb{O}} \to \mathbb{R}\oplus\mathbf{\Gamma}_{1,9}$ (FHSV)

$$J_3^{\mathbb{O}} \;:\; \mathcal{N}=2,\; n_V=27 \longrightarrow \mathcal{N}=2\; \mathbb{R}\oplus\mathbf{\Gamma}_{1,9}\; n_V=11; \tag{8.1}$$
$$E_{7(-25)} \supset \mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(2,10); \tag{8.2}$$
$$\mathbf{56} = (\mathbf{2},\mathbf{12})+(\mathbf{1},\mathbf{32})\,; \tag{8.3}$$
$$\frac{E_{7(-25)}}{E_6\times\mathrm{U}(1)} \supset \frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)}\times\frac{\mathrm{SO}(2,10)}{\mathrm{SO}(2)\times\mathrm{SO}(10)}. \tag{8.4}$$

The truncation condition reads

$$(\mathbf{1},\mathbf{32})=0. \tag{8.5}$$

The resulting theory, the so-called $\mathcal{N}=2$ FHSV model [71], still exhibits a *quartic* U-invariant $\mathcal{I}_4$, but it can be *non-maximally* further truncated to an $\mathcal{N}=2$ $\mathbb{CP}^5$ model with *quadratic* invariant through the procedure considered in section 8.3, to which we address the reader for further elucidation. Note that this case, as well as the cases treated at points 1, 3 and 4 of section 3.4, is based on the maximal (symmetric) Jordan algebraic embedding (see e.g. [72]):

$$J_3^{\mathbb{A}} \supset J_2^{\mathbb{A}}\oplus\mathbb{R},\; \mathbb{A}=\mathbb{O},\mathbb{H},\mathbb{C},\mathbb{R}; \tag{8.6}$$
$$J_2^{\mathbb{A}} \sim \mathbf{\Gamma}_{1,q+1},\; q\equiv \dim_{\mathbb{R}}\mathbb{A}=8,4,2,1. \tag{8.7}$$

8.2 $J_3^{\mathbb{O}} \longrightarrow \mathbb{CP}^6$

Interestingly, the *exceptional magic* theory admits another relevant truncation:

$$J_3^{\mathbb{O}} \;:\; \mathcal{N}=2,\; n_V=27 \longrightarrow \mathcal{N}=2\; \mathbb{CP}^6; \tag{8.8}$$
$$E_{7(-25)} \supset \mathrm{SU}(6,2)\supset\mathrm{SU}(6,1)\times\mathrm{U}(1); \tag{8.9}$$
$$\mathbf{56} = \mathbf{28}+\overline{\mathbf{28}} = \mathbf{21}_{+1}+\mathbf{7}_{-3}+\overline{\mathbf{21}}_{-1}+\overline{\mathbf{7}}_{+3}; \tag{8.10}$$
$$\frac{E_{7(-25)}}{E_6\times\mathrm{U}(1)} \supset \frac{\mathrm{SU}(1,6)}{\mathrm{U}(6)}. \tag{8.11}$$

The $\mathcal{N}=2$ theory is *minimally coupled* to 6 vector multiplets and, on the two-form Abelian field strengths' fluxes, the truncation condition reads

$$\mathbf{21}_{+1}=0. \tag{8.12}$$

It can be proved that the *quartic* invariant $\mathcal{I}_4$ of the $\mathbf{R}=\mathbf{56}$ of $E_{7(-25)}$, under the truncation (8.12), becomes the *square* of the *quadratic* invariant of the $\mathbf{R}=\mathbf{7}$ of $\mathrm{SU}(1,6)$.

8.3 $\mathbb{R}\oplus\mathbf{\Gamma}_{1,2n-1} \longrightarrow \mathbb{CP}^n$

A procedure very similar to the one of section 5 can be considered in this case.

We consider $\mathcal{N}=2$ supergravity based on the rank-3 Jordan algebra $\mathbb{R}\oplus\mathbf{\Gamma}_{1,2n-1}$, with $n_V=2n+1$ vector multiplets, with data

$$\frac{G_4}{H_4} = \frac{\mathrm{SL}_v(2,\mathbb{R})}{\mathrm{U}(1)}\times\frac{\mathrm{SO}(2,2n)}{\mathrm{SO}(2)\times\mathrm{SO}(n)}; \tag{8.13}$$
$$\mathbf{R} = (\mathbf{2},\mathbf{2}+\mathbf{n}). \tag{8.14}$$

The relevant products of electric and magnetic charges read

$$\begin{aligned} & p^2 \equiv p^\Lambda p^\Sigma \eta_{\Lambda\Sigma} = (p^0)^2 + (p^1)^2 - \textstyle\sum_{a=2}^{2n+1} (p^a)^2; \\ & q^2 \equiv q_\Lambda q_\Sigma \eta^{\Lambda\Sigma} = q_0^2 + q_1^2 - \textstyle\sum_{a=2}^{2n+1} q_a^2; \\ & p \cdot q \equiv p^\Lambda q_\Lambda, \end{aligned} \tag{8.15}$$

where η is the symmetric invariant structure of the vector (**Fund**) irrep. $\mathbf{2}+\mathbf{2n}$ of $\mathrm{SO}(2,2n)$, with $\Lambda = 0, 1, \ldots, 2n+1$, where the indices "0" and "1" respectively pertain to the graviphoton and to the axio-dilatonic Maxwell field.

We consider a complexification of the electric and magnetic charge vectors p^Λ and q_Λ as follows:

$$\begin{cases} P^1 \equiv p^0 + ip^1; \\ P^2 \equiv p^2 + ip^3; \\ \ldots . \\ P^{n+1} \equiv p^{2n} + ip^{2n+1}, \end{cases} \tag{8.16}$$

and analogously for the electric charges. Thus (8.15) can be rewritten as

$$p^2 = \left|P^1\right|^2 - \sum_{A=2}^{n+1} \left|P^A\right|^2 = P^i \overline{P}^{\bar{j}} \eta_{i\bar{j}}; \tag{8.17}$$

$$q^2 = |Q_1|^2 - \sum_{A=2}^{n+1} |Q_A|^2 = \eta^{i\bar{j}} Q_i \overline{Q}_{\bar{j}}; \tag{8.18}$$

$$p \cdot q = \sum_{i=1}^{n+1} \mathrm{Re}\left(P^i \overline{Q}_{\bar{i}}\right), \tag{8.19}$$

with η here denoting the invariant rank-2 structure in the product $(\mathbf{1}+\mathbf{n}) \times \left(\overline{\mathbf{1}+\mathbf{n}}\right)$ of $\mathrm{U}(1,n)$, with $i = 1, \ldots, n+1$. Therefore:

$$\frac{1}{4}\mathcal{I}_{4,\mathbb{R}\oplus\mathbf{\Gamma}_{1,2n-1}} = p^2 q^2 - (p \cdot q)^2 \tag{8.20}$$

$$= \eta_{i\bar{j}} \eta^{k\bar{l}} P^i \overline{P}^{\bar{j}} Q_k \overline{Q}_{\bar{l}} - \left(\sum_{i=1}^{n+1} \mathrm{Re}\left(P^i \overline{Q}_{\bar{i}}\right)\right)^2 \tag{8.21}$$

$$= \frac{1}{4}\left(S_1^2 - |S_2|^2\right), \tag{8.22}$$

where, *mutatis mutandis*, S_1^2 and S_2 are given in (5.11)–(5.12) [39, 66].

