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Simplifying one-loop amplitudes in superstring theory

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Abstract: We show that 4-point vector boson one-loop amplitudes, computed in [1] in the RNS formalism, around vacuum configurations with open unoriented strings, preserving at least $\mathcal{N} = 1$ SUSY in $D = 4$, satisfy the correct supersymmetry Ward identities, in that they vanish for non MHV configurations $(++++)$ and $(-+++)$. In the MHV case $(--++)$ we drastically simplify their expressions. We then study factorisation and the limiting IR and UV behaviours and find some unexpected results. In particular no massless poles are exposed at generic values of the modular parameter. Relying on the supersymmetric properties of our bosonic amplitudes, we extend them to manifestly supersymmetric super-amplitudes and compare our results with those obtained in the $D = 4$ hybrid formalism, pointing out difficulties in reconciling the two approaches for contributions from $\mathcal{N} = 1, 2$ sectors.

Keywords: Superstrings and Heterotic Strings, Intersecting branes models, D-branes, Superstring Vacua

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JHEP01(2016)043

Contents

JHEP01(2016)043

JHEP01(2016)043

Introduction

There is a revival of interest in superstring loop amplitudes from different perspectives [1–7]. The original one-loop computations by Brink, Green and Schwarz [8] paved the way to remarkable developments both in string theory and in field theory related contexts. Some time ago the BGS amplitude for four vector bosons in $D = 10$ Type I superstrings was generalised to $D = 4$ in vacuum configurations with open and unoriented strings preserving at least $\mathcal{N} = 1$ supersymmetry [1]. Although the final result could be expressed in a compact form as a sum over the various sectors, only the contribution of the $\mathcal{N} = 4$ sector and the 'irreducible' contributions of the $\mathcal{N} = 1, 2$ sectors could be easily seen to be proportional to the tree-level amplitude. Supersymmetry Ward identities imply that the only non-vanishing 4-point amplitudes be Maximally Helicity Violating (MHV) [9]. Thus loop corrections should reproduce similar structures to tree-level amplitudes in supersymmetric vacuum configurations in $D = 4$.

Aim of the present paper is to simplify the results of [1] using the spinor helicity formalism and to analyse the singularities of the resulting amplitudes. We will also comment on the soft behaviour of the amplitudes and compare our results with the ones obtained in [10] within the $D = 4$ 'hybrid formalism' [11][1] and find it hard to reconcile the two approaches. The main difference is that despite vector boson vertex operators are compactification independent, i.e. they are proportional to the identity operator of the internal SCFT, only in the $\mathcal{N} = 4$ sector one has a complete factorisation of the space-time and internal part, encoded in the sum of KK momenta or alike. In $\mathcal{N} = 1, 2$ sectors the computation does not factorise even for the 'irreducible' contributions, encoded in a function $\mathcal{F}_{\mathcal{N}}$ of the modular parameter and of the compactification moduli, which does not simply coincide with the internal partition function.

[1] Other hybrid approaches have been proposed in diverse dimensions: $D = 6$ [12], $D = 3$ [13], $D = 2$ [14].

The plan of the paper is as follows. In section 1, we will briefly review one-loop open superstring amplitudes in order to fix the notation. In section 2 and 3 we rewrite 2- and 3-point 'amplitudes',[2] which vanish in the case of $\mathcal{N}=4$ sector, in the helicity formalism. We show that they satisfy the correct Ward identities, in that, for instance, 3-point amplitudes with 3 positive helicity vector bosons vanish, and identify their divergences. The results of [1] for 4-point amplitudes are reviewed in section 4 and systematically simplified in section 5, where we show that only MHV amplitudes are non-vanishing. The case of 'regular' branes at orbifold singularities [15, 16] is discussed in section 6. In section 7 we study factorisation and find some unexpected result, i.e. no massless poles appear for generic modular parameter, even in sectors with reduced SUSY. We then discuss the IR and UV behaviours of the independent amplitudes: planar, non-planar (3−1 and 2−2) and unorientable. After extending our bosonic amplitudes to full super-amplitudes, in section 8 we draw a comparison between our results and the results of the $D=4$ hybrid formalism [10] and we discuss potential sources of disagreement. We will conclude with some speculations about higher points, higher loops, soft limits and KLT relations beyond tree-level in section 9.

1 Superstrings at one loop

In theories with open and unoriented strings scattering amplitudes can be computed inserting the corresponding vertex operators on the boundaries [17]. The vector boson vertex (in the super-ghost pictures $q=0$) reads

$$V_B^{(0)} = a_\mu \left(\partial X^\mu + ik\cdot\psi\psi^\mu\right) e^{ikX} = \left(a\cdot\partial X - \frac{i}{2} f_{\mu\nu}\psi^\mu\psi^\nu\right) e^{ikX} \tag{1.1}$$

where $f_{\mu\nu} = a_\mu k_\nu - k_\mu a_\nu$ is the linearised field strength.[3]

The tree-level disk contribution is similar to Veneziano amplitude

$$\mathcal{A}_4^{\text{tree}}(1,2,3,4) = g_s^2 F^4 \left[\frac{\mathcal{B}(s,t)}{st}\mathcal{T}_{1234} + \frac{\mathcal{B}(t,u)}{tu}\mathcal{T}_{1423} + \frac{\mathcal{B}(u,s)}{us}\mathcal{T}_{1342}\right] \tag{1.2}$$

where g_s is the coupling constant for open string. The Veneziano factor $\mathcal{B}(x,y)$ and the Chan-Paton factor $\mathcal{T}_{abcd}$ read

$$\mathcal{B}(x,y) = \frac{\Gamma(1-\alpha' x)\Gamma(1-\alpha' y)}{\Gamma(1-\alpha' x-\alpha' y)} \qquad , \qquad \mathcal{T}_{abcd} = \text{tr}(t_a t_b t_c t_d) \tag{1.3}$$

while the totally symmetric kinematic factor F^4 is given by

$$F^4 = [2(f_1 f_2 f_3 f_4) - \frac{1}{2}(f_1 f_2)(f_3 f_4) + \text{cyclic}(234)] \tag{1.4}$$

In $D=4$, F^4 is non-vanishing only in the Maximally Helicity Violating (MHV) case

$$F^4_{++++} = F^4_{-+++} = 0 \qquad , \qquad F^4_{--++} = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 41\rangle} st \tag{1.5}$$

[2] We put amplitudes in quote because they do not correspond to "scattering" due to collinear momenta.
[3] Henceforth we will refer to $:\psi_\mu\psi_\nu:$ as (fermionic) bilinear.

JHEP01(2016)043

thereby, for a given color ordering, the partial amplitudes read

$$\mathcal{A}_4^{\text{tree}}[1^-,2^-,3^+,4^+] = \mathcal{A}_{\text{YM}}^{\text{tree}}[1^-,2^-,3^+,4^+]\mathcal{B}(s,t) \tag{1.6}$$

while $\mathcal{A}_4^{\text{tree}}[1^+,2^+,3^+,4^+] = 0 = \mathcal{A}_4^{\text{tree}}[1^-,2^+,3^+,4^+]$. Generalization to higher points can be found in [18–21].

The one-loop four-point amplitude in $D = 10$ was computed long ago by Brink, Green and Schwarz [8]. It receives three contributions: planar, non-planar and un-orientable. Setting the modular parameter of the covering torus to be $\tau_A = iT/2$ for the annulus and $\tau_M = iT/2 + 1/2$ for the Möbius strip, all contributions can be written in the form

$$\mathcal{A}_4^{1-\text{loop}}(1,2,3,4) = g_s^4 \mathcal{T}_{CP} \alpha'^2 F^4 \int_0^\infty \frac{dT}{T^{5+1}} \int_{\mathcal{R}_{CP}} d^4 z_i \, \Pi_4(z_i,k_i) \tag{1.7}$$

where $\mathcal{T}_{CP}$ is the Chan-Paton factor, $\mathcal{R}_{CP}$ is the integration region, depending on color ordering, and

$$\Pi_4(z_i,k_i) = \prod_{i<j} \exp[-2\alpha' k_i{\cdot}k_j \mathcal{G}(z_{ij})] \tag{1.8}$$

is the ubiquitous Koba-Nielsen factor with $\mathcal{G}(z_{ij})$ the scalar propagator (Bargmann kernel) for boundary insertions at one-loop

$$\mathcal{G}_{\mathcal{A}}(z_1,z_2;\tau_{\mathcal{A}}) = -\left[\log\frac{\theta_1(z_1-z_2|\tau)}{\theta_1'(0|\tau)} - 2\pi\frac{[\text{Im}\,(z_1-z_2)]^2}{\text{Im}\,\tau}\right] \tag{1.9}$$

We will often write $\mathcal{G}_{ij}$ instead to $\mathcal{G}(z_{ij})$.

In the planar case all vertex operators are inserted on the same boundary of an annulus

$$\mathcal{T}_{1234}^{\text{plan}} = \text{tr}(t_1t_2t_3t_4)\text{tr}(1) \qquad , \qquad \mathcal{R}_{1234}^{\text{plan}} = \{z_1 > z_2 > z_3 > z_4 = 0\} \tag{1.10}$$

plus cyclic permutations of 234. The parametrization of the world-sheet variable on a boundary is $z = iT\nu/2$ with $\nu \in [0,1]$.

In the non-planar case vertex operators are equally distributed among the two boundaries of an annulus

$$\mathcal{T}_{12|34}^{\text{non-pl}} = \text{tr}(t_1t_2)\text{tr}(t_3t_4) \qquad , \qquad \mathcal{R}_{12|34}^{\text{non-pl}} = \{z_1 > z_2; z_3 > z_4 = 0\} \tag{1.11}$$

plus permutations of 2 with 3 and 4. The parametrization of the world-sheet variable on the other boundary is $z = iT\nu/2 + 1/2$ with $\nu \in [0,1]$.

For gauge groups with (anomalous) U(1) factors there is an additional non-planar contribution with 3 vertices inserted on a boundary and the remaining one on the other boundary

$$\mathcal{T}_{123|4}^{\text{anom}} = \text{tr}(t_1t_2t_3)\text{tr}(t_4) \qquad , \qquad \mathcal{R}_{123|4}^{\text{anom}} = \{z_1 > z_2 > z_3 = 0; z_4\} \tag{1.12}$$

plus permutations of 4 with 1,2,3.

In the un-orientable case vertex operators are inserted along the single boundary of a Möbius strip with twice the length of the strip itself

$$\mathcal{T}_{1234}^{\text{un-or}} = 2\text{tr}(t_1t_2t_3t_4)\mathcal{T}_\Omega \qquad , \qquad \mathcal{R}_{1234}^{\text{un-or}} = \{z_1 > z_2 > z_3 > z_4 = 0\} \tag{1.13}$$

JHEP01(2016)043

plus cyclic permutations of 234, with $\mathcal{T}_\Omega$ the tension of the relevant Ω-plane in units of α', quantized charge. The parametrization of the world-sheet variable on the unique boundary is $z = iT\nu/2$ with $\nu \in [0, 2]$.

At low-energies $\alpha'|k_i \cdot k_j| \ll 1$, $\Pi_4(z_i, k_i) \approx 1$ and one can trivially integrate over the insertion points z_i producing a factor of T^4. The remaining integral over the modular parameter $\int dT/T^2$ is IR finite (for $T \to \infty$) but UV divergent (for $T \to 0$), due to the dilaton tadpole associated to the empty boundary and the 'cross-cap'. Yet for SO(32), the dilaton tadpole cancels and the Type I theory if free of both UV divergences and chiral anomalies [22]. Non-planar amplitudes are regulated by momentum flow between the two boundaries.

A subtle issue related to potential anomalies is the role of the odd spin structure in the computation of scattering amplitudes. In order to detect potential anomalies, one of the gauge boson vertex operators should appear with longitudinal polarisation and should decouple thanks to BRST invariance in a consistent theory. The standard procedure requires the insertion of a vertex operator in the $q = -1$ super-ghost picture and an additional world-sheet super-current insertion brought down by integration over the super-modulus associated to the world-sheet gravitino zero-mode. In $D = 10$ hexagon gauge anomaly could be detected this way [23, 24]. In the $D = 4$ case under consideration, 4-point amplitudes are not anomalous and BRST invariance allows to replace the combination of the super-modulus and world-sheet super-current with a picture changing operator [25]. The latter can act on the vertex operator in the $q = -1$ super-ghost picture and change its picture $q = 0$. As in [1], one can then proceed with all vertex operators in the $q = 0$ picture.[4]

1.1 Partition function

In order to generalize BGS formula to (supersymmetric) vacuum configurations for open and unoriented strings in $D = 4$, one has to first recall the structure of the one-loop partition function. As in [1], we will mainly focus here on magnetised or intersecting D-branes at (non-compact) orbifold singularities [15, 16]. Consistency requires local RR tadpole cancellation [26–30].

The partition function depends on the choice of brane configuration, including number and type of intersections or magnetic fluxes thereon, Ω-planes and orbifold group Γ. For simplicity we will focus on $\Gamma = \mathbb{Z}_n \subset \mathrm{SU}(3)$, i.e. $Z^I \approx \exp(2\pi i n_I/n) Z^I$ with $n_1 + n_2 + n_3 = 0 \pmod 1$ in order to preserve SUSY. We will label the branes of type a by $i_a = 1, \ldots, N_a$, and the orbifold sectors by $h = 0, \ldots, n-1$. We define three combinations that express

[4] This argument must be taken with a grain of salt since it has caused a problematic impression on the referee. According to the referee, that we thank for his/her punctual observation, it leads to an incorrect number of fermionic propagators, i.e. S_{ij}^{n-2} instead of S_{ij}^{n-1}. The latter is what one might expect in line with the counting of loop momenta in the field theory limit. Yet, our treatment of the odd spin structure precisely matches the results in the even spin structures in so far as the counting of S_{ij} is concerned. Moreover, we checked that our procedure reproduces the results of [24] since the longitudinal vertex in the $q = 0$ picture is a total derivative that leads to a vanishing result in a consistent theory or to a boundary term as a signal of an anomaly. We plan to return to this issue in the future.

JHEP01(2016)043

the twist or shift in the open string spectrum

$$u^I_{ab} = hv^I_{ab} + i\epsilon^I_{ab}\frac{T}{2} \tag{1.14}$$

where $I = 1, 2, 3$, ϵ^I_{ab} denotes the angles between brane a and brane b or the shift in the string spectrum induced by the relative magnetic flux and v^I_{ab} denotes the twist caused by the orbifold. The combinations u^I_{ab} determine the amount of supersymmetry preserved in each sector.

1.2 $\mathcal{N} = 1$ sectors

The weakest condition one can impose to have 'minimal', *i.e.* $\mathcal{N} = 1$, supersymmetry is $u^1_{ab} + u^2_{ab} + u^3_{ab} = 0$, with $\prod_I u^I_{ab} \neq 0$. In this case the partition function assumes the form

$$\mathcal{Z}^{\mathcal{N}=1}_\alpha = \mathcal{X}^{\mathcal{N}=1}_{ab}\frac{\theta_\alpha(0)}{\eta^3}\prod_{I=1}^{3}\frac{\theta_\alpha(u^I_{ab})}{\theta_1(u^I_{ab})} \qquad \text{with} \qquad \mathcal{X}^{\mathcal{N}=1}_{ab} = \frac{\mathcal{V}_X I_{ab}}{2_{\rm GSO}2_\Omega n_{\rm orb}(\alpha' T)^2} \tag{1.15}$$

where α labels the four spin structures, $\mathcal{V}_X$ represents the 'regulated' volume of space-time (to be replaced by $(2\pi)^4\delta(\Sigma_i k_i)$ in scattering amplitudes), I_{ab} denotes the number of brane intersections or the degeneracy of Landau levels. We have traced the origin of various integers in the denominator, wherein the factor $(\alpha' T)^2$ accounts for integration over loop-momenta in $D = 4$.

1.3 $\mathcal{N} = 2$ sectors

In sectors with $\mathcal{N} = 2$ supersymmetry one of the u^I_{ab} vanishes, let us say $u^3_{ab} = 0$. As a consequence $u^2_{ab} = -u^1_{ab}$. The partition function reads

$$\mathcal{Z}^{\mathcal{N}=2}_\alpha = \mathcal{X}^{\mathcal{N}=2}_{ab}\frac{\theta^2_\alpha(0)\theta^2_\alpha(u_{ab})}{\eta^6\theta^2_1(u_{ab})} \qquad \text{with} \qquad \mathcal{X}^{\mathcal{N}=2}_{ab} = \frac{\mathcal{V}_X\Lambda^{\parallel}_{ab}I^{\perp}_{ab}}{2_{\rm GSO}2_\Omega n_{\rm orb}(\alpha' T)^2} \tag{1.16}$$

Where $u_{ab} = u^1_{ab}$, $I^{\perp}_{ab}$ denotes the number of intersections or degeneracy of Landau levels in the 'twisted/transverse' directions and $\Lambda^{\parallel}_{ab}$ denotes the lattice sum in the two (one complex) 'untwisted/longitudinal' compact directions.

1.4 $\mathcal{N} = 4$ sectors

Sectors with $\mathcal{N} = 4$ maximal supersymmetry correspond to $u^I_{ab} = 0$ and the partition function is simply given by

$$\mathcal{Z}^{\mathcal{N}=4}_\alpha = \mathcal{X}^{\mathcal{N}=4}_{ab}\frac{\theta^4_\alpha(0)}{\eta^{12}} \qquad \text{with} \qquad \mathcal{X}^{\mathcal{N}=4}_{ab} = \frac{\mathcal{V}_X\Lambda_{ab}}{2_{\rm GSO}2_\Omega n_{\rm orb}(\alpha' T)^2} \tag{1.17}$$

where Λ_{ab} denotes the lattice sum in the six internal directions.

Later on, we will compute 2, 3 and 4-point scattering amplitudes. Although the first two formally vanish on-shell due to collinear kinematics, we report their derivation using the spinor helicity formalism since it highlights the meaning of some of the structures that will later appear in the more interesting 4-pt amplitudes. Definitions and notation for elliptic functions and helicity spinors can be found in the appendices.

JHEP01(2016)043

2 Two-point amplitudes

Let us begin with the two-point amplitude without specifying for the time being whether the amplitude is planar, non-planar or un-oriented. We will see that the results are substantially the same up to minor modifications. Although momentum conservation implies $k_1{\cdot}k_2 = 0$ and then $k_1{\cdot}a_2 = 0 = k_2{\cdot}a_1$, we will formally keep $f_1 f_2 \neq 0$. The one-loop amplitude is given by

$$\begin{aligned}\mathcal{A}_2^{\text{1-loop}}[1,2] &= g_s^2 \sum_\alpha c_\alpha \int_0^\infty \frac{dT}{T} \int_0^{iT/2} dz_1 \int_0^{z_1} dz_2 \delta(z_2)\, \langle V_B^{(0)}(z_1)\, V_B^{(0)}(z_2)\rangle_\alpha \\ &= g_s^2 \int_0^\infty \frac{dT}{T} \int d\mu^{(2)} (\mathbb{E}^{(2)} + \mathbb{O}^{(2)})\end{aligned} \tag{2.1}$$

where $\mathbb{E}^{(2)}$ and $\mathbb{O}^{(2)}$ denote the contributions of the even and odd spin structures. Contractions with zero and one bilinear are zero, we have only the two bilinears contribution. In the even spin structures, the reduced contraction of two bilinears yields

$$\begin{aligned}\mathbb{E}^{(2)} &= \sum_\alpha c_\alpha \,\langle\langle V_B^{(0)}(z_1)\, V_B^{(0)}(z_2)\rangle\rangle_\alpha \\ &= -\,\langle\langle {:}k_1{\cdot}\psi_1\, a_1{\cdot}\psi_1{:}\, {:}k_2{\cdot}\psi_2\, a_2{\cdot}\psi_2{:}\rangle\rangle_\alpha = -\alpha'^2 \frac{(f_1 f_2)}{2} S_\alpha^2(z_{12})\end{aligned} \tag{2.2}$$

where $S_\alpha(z_{12})$ is the one-loop fermionic propagator (Szego kernel)[5]

$$S_\alpha(z_1, z_2; \tau) = \begin{cases} -\partial_{z_1}\mathcal{G}(z_1, z_2), & \text{if } \alpha = 1, \quad \text{odd} \\ \dfrac{\theta_\alpha(z_1 - z_2)}{\theta_1(z_1 - z_2)} \dfrac{\theta_1'(0)}{\theta_\alpha(0)}, & \text{if } \alpha \neq 1 \quad \text{even.} \end{cases} \tag{2.3}$$

where $\mathcal{G}$ is the scalar propagator in 1.9. We often use the notation S_{ij} instead of $S_1(z_{ij}) = -\partial_i \mathcal{G}(z_{ij})$. Using the identity $S_\alpha^2 = \mathcal{P} - e_{\alpha-1}$, where $\mathcal{P}$ is Weierstrass function, that does not contribute to the sum over spin structures, and $e_{\alpha-1} = 2\pi i \partial_\tau \log(\theta_\alpha/\eta)$, we have

$$\mathbb{E}^{(2)} = -\frac{\alpha'^2}{2}(f_1 f_2)\mathcal{E}_\mathcal{N} e^{-\alpha' k_1{\cdot}k_2 \mathcal{G}_{12}} \tag{2.4}$$

$$\mathcal{E}_\mathcal{N} = -\sum_\alpha c_\alpha e_{\alpha-1} \mathcal{Z}_\alpha^\mathcal{N} \tag{2.5}$$

where the function $\mathcal{E}_\mathcal{N}$ introduced in [1], labelled by the number $\mathcal{N}$ of preserved SUSY, depends on the world-sheet modular parameter T, on the parameters of the brane configuration coded in u_{ab}^I and the moduli of the compactification. $\mathcal{E}_\mathcal{N}$ vanishes in $\mathcal{N} = 4$ sectors due to Riemann identity.

In the odd spin structure, which only contributes in $\mathcal{N} = 1$ sectors[6] we can absorb the four zero modes in a unique way

$$\mathbb{O}^{(2)} = -c_1^{\text{GSO}} \,\langle {:}k_1{\cdot}\psi_0\, a_1{\cdot}\psi_0{:}\, {:}k_2{\cdot}\psi_0\, a_2{\cdot}\psi_0{:}\rangle = -2\alpha'^2 c_1^{\text{GSO}} \pi^2 \mathcal{X}_{\mathcal{N}=1}(\tilde{f}_1 f_2)/T^2 \tag{2.6}$$

[5] In the space orthogonal to the constant zero-mode.