By imposing the very same *"degeneration"* truncating condition (5.13)–(5.14), and evaluating (8.20)–(8.22) on (5.13)–(5.14), one obtains (in section 2, the complex charge vector (P^i, Q_i) has been indicated by $\mathbf{Q}$):

$$\mathcal{I}_{4,\mathbb{R}\oplus\mathbf{\Gamma}_{1,2n-1}}\big|_{S_2=0} = (S^1)^2 = 4\left(P^i \overline{P}^{\bar{j}} \eta_{i\bar{j}}\right)^2 = \left(\mathcal{I}_{2,\mathbb{CP}^n}\right)^2. \tag{8.23}$$

Namely, the *quartic* invariant $\mathcal{I}_{4,\mathbb{R}\oplus\mathbf{\Gamma}_{1,2n-1}}$ of the real irrep. $\mathbf{R} = (\mathbf{2}, \mathbf{2}+\mathbf{2n})$ of the semisimple group of type E_7 $G_4 = \mathrm{SL}_v(2,\mathbb{R}) \times \mathrm{SO}(2,2n) = Conf\left(\mathbb{R} \oplus \mathbf{\Gamma}_{1,2n-1}\right)$ *"degenerates"* into the square of the *quadratic* invariant $\mathcal{I}_{2,\mathbb{CP}^n}$ of the complex irrep. $\mathbf{R}' = \mathbf{1} + \mathbf{n}$ of the "degenerate" group of type E_7 $G_4' = \mathrm{U}(1,n)$. This latter is the U-duality group of $\mathcal{N} = 2$ supergravity *minimally coupled* to n vector multiplets [22].

In a manifestly $\mathrm{U}(1,n)$-covariant symplectic basis, $\mathcal{I}_{2,\mathbb{CP}^n}$ reads:

$$\mathcal{I}_{2,\mathbb{CP}^n} = (\mathfrak{p}^0)^2 + \mathfrak{q}_0^2 - \sum_{\alpha=1}^{n} \left[(\mathfrak{p}^\alpha)^2 + \mathfrak{q}_\alpha^2 \right]. \tag{8.24}$$

In order to make (8.24) consistent with (8.23), the following *dyonic identification* of charges can be performed:

$$\begin{aligned} P^1 &\equiv \tfrac{1}{\sqrt{2}}(\mathfrak{p}^0 + i\mathfrak{q}_0); \\ P^A &\equiv \tfrac{1}{\sqrt{2}}(\mathfrak{p}^\alpha + i\mathfrak{q}_\alpha). \end{aligned} \tag{8.25}$$

Note that in this case (5.13) manifestly breaks $\mathrm{SL}_v(2,\mathbb{R})$, whereas its solution (5.14) further breaks $\mathrm{SO}(2,2n)$ down to $\mathrm{U}(1,n)$.

The *"degeneration"* of $\mathcal{I}_{4,\mathbb{R}\oplus\mathbf{\Gamma}_{1,2n-1}}$ can also be considered in the scalar-dressed formalism, in which [39, 66, 67]

$$S_1 = |Z|^2 + |Z_s|^2 - Z_I\overline{Z}^I; \tag{8.26}$$

$$S_2 = 2iZ\overline{Z_s} - Z_I Z^I, \tag{8.27}$$

where Z, Z_s and Z_I respectively are the central charge, axio-dilatonic matter charge and non-axio-dilatonic matter charges ($I = 1, \ldots, 2n$ denotes "flatted" local indices, also the index s does). Recall that $Z_s \equiv \mathcal{D}_s Z$, $Z_I \equiv \mathcal{D}_I Z$, $\overline{Z}^I = \overline{Z_I}$, $Z^I \equiv Z_I$, where $\mathcal{D}$ is the Kähler-covariant differential operator in "flatted" local indices. By splitting the index I as $I = \left\{ \widetilde{I}, \widehat{I} \right\}$ with $\widetilde{I} = 1, \ldots, n$ and $\widehat{I} = 1, \ldots, n$, the *"degeneration"* condition (5.13)

$$S_2 = 0 \Leftrightarrow 2iZ\overline{Z_s} = Z_I Z^I \tag{8.28}$$

can be solved by setting

$$Z_s = 0, \quad Z_{\widetilde{I}} = iZ_{\widehat{I}}, \tag{8.29}$$

thus implying (recall (8.22))

$$\mathcal{I}_{4,\mathbb{R}\oplus\mathbf{\Gamma}_{1,2n-1}} = S_1^2 = \left(|Z|^2 - \left|Z_{\widetilde{I}}\right|^2 - \left|Z_{\widehat{I}}\right|^2 \right)^2 = \left(|Z|^2 - 2\left|Z_{\widetilde{I}}\right|^2 \right)^2 = (\mathcal{I}_{2,\mathbb{CP}^n})^2, \tag{8.30}$$

where the re-writing of the invariant $\mathcal{I}_{2,\mathbb{CP}^n}$ in the scalar-dressed formalism reads (see e.g. [39, 66, 67])

$$\mathcal{I}_{2,\mathbb{CP}^n} = |Z|^2 - |Z_\alpha|^2, \tag{8.31}$$

thus yielding the following *identification* of scalar-dressed charges with α-dependent "$\pm$" branches:

$$Z_{\widetilde{I}} \equiv \pm \frac{i}{\sqrt{2}} Z_\alpha. \tag{8.32}$$

It should be stressed that (5.14) and (8.29) are different solutions, in two different (respectively "bare" and "scalar-dressed") formalisms, to the *"degeneration"* condition (5.13) (or, equivalently, (8.28)). Note that the solution (8.29) to the manifestly $\mathrm{SL}_v(2,\mathbb{R})$-breaking *"degeneration"* condition (5.13) (or, equivalently, (8.28)) consistently breaks $\mathrm{SO}(2,2n)$ down to $\mathrm{U}(1,n)$.

Mutatis mutandis, the *"degeneration"* in the scalar-dressed formalism considered above can also be performed for of $\mathcal{I}_{4,\mathbb{R}\oplus\mathbf{\Gamma}_{5,2n-1}}$ of section 5; essentially, one has to identify

$$Z \equiv Z_1, \quad i\overline{Z_s} \equiv Z_2, \tag{8.33}$$

where Z_1 and Z_2 are the skew-eigenvalues of the $\mathcal{N}=4$ central charge matrix Z_{AB} ($A,B = 1,\ldots,4$) (see e.g. [39, 66–68]).

In group-theoretical terms, the *"degeneration"* truncating procedure under consideration goes as follows:

$$\begin{aligned} &\mathrm{SL}_v(2,\mathbb{R}) \times \mathrm{SO}(2,2n) \supset \mathrm{U}(1,n); \\ &\qquad (\mathbf{2},\mathbf{2}+\mathbf{2n}) = 2\cdot\left[(\mathbf{1}+\mathbf{n})_{+1} + \left(\overline{\mathbf{1}+\mathbf{n}}\right)_{-1}\right], \end{aligned} \tag{8.34}$$

with the double-counting eventually removed by the *"degeneration"* truncating condition (5.13)–(5.14), which sets to zero $n+1$ complex, i.e. $2n+2$ real, charge combinations. As mentioned, (5.13) manifestly breaks $\mathrm{SL}_v(2,\mathbb{R})$-invariance, and its various branches, generated by the various possibilities in the choice of "$\pm$" for each index i, are all inter-related by suitable $\mathrm{U}(1,n)$-transformations. At the level of the vector multiplets' scalar manifolds, it holds

$$\underbrace{\frac{\mathrm{SL}_v(2,\mathbb{R})}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(2,2n)}{\mathrm{SO}(2)\times\mathrm{SO}(2n)}}_{\mathcal{N}=2,\ \mathbb{R}\oplus\mathbf{\Gamma}_{1,2n-1}} \supset \underbrace{\frac{\mathrm{SU}(1,n)}{\mathrm{U}(n)}}_{\mathcal{N}=2,\ \mathbb{CP}^n}. \tag{8.35}$$

8.3.1 A remark on $\mathbb{CP}^1$

It is worth pointing out that the $n=1$ case of the *"degeneration"* procedure (8.34)–(8.35) is *different* from the "usual" *truncation* of the $\mathbb{R}\oplus\mathbf{\Gamma}_{1,n-1}$ sequence down to the *axio-dilatonic minimally coupled* 1-modulus $\mathbb{CP}^1$ model, achieved by setting $n=0$:

$$\begin{aligned} &\mathrm{SL}_v(2,\mathbb{R}) \times \mathrm{SO}(2,n) \overset{n=0}{\longrightarrow} \mathrm{SL}_v(2,\mathbb{R}) \times \mathrm{SO}(2) \sim \mathrm{U}(1,1); \\ &(\mathbf{2},\mathbf{2}+\mathbf{2n}) \overset{n=0}{\longrightarrow} (\mathbf{2},\mathbf{2}) \sim \mathbf{2}_{+1} + \overline{\mathbf{2}}_{-1}; \\ &\underbrace{\frac{\mathrm{SL}_v(2,\mathbb{R})}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(2,n)}{\mathrm{SO}(2)\times\mathrm{SO}(n)}}_{\mathcal{N}=2,\ \mathbb{R}\oplus\mathbf{\Gamma}_{1,n-1}} \overset{n=0}{\longrightarrow} \underbrace{\frac{\mathrm{SL}_v(2,\mathbb{R})}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(2)}{\mathrm{SO}(2)}}_{\mathcal{N}=2,\ \mathbb{CP}^1} \sim \underbrace{\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{U}(1)}{\mathrm{U}(1)}}_{\text{axion-dilaton}}. \end{aligned} \tag{8.36}$$

Thus, the U-duality group of the 1-modulus *minimally coupled* $\mathcal{N}=2$ theory is the unbroken *axio-dilatonic* $\mathrm{SL}_v(2,\mathbb{R})$ group times the factor $\mathrm{SO}(2,n=0)=\mathrm{SO}(2)$. On the other hand, the $n=1$ case of the *"degeneration"* procedure described in section 8.3 manifestly breaks $\mathrm{SL}_v(2,\mathbb{R})$, and it determines the U-duality group of the 1-modulus *minimally coupled* $\mathcal{N}=2$ theory as the $n=1$ case of the breaking $\mathrm{SO}(2,2n)\to\mathrm{U}(1,n)$ of the symmetry pertaining to the *non-axio-dilatonic* matter sector.

At the level of invariant polynomials of the symplectic irrep. of the U-duality group, the truncation (8.36) works as (recall (8.16) and (8.25)):

$$\begin{aligned}\mathcal{I}_{4,\mathbb{R}\oplus\mathbf{\Gamma}_{1,n-1}}\big|_{n=0} &= 4\left\{\left[(p^1)^2+(p^2)^2\right](q_1^2+q_2^2)-(p^1q_1+p^2q_2)^2\right\}\\ &= 4(p^1q_2-p^2q_1)^2 = (\mathcal{I}_{2,\mathbb{CP}^1})^2. \end{aligned} \tag{8.37}$$

The $\mathcal{N}=2$ symplectic basis obtained in this truncation is the one in which the holomorphic prepotential reads $F=-iX^1X^2$, and it thus differs from the one pertaining to (8.24) with $n=1$, in which $F=-i\left[(X^0)^2-(X^1)^2\right]$. Indeed, while (8.37) does not vanish iff *both* the graviphoton (index 1) and the matter Maxwell field (index 2) have *at least* one non-vanishing field strength's flux (namely, iff *at least* $p^1,q_2\neq 0$ or $p^2,q_1\neq 0$), (8.24) can be non-vanishing also when the graviphoton (index 0) *or* the matter Maxwell field (index 1) has *both* electric and magnetic zero charges. The $\mathrm{Sp}(4,\mathbb{R})/\mathrm{U}(1,1)$ finite transformation $\mathcal{S}$ relating the two symplectic bases under consideration reads

$$\begin{pmatrix}X^1\\X^2\\F_1\\F_2\end{pmatrix}_{F=-iX^1X^2} = \mathcal{S}\begin{pmatrix}X^0\\X^1\\F_0\\F_1\end{pmatrix}_{F=-i[(X^0)^2-(X^1)^2]}; \tag{8.38}$$

$$\mathcal{S}\equiv\frac{1}{2}\begin{pmatrix}2&2&0&0\\2&-2&0&0\\0&0&1&1\\0&0&1&-1\end{pmatrix}\in \mathrm{Sp}(4,\mathbb{R})/\mathrm{U}(1,1). \tag{8.39}$$

8.4 *"Generalized"* groups of type E_7 and special geometry

As introduced in section 4.3 of [30], special Kähler geometry can be reformulated in order to capture both non-degenerate and degenerate groups of type E_7 in a coordinate-independent (i.e. diffeomorphism-invariant) way. This is achieved by introducing *"generalized"* groups of type E_7, based on a quartic *"entropy functional"*, expressed in terms of the scalar-dressed basis of $\mathcal{N}=2$ central charge Z (graviphoton) and $\mathcal{N}=2$ matter charges $Z_i\equiv D_iZ$ (vector multiplets) as follows:

$$\mathcal{I}_4=(i_1-i_2)^2+4i_4-i_5; \tag{8.40}$$

$$i_1\equiv|Z|^2; \tag{8.41}$$

$$i_2\equiv Z_i\overline{Z}^{\bar{i}}; \tag{8.42}$$

$$i_3\equiv\frac{i}{6}\left[Z\overline{C}_{\bar{i}\bar{j}\bar{k}}Z^{\bar{i}}Z^{\bar{j}}Z^{\bar{k}}+\overline{Z}C_{ijk}\overline{Z}^i\overline{Z}^j\overline{Z}^k\right] \tag{8.43}$$

$$i_4\equiv\frac{i}{6}\left[Z\overline{C}_{\bar{i}\bar{j}\bar{k}}Z^{\bar{i}}Z^{\bar{j}}Z^{\bar{k}}-\overline{Z}C_{ijk}\overline{Z}^i\overline{Z}^j\overline{Z}^k\right]; \tag{8.44}$$

$$i_5\equiv g^{i\bar{i}}C_{ijk}\overline{C}_{\bar{i}\bar{l}\bar{m}}\overline{Z}^j\overline{Z}^kZ^{\bar{l}}Z^{\bar{m}}. \tag{8.45}$$

Note that $\mathcal{I}_4=(i_1-i_2)^2$ if $C_{ijk}=0$; this corresponds to symmetric $\mathcal{N}=2$ $\mathbb{CP}^n$ models, which upon reduction to $\mathcal{N}=1$ yield *minimal coupling*. Another way to obtain $\mathcal{N}=2$ $\mathbb{CP}^n$ models by truncating an $\mathcal{N}=2$ theory with $C_{ijk}\neq 0$ is discussed in subsection 8.3.