[6] In $\mathcal{N} = 2, 4$ sectors one cannot absorb the internal fermionic zero-modes ψ_0^i with vector boson vertex operators.

JHEP01(2016)043

For brevity we define the function

$$\mathcal{C}_{\mathcal{N}=1} = -2c_1^{\rm GSO}\pi^2\mathcal{X}_{\mathcal{N}}/T^2 \tag{2.7}$$

Combining the contributions of even and odd spin structures, the two-point amplitude thus reads

$$\mathcal{A}_2^{\text{1-loop}}[1,2] = g_s^2\alpha'^2\int_0^\infty \frac{dT}{T}\left[\mathcal{C}_{\mathcal{N}=1}(\tilde{f}_1 f_2) + \mathcal{E}_{\mathcal{N}}(f_1 f_2)\right]\int d\mu^{(2)}\, e^{-\alpha' k_1\cdot k_2 \mathcal{G}_{12}} \tag{2.8}$$

Fixing the helicities and noting that $k_1\cdot k_2 = 0$ we have two simple results. Choosing $(\pm\pm)$ and using $\tilde{f}^{\pm} = \pm i f^{\pm}$ we obtain

$$\mathcal{A}_2^{\text{1-loop}}[1^{\pm},2^{\pm}] = -\frac{g_s^2\alpha'^2}{4}(f_1^{\pm}f_2^{\pm})\int_0^\infty \frac{dT}{T}\left[\mathcal{E}_{\mathcal{N}} \pm i\mathcal{C}_{\mathcal{N}=1}\right]T^2 \tag{2.9}$$

while choosing $(\pm\mp)$ the amplitude vanishes, *viz.*

$$\mathcal{A}_2^{\text{1-loop}}[1^{\pm},2^{\mp}] = 0 \tag{2.10}$$

To obtain planar, non-planar and un-oriented contributions one has to choose the specific modular parameter and the corresponding integration domain. The result is generically divergent for both $\mathcal{N} = 1, 2$ sectors, wherein it encodes β-functions and one-loop threshold corrections to the gauge kinetic functions [31, 32]. As already observed, it vanishes in $\mathcal{N} = 4$ sectors, which points to the no-bubble conjecture in $\mathcal{N} = 4$ SYM, i.e. to the absence of one-loop massless amplitudes with two (bunches of) insertion points.

3 Three-point amplitudes

We continue our preliminary analysis and compute the three-point one-loop amplitude that reads:

$$\begin{aligned}
&\mathcal{A}_3^{\text{1-loop}}[1,2,3] \\
&\quad = g_s^3\sum_\alpha c_\alpha\int_0^\infty\frac{dT}{T}\int_0^{iT/2}dz_1\int_0^{z_1}dz_2\int_0^{z_2}dz_3\;\delta(z_3)\,\langle V_B^{(0)}(z_1)\,V_B^{(0)}(z_2)\,V_B^{(0)}(z_3)\rangle_\alpha \\
&\quad = g_s^3\int_0^\infty\frac{dT}{T}\int d\mu^{(3)}\left(\mathbb{E}^{(3)}+\mathbb{O}^{(3)}\right)
\end{aligned} \tag{3.1}$$

Momentum conservation for massless vector bosons implies $k_i\cdot k_j = 0$ i.e. collinear momenta. In order to proceed one could either relax momentum conservation [31, 33–35] yet with $(\sum_i k_i)^2 = 0$ or analytically continue to complex momenta [36]. In the spinor helicity formalism, reviewed in appendix A, one has $2k_i\cdot k_j = -\langle ij\rangle[ij]$ and there are two options: either $\langle ij\rangle \neq 0$, with $[ij] = 0$, convenient for MHV or $(---)$ helicity configurations or the other way around. In the even spin structures, we have two types of contributions from two and three bilinears. The term with three bilinears produces

$$\begin{aligned}
\mathbb{E}^{(3)}_{\text{3-bil}} &= -i\sum_\alpha c_\alpha\,\langle{:}k_1\cdot\psi_1\,a_1\cdot\psi_1{:}\,{:}k_2\cdot\psi_2\,a_2\cdot\psi_2{:}\,{:}k_3\cdot\psi_3\,a_3\cdot\psi_3{:}\rangle_\alpha \\
&= i\alpha'^3(f_1f_2f_3)\sum_\alpha c_\alpha S_\alpha(z_{12})S_\alpha(z_{23})S_\alpha(z_{13})\mathcal{Z}_\alpha^{\mathcal{N}}\Pi_3(z_i,k_i) \\
&= -i\alpha'^3(f_1f_2f_3)\omega_{123}\Pi_3(z_i,k_i)
\end{aligned} \tag{3.2}$$

JHEP01(2016)043

where

$$\omega_{123} = S_{12} + S_{23} + S_{31} \quad , \tag{3.3}$$

$\Pi_3(z_i, k_i) = \prod_{i<j} e^{-\alpha' k_i \cdot k_j \mathcal{G}_{ij}}$ is the Koba-Nielsen factor and in the last step we have used

$$S_\alpha(z_{12}) S_\alpha(z_{23}) = -\omega_{123} S_\alpha(z_{13}) - S'_\alpha(z_{13}) \quad 2 S_\alpha(z_{12}) S'_\alpha(z_{12}) = \mathcal{P}'_{12} \tag{3.4}$$

The terms with two bilinears produce

$$\begin{aligned} \mathbb{E}^{(3)}_{\text{2-bil}} &= -\sum_{\text{cyclic}} \sum_{\alpha} c_\alpha \, \langle : a_1 \cdot \partial X_1 : : k_2 \cdot \psi_2 \, a_2 \cdot \psi_2 : : k_3 \cdot \psi_3 \, a_3 \cdot \psi_3 : \rangle_\alpha \\ &= i \frac{\alpha'^3}{2} \mathcal{E}_{\mathcal{N}} \sum_{\text{cyclic}} a_3 \cdot P_3(f_1 f_2) \Pi_3(z_i, k_i) \end{aligned} \tag{3.5}$$

where $P_i^\mu = \sum_{j \neq i} k_j^\mu S_{ij}$, with $S_{ij} = -\partial_i \mathcal{G}_{ij}$.

In the odd spin structure we have similar contributions, from three bilinears we have three terms depending on the choice of the points where the two zero modes are absorbed[7]

$$\begin{aligned} \mathbb{O}^{(3)}_{\text{3-bil}} &= -i c_1^{\text{GSO}} \sum_{\text{swaps}} \langle : k_1 \cdot \psi_0 \, a_1 \cdot \psi_0 : : k_2 \cdot \psi_0 \, a_2 \cdot \psi : : k_3 \cdot \psi_0 \, a_3 \cdot \psi : \rangle \\ &= -i \alpha'^3 \mathcal{C}_{\mathcal{N}=1} \sum_{\text{cyclical}} (\tilde{f}_1 f_2 f_3) S_{23} \Pi_3(z_i, k_i) \end{aligned} \tag{3.6}$$

where the sum on exchanges means summing terms with ψ and ψ_0 exchanged in the same vertex. From two bilinears we have three terms

$$\begin{aligned} \mathbb{O}^{(3)}_{\text{2-bil}} &= -\sum_{\text{cyclical}} \langle : a_1 \cdot \partial X : : k_2 \cdot \psi_0 \, a_2 \cdot \psi_0 : : k_3 \cdot \psi_0 \, a_3 \cdot \psi_0 : \rangle \\ &= \frac{i}{2} \alpha'^3 \mathcal{C}_{\mathcal{N}=1} \sum_{\text{cyclical}} a_3 \cdot P_3(\tilde{f}_1 f_2) \Pi_3(z_i, k_i) \end{aligned} \tag{3.7}$$

Let us consider the two independent helicity configurations. First the case $(+++)$. We begin from even spin structures, we must compute the scalar products $a_i \cdot P_i$:

$$\begin{aligned} a_3^+ \cdot P_3(f_1^+ f_2^+) &= -\frac{[12]^2}{\sqrt{2}} \left(\frac{[31]\langle 1q \rangle}{\langle 3q \rangle} S_{31} + \frac{[32]\langle 2q \rangle}{\langle 3q \rangle} S_{32} \right) \\ &= (f_1^+ f_2^+ f_3^+) \left(\frac{[12]\langle 1q \rangle}{[23]\langle 3q \rangle} S_{31} - \frac{[12]\langle 2q \rangle}{[31]\langle 3q \rangle} S_{32} \right) \end{aligned} \tag{3.8}$$

Using momentum conservation $[12]\langle 1q \rangle = -[32]\langle 3q \rangle$ and $[12]\langle 2q \rangle = -[13]\langle 3q \rangle$, so that one has $a_3^+ \cdot P_3(f_1^+ f_2^+) = (f_1^+ f_2^+ f_3^+)(S_{31} + S_{23})$ and one can thus factor out $(f_1^+ f_2^+ f_3^+)$ and get

$$\mathbb{E}^{(3)} = \frac{i}{2} \alpha'^3 \mathcal{E}_{\mathcal{N}} (f_1^+ f_2^+ f_3^+) \sum_{\text{cyclic}} (S_{31} - S_{23}) \Pi_3(z_i, k_i) = 0 \tag{3.9}$$

[7] Absorbing all the zero modes at two points would give zero due to normal ordering at the remaining point.

JHEP01(2016)043

The contribution of the odd spin structure becomes proportional to the contribution of the even spin structure after using $\tilde{f} = if$, thus the complete amplitude vanishes for this choice of helicities, as well as for $(---)$,

$$\mathcal{A}_3^{\text{1-loop}}[1^+, 2^+, 3^+] = 0 = \mathcal{A}_3^{\text{1-loop}}[1^-, 2^-, 3^-] \tag{3.10}$$

as expected from SUSY Ward identities.[8] Indeed, barring anomalous U(1)'s, the only supersymmetric invariant for 2- and 3-points is $W^\alpha W_\alpha$ giving rise to the standard F^2 bosonic term, since F^3, though present in the bosonic string even at tree level, does not admit a SUSY completion.

Let us then consider the case $(-++)$. For this helicity configuration one has a single term. The even spin structures produce

$$\mathbb{E}^{(3)} = \frac{i}{2}\alpha'^3 \mathcal{E}_\mathcal{N} a_1^- \cdot P_1(f_2^+ f_3^+)\Pi_3(z_i, k_i) \tag{3.11}$$

It is convenient to compute $a_1^- \cdot P_1(f_2^+ f_3^+)$ factorizing the MHV amplitude. In $a_1^- \cdot k_i$ the factor $\langle 1i \rangle$ would give zero due to collinear kinematics. In order to circumvent these subtleties, one can analytically continue to complex momenta and choose $q_1 = k_3$. This yields the 'right' result:

$$a_1^- \cdot P_1(f_2^+ f_3^+) = -\frac{1}{\sqrt{2}}\frac{\langle 12\rangle[23]}{[13]}[23]^2 S_{12} = -\sqrt{2}\frac{[23]^3}{[12][31]} k_1 \cdot k_2 S_{12} \tag{3.12}$$

Using partial integration one can replace $k_1 \cdot k_2 S_{12}$ with $k_2 \cdot k_3 S_{23}$ to make it look more symmetric and finally find

$$\mathbb{E}^{(3)} = -\frac{i}{\sqrt{2}}\alpha'^3 \mathcal{E}_\mathcal{N} \frac{[23]^3}{[12][31]} k_2 \cdot k_3 S_{23} \Pi_3(z_i, k_i) \tag{3.13}$$

In the odd spin structure, only $\mathcal{N} = 1$ sectors contribute. Following similar steps, one finds a similar result with $\mathcal{E}_\mathcal{N}$ replaced by $-i\mathcal{C}_\mathcal{N}$. Now we can write the complete three-point amplitude

$$\mathcal{A}_3^{\text{1-loop}}[1^-, 2^+, 3^+] = -\frac{i}{\sqrt{2}} g_s^3 \alpha'^3 \frac{[23]^3}{[12][31]} \int_0^\infty \frac{dT}{T} \int d\mu^{(3)} \left(\mathcal{E}_\mathcal{N} - i\mathcal{C}_{\mathcal{N}=1}\right) k_2 \cdot k_3 S_{23} \Pi_3(z_i, k_i) \tag{3.14}$$

To compute planar, non-planar and un-oriented contributions one has to choose the specific modular parameter and the corresponding integration domain [26, 37–40]. The result is generically divergent for both $\mathcal{N} = 1, 2$ sectors, wherein it is related by gauge invariance to the 2-point amplitude, encoding β-functions and one-loop threshold corrections to the gauge kinetic functions. As already observed, it vanishes in $\mathcal{N} = 4$ sectors, in a way reminiscent of the no-triangle conjecture in $\mathcal{N} = 4$ SYM, i.e. the absence of one-loop massless amplitudes with three (bunches of) insertion points.

[8] We thank Nathan Berkovits for raising this issue.

JHEP01(2016)043

4 Four-point amplitudes

We are now ready to compute four-point amplitudes. We start by briefly reviewing and summarising the results of [1] and then analyse them in terms of helicity configurations, color orderings and limiting behaviours.

The starting point is

$$\begin{aligned}\mathcal{A}_4^{\text{1-loop}}[1,2,3,4] &= g_s^4 \sum_\alpha c_\alpha \int_0^\infty \frac{dT}{T} \int d\mu^{(4)} \, \langle V_0(1)V_0(2)V_0(3)V_0(4)\rangle_\alpha \\ &= g_s^4 \int_0^\infty \frac{dT}{T} \int d\mu^{(4)} \left(\mathbb{E}^{(4)} + \mathbb{O}^{(4)}\right)\end{aligned} \tag{4.1}$$

where the integration region $\mathcal{R}_{CP}$ and the Chan-Paton factor depend on the distributions of insertions on the two boundaries for the annulus (planar 4−0, non-planar 3−1 and 2−2). For the un-orientable case there is no choice, except for the relative ordering of the insertions. Here, we will only summarise the results, the details can be found in [1].

4.1 Even spin structures, four bilinears

$$\mathbb{E}^{(4)}_{\text{4-bil}} = \sum_\alpha c_\alpha \, \langle\langle :k_1{\cdot}\psi_1 \, a_1{\cdot}\psi_1 : \ldots : k_4{\cdot}\psi_4 \, a_4{\cdot}\psi_4 :\rangle\rangle_\alpha \, \mathcal{Z}_\alpha^{\mathcal{N}} \tag{4.2}$$

The fermionic contribution consists in two types of terms connected and disconnected. The result for connected contractions is

$$\mathbb{E}^{(4)}_{\text{4-bil,conn}} = \alpha'^4 \sum_{\text{conn}} (f_1 f_2 f_3 f_4) \left[\frac{1}{2}\mathcal{E}_{\mathcal{N}}(\mathcal{P}_{13} + \omega_{123}\omega_{341} + \mathcal{P}_{24} + \omega_{234}\omega_{412}) - \mathcal{F}_{\mathcal{N}}\right] \Pi_4(z_i, k_i) \tag{4.3}$$

where $\mathcal{E}_{\mathcal{N}}$ (vanishing for $\mathcal{N} = 4$) was defined previously in (2.5) and ω_{ijk} are defined in 3.3, while

$$\mathcal{F}_{\mathcal{N}} = \sum_{\alpha=2}^{4} c_\alpha e_{\alpha-1}^2 \mathcal{Z}_\alpha^{\mathcal{N}} \tag{4.4}$$

depends on the number $\mathcal{N}$ of preserved SUSY, on the world-sheet modular parameter T, on the parameters of the brane configuration coded in u^I_{ab} and the moduli of the "compactification". The disconnected contractions yield

$$\mathbb{E}^{(4)}_{\text{4-bil,disconn}} = \alpha'^4 \sum_{\text{disconn}} \frac{1}{4}(f_1 f_2)(f_3 f_4) \left[\mathcal{E}_{\mathcal{N}} \left(\mathcal{P}_{12} + \mathcal{P}_{34}\right) + \mathcal{F}_{\mathcal{N}}\right] \Pi_4(z_i, k_i) \tag{4.5}$$

4.2 Even spin structures, three bilinears

Aside from the bosonic contractions, the fermionic contractions are the same as for three-point amplitudes, discussed previously, thus one finds

$$\begin{aligned}\mathbb{E}^{(4)}_{\text{3-bil}} &= -i \sum_{\text{cyclic}} \sum_\alpha c_\alpha \, \langle\langle :a_1{\cdot}\partial X_1 : : k_2{\cdot}\psi_2 \, a_2{\cdot}\psi_2 \, k_2{\cdot}\psi_2 : : a_3{\cdot}\psi_3 \, k_3{\cdot}\psi_3 : : a_4{\cdot}\psi_4 :\rangle\rangle_\alpha \, \Pi_4(z_i, k_i) \mathcal{Z}_\alpha^{\mathcal{N}} \\ &= -\alpha'^4 \sum_{\text{cyclic}} a_1 P_1 (f_1 f_2 f_3) \omega_{123} \mathcal{E}_{\mathcal{N}} \Pi_4(z_i, k_i)\end{aligned} \tag{4.6}$$

JHEP01(2016)043

4.3 Even spin structures, two bilinears

We already computed the fermionic contractions, thus the contribution to the amplitude is

$$\begin{aligned}\mathbb{E}^{(4)}_{\text{2-bil}} &= -\sum_{\text{pairs}} \langle\langle :a_1\cdot\partial X_1: :a_2\cdot\partial X_2:\rangle\rangle \, \langle\langle :k_3\cdot\psi_3\, a_3\cdot\psi_3: :k_4\cdot\psi_4\, a_4\cdot\psi_4:\rangle\rangle_\alpha \, \mathcal{Z}^{\mathcal{N}}_\alpha \\ &= -\frac{\alpha'^3}{2}\mathcal{E}_{\mathcal{N}} \sum_{\text{pairs}} (f_3 f_4) \left(a_1\cdot a_2 \partial_1\partial_2\mathcal{G}_{12} - \alpha' a_1\cdot P_1 a_2\cdot P_2\right)\Pi_4(z_i,k_i)\end{aligned} \tag{4.7}$$

Notice that each term is gauge invariant *per se* up to total derivatives. For instance, replacing a_1 (or a_2) with the momentum k_1 and noting that $\partial_{z_i}\Pi_4 = -\alpha' k_i\cdot P_i \Pi_4$ (with $\Pi_4 = \Pi_4(z_i,k_i)$ for brevity) and $\partial_1(a_2\cdot k_1 \partial_2\mathcal{G}_{21}) = -\partial_1(a_2\cdot P_2)$, the bosonic contractions in $\mathbb{E}^{(4)}_{\text{2-bil}}$ can be rewritten as a total derivative that vanishes upon integration

$$\left(k_1\cdot a_2 \partial_1\partial_2\mathcal{G}_{12} - \alpha' k_1\cdot P_1 a_2\cdot P_2\right)\Pi_4 = \partial_1(a_2\cdot P_2)\Pi_4 + \partial_1\Pi_4 a_2\cdot P_2 = \partial_1(a_2\cdot P_2\Pi_4) \tag{4.8}$$

4.4 Odd spin structure, four bilinears

Four fermionic bilinears allow three types of contractions.

First one can absorb the four zero modes at two points (for example z_1 and z_2):

$$\begin{aligned}\mathbb{O}^{(4)}_{\text{4-bil,2}} &= c_1^{\text{GSO}} \sum_{\text{pairs}} \langle :k_1\cdot\psi_0\, a_1\cdot\psi_0: :k_2\cdot\psi_0\, a_2\cdot\psi_0: :k_3\cdot\psi\, a_3\cdot\psi: :k_4\cdot\psi\, a_4\cdot\psi:\rangle \\ &= \frac{1}{4}\alpha'^4 \mathcal{C}_{\mathcal{N}=1} \sum_{\text{pairs}} (\tilde{f}_1 f_2)(f_3 f_4) S_{34}^2 \Pi_4(z_i,k_i)\end{aligned} \tag{4.9}$$

Second, one can absorb two zero modes in a point and the others in two separate points. There are twelve ways to do this:

$$\begin{aligned}\mathbb{O}^{(4)}_{\text{4-bil,1}} &= c_1^{\text{GSO}} \sum_{\text{cyclic}} \sum_{\text{subcyclic}} \sum_{\text{swaps}} \langle :k_1\cdot\psi_0\, a_1\cdot\psi_0: :k_2\cdot\psi_0\, a_2\cdot\psi: :k_3\cdot\psi_0\, a_3\cdot\psi: :k_4\cdot\psi\, a_4\cdot\psi:\rangle \\ &= \alpha'^4 \mathcal{C}_{\mathcal{N}=1} \sum_{\text{cyclical}} \sum_{\text{conn}} (\tilde{f}_1 f_2 f_4 f_3) S_{24} S_{43} \Pi_4(z_i,k_i)\end{aligned} \tag{4.10}$$

Third, one can absorb the zero modes in four different points:

$$\begin{aligned}\mathbb{O}^{(4)}_{\text{4-bil,0}} &= c_1^{\text{GSO}} \sum_{\text{swaps}} \langle :k_1\cdot\psi_0\, a_1\cdot\psi: :k_2\cdot\psi_0\, a_2\cdot\psi: :k_3\cdot\psi_0\, a_3\cdot\psi: :k_4\cdot\psi_0\, a_4\cdot\psi:\rangle \\ &= -2\alpha'^4 \mathcal{C}_{\mathcal{N}=1} \sum_{\text{disconn}} \epsilon_{\mu_1\mu_2\mu_3\mu_4} (f_1 f_2)^{\mu_1\mu_2} (f_3 f_4)^{\mu_3\mu_4} S_{12} S_{34} \Pi_4(z_i,k_i)\end{aligned} \tag{4.11}$$

4.5 Odd spin structure, three bilinears

With three bilinears, one has three ways to absorb zero modes and the contractions yield

$$\begin{aligned}\mathbb{O}^{(4)}_{\text{3-bil}} &= -\, i c_1^{\text{GSO}} \sum_{\text{cyclical}} \sum_{\text{swaps}} \langle :a_1\cdot\partial X: :k_2\cdot\psi_0\, a_2\cdot\psi_0: :k_3\cdot\psi_0\, a_3\cdot\psi: :k_4\cdot\psi_0\, a_4\cdot\psi:\rangle \\ &= -\, \alpha'^4 \mathcal{C}_{\mathcal{N}=1} \sum_{\text{cyclical}} a_1\cdot P_1 \sum_{i=2}^{4} (\tilde{f}_i f_{i+1} f_{i+2}) S_1(z_{i+1\, i+2}) \Pi_4(z_i,k_i)\end{aligned} \tag{4.12}$$

JHEP01(2016)043

4.6 Odd spin structure, two bilinears

With two fermionic bilinears, there are six ways to absorb the four fermionic zero-modes

$$\begin{aligned}\mathbb{O}^{(4)}_{\text{2-bil}} &= -c_1^{\text{GSO}} \sum_{\text{pairs}} \langle :a_1{\cdot}\partial X_1: :a_2{\cdot}\partial X_2: :k_3{\cdot}\psi_0\, a_3{\cdot}\psi_0: :k_4{\cdot}\psi_0\, a_4{\cdot}\psi_0:\rangle \\ &= -\frac{1}{2}\alpha'^3 \mathcal{C}_{\mathcal{N}=1} \sum_{\text{pairs}} \left(a_1{\cdot}a_2 \partial_1\partial_2 \mathcal{G}_{12} - \alpha' a_1{\cdot}P_1 a_2{\cdot}P_2\right) (\tilde{f}_3 f_4) \Pi_4(z_i,k_i)\end{aligned} \tag{4.13}$$

5 Simplifying 4-pt amplitudes

Let us now simplify the above results and show that non MHV amplitudes vanish. We will also identify the regions of the integration domain that generically expose singularities and later on discuss which (tadpole) conditions the brane configurations must satisfy in order to cancel or mitigate the singular behaviours.

After analyzing the symmetry properties of the integration variables, that allow to manipulate the integrands and reduce the number of independent contributions, we will study the three independent helicity configurations and check that $\mathcal{A}(++++) = 0$ and $\mathcal{A}(-+++) = 0$.

The $\mathcal{F}$-term is proportional to F^4 thus reproduces the MHV structure, so we will focus on the $\mathcal{E}$-term.

5.1 $\mathcal{A}_4^{\text{1-loop}}[1^+,2^+,3^+,4^+] = 0$

This case is the most laborious because none of the traces over the Lorentz indices of the f_i vanish.

The six terms arising from contractions of two bilinears are separately gauge invariant, thus we can always choose $q_i = q_j = q$ and get $a_i{\cdot}a_j = 0$ and fix q so as to make some other product between momenta and polarizations vanish. For example one can compute $a_1{\cdot}P_1 a_2{\cdot}P_2$ with the choice $q_1 = q_2 = k_4$ and get

$$a_1^+{\cdot}P_1 a_2^+{\cdot}P_2 (f_3^+ f_4^+) = -\frac{1}{2}[12]^2 (S_{12}-S_{13})(S_{21}-S_{23})(f_3^+ f_4^+) = -\frac{1}{2}[12]^2[34]^2 \left(S_{12}^2 + \Omega_{123}\right) \tag{5.1}$$

For brevity we define

$$\Omega_{123} = S_{12}S_{23} + S_{23}S_{31} + S_{31}S_{12} \tag{5.2}$$

Collecting the various Lorentz invariant structures yields

$$\mathbb{E}^{(4)}_{\text{2-bil}} = -\frac{1}{4}\mathcal{E}_{\mathcal{N}} \sum_{\text{disconn}} (f_1^+ f_2^+)(f_3^+ f_4^+) \left(S_{12}^2 + S_{34}^2 + \Omega_{123} + \Omega_{134}\right) \tag{5.3}$$

Using Schouten's identity traces with two and four f's can be related

$$\begin{aligned}(f_1^+ f_2^+)(f_3^+ f_4^+) &= [12]^2[34]^2 \\ &= [12][23][34][41] + [12][24][43][31] \\ &= 2(f_1^+ f_2^+ f_3^+ f_4^+) + 2(f_1^+ f_2^+ f_4^+ f_3^+)\end{aligned} \tag{5.4}$$

JHEP01(2016)043

One can easily obtain two more similar formulae permuting the external legs. These formulae can be inverted to give

$$(f_1^+ f_2^+ f_3^+ f_4^+) = \frac{1}{4}\left[(f_1^+ f_2^+)(f_3^+ f_4^+) - (f_1^+ f_3^+)(f_2^+ f_4^+) + (f_1^+ f_4^+)(f_2^+ f_3^+)\right] \tag{5.5}$$

We rewrite all the traces in terms of single traces of four f's and obtain

$$\mathbb{E}^{(4)}_{\text{2-bil}} = -\frac{1}{2}\mathcal{E}_{\mathcal{N}} \sum_{\text{conn}} (f_1^+ f_2^+ f_3^+ f_4^+) \left(S_{12}^2 + S_{34}^2 + S_{14}^2 + S_{23}^2 + 2\Omega_{123} + 2\Omega_{134}\right) \Pi_4(z_i, k_i) \tag{5.6}$$

The three-bilinear term can be simplified using a cyclic gauge choice, for example $q_i = k_{i+2}$, and momentum conservation $a_i^+ \cdot P_i = a_i \cdot k_{i+1}(S_{i\,i+1} + S_{i+3\,i})$. With this choice all the kinematic factors become equal $a_i^+ \cdot k_{i+1}(f_{i+1}^+ f_{i+2}^+ f_{i+3}^+) = [12][23][34][41]/2 = (f_1^+ f_2^+ f_3^+ f_4^+)$. Thus one finds

$$\mathbb{E}^{(4)}_{\text{3-bil}} = -(f_1^+ f_2^+ f_3^+ f_4^+)\mathcal{E}_{\mathcal{N}} \sum_i \left(S_{i\,i+1} + S_{i+3\,i}\right) \omega_{i+1\,i+2\,i+3} \Pi_4(z_i, k_i) \tag{5.7}$$

Expanding the sum we obtain

$$\begin{aligned}\mathbb{E}^{(4)}_{\text{3-bil}} = -(f_1^+ f_2^+ f_3^+ f_4^+)\mathcal{E}_{\mathcal{N}} \Big[& 4\left(S_{12}S_{34} + S_{23}S_{41}\right) \\ & + \sum_{\text{cyclic}} \left(2S_{41}S_{12} - S_{12}S_{24} - S_{24}S_{41}\right) \Big] \Pi_4(z_i, k_i)\end{aligned} \tag{5.8}$$

The terms $S_{12}S_{34} + S_{23}S_{41}$ can be rewritten in four ways using partial integrations, for example choosing the tern (412):

$$S_{12}S_{34} + S_{23}S_{41} = -2S_{41}S_{12} - \frac{u}{s}\left(S_{41}S_{12} + S_{12}S_{24}\right) - \frac{u}{t}\left(S_{41}S_{12} + S_{24}S_{41}\right) \tag{5.9}$$

To obtain the other three it's enough to perform a cyclic permutation on the indices (412). Replacing last equation in $\mathbb{E}^{(4)}_{\text{3-bil}}$ one gets

$$\begin{aligned}\mathbb{E}^{(4)}_{\text{3-bil}} = (f_1^+ f_2^+ f_3^+ f_4^+)\mathcal{E}_{\mathcal{N}} \sum_{\text{cyclic}} \Big[& \left(S_{12}S_{24} + S_{24}S_{41}\right) \\ & + \frac{u}{s}\left(S_{41}S_{12} + S_{12}S_{24}\right) + \frac{u}{t}\left(S_{41}S_{12} + S_{24}S_{41}\right) \Big] \Pi_4(z_i, k_i)\end{aligned} \tag{5.10}$$

The ratios of s, t, u can be used to transform the traces of f's into one another according to

$$u(f_1^+ f_2^+ f_3^+ f_4^+) = t(f_1^+ f_2^+ f_4^+ f_3^+) = s(f_1^+ f_3^+ f_2^+ f_4^+) \tag{5.11}$$

that can be easily proved using the helicity formalism and momentum conservation. Thus one gets

$$\begin{aligned}(f_1^+ f_2^+ f_3^+ f_4^+) & \left(S_{24}(S_{12} + S_{41}) + \frac{u}{t} S_{41}(S_{12} + S_{24}) + \frac{u}{s} S_{12}(S_{41} + S_{24}) \right) \\ & = (f_1^+ f_2^+ f_3^+ f_4^+) S_{24}(S_{12} + S_{41}) + (f_1^+ f_2^+ f_4^+ f_3^+) S_{41}(S_{12} + S_{24}) \\ & \qquad + (f_1^+ f_3^+ f_2^+ f_4^+) S_{12}(S_{41} + S_{24})\end{aligned} \tag{5.12}$$

JHEP01(2016)043

that can be rewritten as a "connected" sum

$$\mathbb{E}^{(4)}_{\text{3-bil}} = \alpha'^4 \mathcal{E}_{\mathcal{N}} \sum_{\text{conn}} (f_1^+ f_2^+ f_3^+ f_4^+) \sum_{\text{cyclic}} (S_{12}S_{24} + S_{24}S_{41})\, \Pi_4(z_i, k_i) \tag{5.13}$$

It remains to simplify terms arising with four bilinears. We peruse (5.4) to rewrite disconnected terms in terms of traces of four f's and use the identity $\mathcal{P}_{24} + \omega_{234}\omega_{413} = \mathcal{P}_{13} + \omega_{123}\omega_{341}$ to get

$$\mathbb{E}^{(4)}_{\text{4-bil},\mathcal{E}} = \sum_{\text{conn}} (f_1^+ f_2^+ f_3^+ f_4^+) \left[\frac{1}{2}(\mathcal{P}_{12} + \mathcal{P}_{34} + \mathcal{P}_{14} + \mathcal{P}_{23}) + \mathcal{P}_{13} + \omega_{123}\omega_{341}\right] \Pi_4(z_i, k_i) \tag{5.14}$$

Expanding the products $\omega_{123}\omega_{341}$ and using partial integration one gets

$$\begin{aligned}\omega_{123}\omega_{341} &= -S_{13}^2 + S_{12}S_{34} + S_{23}S_{41} - (S_{21}S_{13} + S_{13}S_{32}) - (S_{13}S_{34} + S_{41}S_{13}) + S_{41}S_{12} + S_{23}S_{34} \\ &= -S_{13}^2 - (S_{21}S_{13} + S_{13}S_{32}) - (S_{13}S_{34} + S_{41}S_{13}) \\ &\quad - \frac{u}{s}(S_{12}S_{24} + S_{41}S_{12}) - \frac{u}{t}(S_{42}S_{23} + S_{23}S_{34})\end{aligned} \tag{5.15}$$

Recalling equation (5.4), the ratios u/t and u/s produce further mixing among the various Lorentz invariant structures:

$$\begin{aligned}\mathbb{E}^{(4)}_{\text{4-bil},\mathcal{E}} = \sum_{\text{conn}} (f_1^+ f_2^+ f_3^+ f_4^+) \Bigg[&\frac{1}{2}(\mathcal{P}_{12} + \mathcal{P}_{34} + \mathcal{P}_{14} + \mathcal{P}_{23}) \\ &- 2\mathcal{Y}_{13} - \sum_{\text{cyclic}} (S_{12}S_{24} + S_{24}S_{41})\Bigg] \Pi_4(z_i, k_i)\end{aligned} \tag{5.16}$$

where

$$\mathcal{Y}(z_{ij}) = -2\left[\mathcal{P}(z_{ij}) - S^2(z_{ij})\right] \tag{5.17}$$

Assembling all the terms and rewriting $\mathcal{P} - S^2$ in terms $\mathcal{Y}$,[9] one finally gets

$$\mathbb{E}^{(4)} = -\mathcal{E}_{\mathcal{N}} \sum_{\text{conn}} (f_1^+ f_2^+ f_3^+ f_4^+) \left[\mathcal{Y}_{12} + \mathcal{Y}_{34} + \mathcal{Y}_{14} + \mathcal{Y}_{23} + 2\mathcal{Y}_{13} + \Omega_{123} + \Omega_{134}\right] \Pi_4(z_i, k_i) \tag{5.18}$$

This expression vanishes using a generalized version of Fay 'trisecant' identity [42]:

$$\Omega_{123} = S_{12}S_{23} + S_{23}S_{31} + S_{31}S_{12} = -\mathcal{Y}_{12} - \mathcal{Y}_{23} - \mathcal{Y}_{31} \tag{5.19}$$

Let us now verify that the amplitude vanishes in the odd spin structure too. As usual only $\mathcal{N} = 1$ sectors contribute. The two and three bilinears terms are the same as the sum over even spin structures, up to a by-now familiar constant:

$$\mathbb{O}^{(4)}_{\text{2-bil}} = -\frac{\alpha'^4}{2}(i\mathcal{C}_{\mathcal{N}=1}) \sum_{\text{conn}} (f_1^+ f_2^+ f_3^+ f_4^+) \left(S_{12}^2 + S_{34}^2 + S_{14}^2 + S_{23}^2 + 2\Omega_{123} + 2\Omega_{134}\right) \Pi_4(z_i, k_i) \tag{5.20}$$

$$\mathbb{O}^{(4)}_{\text{3-bil}} = \alpha'^4 (i\mathcal{C}_{\mathcal{N}=1}) \sum_{\text{conn}} (f_1^+ f_2^+ f_3^+ f_4^+) \sum_{\text{cyclic}} (S_{12}S_{24} + S_{24}S_{41})\, \Pi_4(z_i, k_i) \tag{5.21}$$

[9] See [41] for more details on relations between elliptic functions and string one-loop amplitudes.

JHEP01(2016)043

Let us then consider the terms with four bilinears. The term $\mathbb{O}^{(4)}_{\text{4-bil},2}$ is similar to the disconnected part of four bilinears in the even spin structures:

$$\mathbb{O}^{(4)}_{\text{4-bil},2} = \frac{\alpha'^4}{2}(i\mathcal{C}_{\mathcal{N}=1}) \sum_{\text{conn}} (f_1^+ f_2^+ f_3^+ f_4^+) \left(S_{12}^2 + S_{34}^2 + S_{14}^2 + S_{23}^2\right) \Pi_4(z_i, k_i) \tag{5.22}$$

The term $\mathbb{O}^{(4)}_{\text{4-bil},1}$ yields

$$\begin{aligned} \mathbb{O}^{(4)}_{\text{4-bil},1} &= \alpha'^4 (i\mathcal{C}_{\mathcal{N}=1}) \sum_{\text{cyclical}} \sum_{\text{conn}} (f_1^+ f_2^+ f_4^+ f_3^+) S_{34} S_{42} \Pi_4(z_i, k_i) \\ &= \alpha'^4 (i\mathcal{C}_{\mathcal{N}=1}) \sum_{\text{cyclical}} \sum_{\text{conn}} (f_1^+ f_2^+ f_3^+ f_4^+) S_{21} S_{14} \Pi_4(z_i, k_i) \end{aligned} \tag{5.23}$$

The more problematic term is $\mathbb{O}^{(4)}_{\text{4-bil},0}$ but it can be computed using the decomposition of f in definite helicity parts, detailed in appendix A,

$$\begin{aligned} \mathbb{O}^{(4)}_{\text{4-bil},0} &= -2\alpha'^4 \mathcal{C}_{\mathcal{N}=1} \sum_{\text{disconn}} \epsilon_{\mu_1\mu_2\mu_3\mu_4} (f_1^+ f_2^+)^{\mu_1\mu_2} (f_3^+ f_4^+)^{\mu_3\mu_4} S_{12} S_{34} \\ &= -\frac{\alpha'^4}{2} (i\mathcal{C}_{\mathcal{N}=1}) \sum_{\text{disconn}} (f_1^+ f_2^+)(f_3^+ f_4^+) \left(S_{13} S_{24} - S_{41} S_{23}\right) \end{aligned} \tag{5.24}$$

Using equation (5.4) the expression becomes

$$\mathbb{O}^{(4)}_{\text{4-bil},0} = \alpha'^4 (i\mathcal{C}_{\mathcal{N}=1}) \sum_{\text{conn}} (f_1^+ f_2^+ f_3^+ f_4^+) \left(S_{12} S_{34} + S_{41} S_{23}\right) \tag{5.25}$$

Singularly the terms $S_{12}S_{34}$ and $S_{41}S_{23}$ can be transformed in $-\Omega_{234}$ and $-\Omega_{214}$ using partial integration and mixing of Lorentz invariant but reducible structures:

$$\begin{aligned} \sum_{\text{conn}} (f_1^+ f_2^+ f_3^+ f_4^+) S_{12} S_{34} &= -\sum_{\text{conn}} (f_1^+ f_2^+ f_3^+ f_4^+) \left[S_{23} S_{34} + \frac{u}{s} \left(S_{23} S_{34} + S_{34} S_{42}\right)\right] \\ &= -\sum_{\text{conn}} (f_1^+ f_2^+ f_3^+ f_4^+) \left[S_{23} S_{34} + \frac{u}{s} \left(\Omega_{234} - S_{42} S_{23}\right)\right] \\ &= -\sum_{\text{conn}} (f_1^+ f_2^+ f_3^+ f_4^+) \left[S_{23} S_{34} + \Omega_{234} - S_{43} S_{32}\right] \\ &= -\sum_{\text{conn}} (f_1^+ f_2^+ f_3^+ f_4^+) \Omega_{234} \end{aligned} \tag{5.26}$$

Similarly for $S_{41}S_{23}$. The final result is

$$\mathbb{O}^{(4)}_{\text{4-bil},0} = -\alpha'^4 (i\mathcal{C}_{\mathcal{N}=1}) \sum_{\text{conn}} (f_1^+ f_2^+ f_3^+ f_4^+) \left(\Omega_{234} + \Omega_{214}\right) \tag{5.27}$$

Summing all the terms, the contribution of the odd spin structure vanishes too. Contrary to two or three point amplitudes, contributions from even and odd spin structure are not simply proportional. Comparing even and odd spin structures $\mathbb{O}^{(4)}_{\text{2-bil}}$, $\mathbb{O}^{(4)}_{\text{3-bil}}$ and $\mathbb{O}^{(4)}_{\text{4-bil},2}$ are substantially equal to the even ones. $\mathbb{O}^{(4)}_{\text{4-bil},0}$ reproduces the same terms as in $\omega\omega$. We note that S^2 takes the place of Weierstrass function thus all the structures $\mathcal{P} - S^2$ vanish. $\mathbb{O}^{(4)}_{\text{4-bil},1}$ has no counterpart in even sector.

JHEP01(2016)043

5.2 $\mathcal{A}_4^{\text{1-loop}}[1^-,2^+,3^+,4^+]=0$

In this case, all the traces with f_1^- are zero

$$(f_1^- f_i^+)=0 \quad (f_1^- f_i^+ f_j^+)=0 \quad (f_1^- f_i^+ f_j^+ f_k^+)=0 \tag{5.28}$$

As a result the $(ffff)$ terms and the four bilinears terms vanish. Only one term from three bilinears survives and three terms from two bilinears:

$$\begin{aligned}\mathbb{E}^{(4)} = -\alpha'^4 a_1^-\cdot P_1 \mathcal{E}_{\mathcal{N}}\Big[&(f_2^+ f_3^+ f_4^+)\omega_{234} - \frac{1}{2}a_2^+\cdot P_2(f_3^+ f_4^+) \\ &-\frac{1}{2}a_3^+\cdot P_3(f_2^+ f_4^+) - \frac{1}{2}a_4^+\cdot P_4(f_2^+ f_3^+)\Big]\Pi_4(z_i,k_i)\end{aligned} \tag{5.29}$$

We can compute the terms with two bilinears using the gauge choice $q_2=q_3=q_4=k_1$ and $q_1=k_3$ and obtain:

$$a_2^+\cdot P_2 = a_2^+\cdot k_3(S_{23}+S_{42}) \quad , \quad a_3^+\cdot P_3 = a_3^+\cdot k_4(S_{34}+S_{23}) \quad , \quad a_4^+\cdot P_4 = a_4^+\cdot k_2(S_{42}+S_{34}) \tag{5.30}$$

As $a_2^+\cdot k_3(f_3^+ f_4^+)$, $a_3^+\cdot k_4(f_2^+ f_4^+)$ and $a_4^+\cdot k_2(f_2^+ f_3^+)$ are all equal to $(f_2^+ f_3^+ f_4^+)$, we can factor this out and the sum vanishes

$$\begin{aligned}\mathbb{E}^{(4)} = -\,\alpha'^4 \mathcal{E}_{\mathcal{N}} a_1^-\cdot P_1(f_2^+ f_3^+ f_4^+)\Big[&S_{23}+S_{34}+S_{42} \\ &-\frac{1}{2}\left(S_{23}+S_{42}+S_{34}+S_{23}+S_{42}+S_{34}\right)\Big]\Pi_4(z_i,k_i)=0\end{aligned} \tag{5.31}$$

In the odd spin structure $\mathbb{O}^{(4)}_{\text{2-bil}}$ and $\mathbb{O}^{(4)}_{\text{3-bil}}$ are substantially equal to their even counterparts, in fact by direct computation we obtain the same result with the usual replacement of $\mathcal{E}_{\mathcal{N}}$ with $i\mathcal{C}_{\mathcal{N}=1}$, thus the odd spin structure contribution vanishes too. So we conclude that the whole result is zero.