One can make a model-independent analysis holding for any special Kähler geometry, by relating the invariants i_1, i_2, i_3, i_4 and i_5 defined in (8.41)–(8.45) to the three roots λ_1, λ_2, λ_3 of the universal cubic equation (cfr. eqs. (5.11)-(5.18) of [95])

$$\lambda^3 - i_2\lambda^2 + \frac{i_5}{4}\lambda - \frac{(i_3^2 + i_4^2)}{4i_1} = 0. \tag{8.46}$$

Within this formalism, the *"degeneration"* corresponds to truncating the $\mathcal{N} = 2$ vector multiplets such that

$$i_3 = i_4 = i_5 = 0 \tag{8.47}$$

$$\Downarrow$$

$$\mathcal{I}_4 = (i_1 - i_2)^2. \tag{8.48}$$

The condition (8.47) implies that a unique non-vanishing independent root of (8.46) exists, namely $\lambda = i_2$.

All reductions treated in section 8 satisfy the condition (8.47), which can be regarded as a necessary, but not necessarily sufficient, condition for truncating *any* $\mathcal{N} = 2$ model down to an $\mathcal{N} = 2$ $\mathbb{CP}^n$ model, and thus to $\mathcal{N} = 1$ supergravity models with *minimal coupling*.

9 $\mathcal{N} = 2 \to \mathcal{N} = 1$ truncation and *minimal coupling*

Truncation of $\mathcal{N} = 2$ theories to $\mathcal{N} = 1$ theories was studied in [10, 11]. From eq. (1.4) it is clear that, after projecting out the graviphoton, the anti-holomorphic vector kinetic matrix becomes

$$\mathcal{N}_{\alpha\beta} = \overline{\mathcal{F}_{\alpha\beta}} = \overline{\partial}_{\overline{\alpha}}\overline{\partial}_{\overline{\beta}}\overline{F}\left(\overline{X}\right), \tag{9.1}$$

where the projective symplectic sections $t^a \equiv X^a/X^0$ have been split as

$$t^a \equiv (t^\alpha, t^i), \tag{9.2}$$

with index α referring to the scalar directions of the would-be $\mathcal{N} = 1$ vector multiplets, whereas index i refers to the would-be $\mathcal{N} = 1$ chiral multiplets. As pointed out above, *minimal coupling* of vectors requires $F(X)$ to be *quadratic* in the $\mathcal{N} = 2$ symplectic sections corresponding to $\mathcal{N} = 1$ vector multiplets, such that when truncating down to $\mathcal{N} = 1$, the kinetic vector matrix $\mathcal{N}_{\alpha\beta}$ is a *scalar-independent* symmetric rank-2 tensor. Note that we here use a symplectic frame of special Kähler geometry in which an holomorphic prepotential exists

$$F(X) = (X^0)^2 F\left(\frac{X}{X^0}\right) \equiv (X^0)^2 f(t), \tag{9.3}$$

so that the C-tensor of special geometry reads

$$C_{abc} = e^K \partial_a\partial_b\partial_c f(t). \tag{9.4}$$

In particular, in this basis, d-geometries (which include all symmetric special geometries but the $\mathbb{CP}^n$ models) correspond to

$$\partial_a \partial_b \partial_c f = d_{abc} \text{ constant}, \tag{9.5}$$

whereas $\mathbb{CP}^n$ models correspond to

$$f(t) = -\frac{i}{2}\left[1 - \sum_a (t^a)^2\right]. \tag{9.6}$$

It is worth remarking that minimal coupling requires, in addition to

$$C_{\alpha\beta\gamma} = 0 = C_{\alpha ij}, \tag{9.7}$$

also [10, 11]

$$C_{\alpha\beta i} = 0, \tag{9.8}$$

and thus the only non-vanishing components of the C-tensor can lie along the directions C_{ijk} corresponding to the would-be $\mathcal{N} = 1$ chiral multiplets.

For symmetric cosets, this is only possible for $\mathbb{CP}^n$ scalar manifolds, with $n = n_c + n_V$ (with n_c and n_V here denoting the number of $\mathcal{N} = 1$ chiral and vector multiplets, respectively). The only other possibility would consist in taking the models based on the *semi-simple* U-duality group $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(2,n)$, and considering only one vector multiplet, but this is nothing but the $\mathbb{CP}^1$ model itself (see the comment in subsubsection 8.3.1).

For non-symmetric special geometry, other solutions exist. In Calabi-Yau compactifications, the effective $\mathcal{N} = 2$ prepotential for particular orbifold realizations can have a cubic dependence on the *untwisted* moduli X_U and a quadratic dependence on the *twisted* moduli X_T (see e.g. [90], and refs. therein):

$$F(X_U, X_T) = C_{ijk} X_U^i X_U^j X_U^k + C_{\alpha\beta} X_T^\alpha X_T^\beta. \tag{9.9}$$

If one performs a truncation in which the $\mathcal{N} = 1$ chiral multiplets correspond to *untwisted* moduli and $\mathcal{N} = 1$ vector multiplets correspond to *twisted* ones (as suggested by the index splitting in (9.9), one obtains a scalar-independent kinetic vector matrix: $f_{\alpha\beta} = C_{\alpha\beta}$ (*minimal* $\mathcal{N} = 1$ vector coupling).

Theories which exhibit *minimal coupling* under truncation can for instance be given by suitable projections of an original $\mathcal{N} = 3$ theory down to $\mathcal{N} = 1$. Indeed, if some vector multiplets survive the truncation down to $\mathcal{N} = 1$, they necessarily exhibit a minimal coupling, because the matrix $f_{\alpha\beta}$ is independent of the remaining $\mathcal{N} = 1$ chiral multiplets' complex scalar fields. This can be understood by considering the intermediate truncation $\mathcal{N} = 3 \to \mathcal{N} = 2$, corresponding to the following branching of the U-duality group (see section 7):

$$\mathrm{U}(3,n) \supset \mathrm{U}(1,n_V) \times \mathrm{SU}(2,n_H) \times \mathrm{U}(1), \; n = n_V + n_H. \tag{9.10}$$

The kinetic matrix of the $\mathcal{N} = 2$ n_V vector multiplets is independent of the n_H $\mathcal{N} = 2$ hyperscalars, and after projecting out the $\mathcal{N} = 2$ graviphoton and thus reducing to $\mathcal{N} = 1$, it also becomes independent of the scalars corresponding to the $\mathcal{N}=2$ vector multiplets, thus becoming constant and giving rise to an $\mathcal{N}=1$ *minimal* vector coupling.