5.3 $\mathcal{A}_4^{\text{1-loop}}[1^-,2^-,3^+,4^+]\neq 0$

In this case, terms with traces of three f's vanish. The only non-vanishing irreducible traces are $(f_1^- f_2^-)=-\langle 12\rangle^2$, $(f_3^+ f_4^+)=-[34]^2$ and

$$(f_1^- f_2^- f_3^+ f_4^+)=(f_1^- f_3^+ f_2^- f_4^+)=\frac{1}{4}\langle 12\rangle^2[34]^2=-F^4_{--++} \tag{5.32}$$

thanks to $f_i^+ f_j^- = f_j^- f_i^+$. For brevity we use F^4 instead of F^4_{--++}. For terms with two bilinears extra simplifications take place with the gauge choices $q_1=q_2=k_3,k_4$ and $q_3=q_4=k_1,k_2$, while terms from contractions of three bilinears vanish in this case. The $\mathcal{F}$-term is non zero

$$\mathbb{E}^{(4)}_{\mathcal{F}}=\frac{1}{2}F^4\Pi_4(z_i,k_i) \tag{5.33}$$

Contributions derived from two bilinear contractions can be reduced to two terms:

$$\mathbb{E}^{(4)}_{\text{2-bil}}=\frac{\mathcal{E}_{\mathcal{N}}}{2}(f_3^+ f_4^+)a_1^-\cdot P_1 a_2^-\cdot P_2\Pi_4(z_i,k_i)+\frac{\mathcal{E}_{\mathcal{N}}}{2}(f_1^- f_2^-)a_3^+\cdot P_3 a_4^+\cdot P_4\Pi_4(z_i,k_i) \tag{5.34}$$

JHEP01(2016)043

Using the gauge choices $q_1 = q_2 = k_4$, one finds $a_1^- \cdot P_1 a_2^- \cdot P_2 = -1/2(f_1^- f_2^-)(S_{12}^2 + \Omega_{123})$. If we choose $q_1 = q_2 = k_3$ we obtain $a_1^- \cdot P_1 a_2^- \cdot P_2 = -1/2(f_1^- f_2^-)(S_{12}^2 + \Omega_{412})$. Symmetrizing $a_1^- \cdot P_1 a_2^- \cdot P_2$ we have

$$a_1^- \cdot P_1 a_2^- \cdot P_2 = -\frac{1}{4}(f_1^- f_2^-)\left(2S_{12}^2 + \Omega_{123} + \Omega_{412}\right) \tag{5.35}$$

and a similar one for $a_3^+ \cdot P_3 a_4^+ \cdot P_4$. Summing these two terms yields

$$\mathbb{E}^{(4)}_{\text{2-bil}} = \frac{1}{16}\mathcal{E}_{\mathcal{N}} F^4 \left(2S_{12}^2 + 2S_{34}^2 + \Omega_{123} + \Omega_{234} + \Omega_{341} + \Omega_{412}\right) \Pi_4(z_i, k_i) \tag{5.36}$$

The $\mathcal{E}$-terms of four bilinear produce

$$\begin{aligned}\mathbb{E}^{(4)}_{\text{4-bil},\mathcal{E}} = -\frac{\alpha'^4}{8} F^4 \Big[&2\mathcal{P}_{12} + 2\mathcal{P}_{34} + \mathcal{P}_{13} + \mathcal{P}_{24} + \mathcal{P}_{14} + \mathcal{P}_{23} \\ &+ \omega_{123}\omega_{341} + \omega_{124}\omega_{431} + \omega_{132}\omega_{241} + \omega_{234}\omega_{412} + \omega_{243}\omega_{312} + \omega_{324}\omega_{413}\Big] \Pi_4(z_i, k_i)\end{aligned} \tag{5.37}$$

Expanding and collecting the sums of ω's

$$\begin{aligned}&\omega_{123}\omega_{341} + \omega_{124}\omega_{431} + \omega_{132}\omega_{241} + \omega_{234}\omega_{412} + \omega_{243}\omega_{312} + \omega_{324}\omega_{413} \\ &\quad = -S_{12}^2 - S_{34}^2 - S_{13}^2 - S_{24}^2 - S_{14}^2 - S_{23}^2 - \Omega_{123} - \Omega_{234} - \Omega_{341} - \Omega_{412}\end{aligned} \tag{5.38}$$

Using formula (5.19) and summing all the contributions we have

$$\mathbb{E}^{(4)} = \frac{\alpha'^4}{8} F^4 \left[4\mathcal{F}_{\mathcal{N}} + \mathcal{E}_{\mathcal{N}} \left(\mathcal{Y}_{12} + \mathcal{Y}_{34} - \mathcal{Y}_{13} - \mathcal{Y}_{24} - \mathcal{Y}_{14} - \mathcal{Y}_{23}\right)\right] \Pi_4(z_i, k_i) \tag{5.39}$$

The contribution of the odd spin structure admits the same simplifications. The two bilinears contribution is similar to the even one except for a minus sign due to $\tilde{f}$:

$$\mathbb{O}^{(4)}_{\text{2-bil}} = \frac{\alpha'^4}{16}(i\mathcal{C}_{\mathcal{N}=1}) F^4 \left(-2S_{12}^2 + 2S_{34}^2 - \Omega_{123} - \Omega_{412} + \Omega_{143} + \Omega_{234}\right) \Pi_4(z_i, k_i) \tag{5.40}$$

As in the $(++++)$ case, $\mathbb{O}^{(4)}_{\text{4-bil},2}$ cancels the S^2 in $\mathbb{O}^{(4)}_{\text{2-bil}}$

$$\mathbb{O}^{(4)}_{\text{4-bil},2} = -\frac{\alpha'^4}{8}(i\mathcal{C}_{\mathcal{N}=1}) F^4 \left(-S_{12}^2 + S_{34}^2\right) \Pi_4(z_i, k_i) \tag{5.41}$$

Terms with one bilinear yield

$$\begin{aligned}\mathbb{O}^{(4)}_{\text{4-bil},1} &= \alpha'^4 \mathcal{C}_{\mathcal{N}=1} \sum_{\text{cyclical}} \sum_{\text{conn}} (\tilde{f}_1^- f_2^- f_4^+ f_3^+) S_{24} S_{43} \Pi_4(z_i, k_i) \\ &= \alpha'^4 \mathcal{C}_{\mathcal{N}=1} \sum_{\text{cyclical}} (\tilde{f}_1^- f_2^- f_3^+ f_4^+) \Omega_{234} \Pi_4(z_i, k_i) \\ &= -\frac{\alpha'^4}{4}(i\mathcal{C}_{\mathcal{N}=1}) F^4 \left(-\Omega_{234} - \Omega_{341} + \Omega_{412} + \Omega_{123}\right) \Pi_4(z_i, k_i)\end{aligned}$$

JHEP01(2016)043

Terms with no bilinears yield

$$\begin{aligned}\mathbb{O}^{(4)}_{\text{4-bil,0}} &= -2\alpha'^4\mathcal{C}_{\mathcal{N}}\sum_{\text{disconn}}\epsilon_{\mu_1\mu_2\mu_3\mu_4}(f_1^+f_2^+)^{\mu_1\mu_2}(f_3^+f_4^+)^{\mu_3\mu_4}S_{12}S_{34}\Pi_4(z_i,k_i)\\ &= -2\alpha'^4\mathcal{C}_{\mathcal{N}}\epsilon_{\mu_1\mu_2\mu_3\mu_4}(f_1^-)_{i'}(f_2^-)_{j'}(f_3^+)_i(f_4^+)_j\\ &\quad\times\Big[(\Sigma^{i'}\Sigma^{j'})^{\mu_1\mu_2}(\bar\Sigma^i\bar\Sigma^j)^{\mu_3\mu_4}S_{12}S_{34}+(\Sigma^{i'}\bar\Sigma^i)^{\mu_1\mu_3}(\Sigma^{j'}\bar\Sigma^j)^{\mu_2\mu_4}S_{13}S_{24}\\ &\quad+(\Sigma^{i'}\bar\Sigma^j)^{\mu_1\mu_4}(\bar\Sigma^i\Sigma^{j'})^{\mu_3\mu_2}S_{14}S_{32}\Big]\Pi_4(z_i,k_i)\end{aligned}\tag{5.42}$$

The matrices $(\Sigma^{i'}\bar\Sigma^i)^{\mu\nu}$ are symmetric in μ, ν thus contractions with the Levi-Civita tensor give zero. The remaining term vanishes too

$$\epsilon_{\mu_1\mu_2\mu_3\mu_4}(\Sigma^{i'}\Sigma^{j'})^{\mu_1\mu_2}(\bar\Sigma^i\bar\Sigma^j)^{\mu_3\mu_4} = -i\epsilon^{i'j'}{}_{k'}\epsilon^{ij}{}_k\,\mathrm{Tr}(\Sigma^{k'}\bar\Sigma^k) = 0\tag{5.43}$$

Summing all the terms in the odd spin structure and using the generalized Fay identity, we have

$$\mathbb{O}^{(4)} = \alpha'^4(i\mathcal{C}_{\mathcal{N}=1})F^4\frac{5}{8}\left(\mathcal{Y}_{12}-\mathcal{Y}_{34}\right)\Pi_4(z_i,k_i)\tag{5.44}$$

Finally

$$\begin{aligned}\mathcal{A}_4^{\text{1-loop}}[1^-,2^-,3^+,4^+] = \frac{\alpha'^4g_s^4}{8}F^4\int_0^\infty\frac{dT}{T}\int d\mu^{(4)}\Big[&4\mathcal{F}_{\mathcal{N}}+5i\mathcal{C}_{\mathcal{N}=1}\left(\mathcal{Y}_{12}-\mathcal{Y}_{34}\right)\\ &+\mathcal{E}_{\mathcal{N}}\left(\mathcal{Y}_{12}+\mathcal{Y}_{34}-\mathcal{Y}_{13}-\mathcal{Y}_{24}-\mathcal{Y}_{14}-\mathcal{Y}_{23}\right)\Big]\Pi_4(z_i,k_i)\end{aligned}\tag{5.45}$$

A more symmetric result obtains for $\mathcal{A}_4^{\text{1-loop}}[1^-,2^+,3^-,4^+]$ that is invariant under exchanges of 1 with 3 and of 2 with 4.

5.4 Permutation properties

The integrands of four-point amplitude have interesting transformation properties under specific permutations of the variables z_i. The explicit integration measure in the planar/un-oriented case reads

$$\int d\mu^{(4)}_{1234} = \int_0^{\kappa\cdot iT/2}dz_1\int_0^{z_1}dz_2\int_0^{z_2}dz_3\int_0^{z_3}dz_4\delta(z_4)\tag{5.46}$$

Where $\kappa=1$ for the annulus and $\kappa=2$ for the Möbius-strip, due to the double length of the boundary in the un-oriented case. The T dependence of the domain can be removed changing variables to $z_i=\kappa\cdot iT\nu_i/2$. The measure can be written in a form in which it is easy to recognize its permutation properties. To this end we can introduce the necessary step functions

$$\int d\mu^{(4)}_{1234} = \left(\frac{iT\kappa}{2}\right)^4\int_{\mathbb{R}^4}d^4\nu\theta(1-\nu_1)\theta(\nu_1-\nu_2)\ldots\theta(\nu_3-\nu_4)\theta(\nu_1)\ldots\theta(\nu_4)\delta(\nu_4)\tag{5.47}$$

JHEP01(2016)043

We may use the arguments of the first four step functions as new integration variables α_i that read

$$\alpha_i = \nu_i - \nu_{i+1} \quad \text{for } i = 1,2,3 \quad , \quad \alpha_4 = 1 - \nu_1 \tag{5.48}$$

The measure in the new variables is manifestly invariant under all permutations

$$\int d\mu^{(4)}_{1234} = \left(\frac{iT\kappa}{2}\right)^4 \int_0^1 d\alpha_1 \dots \int_0^1 d\alpha_4 \delta\left(1 - \textstyle\sum_{i=1}^4 \alpha_i\right) \tag{5.49}$$

In the non-planar 3−1 case, the measure is substantially equivalent to the one for planar three-point amplitudes, thus we can define variables similar to the previous case

$$\beta_1 = \nu_1 - \nu_2 \quad , \quad \beta_2 = \nu_2 - \nu_3 \quad , \quad \beta_3 = 1 - \nu_3 \quad , \quad \beta_4 = \nu_4 \tag{5.50}$$

The new measure is invariant under all the permutation of the first three variables

$$\int d\mu^{(4)}_{123|4} = \left(\frac{iT}{2}\right)^4 \int_0^1 d\beta_1 \int_0^1 d\beta_2 \int_0^1 d\beta_3 \theta(1 - \beta_1 - \beta_2 - \beta_3) \int_0^1 d\beta_4 \delta(\beta_4) \tag{5.51}$$

Without changing variables and using the symmetry of the Chan-Paton factors, in the 2−2 non-planar case the measure can be rewritten as

$$\int d\mu^{(4)}_{12|34} = \frac{1}{4}\left(\frac{iT}{2}\right)^4 \int_{[0,1]^4} d^4\nu \delta(\nu_4) = \frac{1}{16}\left(\frac{iT}{2}\right)^4 \int_{[0,1]^4} d^4\nu \sum_i \delta(\nu_i) \tag{5.52}$$

The last identity following from the arbitrariness in the choice of the point that can be fixed at the origin. In this form it is clearly invariant under all the permutations of the ν_i variables.

Symmetry properties of the measures can be used to simplify the computations. All world-sheet integrals assume the schematic form

$$\int d\mu^{(4)}\, \mathcal{I}(z_i, k_i) \Pi_4(z_i, k_i) \tag{5.53}$$

The idea is to find the permutations that leave the Koba-Nielsen factor $\Pi_4(z_i, k_i)$ invariant and use them to act on the function $\mathcal{I}(z_i)$ in order to simplify it. If we explicitly write $\Pi_4(z_i, k_i)$ in the cases 4−0, 3−1 and 2−2 we find

$$\Pi_{4-0}(z_i, k_i) = e^{-\alpha' k_1\cdot k_2(\mathcal{G}_{12}+\mathcal{G}_{34}) - \alpha' k_1\cdot k_3(\mathcal{G}_{13}+\mathcal{G}_{24}) - \alpha' k_1\cdot k_4(\mathcal{G}_{14}+\mathcal{G}_{23})} \tag{5.54}$$

$$\Pi_{3-1}(z_i, k_i) = e^{-\alpha' k_1\cdot k_2(\mathcal{G}_{12}+\mathcal{G}^T_{34}) - \alpha' k_1\cdot k_3(\mathcal{G}_{13}+\mathcal{G}^T_{24}) - \alpha' k_1\cdot k_4(\mathcal{G}^T_{14}+\mathcal{G}_{23})} \tag{5.55}$$

$$\Pi_{2-2}(z_i, k_i) = e^{-\alpha' k_1\cdot k_2(\mathcal{G}_{12}+\mathcal{G}_{34}) - \alpha' k_1\cdot k_3(\mathcal{G}^T_{13}+\mathcal{G}^T_{24}) - \alpha' k_1\cdot k_4(\mathcal{G}^T_{14}+\mathcal{G}^T_{23})} \tag{5.56}$$

where $\mathcal{G}^T(z_{12}) = \mathcal{G}(z_{12}+1/2)$. We note that 4−0 and 2−2 are invariant under permutations

$$g_u = \left(\begin{smallmatrix}1\leftrightarrow 3\\ 2\leftrightarrow 4\end{smallmatrix}\right) \quad g_t = \left(\begin{smallmatrix}1\leftrightarrow 4\\ 2\leftrightarrow 3\end{smallmatrix}\right) \quad g_s = \left(\begin{smallmatrix}1\leftrightarrow 2\\ 3\leftrightarrow 4\end{smallmatrix}\right) \tag{5.57}$$

The symmetry group is $\mathbb{Z}_2 \times \mathbb{Z}_2$. This permutations are also symmetries of the 4−0 and 2−2 measures. In the 2−2 case that is evident. To see this in the 4−0 case, we express z_i in terms of the variables α_i and the Koba-Nielsen factor becomes

$$\Pi_{4-0}(\alpha_i, k_i) = e^{-\alpha' k_1\cdot k_2(\mathcal{G}(\alpha_1)+\mathcal{G}(\alpha_3)) - \alpha' k_1\cdot k_3(\mathcal{G}(\alpha_1+\alpha_2)+\mathcal{G}(\alpha_2+\alpha_3)) - \alpha' k_1\cdot k_4(\mathcal{G}(\alpha_2)+\mathcal{G}(\alpha_4))} \tag{5.58}$$

JHEP01(2016)043

In terms of α_i the permutations that leave the measure invariant are generated by

$$g_s : \alpha_1 \leftrightarrow \alpha_3 \quad , \quad g_u : \alpha_2 \leftrightarrow \alpha_4 \quad , \quad g_t : \alpha_1 \leftrightarrow \alpha_3, \alpha_2 \leftrightarrow \alpha_4 \tag{5.59}$$

In the 3−1 case there are no common permutations between the measure and Π_{3-1}.

Recalling the result (5.45) for the four-point amplitude, using the permutation property for the cases 4−0 (including the un-oriented case) and 2−2 we can identify some $\mathcal{Y}$ functions with one another:

$$\mathcal{Y}_{34} \sim \mathcal{Y}_{12} \quad \mathcal{Y}_{24} \sim \mathcal{Y}_{13} \quad \mathcal{Y}_{23} \sim \mathcal{Y}_{14} \tag{5.60}$$

and find a simpler expression for (5.45)

$$\mathcal{A}_4^{\text{1-loop}}[1^-,2^-,3^+,4^+] \tag{5.61}$$
$$= \frac{\alpha'^4 g_s^4}{8} F^4 \int_0^\infty \frac{dT}{T} \int d\mu^{(4)} \Big[4\mathcal{F}_{\mathcal{N}} + \mathcal{E}_{\mathcal{N}} \left(\mathcal{Y}_{12} + \mathcal{Y}_{34} - \mathcal{Y}_{13} - \mathcal{Y}_{24} - \mathcal{Y}_{14} - \mathcal{Y}_{23} \right) \Big] \Pi_4(z_i, k_i)$$

The contribution of the odd spin structure vanishes. In the 3−1 case no further simplification of (5.45) seems possible.

5.5 Factorization

In string theory, OPE of vertex operators produce singularities that are related to factorization of the amplitudes on both massless and massive poles in intermediate channels. In sectors with $\mathcal{N} = 4$ susy no massless poles are expected to be exposed in two-particle channels of 4-point amplitudes, since 2- and 3-point 'amplitudes' of massless states do not receive quantum corrections. For four-point amplitudes, in sectors with $\mathcal{N} = 1, 2$ susy, one may expect factorization into sub-amplitudes of massless vectors connected by massless or massive propagators. However since the function $\mathcal{Y}$ has no poles, the four-point one-loop amplitude doesn't seem to factorize into two- and three-point one-loop sub-amplitudes of massless states but can only expose massive poles for generic values of the modular parameter T. Indeed, the series expansion of $\mathcal{Y}(z)$ produces

$$\mathcal{Y}(z) = -8 \left(\eta_1 + \frac{2\pi}{T} \right) - 8 \left(\frac{T^2}{10} (\eta_2 + 3\eta_1^2) + \eta_1 T + \pi^2 \right) \nu^2 + \mathcal{O}(\nu^4) \tag{5.62}$$

In order to exposed singular behaviours associated to massless open and closed string states one has to consider the boundaries of the moduli space in T, capturing the UV ($T = 0$) and IR ($T = \infty$) limits, we will address this issue after an interlude on 'regular' branes, that give rise to super-conformal theories in the low energy limit [15, 16].

5.6 *Caveat*

Although our results for 4-point one-loop amplitudes look perfectly consistent in that they satisfy the expected Ward identities for gauge invariance and supersymmetry and show no 'unphysical' singularities (violation of unitarity), additional subtleties may occur in the cases with reduced $\mathcal{N} = 1, 2$ supersymmetry.[10] In contrast to the maximally supersymmetric $\mathcal{N} = 4$ case, due to the presence of fermionic propagators S_{ij} in the integrand,

[10] We would like to thank the anonymous referee for pointing out this issue.

JHEP01(2016)043

singularities in vanishing three-particle Mandelstam invariants such as $s_{123} = k_4^2 = 0$ may be exposed using a regulator that relaxes momentum conservation, following the pioneering paper by Minahan on one-loop beta functions in heterotic compactifications [33]. Consistency of the procedure would require imposing $\sum_{i<j} s_{ij} = 0$. Yet finite contributions, resulting from $s_{123}/s_{123} = 1$ may appear that are completely absent in our approach. We believe that these finite contributions may be an artefact of the procedure that, though perfectly justified for 2- and 3-pt scattering amplitudes, that would vanish due to collinear kinetics for mass-less external states preventing any form of 'scattering', is un-necessary in the 4-point case. In order to clarify this issue, one should carefully analyse further constraints on the 4-pt amplitudes such as their field-theory limit. This is beyond the scope of the present investigation. We would like to add that even in case the relevant field-theory limit of our amplitudes showed a finite discrepancy of the form $\alpha'/\alpha' = 1$ with available field theory results at one-loop, one can put the blame on higher spin states running in the loop that are obviously absent in standard field theories. In fact one could reverse the argument upside down or inside out and use loop corrections to probe string effects. Agreement with the field theory limit is only guaranteed at tree-level.

6 Regular branes and super-conformal theories

So far we have not specified the open string vacuum configuration around which the vector boson scattering amplitude is computed. For illustrative purposes, we would like to focus on the simple but very interesting case of regular branes at a $\mathbb{Z}_n$ orbifold singularity [15, 16]. For given n there might be several inequivalent choices (in fact at least two i.e. (1,-1,0) and (1,1,-2)) for the action of the $\mathbb{Z}_n$ on the three complex (six real) transverse coordinates $Z^I \approx \omega_n^{h_I} Z^I$, with $\omega_n = \exp(2\pi i/n)$ and $h_1 + h_2 + h_3 = 0 \pmod n$. There are n different kinds of fractional branes transforming according to the n irreducible (one-dimensional) representations of $\mathbb{Z}_n$. The low-energy dynamics is governed by a quiver field theory with n nodes, corresponding to the n gauge groups, and matter in bi-fundamental or adjoint representation, represented by arrows connecting the nodes.

N regular branes are collections of the same number N of fractional branes of each kind. The resulting gauge group is $\mathrm{U}(N)^n$. At low energies, i.e. in the IR, the 'anomalous' U(1)'s decouple and the dynamics is governed by a super-conformal field theory. The discrete Wilson line, representing the embedding of $\mathbb{Z}_n$ in the Chan-Paton group is given by

$$\gamma = \oplus_{h=0}^{n-1} \omega_n^h \mathbf{1}_{N\times N} \tag{6.1}$$

so much so that

$$\mathrm{tr}(\gamma^\ell) = 0 \quad \forall \ell \neq 0 \tag{6.2}$$

This is enough to guarantee that planar amplitudes for the states $\Phi^{(0)}$ surviving the orbifold projection be identical to the ones for N D3-branes in flat space-time and vanish for the states that have been projected out [15, 16].

JHEP01(2016)043

6.1 Tree level, disk

This is obviously true at tree level where the amplitudes are given by

$$\begin{aligned}
\mathcal{A}_r^{\text{disk}} &= \frac{1}{n^r}\sum_{h_i=0}^{n-1}\text{tr}(\gamma^{h_1}\Phi_1\gamma^{h_2}\Phi_2\dots\gamma^{h_r}\Phi_r)\\
&= \frac{1}{n^r}\sum_{h_i=0}^{n-1}\text{tr}(\gamma^{h_1}\Phi_1\gamma^{-h_1}\gamma^{h_1+h_2}\Phi_2\dots\gamma^{h_r}\Phi_r)\\
&= \frac{1}{n^r}\sum_{h_i=0}^{n-1}\text{tr}(\gamma^{h_1}\Phi_1\gamma^{-h_1}\gamma^{h_1+h_2}\Phi_2\gamma^{-h_1-h_2}\gamma^{h_1+h_2+h_3}\dots\gamma^{h_r}\Phi_r)\\
&= \frac{1}{n^r}\sum_{h_i=0}^{n-1}\text{tr}(\gamma^{h_1}\Phi_1\gamma^{-h_1}\gamma^{h_1+h_2}\Phi_2\gamma^{-h_1-h_2}\dots\gamma^{\sum_i h_i}\Phi_r\gamma^{\sum_i h_i}\gamma^{-\sum_i h_i})\\
&= \text{tr}(\Phi_1^{(0)}\Phi_2^{(0)}\dots\Phi_r^{(0)})
\end{aligned} \tag{6.3}$$

At one-loop only $\mathcal{N}=4$ sectors contribute, $\mathcal{N}=1$ and $\mathcal{N}=2$ sectors give zero, since for the latter the 'empty' boundary would contribute $\text{tr}(\gamma^\ell)=0$. At higher loop, the $b-1$ 'empty' boundaries would contribute $\prod_{i=1}^{b-1}\text{tr}(\gamma^{\ell_i})=0$ unless $\ell_i=0$ for all $i=1,\dots b-1$.