Other non-symmetric special geometries are obtained in $\mathcal{N}=1$ Calabi-Yau orientifold compactifications [5, 91–93]. The kinetic vector matrix generally depends on the moduli, and in the simplest case reads as

$$\overline{\mathcal{N}}_{\alpha\beta} = d_{\alpha\beta i} z^i, \tag{9.11}$$

where as above α, β run over $\mathcal{N}=1$ vector multiplets, and i runs over $\mathcal{N}=1$ chiral multiplets. (9.11) corresponds to orientifold projections of $\mathcal{N}=2$ special d-geometries [94], as they naturally occur in Calabi-Yau compactifications (where the d-tensor is related to the triple intersection numbers).

10 On *Freudenthal duality* and its *"degeneration"*

All the cases in which $\mathcal{I}_4$ degenerates to $(\mathcal{I}_2)^2$ provide instances of the so-called *Freudenthal duality* [17, 73], whose manifest invariance (by construction, and apart from possible "hidden" symmetries) is given by the U-duality group of the theory obtained *after* truncation.

In the *"degenerative"* truncations under consideration, the corresponding *"degeneration"* of the (*on-shell*, non-polynomial) *Freudenthal duality* is given by the (*on-shell*, linear) formula:

$$\widetilde{\mathcal{Q}}^M \equiv \mathbb{C}^{MN} \frac{\partial \mathcal{I}_2}{\partial \mathcal{Q}^N}, \tag{10.1}$$

where $\mathcal{Q}$ is the dyonic charge vector, and

$$\mathbb{C}^{MN} \equiv \begin{pmatrix} 0^{\Lambda\Sigma} & -\delta^\Lambda_\Sigma \\ \delta^\Sigma_\Lambda & 0_{\Lambda\Sigma} \end{pmatrix} \tag{10.2}$$

is the symplectic metric. Due to the very structure of $\mathcal{I}_2$, it holds that

$$\widetilde{\mathcal{I}}_2(\mathcal{Q}) \equiv \mathcal{I}_2\left(\widetilde{\mathcal{Q}}\right) = \mathcal{I}_2(\mathcal{Q}). \tag{10.3}$$

In the manifestly $\mathrm{U}(1,n)$-covariant $\mathcal{N}=2$ symplectic basis specified by (8.24), the *"degenerate"* Freudenthal duality (10.1) can be made explicit as follows:

$$\widetilde{\mathcal{Q}}^M \equiv \mathbb{C}^{MN} \mathcal{A}_{NP} \mathcal{Q}^P; \tag{10.4}$$

$$\mathcal{A}_{MN} \equiv \begin{pmatrix} \eta_{\Lambda\Sigma} & 0^\Sigma_\Lambda \\ 0^\Lambda_\Sigma & -\eta^{\Lambda\Sigma} \end{pmatrix}, \tag{10.5}$$

namely, in components ($\mathcal{Q} = \left(\mathfrak{p}^\Lambda, \mathfrak{q}_\Lambda\right)^T$, consistent with (8.24)):

$$\begin{pmatrix} \widetilde{\mathfrak{p}}^\Lambda \\ \\ \widetilde{\mathfrak{q}}_\Lambda \end{pmatrix} = \begin{pmatrix} -\eta^{\Lambda\Sigma}\mathfrak{q}_\Sigma \\ \\ \eta_{\Lambda\Sigma}\mathfrak{p}^\Sigma \end{pmatrix}, \tag{10.6}$$

where η is the metric of (the fundamental irrep. of) $\mathrm{SO}(1,n)$. Note that this explicit treatment can be generalized to $\mathcal{N}=3$ supergravity in the manifestly $\mathrm{U}(3,n)$-covariant symplectic basis specified by (5.16) by simply considering η as the metric of (the fundamental irrep. of) $\mathrm{SO}(3,n)$.

It can be easily checked that the *"degenerate"* Freudenthal duality transformation $\mathbb{C}\mathcal{A}$ (10.4)–(10.6) is nothing but a particular *anti-involutive* symplectic transformation of the relevant U-duality group G_4. Thus, the invariance (10.3) is trivial, and in the simple, *degenerate* groups of type E_7 relevant to $D = 4$ supergravity (namely, $\mathrm{U}(1,n)$ or $\mathrm{U}(3,n)$) the corresponding Freudenthal duality is an *anti-involutive* U-duality transformation.

11 *Non-minimal* coupling and fermions

Certain aspects of *non-minimal* vector coupling reflect on fermions and their interactions. In particular, one finds that in case that the holomorphic function $f_{\alpha\beta}(z)$ depends on z the mass of gaugino's may have a non-vanishing tree level contribution of the form (in the notation of [97])

$$\frac{1}{4} f_{\alpha\beta i}\, g^{-1i}{}_j e^{K/2} D^j W \bar{\lambda}_R^\alpha \lambda_R^\beta + (R \Leftrightarrow L). \tag{11.1}$$

Such a mass term for $D^j W \neq 0$ may play an important role in particle physics. In the *minimal coupling* case, $f_{\alpha\beta i} \equiv \frac{\partial f_{\alpha\beta}}{\partial z^i} = 0$, and the mass of gaugino's may only come from soft breaking terms and from quantum effects.

Another case of *non-minimal* coupling in the fermion sector involves a Pauli coupling of a vector to a fermion of the chiral multiplet and a gaugino (see also appendix A further below)

$$\frac{1}{4} f_{\alpha\beta}{}^i \bar{\chi}_i \gamma^{\mu\nu} F_{\mu\nu}^{-\alpha} \lambda_L^\beta + \text{h.c.}\,. \tag{11.2}$$

This process is interesting in the context of creation of matter in the Universe, after inflation. The bosonic cubic vertices ϕF^2 or $aF\tilde{F}$ provide a possibility of creation of vectors fields from the inflaton (scalar ϕ, or the axion a). A Pauli coupling above will allow the fermionic partner of the inflaton, χ to decay and create a vector and a gaugino, standard model particles. Thus the dependence of the vector coupling on scalars due to supersymmetry is present also in the fermionic sector of the theory and may also be useful. Clearly, both terms in (11.1) and in (11.2) are absent in models of $\mathcal{N} = 1$ supergravity with minimal coupling, but necessarily present in models originating from higher supersymmetries.

12 Conclusion

The minimal vector coupling in $\mathcal{N} = 1$ supergravity corresponds to the choice of the constant vector kinetic term as shown in eq. (1.1), when instead of a holomorphic function of scalars, $f_{\alpha\beta}(z)$, as in eq. (1.2), one has $f_{\alpha\beta} = \delta_{\alpha\beta}$. Meanwhile, there is an interesting possibility to use the couplings like ϕF^2, and $aF\tilde{F}$ and the ones with fermions, for cosmological applications, see for example [89].