Let us consider four-point amplitudes at one-loop.

6.2 Planar amplitudes

Let us consider first the planar $4-0$ case. For a given color ordering one has

$$\mathcal{A}_{4-0}^{1-\text{loop}} = \frac{1}{n}\sum_{h=0}^{n-1}\text{tr}(\gamma^h t_1t_2t_3t_4)\text{tr}(\gamma^h)\mathcal{A}_{4-0}^{(h)} \tag{6.4}$$

Vector bosons have both ends on the same kind of fractional brane, let us say the ℓ-th. For this choice

$$\mathcal{A}_{4-0}^{1-\text{loop}} = \frac{N}{n}\text{tr}_\ell(t_1t_2t_3t_4)\sum_{h=0}^{n-1}\omega_n^{(\ell+1)h}\mathcal{A}_{4-0}^{(h)} \tag{6.5}$$

The situation is more involved for un-oriented and non-planar amplitudes, that are however suppressed at large N [15, 16].

6.3 Non-planar amplitudes

Non-planar amplitudes differ in principle from the ones in the parent $\mathcal{N}=4$ theory, since the contributions of $\mathcal{N}=1$ and $\mathcal{N}=2$ sectors with $h\neq 0$ weighted by $\text{tr}(\gamma^{\ell_i}t_1\dots)\text{tr}(\gamma^{\ell'_i}t'_1\dots)$ are generically non-zero. Let us focus on the two cases $2-2$ and $3-1$ in turn.

In the $2-2$ case one has

$$\mathcal{A}_{2-2}^{1-\text{loop}} = \frac{1}{n}\sum_{h=0}^{n-1}\text{tr}(\gamma^h t_1t_2)\text{tr}(\gamma^h t_3t_4)\mathcal{A}_{2-2,h}^{1-\text{loop}} \tag{6.6}$$

JHEP01(2016)043

In the 3−1 case one has

$$\mathcal{A}_{3-1}^{1-\text{loop}} = \frac{1}{n}\sum_{h=0}^{n-1} \text{tr}(\gamma^h t_1 t_2 t_3)\text{tr}(\gamma^h t_4)\mathcal{A}_{3-1,h}^{1-\text{loop}} \tag{6.7}$$

Factorization on massless intermediate closed string states accounts for the generalised Stückelberg mechanism giving mass to the anomalous U(1)'s for which $\text{tr}(\gamma^h t_a) \neq 0$ [32, 43–48].

6.4 Un-oriented amplitudes

The presence of Ω-planes tends to generate local tadpoles that require a net number of fractional branes, whenever $n \neq 1$. In the case of N D3's in flat space-time, there are 4 different kinds of Ω3-planes one can add, depending on the quantised values of B_2 and C_2 [49–51]: $\Omega 3^-$ leading to SO($2N$), $\widetilde{\Omega 3}^-$ leading to SO($2N+1$), $\Omega 3^+$ leading to Sp($2N$), $\widetilde{\Omega 3}^+$ leading to Sp($2N$)′ (with $\vartheta' = \vartheta + \pi$). Only $\Omega 3^-$ and $\Omega 3^+$ admit a perturbative world-sheet description as the one used in the present analysis.

For D3's at orbifold singularities one can balance the tadpole contribution of the Ω-planes with the contribution of flavour branes [52]. The prototypical example is N D3's at the un-oriented $\mathbb{C}/\mathbb{Z}^2$ singularity with 4 D7's and an $\Omega 7^-$ [53]. The low energy dynamics is governed by and $\mathcal{N} = 2$ SCFT with gauge group Sp($2N$) and 8 (half) hypermultiplets in the fundamental **2N** and one hypermultiplets in the anti-symmetric skew-traceless tensor representation **N(2N−1)**. The global symmetry is SO(8). The spectra and vacuum configurations of un-oriented $\mathcal{N} = 1$ (super-conformal) quiver theories with flavour symmetries have been studied in some details in [52]. In principle one can study scattering amplitudes along the lines of the present analysis or even include the effect of closed-string fluxes leading to mass deformations of the quivers [54]. We refrain to do so here.

7 UV and IR behaviours

In this section we will analyse the potential divergences of the 1-loop amplitudes. We are interested in studying such conditions as tadpole cancellation under which the amplitudes are finite. The amplitude we computed assume the schematic form

$$\mathcal{A}_N^{\text{1-loop}} = F^N \int_0^\infty \frac{dT}{T}\Phi(T) \int d\mu^{(N)}\, f(\nu_i, T)\Pi_4(\nu_i, k_i, T) \tag{7.1}$$

where Φ represents $\mathcal{F}$, $\mathcal{E}$ or $\mathcal{C}$. For open strings the limits $T \to 0$ and $T \to \infty$ encode respectively the UV and the IR behaviours. Using modular transformations one can transform one-loop open-string amplitudes (direct channel) into tree-level closed-string exchange amplitudes (transverse channel) [26, 37–40].

7.1 Bosonic propagators

In all cases we must manipulate Koba-Nielsen factors and study their limits. To this end we need to study the limiting behaviours of the bosonic propagator on the annulus and the Möbius strip.

JHEP01(2016)043

7.1.1 Direct channel

For the annulus the propagator between two points, $z_1 = \tau\nu_1$ and $z_2 = \tau\nu_2 + x$, is given by

$$\mathcal{G}_{\mathcal{A}}(\tau\nu_1, \tau\nu_2 + x) = -2\log\frac{\theta_1(\tau\nu_{12} + x)}{\theta_1'(0)} + 2\pi\nu_{12}^2 \mathrm{Im}\,\tau \tag{7.2}$$

where $x = 0, 1/2$ correspond to insertions on the same or different boundaries, respectively. For $x = 0$, the logarithmic derivative of θ_1 yields

$$\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)} = \frac{\sin\pi\tau\nu}{\pi}\prod_{n=1}^{\infty}\frac{(1-q^{n+\nu})(1-q^{n-\nu})}{(1-q^n)^2} = \frac{i}{\pi}q^{-\nu/2}\prod_{n=1}^{\infty}\frac{(1-q^{n+\nu-1})(1-q^{n-\nu})}{(1-q^n)^2} \tag{7.3}$$

where $q = e^{2\pi i\tau}$. We find convenient to define the functions

$$h_{\pm}(\nu,\tau) = \prod_{n=1}^{\infty}\frac{1\pm q^{n+\nu-1}}{1-q^n}\frac{1\pm q^{n-\nu}}{1-q^n} \tag{7.4}$$

that satisfy $h_{\pm}(1-\nu) = h_{\pm}(\nu)$. For $x = 1/2$ h_+ gets replaced by h_-. The propagators become

$$\begin{aligned}\mathcal{G}_{\mathcal{A}}(\tau\nu_1, \tau\nu_2) &= -2\log\left(\frac{i}{\pi}q^{-\nu_{12}(1-\nu_{12})/2}h_-(\nu_{12})\right)\\ \mathcal{G}^T_{\mathcal{A}}(\tau\nu_1, \tau\nu_2) &= -2\log\left(\frac{1}{\pi}q^{-\nu_{12}(1-\nu_{12})/2}h_+(\nu_{12})\right)\end{aligned} \tag{7.5}$$

that are invariant under $\nu \leftrightarrow 1-\nu$ in both cases. Using similar manipulations one can easily obtain the propagator on the Möbius strip

$$\mathcal{G}_{\mathcal{M}}(\tau\nu_1, \tau\nu_2) = -2\log\left(\frac{i}{\pi}q^{-\nu_{12}(1-\nu_{12})/2}h_-(\nu_{12})e^{-\frac{i\pi\nu_{12}^2}{2}}\right) \tag{7.6}$$

7.1.2 Transverse channel

The transverse channel description results from the modular transformation S for the annulus and $P = TST^2S$ for the Möbius-strip [26, 37–40]. Denoting by $\tilde{\tau}$ the transformed modular parameter, one has

$$\tilde{z}_{\mathcal{A}} = \frac{z}{\tau_{\mathcal{A}}} \quad , \quad \tilde{\tau}_{\mathcal{A}} = -\frac{1}{\tau_{\mathcal{A}}} \qquad \text{and} \qquad \tilde{z}_{\mathcal{M}} = \frac{z}{2\tau_{\mathcal{M}}-1} \quad , \quad \tilde{\tau}_{\mathcal{M}} = \frac{\tau_{\mathcal{M}}-1}{2\tau_{\mathcal{M}}-1} \tag{7.7}$$

Defining $\ell = 2/T$ and parametrizing $z = x + i\kappa T y/2$ and $\tau = i\kappa T/2 + (\kappa-1)/2$ ($\kappa = 1$ for the annulus and $\kappa = 2$ for the Möbius strip), one finds

$$\tilde{\tau}_{\mathcal{A}} = i\ell \quad , \quad \tilde{z}_{\mathcal{A}} = y - ix\ell \qquad \text{and} \qquad \tilde{\tau}_{\mathcal{M}} = \frac{1}{2} + \frac{i\ell}{4} \quad , \quad \tilde{z}_{\mathcal{M}} = \frac{1}{2}(y - ix\ell) \tag{7.8}$$

Under these transformations the propagator gets shifted by a function of τ. In the Koba-Nielsen factor, the shift is innocuous as it cancels thanks to momentum conservation.

On the transverse annulus, the propagator for points on the same boundary reads

$$\mathcal{G}_{\mathcal{A}}(\nu_1, \nu_2|\tilde{\tau}) = -2\log\frac{\theta_1(\nu_{12}|\tilde{\tau})}{\theta_1'(0|\tilde{\tau})} = -2\log\left[\frac{\sin\pi\nu_{12}}{\pi}g_-(\nu_{12})\right] \tag{7.9}$$

JHEP01(2016)043

where

$$g_{\pm}(\nu,\tau) = \prod_{n=1}^{\infty} \frac{1 \pm e^{2\pi i\nu} q^n}{1-q^n} \frac{1 \pm e^{-2\pi i\nu} q^n}{1-q^n} \tag{7.10}$$

For points on different boundaries, θ_1 gets replaced by θ_2:

$$\mathcal{G}_{\mathcal{A}}(\nu_1, \nu_2 + 1/2|\tau) = \mathcal{G}^T_{\mathcal{A}}(\tau\nu_1, \tau\nu_2|\tau) = -2\log\frac{\theta_2(\nu_{12})}{\theta_1'(0)} + 2\pi\nu_{12}^2 \mathrm{Im}\,\tau \tag{7.11}$$

Under S modular transformations the propagator becomes

$$\mathcal{G}^T_{\mathcal{A}}(\nu_1, \nu_2|\tilde{\tau}) = -2\log\frac{\theta_2(\nu_{12}|\tilde{\tau})}{\theta_1'(0|\tilde{\tau})} = -2\log\left[\frac{\cos\pi\nu_{12}}{\pi} g_+(\nu_{12})\right] \tag{7.12}$$

For the Möbius strip the propagator is

$$\mathcal{G}_{\mathcal{M}}(\nu_1, \nu_2|\tilde{\tau}_{\mathcal{M}}) = -2\log\left[\frac{\sin\pi\nu_{12}/2}{\pi} g_+\left(\frac{\nu_{12}}{2}, \tilde{\tau}_{\mathcal{M}}\right)\right] \tag{7.13}$$

7.2 Functions $\mathcal{F}_{\mathcal{N}}$, $\mathcal{E}_{\mathcal{N}}$, $\mathcal{C}_{\mathcal{N}}$ and $\mathcal{Y}$

In addition to the propagators, one needs the limiting behaviours of the functions $\mathcal{F}_{\mathcal{N}}$, $\mathcal{E}_{\mathcal{N}}$, $\mathcal{C}_{\mathcal{N}}$ and $\mathcal{Y}$. For simplicity we will only consider configurations of branes at orbifold singularities, in particular 'regular' branes [15, 16], thus we can set $\varepsilon^I_{ab} = 0$ in u^I_{ab}.

7.2.1 $\mathcal{N} = 4$ sectors

It is easy to see that $\mathcal{E}_{\mathcal{N}=4} = 0$ due to Riemann identity. For the same reason $\mathcal{F}_{\mathcal{N}=4}$ is simply proportional to $\Lambda^{(6)}/T^2$ with

$$\Lambda^{(6)} = \sum_{\{p\}} e^{-2\pi \mathrm{Im}\,\tau \alpha' p^2/R^2} \tag{7.14}$$

for D9-branes. Using Poisson resummation, for a lattice with dimension $2r$ one finds

$$\Lambda^{(2r)}(\tau_2) = \left(\frac{2\alpha'\tau_2}{R^2}\right)^r \Lambda^{(2r)}(\tau_2^{-1}) \tag{7.15}$$

Using T-duality along all 6 internal directions one gets D3-branes and the lattice sum becomes

$$\Lambda^{(6)}_{\mathrm{D3}} = \sum_{\{w\}} e^{-2\pi \mathrm{Im}\,\tau w^2 \tilde{R}^2/\alpha'} \tag{7.16}$$

Introducing a non zero separation Δx between the D3-branes one can regularize IR divergences. Δx is related to the mass of the lowest states by $\Delta x = \alpha' M$. The separation Δx can be chosen in many ways in the $\mathcal{N} = 4$ sector: the branes are parallel in all the six compact dimension thus they can be displaced along one, two or three complex dimensions D_s.

$$\Lambda^{(6)}_{\mathrm{D3}} = \sum_{\{w\}} e^{-2\pi \mathrm{Im}\,\tau (w\tilde{R} + \Delta x)^2/\alpha'} \quad , \quad \Lambda^{(6)}_{\mathrm{D9}} = \sum_{\{p\}} e^{-2\pi \mathrm{Im}\,\tau \alpha' (p+a)^2/R^2} \tag{7.17}$$

JHEP01(2016)043

where the second formula is for the D9-branes description with a the Wilson line related to branes' separation by $\Delta x = a\alpha'/R$. In the limit $T \to \infty$ the behaviour of $\Lambda^{(6)}$ is dominated by the exponential

$$\Lambda^{(6)}_{\rm D9} \xrightarrow{T\to\infty} b_p e^{-2\pi\alpha' {\rm Im}\,\tau \min_p (p+a)^2/R^2} \tag{7.18}$$

where b_p accounts for possible degeneracies. For $|a| < 1/2$ the minimum corresponds to $p = 0$ and its value is M^2. In the transverse channel the limit $\ell \to \infty$ produces

$$\Lambda^{(6)}_{\rm D9} = \sum_{\{p\}} e^{-2\pi\alpha'(p+a)^2/\ell R^2} \xrightarrow{l\to\infty} \left(\frac{\ell R^2}{2\alpha'}\right)^3 \tag{7.19}$$

In the partial decompactification limit $R \to \infty$ with separation along D_s directions, one has

$$\Lambda^{(6)} = \frac{V_6}{(2\pi\alpha' {\rm Im}\,\tau)^{3-D_s}} e^{-2\pi\alpha' {\rm Im}\,\tau M^2} \tag{7.20}$$

where V_6 is the regulated volume of $\mathbb{R}^6$. The limits in this case yield

$$\Lambda^{(6)}_{\rm D9}(R=\infty) \xrightarrow{T\to\infty} \frac{V_6}{(\pi\alpha' T)^{3-D_s}} e^{-\pi\alpha' T M^2}, \quad \Lambda^{(6)}_{\rm D9}(R=\infty) \xrightarrow{\ell\to\infty} V_6 \left(\frac{\ell}{2\pi\alpha'}\right)^{3-D_s} \tag{7.21}$$

In conclusion the limits of $\mathcal{F}_{\mathcal{N}=4}$ produce

$$\mathcal{F}_{\mathcal{N}=4} \xrightarrow{T\to\infty} \frac{1}{4n\alpha'^2} \frac{(2\pi)^4}{T^2} e^{-\pi\alpha' T M^2}, \quad \mathcal{F}_{\mathcal{N}=4} \xrightarrow{\ell\to\infty} (2\pi)^4 \frac{1}{4n\alpha'^2} \frac{\ell^2}{\kappa^4} \left(\frac{\ell R^2}{2\alpha'}\right)^3 \tag{7.22}$$

7.2.2 $\mathcal{N} = 2$ sectors

In this case $\mathcal{E}_{\mathcal{N}=2}$ is non-zero, in fact

$$\mathcal{F}_{\mathcal{N}=2} = -\mathcal{E}_{\mathcal{N}=2} \mathcal{P}(u) \qquad \mathcal{E}_{\mathcal{N}=2} = (2\pi)^2 \mathcal{X}_{\mathcal{N}=2} = \frac{(2\pi)^2 \Lambda^{(2)}(\tau) I^{(4)}}{4n(2\alpha' {\rm Im}\,\tau)^2} \tag{7.23}$$

$I^{(4)}$ is a constant in the limits $\ell, T \to \infty$, $\Lambda^{(2)}$ is substantially equal to $\Lambda^{(6)}$ with the restriction that brane separation can take place only along one complex direction.

$$\Lambda^{(2)}_{\rm D9} \xrightarrow{T\to\infty} e^{-2\pi\alpha' {\rm Im}\,\tau M^2} \quad \Lambda^{(2)}_{\rm D9} \xrightarrow{l\to\infty} \frac{\ell R^2}{2\alpha'} \tag{7.24}$$

In the amplitudes terms like $\mathcal{E}_{\mathcal{N}=2}$, often appear in combination with Weierstrass $\mathcal{P}$ function or the function $\mathcal{Y}$.

Let us focus first on the $\mathcal{P}(z,\tau)$ and consider $z = x + iy\tau_2$ ($\tau_2 = {\rm Im}\,\tau \propto T$) with $y = 0$ and $y \neq 0$. The variable z can either play the role of $u^I_{ab} = h v^I_{ab}$ or of a world-sheet coordinate. In general $z = i\tau_2\nu + (1/2)$. Anyway the in front of $i\tau_2$ is a real number between zero and one. Expanding Weierstrass function as a power series in q yields

$$\mathcal{P}(z,\tau) = \frac{\pi^2}{\sin^2 \pi z} - \frac{\pi^2}{3} + 8\pi^2 \sum_{n=1}^{\infty} \sum_{d_n|n} q^n d_n \left[1 - \cos(2\pi d_n z)\right] \tag{7.25}$$

JHEP01(2016)043

where d_n are the divisors of n. In the limit $T \to \infty$ the cosines diverge as $e^{2\pi\tau_2 d_n y}$, but this contribution is suppressed by $q^n \sim e^{-2\pi\tau_2 n}$ since $d_n y < n$,

$$\mathcal{P}(z,\tau) \xrightarrow{T\to\infty} \frac{\pi^2}{\sin^2\pi z} - \frac{\pi^2}{3} \to \begin{cases} -\pi^2/3 & \text{if } y \neq 0 \\ \pi^2/\sin^2(\pi x) - \pi^2/3 & \text{if } y = 0 \end{cases} \tag{7.26}$$

In the limit $\ell \to \infty$ Weierstrass function becomes

$$\mathcal{P}(z,\tau) = \tilde{\tau}^2 \mathcal{P}(\tilde{z},\tilde{\tau}) \xrightarrow{\ell\to\infty} \tilde{\tau}^2 \left(\frac{\pi^2}{\sin^2\pi\tilde{z}} - \frac{\pi^2}{3}\right) \to -\frac{\ell^2\pi^2}{\kappa^4} \begin{cases} -1/3 & \text{if } x \neq 0 \\ 1/\sin^2(\pi y/\kappa) - 1/3 & \text{if } x = 0 \end{cases} \tag{7.27}$$

Let us now consider the function $\mathcal{Y}$ only in the case $y > 0$ and $x = 0, 1/2$. We need $S(z)$ and the series expansion of $\partial_z \log\theta_1$:

$$\partial_z \log\theta_1(z|\tau) = \frac{\theta_1'(z|\tau)}{\theta_1(z|\tau)} = \pi\cot(\pi z) + 4\pi \sum_{n=1}^{\infty} \sum_{d_n} q^n \sin(2\pi d_n z) \tag{7.28}$$

The limits $T \to \infty$ and $\ell \to \infty$ produce

$$\begin{aligned} S(z,\tau) &\xrightarrow{T\to\infty} \begin{cases} -i\pi y\,(1/|y|+2) & \text{if } y{\neq}0 \\ 1/x & \text{if } y = 0, x{\to}0 \end{cases} \\ S(z,\tau) &\xrightarrow{\ell\to\infty} \frac{i\pi\ell}{\kappa^2} \begin{cases} 2i & \text{if } x = 1/2 \\ \cot\pi y/\kappa & \text{if } x = 0 \end{cases} \end{aligned} \tag{7.29}$$

Now it's easy to compute the limits for the functions $\mathcal{E}_{\mathcal{N}}$, $\mathcal{F}_{\mathcal{N}}$ and $\mathcal{Y}(z)$ that read

$$\mathcal{E}_{\mathcal{N}=2} \xrightarrow{T\to\infty} \frac{I^{(4)}}{4n\alpha'^2}\frac{4\pi^2}{T^2} e^{-\pi\alpha' TM^2}, \tag{7.30}$$

$$\mathcal{E}_{\mathcal{N}=2} \xrightarrow{\ell\to\infty} \frac{I^{(4)}}{4n\alpha'^2}\pi^2\ell^2\frac{\ell R^2}{2\alpha'} \tag{7.31}$$

$$\mathcal{F}_{\mathcal{N}=2} \xrightarrow{T\to\infty} \frac{I^{(4)}}{4n\alpha'^2}\frac{4\pi^4}{T^2}\left(\frac{1}{\sin^2(h\pi v)} - \frac{1}{3}\right) e^{-\pi\alpha' TM^2}, \tag{7.32}$$

$$\mathcal{F}_{\mathcal{N}=2} \xrightarrow{\ell\to\infty} -\frac{\ell^2\pi^2}{\kappa^4}\frac{I^{(4)}}{4n\alpha'^2}\frac{\pi^2\ell^2}{3}\frac{\ell R^2}{2\alpha'} \tag{7.33}$$

$$\mathcal{Y}(z) \xrightarrow{T\to\infty} -4\pi^2\left(\frac{1}{3} + 2|y|(1+|y|)\right) \quad , \quad \mathcal{Y}(z) \xrightarrow{\ell\to\infty} \frac{\ell^2\pi^2}{\kappa^4} \begin{cases} 11 & \text{if } x = 1/2 \\ 2 & \text{if } x = 0 \end{cases} \tag{7.34}$$

7.2.3 $\mathcal{N} = 1$ sectors

This is the most laborious case, one needs to study the function

$$\mathcal{H}(z) = \prod_{I=1}^{3} \theta_1(z + u_{ab}^I) \tag{7.35}$$

since $\mathcal{F}_{\mathcal{N}=1}$ and $\mathcal{E}_{\mathcal{N}=1}$ are given by

$$\mathcal{F}_{\mathcal{N}=1} = \mathcal{E}_{\mathcal{N}=1}\left(\frac{1}{6}\frac{\mathcal{H}'''(0)}{\mathcal{H}'(0)} + 3\eta_1\right) \qquad \mathcal{E}_{\mathcal{N}=1} = 2\pi\frac{\mathcal{H}'(0)}{\mathcal{H}(0)}\mathcal{X}_{\mathcal{N}=1} \tag{7.36}$$

JHEP01(2016)043

Let us start with $\mathcal{E}_{\mathcal{N}=1}$, the ratio $\mathcal{H}'/\mathcal{H}$ can be seen as a sum of logarithmic derivatives

$$\frac{\mathcal{H}'(0)}{\mathcal{H}(0)} = \sum_{I=1}^{3} \partial_z \log\theta_1\big|_{z=u_I} \tag{7.37}$$

It is easy to see that this function is a modular form of weight one. In the limits the q-series vanish thus only cotangents remain in the direct channel

$$\frac{\mathcal{H}'(0)}{\mathcal{H}(0)} \xrightarrow{T\to\infty} \pi\sum_I \cot(\pi v_I h) \tag{7.38}$$

In the transverse channel one has

$$\frac{\mathcal{H}'(0)}{\mathcal{H}(0)} \xrightarrow{\ell\to\infty} \frac{i\pi\ell}{\kappa^2}\sum_I \cot(\pi u_I) \to -\frac{\pi\ell}{\kappa^2}\sum_I \tfrac{u_I}{|u_I|} \tag{7.39}$$

$\mathcal{F}_{\mathcal{N}=1}$ is even more laborious to analyse

$$\mathcal{F}_{\mathcal{N}=1} = 2\pi\left(\frac{1}{6}\frac{\mathcal{H}'''(0)}{\mathcal{H}(0)} + 3\eta_1\frac{\mathcal{H}'(0)}{\mathcal{H}(0)}\right)\mathcal{X}_{\mathcal{N}=1} = 2\pi\Phi(u_I)\mathcal{X}_{\mathcal{N}=1} \tag{7.40}$$

One can expand the expression in brackets using the logarithm derivatives of θ_1, that we call $\phi(z|\tau) = \partial_z \log\theta_1(z|\tau)$ for brevity:

$$\begin{aligned}\Phi(u_I) = &\prod_I \phi(u_I) + \frac{1}{2}\sum_{I_1\neq I_2}\phi(u_{I_1})\left(\phi'(u_{I_2}) + \phi^2(u_{I_2})\right)\\ &+ \frac{1}{6}\sum_I\left(\phi''(u_I) + 3\phi'(u_I)\phi(u_I) + \phi^3(u_I)\right) + 3\eta_1\sum_I\phi(u_I)\end{aligned} \tag{7.41}$$

Under modular transformations the functions $\phi(z)$ and η_1 are modular forms of weight one and two, respectively. The function $\Phi(z)$ is a modular form of weight three. The limit $T\to\infty$ of $\Phi(u_I)$ can be computed similarly to $\mathcal{H}'/\mathcal{H}$ and the limit $\ell\to\infty$ in the transverse channel can be taken using $\Phi(u) = \tilde{\tau}^3\Phi(\tilde{u})$

$$\Phi(z) \xrightarrow{T\to\infty} \pi^3\left(\prod_I \cot(h\pi v_I) - \frac{2}{3}\sum_I \cot(h\pi v_I)\right) \quad , \quad \Phi(z) \xrightarrow{l\to\infty} -i\left(\frac{i\pi\ell}{\kappa^2}\right)^3\sum_I \frac{u_I}{|u_I|} \tag{7.42}$$

In the $\mathcal{N}=1$ case the function $\mathcal{C}_{\mathcal{N}}$ also appears that has a simple form. For future use, we list here all the limits

$$\mathcal{E}_{\mathcal{N}=1} \xrightarrow{T\to\infty} \frac{I^{(6)}}{4n\alpha'^2}\frac{2\pi^2}{T^2}\sum_I \cot(h\pi v_I), \tag{7.43}$$

$$\mathcal{E}_{\mathcal{N}=1} \xrightarrow{\ell\to\infty} -\frac{I^{(6)}}{4n\alpha'^2}\frac{\pi^2}{2}\frac{\ell^3}{\kappa^2}\sum_I \frac{u_I}{|u_I|} \tag{7.44}$$

$$\mathcal{F}_{\mathcal{N}=1} \xrightarrow{T\to\infty} \frac{I^{(6)}}{4n\alpha'^2}\frac{2\pi^4}{T^2}\left[\prod_I \cot(h\pi v_I) - \frac{2}{3}\sum_I \cot(h\pi v_I)\right], \tag{7.45}$$

$$\mathcal{F}_{\mathcal{N}=1} \xrightarrow{\ell\to\infty} -\frac{I^{(6)}}{4n\alpha'^2}\frac{\pi^4\ell^5}{2\kappa^6}\sum_I \tfrac{u_I}{|u_I|} \tag{7.46}$$

$$\mathcal{C}_{\mathcal{N}}(T) = -\frac{c_1^{\rm GSO} I^{(6)}}{4n\alpha'^2}\frac{2\pi^2}{T^4} \quad , \quad \mathcal{C}_{\mathcal{N}}(\ell) = -\frac{c_1^{\rm GSO} I^{(6)}}{4n\alpha'^2}\frac{\ell^4\pi^2}{8} \tag{7.47}$$

JHEP01(2016)043

7.3 Two- and three-point 'amplitudes'

We now have all the tools to compute the UV and IR limits of two- and three-point 'amplitudes'.

In $\mathcal{N}=4$ sectors both amplitudes are zero because $\mathcal{E}_{\mathcal{N}=4}=0$. This is an expected result in fact in $D=4$ SYM $\mathcal{N}=4$ theory has a vanish β function and no threshold corrections to the gauge kinetic function.

Let us start with the two-point 'amplitude' that reads

$$\mathcal{A}_2^{\text{1-loop}}[1^+,2^+]=\frac{g_s^2\alpha'^2}{2}[12]^2\int_0^\infty\frac{dT}{T}\left(\mathcal{E}_{\mathcal{N}}+i\mathcal{C}_{\mathcal{N}=1}\right)\int d\mu^{(2)}\,e^{-\alpha' k_1\cdot k_2\mathcal{G}_{12}} \tag{7.48}$$

Despite analytic continuation of momenta, $k_1\cdot k_2=0$ thus the integral over the world-sheet insertions gives $(iT/2)^2$. For this reason the color-ordered amplitude is the same in the planar and non-planar cases, they differ in the Chan-Paton factor. In the IR limit in $\mathcal{N}=2$ sectors one has a logarithmic divergence due to $\Lambda^{(2)}$

$$\mathcal{A}_2^{\text{1-loop}}[1^+,2^+]\xrightarrow{T\to\infty}-g_s^2\frac{\pi^2R^2I^{(2)}}{8\alpha' n}[12]^2\log L \tag{7.49}$$

One can regularize IR divergences introducing brane separation in the amplitude

$$\mathcal{A}_2^{\text{1-loop}}[1^+,2^+]\xrightarrow{T\to\infty}-g_s^2\frac{\pi^2I^{(2)}}{8n}[12]^2\int^\infty\frac{dT}{T}e^{-\pi\alpha' TM^2} \tag{7.50}$$

In the $\mathcal{N}=1$ case there are no directions to separate the branes and the integral over T diverges logarithmically

$$\mathcal{A}_2^{\text{1-loop}}[1^+,2^+]\xrightarrow{T\to\infty}g_s^2\frac{\pi^2I^{(6)}}{n}[12]^2\textstyle\sum_I\cot(h\pi v_I)\log L \tag{7.51}$$

where L is the IR cut-off. We note that the odd spin structure does not contribute to the limit.

In the transverse channel we treat separately planar and un-oriented amplitudes, related to massless tadpoles and their cancellation, and the non-planar amplitude, related to the masses of anomalous U(1) vector bosons. For momentum conservation $k_1\cdot k_2=0$ and the integral on the world-sheet in the direct channel produces $-T^2/4$ in any case, while in the transverse channel it gives $-1/\ell^2$ in the planar and non-planar case and $-16/\ell^2$ in the un-oriented case. Summing planar and un-oriented contributions produces

$$\mathcal{A}_{2-0}^{\text{1-loop}}(1^+,2^+)=-\frac{g_s^2\alpha'^2}{2}[12]^2\left[\text{tr}(\gamma^h)\text{tr}(t_1t_2\gamma^{-h})+32\text{tr}(t_1t_2W_{2h}^\Omega)\right]\int_0^\infty\frac{d\ell}{\ell^3}\left(\mathcal{E}_{\mathcal{N}}+i\mathcal{C}_{\mathcal{N}=1}\right) \tag{7.52}$$

where in the integral over ℓ, the measure descends from $\tau=\tau_{\mathcal{A}}=i\ell$. In the non-planar case

$$\mathcal{A}_{1-1}^{\text{1-loop}}(1^+,2^+)=-\frac{g_s^2\alpha'^2}{2}[12]^2\text{tr}(\gamma^h t_1)\text{tr}(t_2\gamma^{-h})\int_0^\infty\frac{d\ell}{\ell^3}\left(\mathcal{E}_{\mathcal{N}}+i\mathcal{C}_{\mathcal{N}=1}\right) \tag{7.53}$$

The behaviour of both amplitudes is coded in color-ordered amplitudes, their limit in the direct channel for $\mathcal{N}=2$ sectors reads

$$\mathcal{A}_{1-1}^{\text{1-loop}}[1^+,2^+]\xrightarrow{\ell\to\infty}-g_s^2[12]^2\frac{I^{(4)}}{4n}\frac{\pi^2}{2}\frac{R^2}{2\alpha'}L \tag{7.54}$$

JHEP01(2016)043

where L is the IR cutoff. In $\mathcal{N}=1$ sectors the limit is dominated by the contribution of the odd spin structure

$$\mathcal{A}_{1-1}^{\text{1-loop}}[1^+,2^+] \xrightarrow{\ell\to\infty} i g_s^2 [12]^2 \frac{c_1^{\text{GSO}} I^{(6)}}{4n} \frac{L^2\pi^2}{32} \tag{7.55}$$

Let us now consider the (color-ordered) three-point amplitude with helicities $(-++)$ that reads

$$\mathcal{A}_3^{\text{1-loop}}[1^-,2^+,3^+] = -\frac{i}{\sqrt{2}} g_s \alpha'^2 \frac{[23]^3}{[12][31]} \int_0^\infty \frac{dT}{T} \left(\mathcal{E}_{\mathcal{N}} - i\mathcal{C}_{\mathcal{N}=1}\right) \alpha' k_2\cdot k_3 \int d\mu^{(3)} S_{23} \Pi_3(z_i,k_i) \tag{7.56}$$

As for two-point amplitudes $\mathcal{N}=4$ sectors do not contribute. The form of the color-ordered amplitude is different in the planar and non-planar cases. Actually there are two non-planar cases: one with 2^+ and 3^+ in the second boundary, that we call $1{-}2$, and one with 1^- and 2^+ in the first boundary, that we call $2{-}1$.

Let us focus on the world-sheet integral, that we need to compute for $k_i\cdot k_j \to 0$. In the planar case, using the variables $\alpha_1=\nu_{12}$, $\alpha_2=\nu_{23}$ and $\alpha_3=1-\nu_3$ one gets

$$\begin{aligned} &\int d\mu_{123}^{(3)} \Pi_3(z_i,k_i) \\ &= \tau^3 \int d\alpha_1 d\alpha_2 d\alpha_3\, \delta(\textstyle\sum_i \alpha_i - 1) \alpha' k_2\cdot k_3 S(\alpha_2\tau) e^{-\alpha' k_1\cdot k_2 \mathcal{G}(\alpha_1\tau) - \alpha' k_2\cdot k_3 \mathcal{G}(\alpha_2\tau) - \alpha' k_1\cdot k_3 \mathcal{G}(\alpha_3\tau)} \end{aligned} \tag{7.57}$$

In this form it is not clear what happens when $\alpha_2\to 0$ and $k_2\cdot k_3\to 0$. As in [33], if we call $\varepsilon = k_2\cdot k_3$, when $\varepsilon\to 0$ the leading term is finite

$$\lim_{\varepsilon\to 0} \left(\frac{iT}{2}\right)^2 \int d\alpha_1 d\alpha_3\, \delta(1-\alpha_1-\alpha_3) e^{-\alpha' k_1\cdot k_2 \mathcal{G}(\alpha_1\tau) - \alpha' k_1\cdot k_3 \mathcal{G}(\alpha_3\tau)} \int d\alpha_2 \varepsilon (\alpha_2)^{\varepsilon-1} \tag{7.58}$$

As a result the integral gives $(iT/2)^2$. This holds true for $\mathcal{A}_{1-2}^{\text{1-loop}}[1^-,2^+,3^+]$ too, in fact using the parametrization $\beta_1=\nu_1$, $\beta_2=\nu_{23}$ and $\beta_3=1-\nu_3$ yields a similar integral. In the case $\mathcal{A}_{2-1}^{\text{1-loop}}[1^-,2^+,3^+]$, the amplitude does not exhibit the kinematical pole. This is in line with the Chan-Paton factor that does not involve the structure constants f_{abc} but rather a product of a δ for 1 and 2 combined with an anomalous U(1) factor for 3. The kinematical factor in the numerator does not cancel and makes this IR contribution vanish.

In the direct and transverse channel, the behaviour of the amplitude is very similar to the two-point amplitude in fact $\mathcal{E}_{\mathcal{N}}$ appears in the both cases. The world-sheet integral give us the same contribution in terms of modular parameter T (or ℓ). This should be interpreted in terms of the running of the field-dependent gauge couplings [32] and is to be expected as a result of supersymmetry Ward identities that only allow the super-invariant W^2 for 2- and 3-point amplitudes, barring anomalous U(1)'s.

7.4 Four-point amplitude, 4−0

Let us now study the limiting behaviours of 4-point amplitudes. In the direct channel, relying on $u=-s-t$, the Koba-Nielsen factor can be rewritten in terms of the variables

JHEP01(2016)043

α_i as

$$\Pi_{4-0} = \exp\Big[\frac{\alpha' s}{2}\left[\mathcal{G}(\alpha_1)+\mathcal{G}(\alpha_3)-\mathcal{G}(\alpha_1+\alpha_2)-\mathcal{G}(\alpha_2+\alpha_3)\right] + \frac{\alpha' t}{2}\left[\mathcal{G}(\alpha_2)+\mathcal{G}(\alpha_4)-\mathcal{G}(\alpha_1+\alpha_2)-\mathcal{G}(\alpha_2+\alpha_3)\right]\Big] \tag{7.59}$$

Inserting in this expression the explicit form of the propagator found in the previous section, the α-independent terms cancel and using $\alpha_1+\alpha_2+\alpha_3+\alpha_4 = 1$ many terms simplify so that

$$\Pi_{4-0}(\alpha_i, k_i) = \left[\frac{h_-(\alpha_1)h_-(\alpha_3)}{h_-(\alpha_1+\alpha_2)h_-(\alpha_2+\alpha_3)}\right]^{-\alpha' s} \times \left[\frac{h_-(\alpha_2)h_-(\alpha_4)}{h_-(\alpha_1+\alpha_2)h_-(\alpha_2+\alpha_3)}\right]^{-\alpha' t} q^{-\alpha'(s\alpha_2\alpha_4+t\alpha_1\alpha_3)} \tag{7.60}$$

In the limit $T \to \infty$, $q \to 0$ all the functions $h(\alpha) \to 1$ while the s,t dependent q-exponential, using the saddle point method, behaves as

$$\int d^4\alpha\delta(1-\textstyle\sum\alpha) e^{\pi T\alpha'(s\alpha_2\alpha_4+t\alpha_1\alpha_3)} \xrightarrow{T\to+\infty} e^{\frac{\pi T\alpha' st}{4(s+t)}} \tag{7.61}$$

In the physical region $s > 0$, $t < 0$ (in the Regge limit $|s| \gg |t|$), with this condition the exponent is $-\pi T\alpha'|t|/4$ thus the integral on T is always convergent in the limit $T \to \infty$.

As in [55], if we expand the functions h_- in powers of q. The form of the exponential suggests that one can interpret the modulus T as a Schwinger parameter and the integral over the variables α_i as a Feynman parametrization of a box integral. The string amplitude can be seen as an infinite sum of massive box amplitudes.

In the transverse channel we find the usual tadpole, in fact we can obtain the same world-sheet integral rescaling the world-sheet coordinate on the Möbius strip by a factor of two and obtaining a 2^4 factor from $\mathcal{F}_\mathcal{N}$ or $\mathcal{YE}_\mathcal{N}$. In the limit $\ell \to \infty$ the Koba-Nielsen factor reduces to products or ratios of sines

$$\mathcal{A}_{4-0}^{\text{1-loop}}[1^-,2^-,3^+,4^+] \xrightarrow{\ell\to\infty} \frac{g_s^4\alpha'^4}{2}F^4\int_0^\infty\frac{d\ell}{\ell}\left(\mathcal{F}_\mathcal{N}-\frac{\pi^2\ell^2}{\kappa^4}\mathcal{E}_\mathcal{N}\right) \int d\mu_{1234}^{(4)}\left[\frac{\sin\pi\alpha_1\sin\pi\alpha_3}{\sin\pi(\alpha_1+\alpha_2)\sin\pi(\alpha_2+\alpha_3)}\right]^{-\alpha' s}\left[\frac{\sin\pi\alpha_2\sin\pi\alpha_4}{\sin\pi(\alpha_1+\alpha_2)\sin\pi(\alpha_2+\alpha_3)}\right]^{-\alpha' t} \tag{7.62}$$

In the $\ell \to \infty$ limit, it is useful to define a combination of $\mathcal{F}$ and $\mathcal{E}$ and evaluate its limit in the various sectors $\mathcal{N} = 1, 2, 4$

$$\left(2\mathcal{F}_\mathcal{N}+a\frac{\pi^2\ell^2}{\kappa^4}\mathcal{E}_\mathcal{N}\right) \xrightarrow{\ell\to\infty} \frac{\pi^4}{4n\alpha'^2}\frac{\ell^2}{\kappa^4}\begin{cases} 16(\ell R^2/2\alpha')^3 & \text{if } \mathcal{N}=4 \\ (a-2/3)I^{(4)}\ell^2(\ell R^2/2\alpha') & \text{if } \mathcal{N}=2 \\ -(2+a)\dfrac{\ell^3 I^{(6)}}{2\kappa^2}\sum_I \frac{u_I}{|u_I|} & \text{if } \mathcal{N}=1 \end{cases} \tag{7.63}$$

JHEP01(2016)043

In the 4−0 case $a = -2$, thus the contribution to the amplitude from $\mathcal{N} = 1$ sectors vanishes in the limit.

The world-sheet integral can be rewritten as a derivative of the Veneziano-like amplitude w.r.t. the string tension as in [22]

$$\int d\mu_{1234}^{(4)} \left[\frac{s_1 s_3}{s_{1+2}s_{2+3}}\right]^{-\alpha' s} \left[\frac{s_2 s_4}{s_{1+2}s_{2+3}}\right]^{-\alpha' t} = -\frac{1}{2\pi^2}\int_0^1 \left(\frac{\log x}{1-x} + \frac{\log(1-x)}{x}\right) x^{-\alpha' s}(1-x)^{-\alpha' t} \tag{7.64}$$

As shown in [22], the one-loop dilaton tadpole is proportional to the logarithmic derivative of the tree level amplitude with respect to the (inverse) tension α'.

7.5 Four-point amplitude, 2−2

In this case one has two independent color-ordered amplitudes, in fact fixing legs 1 and 2 on a boundary and legs 3 and 4 on the other boundary, one has $\mathcal{A}_{2-2}^{\text{1-loop}}[1^-, 2^-, 3^+, 4^+]$ and $\mathcal{A}_{2-2}^{\text{1-loop}}[1^+, 2^-, 3^-, 4^+]$. In the direct channel we have the same limits, $\mathcal{N} = 2, 4$ is regularized by brane separation and $\mathcal{N} = 1$ by momentum flow.

In the transverse channel, one has a behaviour similar to 4−0, the unique changes affect the value of the parameter a that we have defined in the equation (7.63): for $\mathcal{A}_{2-2}^{\text{1-loop}}[1^-, 2^-, 3^+, 4^+]$ one has $a = -20$ while for $\mathcal{A}_{2-2}^{\text{1-loop}}[1^+, 2^-, 3^-, 4^+]$ $a = -2$ again, thus the last amplitude vanishes for $\mathcal{N} = 1$. For instance, for $[1^-, 2^-, 3^+, 4^+]$ one finds

$$\mathcal{A}_{2-2}^{\text{1-loop}}[1^-, 2^-, 3^+, 4^+] \xrightarrow{\ell\to\infty} \frac{g_s^4 \alpha'^4}{4} F^4 \int_0^\infty \frac{d\ell}{\ell}\left(2\mathcal{F}_{\mathcal{N}} + a\frac{\pi^2\ell^2}{\kappa^4}\mathcal{E}_{\mathcal{N}}\right)$$
$$\int d\mu_{123|4}^{(4)} \left[\frac{\sin\pi\beta_1 \sin\pi\beta_3}{\cos\pi(\beta_1+\beta_2)\cos\pi(\beta_2+\beta_3)}\right]^{-\alpha' s} \left[\frac{\cos\pi\beta_2 \cos\pi(\beta_1+\beta_2+\beta_4)}{\cos\pi(\beta_1+\beta_2)\cos\pi(\beta_2+\beta_3)}\right]^{-\alpha' t} \tag{7.65}$$

where $\beta_1 = \nu_1 - \nu_2$, $\beta_2 = 1 - \nu_1$ and similarly for β_3 and β_4. For $[1^+, 2^-, 3^-, 4^+]$ one can easily adapt the formula.