It is therefore interesting to study the origin of such couplings, attractive for cosmology and for particle physics, from well motivated superstring theory and their compactification, and related to these four-dimensional supergravities with higher suppersymmetries.

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As resulting from the present paper, generalizing and refining the investigation carried out in [5], the answer to this question follows from duality symmetry and has a group theoretical origin. The question is why the vector kinetic matrix $\mathcal{N}_{\Lambda\Sigma}(\varphi)$ in $\mathrm{Im}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda}_{\mu\nu}F^{\mu\nu\Sigma} + i\mathrm{Re}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda}_{\mu\nu}\tilde{F}^{\Sigma\mu\nu}$ (1.3) in $\mathcal{N} \geqslant 2$ depends, generically, or does not depend, in degenerate cases, on scalars, when the theory is reduced to $\mathcal{N} = 1$ case. In $\mathcal{N} \geqslant 2$ there is a duality symmetry group G, embedded into an $\mathrm{Sp}(2n_v, \mathbb{R})$, such that the n_v vector 2-form field strengths and their duals fit into a symplectic representation

$$\mathbf{R}' = \mathcal{S}\mathbf{R}\,, \qquad \mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad \mathcal{S}^t\Omega\,\mathcal{S} = \Omega\,, \qquad \Omega = \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}\,. \tag{12.1}$$

The gauge kinetic term $\mathcal{N}$ generically depends on scalars since it transforms via fractional transformations

$$\mathcal{N}' = (C + D\mathcal{N})(A + B\mathcal{N})^{-1}\,. \tag{12.2}$$

The symplectic symmetric tensor (see e.g. [13], and refs. therein)

$$\mathcal{M}_{MN}(\mathcal{N}) \equiv \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}; \tag{12.3}$$

$$\mathcal{A} \equiv \mathrm{Im}\mathcal{N} + \mathrm{Re}\mathcal{N}\,(\mathrm{Im}\mathcal{N})^{-1}\,\mathrm{Re}\mathcal{N}; \quad \mathcal{B} \equiv -\mathrm{Re}\mathcal{N}\,(\mathrm{Im}\mathcal{N})^{-1}\,;$$
$$\mathcal{C} \equiv -\,(\mathrm{Im}\mathcal{N})^{-1}\,\mathrm{Re}\mathcal{N}; \quad \mathcal{D} \equiv (\mathrm{Im}\mathcal{N})^{-1}$$

is never constant (i.e. scalar-independent) in $\mathcal{N} \geqslant 2$ supergravity, because, as shown in [5], this would imply the existence of an invariant quadratic form with Euclidean signature (due to the negative definiteness of $\mathcal{M}$ (12.3)–(12.4) itself). However, in the present investigation we exploited a systematic investigation of the cases in which *degenerate* groups of type E_7, when reduced to $\mathcal{N} = 1$, may provide a *scalar-independent* kinetic vector matrix $\mathcal{N}$, and thus a *scalar-independent* $\mathcal{M}$. For $\mathcal{N} = 2$ theories, this can only occur when the matrix $\mathcal{F}_{\Lambda\Sigma} \equiv \partial_\Lambda\partial_\Sigma F$ projected onto the directions pertaining to the would-be $\mathcal{N} = 1$ vector multiplets, is constant, namely when the holomorphic prepotential F is *quadratic* in the scalar degrees of freedom corresponding to the would-be $\mathcal{N} = 1$ vector multiplets. In *symmetric* special Kähler geometry, this implies that $\mathcal{M}(\mathcal{F})$ (defined as (12.3)–(12.4) with $\mathcal{N}_{\Lambda\Sigma} \to \mathcal{F}_{\Lambda\Sigma}$) is a scalar-independent matrix with Lorentzian signature, and the corresponding quadratic form $\mathcal{Q}\mathcal{M}(\mathcal{F})\mathcal{Q}^T$ defines the quadratic symmetric invariant structure of *degenerate* groups of type E_7 (recall (8.31) and eqs. (34) and (35) of [96])

$$\mathcal{I}_{2,\mathbb{CP}^n} = i_1 - i_2 = -\frac{1}{2}\mathcal{Q}\mathcal{M}(\mathcal{F})\mathcal{Q}^T. \tag{12.4}$$

For *non-degenerate* groups of type E_7, $\mathcal{M}(\mathcal{F})$ is never scalar-independent, and thus *minimal coupling* is not allowed.

In the present paper, we carried out a detailed classification and analysis of all cases of degeneration of groups of type E_7 responsible for the duality symmetry of extended supergravity: in this way, our investigation provides an explanation for the fact that the *minimal coupling* case is *non-generic* in $\mathcal{N} = 1$ supergravity originating from higher supersymmetries, thus supporting the proposal to use a *non-minimal* vector coupling for applications in particle physics and cosmology.

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A Pauli terms

A.1 General structure

In a $D = 4$ $\mathcal{N}$-extended supergravity theory, the general structure of Pauli terms read (we use the notation and conventions of [74], to which the reader is addressed for further elucidation):

$$\left[(\sqrt{-g})^{-1}\mathcal{L}\right]_{\text{Pauli}} = \mathcal{F}^{-\Lambda}_{\mu\nu}\text{Im}\mathcal{N}_{\Lambda\Sigma}\left(L^{\Sigma}_{AB}\overline{\psi}^{\mu A}\psi^{\nu B} + L^{\Sigma}_{IA}\overline{\psi}^{\mu A}\gamma^{\nu}\lambda^{I} + L^{\Sigma}_{IJ}\overline{\lambda}^{I}\gamma^{\mu\nu}\lambda^{J}\right) + \text{h.c.}, \tag{A.1}$$

where λ_I and $\psi_{A\mu}$ respectively denote the spin-$\frac{1}{2}$ fermions and the gravitino fields, and $\mathcal{F}^{(\mp)\Lambda}_{\mu\nu}$ are the self-dual/anti-self-dual combinations of the vector field strengths:

$$\begin{aligned}
\mathcal{F}^{(\mp)\Lambda}_{\mu\nu} &\equiv \frac{1}{2}\left(\mathcal{F}^{\Lambda}_{\mu\nu} \mp i\star\mathcal{F}^{\Lambda}_{\mu\nu}\right);\\
\star\mathcal{F}^{\Lambda}{}_{\mu\nu} &\equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathcal{F}^{\rho\sigma\Lambda},\\
\star\mathcal{F}^{\Lambda(\pm)}{}_{\mu\nu} &= \mp i\mathcal{F}^{\Lambda(\pm)}_{\mu\nu}.
\end{aligned} \tag{A.2}$$

$A, B, \ldots$ indices range in the fundamental representation of the $\mathcal{R}$-symmetry $\text{SU}(\mathcal{N}) \times \text{U}(1)$ (the U(1) term is missing in the maximal case $\mathcal{N} = 8$), their lower (upper) position denoting left (right) chirality. Besides enumerating the fields, the indices I actually are a short-hand notation, which encompasses various possibilities: if the fermions belong to vector multiplets $I \to IA$, since they also transform under $\mathcal{R}$-symmetry; if they refer to fermions of the gravitational multiplet they are a set of three $\text{SU}(\mathcal{N})$ antisymmetric indices: $I \to [ABC]$. (In the particular case of $\mathcal{N} = 2$ n_H hypermultiplets: $I \to \alpha$, where α is in the fundamental of $\text{USp}(2n_H)$).