7.6 Four-point amplitude, 3−1

The non-planar amplitude 3−1 has a unique independent color-ordered amplitude, $\mathcal{A}_{3-1}^{\text{1-loop}}$ $[1^-, 2^-, 3^+, 4^+]$ with the leg 4 on the second boundary.

In the transverse channel, the contribution $\mathcal{C}_{\mathcal{N}}$ of the odd spin structure dominates in limit $\ell \to \infty$ and one gets

$$\mathcal{A}_{3-1}^{\text{1-loop}}[1^-, 2^-, 3^+, 4^+] \xrightarrow{\ell\to\infty} \frac{g_s^4 \alpha'^2}{8}\pi^4 \frac{c_1^{\text{GSO}} I^{(6)}}{4n}\frac{9}{8}F^4 \int_0^\infty \frac{d\ell}{\ell}\ell^5$$
$$\int d\mu_{123|4}^{(4)} \left[\frac{\sin\pi\beta_1 \cos\pi\beta_3}{\sin\pi(\beta_1+\beta_2)\cos\pi(\beta_2+\beta_3)}\right]^{-\alpha' s} \left[\frac{\sin\pi\beta_2 \cos\pi\beta_4}{\sin\pi(\beta_1+\beta_2)\cos\pi(\beta_2+\beta_3)}\right]^{-\alpha' t} \tag{7.66}$$

JHEP01(2016)043

8 Supersymmetry vs Hybrid formalism

As we have checked that the 4-point amplitudes in the RNS formalism are MHV, thus satisfying the expected SUSY Ward identities, we can super-symmetrize them.

In the $\mathcal{N}=1$ case, the relevant super-invariant is $W^2\bar{W}^2$ where $W_\alpha = 1/2\bar{D}^2 D_\alpha U$ is the linearised super-field strength of the vector multiplet described by a real scalar super-field U. The full $\mathcal{N}=1$ super-amplitude would read

$$\mathcal{A}^{\mathcal{N}=1}_{4\text{-pt}} = \int d^2\vartheta d^2\bar{\vartheta}\{(W_1W_2)(\bar{W}_3\bar{W}_4)\mathcal{A}^{Bose}_{4-pt} + \text{perms}\} \tag{8.1}$$

In addition to 4 vector boson amplitudes $\mathcal{A}(-,-,+,+)$, it encodes also 2 vector 2 gluino amplitudes $\mathcal{A}(-1/2,-,+1/2,+)$ and 4 gluino amplitudes $\mathcal{A}(-1/2,+1/2,-1/2,+1/2)$. A direct computation of the one-loop amplitude with 2 or 4 gluini looks quite laborious since in addition to the vertex operator in the 'canonical' -1/2 super-ghost picture

$$V_F^{(-1/2)} = u^\alpha(k) S_\alpha \Sigma e^{-\varphi/2} e^{ikX} \tag{8.2}$$

where Σ is the internal spin field with R-charge $+3/2$, one should use also the vertex in the higher $+1/2$ picture

$$V_F^{(+1/2)} = u^\alpha(k)\partial X_\mu \sigma^\mu_{\alpha\dot{\alpha}} C^{\dot{\alpha}} \Sigma e^{+\varphi/2} e^{ikX} + \ldots \tag{8.3}$$

In the $\mathcal{N}=2$ case, the relevant super-invariant is again $\mathcal{W}^2\bar{\mathcal{W}}^2$ where now $\mathcal{W}$ is the chiral super-field describing $\mathcal{N}=2$ vector multiplets. The full $\mathcal{N}=2$ super-amplitude would read

$$\mathcal{A}^{\mathcal{N}=2}_{4\text{-pt}} = \int d^4\vartheta d^4\bar{\vartheta}\{(\mathcal{W}_1\mathcal{W}_2)(\bar{\mathcal{W}}_3\bar{\mathcal{W}}_4)\mathcal{A}^{Bose}_{4-pt} + \text{perms}\} \tag{8.4}$$

In addition to 4 vector boson amplitudes, it encodes also 2 vector 2 gluino amplitudes $\mathcal{A}(-\frac{1}{2},-,+\frac{1}{2},+)$, 4 gluino amplitudes $\mathcal{A}(-1/2,+1/2,-1/2,+1/2)$, 2 vector 2 scalar amplitudes $\mathcal{A}(-,+,0,0)$, 2 gluino 2 scalar amplitudes $\mathcal{A}(-1/2,+1/2,0,0)$ and 4 scalar amplitudes $\mathcal{A}(0,0,0,0)$. Once again a direct computation of the one-loop amplitudes with gluini looks more involved, while amplitudes with scalars look feasible, since the vertex operator for scalars in vector multiplets are simply

$$V_\phi = \phi(k)(i\partial Z_3 + k\cdot\psi\Psi_3)e^{ikX} \tag{8.5}$$

where Z_3 is the complex 'untwisted' coordinate and Ψ_3 its world-sheet super-partner.

Different manifestly supersymmetric formalisms for the quantisation of the superstring have been proposed, depending on the number of space-time and internal dimensions [11–14]. The one we will focus on here is suitable for compactifications on Calabi-Yau spaces or orbifolds, described by internal $\mathcal{N}=2$ SCFT's [11]. In the Type II case these yield $\mathcal{N}=2$ supersymmetry in $D=4$. In the Type I or Heterotic case these yield $\mathcal{N}=1$ supersymmetry in $D=4$. Another one is suitable for compactifications on manifolds with SU(2) holonomy (e.g. $K3\times T2$) spaces or orbifolds of T^4, described by internal $\mathcal{N}=4$ SCFT's [12]. In the Type II case these yield $\mathcal{N}=4$ supersymmetry in $D=4$. In the Type

JHEP01(2016)043

I or Heterotic case these yield $\mathcal{N}=2$ supersymmetry in $D=4$. We will neither deal with the second approach any further here, nor with the pure spinor formalism [56–59].

We would like to compare our results with those of the hybrid formalism with minimal supersymmetry in $D=4$ [10, 11]. The construction is based on two observations. First one can twist $\mathcal{N}=2$ SCFT's redefining the worldsheet stress tensor T according to $T'=T-J/2$ where J is the U(1) worldsheet current. As a result $c'=0$ and $h'=h-q/2$. Second one can identify the dimension 0 spin fields $\vartheta_a=S_a e^{+\varphi/2}$ with the Grassmann coordinates of superspace. The construction works for $D=4$ whereby $\vartheta_a\to\vartheta_\alpha,\bar{\vartheta}_{\dot{\alpha}}$ denoting by $p_\alpha,\bar{p}_{\dot{\alpha}}$ their dimension 1 conjugate momenta, one has 4 (η,ξ) systems with $c=-8=4\times(-2)$. Including the $c=4$ contribution of the bosonic coordinates X^μ and the $c=0$ contribution of the twisted internal $\mathcal{N}=2$ SCFT, one has a defect $\Delta c=-4$ that can be compensated by an additional chiral boson ρ with $\varepsilon=1$ (as for commuting ghosts like φ for β,γ), background charge $Q_\rho=1$ (instead of $Q_\varphi=2$) and central charge $c=\varepsilon(1+3Q^2)=+4$ (instead of $c_\varphi=13$). After twisting, the mapping of the generators of the twisted $\mathcal{N}=2$ SCFT read

$$T_{\rm hyb}=T_X+T_\vartheta+T_\rho+T_{\rm tSCFT}=T_{\rm RNS}=T_{X\psi}+T_{\rm SCFT}+T_{\rm gh}\quad , \tag{8.6}$$

$$J_{\rm hyb}=-\partial\rho+J_{\rm tSCFT}=bc+\xi\eta \tag{8.7}$$

$$G^+_{\rm hyb}=e^\rho d^2+G^+_{\rm tSCFT}=J_{\rm BRST}\ , \tag{8.8}$$

$$G^-_{\rm hyb}=e^{-\rho}\bar{d}^2+G^-_{\rm tSCFT}=b \tag{8.9}$$

where

$$d_\alpha=p_\alpha+\frac{i}{2}\partial X_{\alpha\dot{\alpha}}\bar{\vartheta}^{\dot{\alpha}}-\frac{1}{4}\bar{\vartheta}^2\partial\vartheta_\alpha+\frac{1}{8}\vartheta_\alpha\partial(\bar{\vartheta}^2) \tag{8.10}$$

$$\bar{d}_{\dot{\alpha}}=\bar{p}_{\dot{\alpha}}+\frac{i}{2}\vartheta^\alpha\partial X_{\alpha\dot{\alpha}}-\frac{1}{4}\vartheta^2\partial\bar{\vartheta}_{\dot{\alpha}}+\frac{1}{8}\bar{\vartheta}_{\dot{\alpha}}\partial(\vartheta^2) \tag{8.11}$$

Unintegrated ('c-ghost-number 1' in a sense) vertex operators for compactification independent states, such as an open string gauge boson or the closed string graviton, can be expressed in terms of a real scalar super-field $U(x,\vartheta,\bar{\vartheta})$ (or $U(x,\vartheta_L,\bar{\vartheta}_L;\vartheta_R,\bar{\vartheta}_R)$ for closed strings). In order for U to be a world-sheet super-primary it must satisfy $D^2U=\bar{D}^2U=\partial_\mu\partial^\mu U=0$. Component fields obtain by acting with super-derivatives. For the vector boson one has

$$A_{\alpha\dot{\alpha}}=[D_\alpha,\bar{D}_{\dot{\alpha}}]U|_{\vartheta=\bar{\vartheta}=0} \tag{8.12}$$

and for the gaugino one finds

$$\lambda_\alpha=\frac{1}{2}\bar{D}^2D_\alpha U|_{\vartheta=\bar{\vartheta}=0} \tag{8.13}$$

Integrated vertex operators follow the twisting prescription

$$\int dzV=\int dzG^-_{\rm hyb}G^+_{\rm hyb}U=\int dzHU \tag{8.14}$$

where the field-dependent super-differential operator H reads

$$H=d^\alpha\bar{D}^2D_\alpha+\bar{d}_{\dot{\alpha}}D^2\bar{D}^{\dot{\alpha}}+\partial\vartheta^\alpha D_\alpha+\partial\bar{\vartheta}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}+\frac{i}{2}\Pi^{\dot{\alpha}\alpha}[D_\alpha,\bar{D}_{\dot{\alpha}}] \tag{8.15}$$

JHEP01(2016)043

with $\Pi^{\dot\alpha\alpha} = \partial X^{\dot\alpha\alpha} - i\bar\vartheta^{\dot\alpha}\partial\vartheta^{\alpha} + i\partial\bar\vartheta^{\dot\alpha}\vartheta^{\alpha}$. The formalism is manifestly supersymmetric since d^{α}, $\bar d_{\dot\alpha}$ and $\Pi^{\dot\alpha\alpha}$ commute with the space-time supersymmetry generators $\mathcal{Q}_{\alpha}$ and $\bar{\mathcal{Q}}_{\dot\alpha}$. Gauge invariance corresponds to $\delta U = \bar D^2\bar\Lambda + D^2\Lambda = \Omega + \bar\Omega$ that adds a total derivative to V.

The 'topological/twisted' prescription for computing one-loop Type II scattering amplitudes [10] reads

$$\mathcal{M}_n = \int \frac{d^2\tau}{\tau_2^2} \int \prod_{i=1}^{n} d^2 z_i \left\langle \left(\int J_L \wedge J_R \right)^2 V_1 V_2 \dots V_n \right\rangle \tag{8.16}$$

where $\int J_L \wedge J_R = \int d^2 w (J_L^{tSCFT} - \partial\rho_L)(J_R^{tSCFT} - \partial\rho_R)$, constructed from the twisted U(1) current, is needed to provide the correct number of zero-modes in the large Hilbert space once translated into RNS fields.

For Type I amplitudes the analogous prescription would be

$$\mathcal{A}_n = \int \frac{dT}{T} \int \prod_{i=1}^{n} dz_i \int dw_1 dw_2 \langle J(w_1) J(w_2) V_1 V_2 \dots V_n \rangle \tag{8.17}$$

As observed in [10] when the external states, such as gauge bosons or gravitons, are ρ and compactification independent, the internal contribution factorizes and one has to simply compute contractions of the space-time super-coordinate fields.

Indeed after dealing with subtleties associated to the integration over the chiral boson ρ one arrives at a manifestly supersymmetric result,

$$\begin{aligned}
\mathcal{M}_n = & \int d^2\vartheta_L d^2\bar\vartheta_L d^2\vartheta_R d^2\bar\vartheta_R \int \frac{d^2\tau}{\tau_2^6} \\
& \times \int \prod_{i=1}^{n} d^2 z_i \frac{\partial}{d\zeta_i} \frac{\partial}{d\bar\zeta_i} \Bigg[\left(\sum \zeta_i W_L^i \right)^2 \left(\sum \zeta_i \bar W_L^i \right)^2 \left(\sum \bar\zeta_i W_R^i \right)^2 \left(\sum \bar\zeta_i \bar W_R^i \right)^2 \\
& \qquad \times |{:}e^{\mathcal{S}}{:}|^2 \exp\left(-2\pi \left[\mathcal{K} + i \sum_i z_i k_i \right]^2 / \tau_2 \right) \prod_{i<j} |\theta_1(z_{ij})|^{k_i \cdot k_j} \times \\
& \qquad \times \prod_i \tilde U(k_i, \vartheta_L, \bar\vartheta_L, \vartheta_R, \bar\vartheta_R) \Bigg]_{\zeta_i = \bar\zeta_i = 0}
\end{aligned} \tag{8.18}$$

where ζ_i and $\bar\zeta_i$ are auxiliary Grassmann variables that serve the purpose to select the multi-linear term in the 'external polarisations'

$$\mathcal{K}^{\mu} = \sum_i \zeta_i B_i^{\mu} - i\sigma^{\mu}_{\alpha\dot\alpha} \sum_{i,j} S_{ij} \zeta_i \zeta_j W_i^{\alpha} \bar W_j^{\dot\alpha} \tag{8.19}$$

and

$$\mathcal{S} = \sum_{i,j} \zeta_i S_{ij} (D_j^{\alpha} W_{\alpha}^i + \bar D_j^{\dot\alpha} \bar W_{\dot\alpha}^i - i k_j^{\mu} B_{\mu}^i) + \mathcal{O}(\zeta^2) + \mathcal{O}(\zeta^3) + \mathcal{O}(\zeta^4) \tag{8.20}$$

whose consistency has been tested at least for the uncompactified case against gauge invariance, modular invariance and periodicity and equivalence with the RNS formalism.

The above formula drastically simplifies for $n = 4$, since all the derivatives with respect to ζ_i and $\bar\zeta_i$ must act on the explicit factor not on the exponents and produce $W_L^2 \bar W_L^2 W_R^2 \bar W_R^2$.

JHEP01(2016)043

One can safely set $\mathcal{K}=0$ and $\mathcal{S}=0$ and get the expected result, i.e. the BGS formula for scattering of Type II super-gravitons in $D=10$ written in a notation suitable for $\mathcal{N}=2$ $D=4$, i.e. for 'compactification independent' states: $\mathcal{N}=2$ supergavity $\{g_{\mu\nu}, 2\psi_\mu, A_\mu\}$ and $\mathcal{N}=2$ dilaton hyper-multiplet $\{\varphi, b_{\mu\nu}, 2\zeta, c_{\mu\nu}, \alpha\}$. These are precisely the states that one gets combining two $\mathcal{N}=1$ vector multiplets, one for the Left- and one for the Right-movers.

Taking the 'square root' of the closed-string result one gets the Type I superstring super-amplitude

$$\mathcal{A}_n^{\text{super}} = \int d^2\vartheta d^2\bar{\vartheta} \int \frac{dT}{T^6} \int_{\mathcal{R}} \prod_{i=1}^{n} dz_i \frac{\partial}{d\zeta_i} \tag{8.21}$$
$$\left[(\Sigma_i \zeta_i W^i)^2 (\Sigma_i \zeta_i \bar{W}^i)^2 : e^{\mathcal{S}} : e^{-\frac{2\pi}{T}[i\mathcal{K}+\Sigma_i z_i k_i]^2} \prod_{i<j} |\theta_1(z_{ij})|^{k_i\cdot k_j} \prod_i \tilde{U}(k_i, \vartheta, \bar{\vartheta}) \right]_{\zeta_i=0}$$

Focussing on the $n=4$ case, one can safely set $\mathcal{K}$ and $\mathcal{S}$ to zero and get

$$\mathcal{A}_n^{\text{super}} = \int d^2\vartheta d^2\bar{\vartheta} W^2 \bar{W}^2 \int \frac{dT}{T^6} \int_{\mathcal{R}} \prod_{i=1}^{n} dz_i e^{-\frac{2\pi}{T}[\Sigma_i z_i k_i]^2} \prod_{i<j} |\theta_1(z_{ij})|^{k_i\cdot k_j} \tag{8.22}$$

This precisely coincides with our result for the $\mathcal{N}=4$ sector in the decompactification limit, where $\Lambda \approx 1/T^3$ (in the absence of an IR regulator) and indeed the internal SCFT is free and decouples from the space-time part.

Since for $\mathcal{N}=1,2$ sectors, the internal contribution does not simply factorize, even when the external states, such as gauge bosons or gravitons, are ρ and compactification independent, but rather produce derivatives of the Witten index, we expect the hybrid approach to fail to give the correct result if not properly amended. The success at tree level is largely due to the fact that tree-level amplitudes for gluons or gravitons are independent of the amount of supersymmetry (in so far as only minimal couplings are present) and the super-symmetrization is unique when the number of supersymmetry is chosen / given.

The probable source of the disagreement with the $D=4$ hybrid formalism [10] are the subtleties in defining the functional integration over the chiral boson ρ.[11] The difference between the $\mathcal{N}=1,2$ and the $\mathcal{N}=4$ contributions to 1-loop amplitudes is that different numbers of fermionic zero-modes can come from the compactification-dependent part of the world-sheet action through the term $R_{mnpq}\Psi_L^m\Psi_L^n\Psi_R^p\Psi_R^q$ where R_{mnpq} is the CY curvature which couples to the left and right-moving internal fermions Ψ_L^m and Ψ_R^n. In the $\mathcal{N}=4$ sector, there is a cancellation (see eq. (3.2) in [10]) between the functional integral over the ρ field and over fermionic zero modes coming from the compactification. Probably this cancellation does not occur in the $\mathcal{N}=1$ and $\mathcal{N}=2$ sectors and the coupling of Ψ_L^m and Ψ_R^n through the CY curvature affects the factorization properties.

9 Conclusions

We have shown that the 4-point vector boson amplitudes computed in [1] satisfy the correct supersymmetry Ward identities in that they vanish for non MHV helicity configurations

[11] We thank Nathan Berkovits for suggesting this interpretation.

JHEP01(2016)043

$(++++)$ and $(-+++)$. In the MHV case $(++--)$ we have simplified their expressions to an extremely compact form. The integrands only involve three functions $\mathcal{E}_{\mathcal{N}}$, $\mathcal{C}_{\mathcal{N}=1}$ and $\mathcal{F}_{\mathcal{N}}$ of the relevant modular parameter T, of the brane configuration u^I_{ab} and the 'compactification' moduli, the ubiquitous Koba-Nielsen factor $\Pi(z_i, k_i)$ and the non-holomorphic function $\mathcal{Y}(z_{ij}) = -2[\mathcal{P}(z_{ij}) - S^2(z_{ij})]$ with no poles in z_{ij}. In $\mathcal{N} = 4$ sectors $\mathcal{E}_{\mathcal{N}=4} = 0$ and only $\mathcal{F}_{\mathcal{N}=4} \approx \Lambda_{\|}$ plays a role. Somewhat unexpectedly we have found that no massless poles in two-particle channels are exposed in the $\mathcal{N} = 1, 2$ sectors either, thanks to the regular behaviour of $\mathcal{Y}$.

We have then studied the limiting IR and UV behaviour, confirming standard expectations.

Relying on the supersymmetric properties of the result we have generalised our bosonic amplitudes to manifestly supersymmetric super-amplitudes and compared the results with those obtained in the hybrid formalism and found it hard to reconcile the contributions of $\mathcal{N} = 1, 2$ sectors that can be ascribed to subtleties in performing the functional integral of the chiral boson ρ. We hope one could find a way to overcome this problem and reproduce the results we found in the RNS formalism within the (minimal) hybrid formalism. Alternatively, one could address the same issues within the pure spinor approach [18, 19, 56–59] if one could find a reliable way to partially break supersymmetry.

Based on our present analysis, there are various directions that one can explore: higher number of insertion points, higher loops, more realistic brane configurations or closed string amplitudes. Let's comment on these extensions.

Concerning higher points, 5-points look feasible since only MHV or anti-MHV amplitudes should be non vanishing, 6-points looks harder since also NMHV amplitudes corresponding to $(+++---)$ helicity configurations should be non-vanishing. Factorizations in three- and higher-particle channels could be analysed and the soft behaviour in string theory could be studied more systematically, extending the tree level analyses [60–65].

Some two-loop results are accessible and at three loops there is some work [39, 40] but starting at four loops one should expect conceptual problems in addition to practical ones [66, 67].

Barring some subtleties, it should be almost straightforward to generalise our manifestly supersymmetric results to closed superstring amplitudes and explicitly check if any form of KLT relations may be hidden in the connection.

Last but not least, phenomenologically more appealing configurations than 'regular' branes should be easy to address [30, 68–72].