The matrices entering the Lagrangian are in general all dependent on the scalar fields q^i. $\mathcal{N}_{\Lambda\Sigma}$ is the kinetic vector matrix, generally depending on (a subset q^i of) the scalar fields q^u. According to [6, 7], the indices Λ, Σ sit in the relevant symplectic representation of the U-duality group G. The structures L^{Σ}_{AB}, L^{Σ}_{IA}, L^{Σ}_{IJ} are coset representatives of the σ-model G/H for $\mathcal{N} > 2$, while they are objects of special Kähler geometry for $\mathcal{N} = 2$. For $\mathcal{N} = 1$, they are related to the kinetic matrix of the vectors (with $L^{\Sigma}_{AB} = 0$, because there are no vectors in the $\mathcal{N} = 1$ gravity multiplet).

In the following, we will specify (A.1) to $\mathcal{N} = 8$, to $\mathcal{N} = 2$ (in particular, when G is a "degenerate" group *"of type E_7"*) and to $\mathcal{N} = 1$ theories (also in presence of *minimal coupling*).

A.2 $\mathcal{N} = 8$

In this case, $A = 1, \ldots, 8$ range in the **8** of the $\mathcal{R}$-symmetry SU(8). Only gravitational multiplet is present; the gauginos $\lambda_{[ABC]}$ are in the rank-3 antisymmetric irrep. **56** of SU(8), whereas the scalars $q^{[ABCD]}$ sit into the rank-4 antisymmetric *self-real* irrep. **70** of SU(8). (A.1) thus specifies to:

$$\begin{aligned}\mathcal{N} = 8 : \left[(\sqrt{-g})^{-1}\mathcal{L}\right]_{\text{Pauli}} &= \mathcal{F}^{-\Lambda}_{\mu\nu}\text{Im}\mathcal{N}_{\Lambda\Sigma}L^{\Sigma}_{AB}\overline{\psi}^{\mu A}\psi^{\nu B} \\ &+\mathcal{F}^{-\Lambda}_{\mu\nu}\text{Im}\mathcal{N}_{\Lambda\Sigma}L^{\Sigma}_{AB}\overline{\psi}^{\mu}{}_{C}\gamma^{\nu}\lambda^{ABC} \\ &+\mathcal{F}^{-\Lambda}_{\mu\nu}\text{Im}\mathcal{N}_{\Lambda\Sigma}\epsilon_{ABCDEFGH}\overline{\lambda}^{ABC}\gamma^{\mu\nu}\lambda^{DEF}\overline{L}^{\Sigma|GH} + \text{h.c.}\ . \end{aligned} \tag{A.3}$$

Thus, by introducing

$$T^{-}_{\mu\nu,\ AB} \equiv \mathcal{F}^{-\Lambda}_{\mu\nu}\text{Im}\mathcal{N}_{\Lambda\Sigma}L^{\Sigma}_{AB}; \tag{A.4}$$

$$T^{-|AB}_{\mu\nu} \equiv \mathcal{F}^{-\Lambda}_{\mu\nu}\text{Im}\mathcal{N}_{\Lambda\Sigma}\overline{L}^{\Sigma|AB}, \tag{A.5}$$

(A.3) can be rewritten as

$$\begin{aligned}&\mathcal{N} = 8 : \left[(\sqrt{-g})^{-1}\mathcal{L}\right]_{\text{Pauli}} \\ &\quad= T^{-}_{\mu\nu,\ AB}\overline{\psi}^{\mu A}\psi^{\nu B} + T^{-}_{\mu\nu,\ AB}\overline{\psi}^{\mu}{}_{C}\gamma^{\nu}\lambda^{ABC} \\ &\qquad+\epsilon_{ABCDEFGH}\overline{\lambda}^{ABC}\gamma^{\mu\nu}\lambda^{DEF}T^{-|GH}_{\mu\nu} + \text{h.c.}\ . \end{aligned} \tag{A.6}$$

A.3 $\mathcal{N} = 2$

$\mathcal{N} = 2$ supergravity the scalar manifold is a product manifold [65, 75, 76],

$$\mathcal{M}_{scalar} = \mathcal{M}_{vec} \times \mathcal{M}_{hyper} \tag{A.7}$$

since there are two kinds of matter multiplets, the vector multiplets and the hypermultiplets. The geometry of $\mathcal{M}_{vec}$ is described by the *special Kähler geometry* [75, 77–80], while the geometry of $\mathcal{M}_{hyper}$ is described by *quaternionic geometry* [75, 76, 81–87]; for a thorough geometric treatment, see e.g. [8].
With respect to the general case (A.1)

$$\Lambda = 0, 1, \ldots, n_V;\ \ A, B = 1, 2;\ \ i = 1, \ldots, 4n_H + 2n_V;\ \ I = 1, \ldots n_H + n_V, \tag{A.8}$$

where the index 0 pertains to the *graviphoton*.

As it will be the case in $\mathcal{N} = 1$ supergravity, we denote the complex scalars parameterizing $^{(vec)}$ by $z^i, \bar{z}^{\bar{\imath}}$, while the scalars parameterizing $\mathcal{M}_{hyper}$ will be denoted by q^u. When the index I runs over the vector multiplets it must be substituted by IA in all the formulae relevant to the vector multiplet, since the fermions λ^{IA} are in the fundamental of the $\mathcal{R}$-symmetry group U(2).

$L^{\Lambda}(z, \bar{z})$ and its "magnetic" counterpart $M_{\Lambda}(z, \bar{z}) = \mathcal{N}_{\Lambda\Sigma}L^{\Sigma}$ actually form a $2n_V$ dimensional covariantly holomorphic section $V = (L^{\Lambda}, M_{\Lambda})$ of a flat symplectic bundle.

When the index I runs over the hypermultiplets, we rename them as follows: $(I, J) \rightarrow (\alpha, \beta)$ and since there are no vectors in the hypermultiplets we have $f^{\Lambda A}_{\alpha} = 0$

The *Vielbein* of the quaternionic manifold $\mathcal{M}_{hyper}$ are usually denoted by $\mathcal{U}^{\alpha A} \equiv \mathcal{U}^{\alpha A}_u dq^u$, where $\alpha = 1, \dots, 2n_H$ is an index labelling the fundamental representation of $\mathrm{USp}(2n_H)$. The inverse matrix *Vielbein* is ${}^{u}_{\alpha A}$. We raise and lower the indices $\alpha, \beta, \dots$ and $A, B, \dots$ with the symplectic matrices $\mathbb{C}^{\alpha\beta}$ and ϵ_{AB}.

Thus, (A.1) specifies to:

$$
\begin{aligned}
\mathcal{N} &= 2 : \left[(\sqrt{-g})^{-1}\mathcal{L}\right]_{\text{Pauli}} \\
&= \mathcal{F}^{-\Lambda}_{\mu\nu}\mathrm{Im}\mathcal{N}_{\Lambda\Sigma}\left[\begin{array}{l} 4L^{\Sigma}\overline{\psi}^{A\mu}\psi^{B\nu}\epsilon_{AB} - 4i\overline{D}_{\bar{i}}\overline{L}^{\Sigma}\overline{\lambda}^{\bar{i}}_{A}\gamma^{\nu}\psi^{\mu}_{B}\epsilon^{AB} \\ \\ +\frac{i}{2}C_{ijk}g^{k\bar{k}}\overline{D}_{\bar{k}}\overline{L}^{\Sigma}\overline{\lambda}^{iA}\gamma^{\mu\nu}\lambda^{jB}\epsilon_{AB} - L^{\Sigma}\overline{\zeta}_{\alpha}\gamma^{\mu\nu}\zeta_{\beta}\mathbb{C}^{\alpha\beta} \end{array}\right] + \text{h.c.,}
\end{aligned}
\tag{A.9}
$$

where ζ_α, $\overline{\zeta}_\alpha$ denote the spin-$\frac{1}{2}$ fermions of the hypermultiplets (*hyperinos*). The kinetic vector matrix $\mathcal{N}_{\Lambda\Sigma}$ can be constructed in terms of L^Λ through the procedure e.g. given in [8].

By introducing the gravity- and matter-*vector projectors*

$$T^{-}_{\mu\nu} \equiv 2i\mathrm{Im}\mathcal{N}_{\Lambda\Sigma}L^{\Sigma}\mathcal{F}^{-\Lambda}_{\mu\nu}; \tag{A.10}$$

$$T^{-i}_{\mu\nu} \equiv -\mathrm{Im}\mathcal{N}_{\Lambda\Sigma}\mathcal{F}^{-\Lambda}_{\mu\nu}g^{i\bar{j}}\overline{D}_{\bar{j}}\overline{L}^{\Sigma}, \tag{A.11}$$

(A.9) can be rewritten as

$$
\begin{aligned}
\mathcal{N} &= 2 : \left[(\sqrt{-g})^{-1}\mathcal{L}\right]_{\text{Pauli}} \\
&= -\frac{i}{2}T^{-}_{\mu\nu}\left[4\overline{\psi}^{A\mu}\psi^{B\nu}\epsilon_{AB} - \overline{\zeta}_{\alpha}\gamma^{\mu\nu}\zeta_{\beta}\mathbb{C}^{\alpha\beta}\right] && \text{(A.12)} \\
&\quad +\frac{i}{2}T^{-k}_{\mu\nu}\left[8g_{k\bar{i}}\overline{\lambda}^{\bar{i}}_{A}\gamma^{\nu}\psi^{\mu}_{B}\epsilon^{AB} - C_{ijk}\overline{\lambda}^{iA}\gamma^{\mu\nu}\lambda^{jB}\epsilon_{AB}\right] + \text{h.c.}\,. && \text{(A.13)}
\end{aligned}
$$

Note that for $\mathcal{N} = 2$ *minimally coupled* theories, whose U-duality group is a *degenerate* group of type E_7: $G_4 = \mathrm{U}(1, n_V)$, it holds that $C_{ijk} = 0$, and thus the second Pauli term in the "matter sector" (A.13) is absent.

A.4 $\mathcal{N} = 1$

In order to specify the general formula (A.1) to $\mathcal{N} = 1$, we recall that the scalar manifold is in this case a Kähler-Hodge manifold and that the $\mathcal{R}$-symmetry reduces simply to $\mathrm{U}(1)$.; for a general treatment, see e.g. [9, 88]. It is convenient in this case to use as *"Vielbeins"* the differentials of the complex coordinates $dz^i, d\bar{z}^{\bar{i}}$, where $z^i(x)$ are the complex scalar fields parameterizing the Kähler-Hodge manifold of (complex) dimension n_C; thus, in this case we set $q^u \to (z^i, \bar{z}^{\bar{i}})$. The spin $\frac{1}{2}$ fermions are either in chiral or in vector multiplets; so, the index I runs over the number $n_V + n_C$ of vector and chiral multiplets: $I = 1, \dots, n_V + n_C$. Furthermore, it is convenient to assign the index Λ, the same as for the vectors, to the fermions of the vector multiplets: we will denote them as λ^Λ, $\Lambda = 1, \dots, n_V$; the fermions of the chiral multiplets will instead be denoted by $\chi^i, \chi^{\bar{i}}$ in the case of left-handed or right-handed spinors, respectively. Since the gravitino and the gaugino fermions have no $\mathrm{SU}(\mathcal{N})$

indices, their chirality will be denoted by a lower or an upper dot for left-handed or right handed fermions respectively, namely ($\psi_\bullet$, $\psi^\bullet$) and ($\lambda^\Lambda_\bullet$, $\lambda^{\bullet\Lambda}$). Thus, (A.1) specifies to:

$$\mathcal{N}=1: \left[(\sqrt{-g})^{-1}\mathcal{L}\right]_{\text{Pauli}} = \text{Im}\mathcal{N}_{\Lambda\Sigma}\mathcal{F}^{-\Lambda}_{\mu\nu}\bar{\lambda}^{\bullet\Sigma}\gamma^\mu\psi^\nu_\bullet - \frac{i}{8}\partial_i\overline{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}^{-\Lambda}_{\mu\nu}\bar{\chi}^i\gamma^{\mu\nu}\lambda^\Sigma_\bullet + \text{h.c.}, \tag{A.14}$$

where $\mathcal{F}^{(\mp)\Lambda}_{\mu\nu}$ are defined in (A.2). Within the adopted conventions, $\mathcal{N}_{\Lambda\Sigma}$ is *anti-holomorphic* in the chiral multiplets' complex scalars:

$$\partial_i\mathcal{N}_{\Lambda\Sigma} = 0. \tag{A.15}$$

It is instructive to compare (A.14) with its $\mathcal{N}=2$ counterpart (A.12)–(A.13). When performing the supersymmetry reduction $\mathcal{N}=2 \to \mathcal{N}=1$, the "gravity sector" (A.12) of the $\mathcal{N}=2$ Pauli terms is projected out because, as mentioned, the $\mathcal{N}=1$ gravity multiplet des *not* contain any *graviphoton*. On the other hand, the "matter sector" (A.13) of the $\mathcal{N}=2$ Pauli terms (simpler in the $\mathcal{N}=2$ *minimally coupled* theory due to $C_{ijk}=0$) becomes (A.14) itself.

Furthermore, it should be noted that when the $\mathcal{N}=1$ scalars are *minimally coupled* to the vectors ($\partial_i\overline{\mathcal{N}}_{\Lambda\Sigma}=0$; thus, from (A.15)), the second term in (A.14) vanishes, and the Pauli term (A.14) acquires its *minimally coupled* form

$$\mathcal{N}=1 \textit{ minimal coupling}: \left[(\sqrt{-g})^{-1}\mathcal{L}\right]_{\text{Pauli}} = \text{Im}\mathcal{N}_{\Lambda\Sigma}\mathcal{F}^{-\Lambda}_{\mu\nu}\bar{\lambda}^{\bullet\Sigma}\gamma^\mu\psi^\nu_\bullet + \text{h.c.}\,. \tag{A.16}$$

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