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JHEP01(2016)043

A Spinor helicity formalism

In $D = 4$ one can introduce helicity spinors u^i_α and $\bar{u}^i_{\dot\alpha}$ such that $k_{i\alpha\dot\alpha} = u^i_\alpha \bar{u}^i_{\dot\alpha}$ and $a^{i+}_{\alpha\dot\alpha} = v^i_\alpha \bar{u}^i_{\dot\alpha}/\sqrt{2} v^{i\beta} u^i_\beta$ as well as $a^{i-}_{\alpha\dot\alpha} = u^i_\alpha \bar{v}^i_{\dot\alpha}/\sqrt{2}\bar{u}^{i\beta}\bar{v}^i_\beta$, with v^i_α and $\bar{v}^i_{\dot\alpha}$ arbitrary Weyl spinors associated to gauge variations in that $v^i_\alpha \to v^i_\alpha + \lambda u^i_\alpha$ yields

$$a^{i+}_{\alpha\dot\alpha} \to a^{i+}_{\alpha\dot\alpha} + \lambda \frac{u^i_\alpha \bar{u}^i_{\dot\alpha}}{v^{i\beta} u^i_\beta} = a^{i+}_{\alpha\dot\alpha} + \frac{\lambda}{v^{i\beta} u^i_\beta} k^i_{\alpha\dot\alpha} \tag{A.1}$$

and similarly for $a^{i-}_{\alpha\dot\alpha}$. The bilinears can be written as $u^\alpha_i u_{j\alpha} = \langle ij\rangle = -\langle ji\rangle$, $\bar{u}_{i\dot\alpha}\bar{u}^{\dot\alpha}_j = [ij] = -[ji]$ and $2k_i{\cdot}k_j = -\langle ij\rangle[ij]$, momentum conservation $\sum_i |i\rangle[i| = 0 = \sum_i |i]\langle i|$. For more details see *e. g.* [36].

A fundamental ingredient in our analysis are the traces of the tensors f's representing the linearised field-strengths. We introduce a compact notation: $(f_1 \dots f_n) = f^{\mu_1}{}_{\mu_2} \dots f^{\mu_n}{}_{\mu_1}$. This quantities are gauge and Lorentz invariant by definition. In our computation we meet traces with two, three and four f's, we need to calculate their values for fixed helicity configurations. It is also useful to decompose the tensors f in positive and negative helicity part $f_{\mu\nu} = f^+_i \bar{\Sigma}^i_{\mu\nu} + f^-_{i'} \Sigma^{i'}_{\mu\nu}$ with $i, i' = 1, 2, 3$. The matrices Σ and $\bar{\Sigma}$ provide the bases of the representations 3_L and 3_R of $\mathrm{SL}(2,\mathbb{C})$ thus they are self-dual and antiself-dual matrices and satisfy

$$\epsilon_{\mu\nu}{}^{\rho\sigma}\bar{\Sigma}^i_{\rho\sigma} = +i\bar{\Sigma}^i_{\mu\nu} \quad \epsilon_{\mu\nu}{}^{\rho\sigma}\Sigma^{i'}_{\rho\sigma} = -i\Sigma^{i'}_{\mu\nu} \quad \bar{\Sigma}^i\Sigma^{i'} = \Sigma^{i'}\bar{\Sigma}^i \qquad \Sigma^{i'}\Sigma^{j'} = \delta^{i'j'}\mathbb{1} + i\epsilon^{i'j'}{}_{k'}\Sigma^{k'} \tag{A.2}$$

$$\mathrm{Tr}\Sigma^{i'} = \mathrm{Tr}\bar{\Sigma}^i = 0 \quad \mathrm{Tr}\Sigma^{i'}\bar{\Sigma}^i = 0 \quad \mathrm{Tr}\Sigma^{i'}\Sigma^{j'} = 4\delta^{i'j'} \quad \mathrm{Tr}\bar{\Sigma}^i\bar{\Sigma}^j = 4\delta^{ij} \tag{A.3}$$

The product of two vectors in $(1,0)$ and $(0,1)$ can be related to traces in the Lorentz indices: $4f^+{\cdot}g^+ = 4f^+_i(g^+)^i = (f^+g^+)$ and $4f^-{\cdot}g^- = (f^-g^-)$. Lorentz invariance helps to recognize when a trace vanishes: self-dual and antiself-dual matrices cannot contract with one another. In particular traces with an odd number of f's with positive (or negative) helicity vanish.

We start to compute traces from products of two f's. We have two independent cases: $(-+)$ and $(++)$. $(f^-_1 f^+_2) = 0$ is zero for Lorentz invariance. In the second case we can use the gauge choice $q_1 = q_2 = q$ to cancel some terms and obtain

$$(f^+_1 f^+_2) = 2a^+_1{\cdot}k_2\, a^+_2{\cdot}k_1 = \frac{[12]\langle 2q\rangle}{\langle 1q\rangle}\frac{[21]\langle 1q\rangle}{\langle 2q\rangle} = -[12]^2 \tag{A.4}$$

For traces with three f's, we have two independent cases: $(-++)$ and $(+++)$. $(f^-_1 f^+_2 f^+_3) = 0$ is zero for Lorentz invariance. For $(+++)$ we use the gauge $q_1 = q_2 = q_3 = q$

$$(f^+_1 f^+_2 f^+_3) = a^+_1{\cdot}k_3\, a^+_2{\cdot}k_1\, a^+_3{\cdot}k_2 - a^+_1{\cdot}k_2\, a^+_2{\cdot}k_3\, a^+_3{\cdot}k_1 = -\frac{1}{\sqrt{2}}[12][23][31] \tag{A.5}$$

For traces with four f's, we have three independent cases: $(++++)$, $(-+++)$ and $(--++)$. $(f^-_1 f^+_2 f^+_3 f^+_4) = 0$ is zero for Lorentz invariance. For $(++++)$ we use the gauge $q_1 = q_2 = q_3 = q_4 = q$

$$(f^+_1 f^+_2 f^+_3 f^+_4) = a^+_1{\cdot}k_4\, a^+_2{\cdot}k_1\, a^+_3{\cdot}k_2\, a^+_4{\cdot}k_3 + a^+_1{\cdot}k_2\, a^+_2{\cdot}k_3\, a^+_3{\cdot}k_4\, a^+_4{\cdot}k_1 = \frac{1}{2}[12][23][34][41] \tag{A.6}$$

JHEP01(2016)043

To compute $(--++)$ we choose the gauge $q_1 = q_2 = k_3$ and $q_3 = q_4 = k_2$

$$(f_1^- f_2^- f_3^+ f_4^+) = a_1^- \cdot a_4^+ \, a_2^- \cdot k_1 \, a_3^+ \cdot k_4 \, k_2 \cdot k_3 = \frac{1}{4}\langle 12\rangle^2 [34]^2 = -\frac{1}{4}\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 41\rangle} st \tag{A.7}$$

B Elliptic functions

Let $q = e^{2\pi i \tau}$, with τ complex, the Jacobi θ functions are defined as

$$\theta[^{\alpha}_{\beta}](z|\tau) = \sum_{k\in\mathbb{Z}} q^{(k-\alpha)^2/2} e^{2\pi i (z-\beta)(k-\alpha)} \tag{B.1}$$

with α and β real numbers, representing the spin structures. θ functions solve the heat equation $4\pi i \partial_\tau \theta = \partial_z^2 \theta$. There are identifications between different values of α and β

$$\theta[^{\alpha+k}_{\beta}](z|\tau) = \theta[^{\alpha}_{\beta}](z|\tau) \qquad \theta[_{-\beta}^{\alpha}](z|\tau) = \theta[^{-\alpha}_{\beta}](-z|\tau) \qquad \theta[_{\beta+k}^{\alpha}](z|\tau) = \theta[^{\alpha}_{\beta}](z-k|\tau) \tag{B.2}$$

This functions enjoy pseudo-periodicity properties

$$\theta[^{\alpha}_{\beta}](z+k|\tau) = e^{-2\pi i k\alpha}\theta[^{\alpha}_{\beta}](z|\tau) \quad \theta[^{\alpha}_{\beta}](z+k\tau|\tau) = e^{-2\pi i k(z-\beta+\tau/2)}\theta[^{\alpha}_{\beta}](z|\tau) \tag{B.3}$$

Under the modular transformations T and S, one finds

$$\theta[^{\alpha}_{\beta}](z|\tau+k) = e^{-i\pi k\alpha(\alpha-1)}\theta[_{\beta+k(\alpha-1/2)}^{\alpha}](z|\tau) \tag{B.4}$$

$$\theta[^{\alpha}_{\beta}]\left(\frac{z}{\tau}\middle| -\frac{1}{\tau}\right) = (-i\tau)^{1/2} e^{i\pi(2\alpha\beta + z^2/\tau)}\theta[_{-\alpha}^{\beta}](z|\tau) \tag{B.5}$$

Often we omit the dependence on τ. For our purposes, we are interested in four particular θ functions

$$\begin{aligned}
\theta[^{1/2}_{1/2}](z|\tau) &= \theta_1(z|\tau) = 2q^{1/8}\sin(\pi z)\prod_{n=1}^{\infty}(1-q^n)(1-e^{2\pi i z}q^n)(1-e^{-2\pi i z}q^n)\\
\theta[^{1/2}_{0}](z|\tau) &= \theta_2(z|\tau) = 2q^{1/8}\cos(\pi z)\prod_{n=1}^{\infty}(1-q^n)(1+e^{2\pi i z}q^n)(1+e^{-2\pi i z}q^n)\\
\theta[^{0}_{0}](z|\tau) &= \theta_3(z|\tau) = \prod_{n=1}^{\infty}(1-q^n)(1+e^{2\pi i z}q^{n-1/2})(1+e^{-2\pi i z}q^{n-1/2})\\
\theta[_{1/2}^{0}](z|\tau) &= \theta_4(z|\tau) = \prod_{n=1}^{\infty}(1-q^n)(1-e^{2\pi i z}q^{n-1/2})(1-e^{-2\pi i z}q^{n-1/2})
\end{aligned} \tag{B.6}$$

Under $z \leftrightarrow -z$ θ_1 is odd and θ_2, θ_3 and θ_4 are even.

There are other two other ubiquitous elliptic functions: the Dedekind function $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ and the Weierstrass function $\mathcal{P}(z,\tau) = \partial_z^2 \log\theta_1(z|\tau) - 2\eta_1(\tau)$, where $\eta_1(\tau) = -2\pi i\partial_\tau \log\eta(\tau)$. Dedekind function is related also to θ_1' by $\theta_1'(0) = 2\pi\eta^3$. Under modular transformations, Weierstrass function is a modular form with weight two and Dedekind function transforms as

$$\mathcal{P}\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2\mathcal{P}(z,\tau) \quad \eta(\tau+1) = e^{i\pi/12}\eta(\tau) \quad \eta\left(-\frac{1}{\tau}\right) = (-i\tau)^{1/2}\eta(\tau) \tag{B.7}$$

JHEP01(2016)043

The contractions of free bosons and fermions at one-loop are:

$$\langle\langle X_{\mu_1}\, X_{\mu_2}\rangle\rangle = \alpha' \eta_{\mu_1\mu_2} \mathcal{G}_\Sigma(z_{12}) \qquad \langle\langle \psi_{\mu_1}\, \psi_{\mu_2}\rangle\rangle_\alpha = \alpha' \eta_{\mu_1\mu_2} S_\alpha(z_{12}) \tag{B.8}$$

$$\langle\langle \partial X^{\mu_1}\, e^{ik_2 X_2}\rangle\rangle = i\alpha' k^{\mu_1} \partial_1 \mathcal{G}_\Sigma(z_{12}) \qquad \langle\langle \partial X^{\mu_1}\, \partial X^{\mu_2}\rangle\rangle = \alpha' \eta^{\mu_1\mu_2} \partial_1\partial_2 \mathcal{G}_\Sigma(z_{12}) \tag{B.9}$$

Where $\mathcal{G}_\Sigma$ is the bosonic propagator (Bargmann kernel) on the surface and S_α is the fermion propagator (Szego kernel). The bosonic propagator on the torus is

$$\mathcal{G}_T(z_1, z_2; \tau) = -\frac{1}{2}\left[\log\left|\frac{\theta_1(z_1 - z_2|\tau)}{\theta_1'(0|\tau)}\right|^2 - 2\pi \frac{[\mathrm{Im}\,(z_1 - z_2)]^2}{\mathrm{Im}\,\tau}\right] \tag{B.10}$$

It's easy to see that $\mathcal{G}_T(z_1, z_2; \tau)$ is even under $z_1 \leftrightarrow z_2$, bi-periodic and quasi-invariant under modular transformation: the unique non-trivial transformation is $\mathcal{G}_T(z_1/\tau, z_2/\tau; -1/\tau) = \mathcal{G}_T(z_1, z_2; \tau) + \frac{1}{2}\log|\tau|^2$. We can say that the propagator is an "inhomogeneous" modular form of degree zero. The annulus obtains from the torus by means of the involution $\tilde{z} = 1 - \bar{z}$ and the modular parameter $\tau_\mathcal{A} = iT/2$ [26, 37–40], so the propagator between two points z_1 and z_2 is the sum of the propagators on the torus between z_1 and all the images of z_2, i.e. z_2 itself and $1 - \bar{z}_2$.

$$\begin{aligned}\mathcal{G}_\mathcal{A}(z_1, z_2; \tau_\mathcal{A}) = \frac{1}{2}\big[&\mathcal{G}_T(z_1, z_2; \tau_\mathcal{A}) + \mathcal{G}_T(z_1, 1 - \bar{z}_2; \tau_\mathcal{A}) \\ &+ \mathcal{G}_T(1 - \bar{z}_1, z_2; \tau_\mathcal{A}) + \mathcal{G}_T(1 - \bar{z}_1, 1 - \bar{z}_2; \tau_\mathcal{A})\big]\end{aligned} \tag{B.11}$$

On the annulus boundaries ($z = \tau\nu$ or $z = \tau\nu + 1/2$), the annulus propagator takes the form

$$\mathcal{G}_\mathcal{A}(z_1, z_2; \tau_\mathcal{A}) = -2\left[\log\frac{\theta_1(z_1 - z_2|\tau)}{\theta_1'(0|\tau)} - \pi\frac{[\mathrm{Im}\,(z_1 - z_2)]^2}{\mathrm{Im}\,\tau}\right] \tag{B.12}$$

The Möbius strip can be obtained in may ways, one consists in using the same involution used to construct the annulus but considering a torus with modular parameter $\tau_\mathcal{M} = \tau_\mathcal{A} + 1/2$ [26, 37–40]. The propagator is similar to the annulus propagator

$$\mathcal{G}_\mathcal{M}(z_1, z_2; \tau_\mathcal{M}) = \mathcal{G}_\mathcal{A}(z_1, z_2; \tau_\mathcal{M}) \tag{B.13}$$

The explicit formula of the Szego kernel or fermionic propagator is

$$S_\alpha(z_1, z_2; \tau) = \begin{cases} -\partial_1 \mathcal{G}_\mathcal{A}(z_1, z_2), & \text{if } \alpha = 1, \\ \dfrac{\theta_\alpha(z_1 - z_2)}{\theta_1(z_1 - z_2)}\dfrac{\theta_1'(0)}{\theta_\alpha(0)}, & \text{if } \alpha \neq 1. \end{cases} \tag{B.14}$$

where $\mathcal{G}_\mathcal{A}$ is not computed on the boundaries. S_α is an odd function under $z_1 \leftrightarrow z_2$. We often use S_{ij} instead $S_1(z_{ij}) = -\partial_i \mathcal{G}_\mathcal{A}(z_{ij})$. The fermionic propagator is a bi-periodic modular form with weight one. When the insertions are on a boundary of the annulus, the explicit form of S_{ij} is

$$S_{ij} = -\frac{\partial}{\partial z_i}\left(\mathcal{G}_\mathcal{A}(z_i, z_j)\big|_{\bar{z}=-z}\right) = 2\left[\frac{\theta_1'(z_{ij})}{\theta_1(z_{ij})} - 2\pi i \frac{\mathrm{Im}\, z_{ij}}{\mathrm{Im}\,\tau_\mathcal{A}}\right] \tag{B.15}$$

JHEP01(2016)043

For non-planar amplitude it is necessary to compute a propagator between two points on different boundaries. The propagator can be obtained with the parametrization $z = \tau\nu + 1/2$

$$\begin{aligned}\mathcal{G}^T_{\mathcal{A}}(z_1,z_2) = \mathcal{G}_{\mathcal{A}}(\tau\nu_1,\tau\nu_2+1/2) &= -2\left[\log\frac{\theta_2(\tau\nu_{12}|\tau)}{\theta_1'(0|\tau)} - \pi\frac{\mathrm{Im}^{\,2} z_{12}}{\mathrm{Im}\,\tau}\right] S^T_{ij} \\ &= 2\left[\frac{\theta_2'(\tau\nu_{ij})}{\theta_2(\tau\nu_{ij})} - 2\pi i\frac{\mathrm{Im}\, z_{ij}}{\mathrm{Im}\,\tau_{\mathcal{A}}}\right]\end{aligned} \tag{B.16}$$

In the computation we bear in mind some properties of θ functions and propagators. In the even spin structures, products of fermionic propagators can be simplified using the two identities

$$S^2_\alpha(z,\tau) = \mathcal{P}(z,\tau) - e_{\alpha-1}(\tau) \quad , \quad S_\alpha(z_{12})S_\alpha(z_{23}) = -S_\alpha(z_{13})\omega_{123} - S'_\alpha(z_{13}) \tag{B.17}$$

where $e_{\alpha-1}(\tau) = 4\pi i\partial_\tau \ln[\theta_{\alpha-1}(0|\tau)/\eta(\tau)]$ and $\omega_{123} = S_{12}+S_{23}+S_{31}$. Deriving w.r.t. z the function $\mathcal{P}-S^2_\alpha$ we find $2S_\alpha(z)S'_\alpha(z) = \partial_z\mathcal{P}(z)$, thus the derivative of S^2_α is independent of α.

There is a formula that links the Weierstrass function and the square of the propagator derivative, it's linked to a function that we call $\mathcal{Y}(z)$

$$\mathcal{Y}(z) = -2\left[\mathcal{P}(z) - S^2(z)\right] \tag{B.18}$$

where $S(z)$ is in its general form, not computed on a boundary. This function is a modular form with weight two and has no poles. One can prove a generalized Fay identity [41, 42]

$$\Omega_{123} = S_{12}S_{23} + S_{23}S_{31} + S_{31}S_{12} = -\mathcal{Y}_{12} - \mathcal{Y}_{23} - \mathcal{Y}_{31} \tag{B.19}$$

In order to study the limiting behaviour of $\mathcal{Y}$ it is convenient to first consider the expansion of θ_1 in powers of z

$$\theta_1(z) = z\theta_1'(0)\left[1 - \eta_1 z^2 + \frac{1}{5}\left(\eta_2 + 3\eta_1^2\right)z^4\right] + \mathcal{O}(z^7) \tag{B.20}$$

We use this expansions to compute $\mathcal{Y}(z)$

$$\begin{aligned}\mathcal{Y}(z = iT\nu/2) &= -\frac{1}{2}\left[-\partial_z^2\log\theta_1(z) - 2\eta_1 - \left(\partial_z\theta_1(z) - 2\pi i\frac{\mathrm{Im}\, z}{\mathrm{Im}\,\tau}\right)^2\right] \\ &= -8\left(\eta_1 + \frac{2\pi}{T}\right) - 8\left(\frac{T^2}{10}(\eta_2+3\eta_1^2) + \pi T\eta_1 + \pi^2\right)\nu^2 + \mathcal{O}(\nu^4)\end{aligned} \tag{B.21}$$

See [41] for more details on relations between elliptic functions and string one-loop amplitudes.

B.1 Vanishing contractions due to Riemann identities

Amplitudes assume the form $\sum_\alpha c_\alpha A_\alpha \mathcal{Z}^{\mathcal{N}}_\alpha$, with A_α some function of z and τ. Contractions with one bilinear are zero because of normal ordering, $\langle{:}\psi_1^{\mu_1}\psi_2^{\mu_1}{:}\rangle_\alpha = 0$. In the even sector contractions of bosonic operators only give zero. To see that we use the Riemann identity:

$$\sum_{\alpha\neq 1} c_\alpha\theta_\alpha(z_1)\theta_\alpha(z_2)\theta_\alpha(z_3)\theta_\alpha(z_4) = \theta_1(z_1')\theta_1(z_2')\theta_1(z_3')\theta_1(z_4') - \theta_1(z_1'')\theta_1(z_2'')\theta_1(z_3'')\theta_1(z_4'') \tag{B.22}$$

JHEP01(2016)043

Where the variables z'_i and z''_i are linear combinations of the z_i

$$\begin{pmatrix} z'_1 \\ z'_2 \\ z'_3 \\ z'_4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \qquad \begin{pmatrix} z''_1 \\ z''_2 \\ z''_3 \\ z''_4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \tag{B.23}$$

The $\theta_1(z)$ functions vanish when $z = 0 \mod 1$, thus this quantity is zero if at least one of the z'_i and one of the z''_i are zero. Considering a generic purely bosonic correlator $\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle_\alpha = \langle\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle\rangle \mathcal{Z}^{\mathcal{N}}_\alpha$, the sum over the even spin structures produces a vanishing result in the most general case with $\mathcal{N} = 1$ supersymmetry, in that

$$\sum_{\alpha \neq 1} c_\alpha \mathcal{Z}^{\mathcal{N}}_\alpha \propto \sum_{\alpha \neq 1} c_\alpha \theta_\alpha(0) \theta_\alpha(u^1_{ab}) \theta_\alpha(u^2_{ab}) \theta_\alpha(u^3_{ab}) = 0 \tag{B.24}$$

To understand why this expression is zero it is enough to consider the first line of the two matrices: $2z'_1 = 2z''_1 = u^1_{ab} + u^2_{ab} + u^3_{ab} = 0 \mod 1$.

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References

[1] M. Bianchi and A.V. Santini, *String predictions for near future colliders from one-loop scattering amplitudes around D-brane worlds*, *JHEP* **12** (2006) 010 [hep-th/0607224] [INSPIRE].

[2] M. Matone and R. Volpato, *Higher genus superstring amplitudes from the geometry of moduli space*, *Nucl. Phys.* **B 732** (2006) 321 [hep-th/0506231] [INSPIRE].

[3] P. Tourkine, *Tropical amplitudes*, arXiv:1309.3551 [INSPIRE].

[4] E. Witten, *Notes on holomorphic string and superstring theory measures of low genus*, arXiv:1306.3621 [INSPIRE].

[5] E. Witten, *More on superstring perturbation theory*, arXiv:1304.2832 [INSPIRE].

[6] E. Witten, *Superstring perturbation theory revisited*, arXiv:1209.5461 [INSPIRE].

[7] M.B. Green, C.R. Mafra and O. Schlotterer, *Multiparticle one-loop amplitudes and S-duality in closed superstring theory*, *JHEP* **10** (2013) 188 [arXiv:1307.3534] [INSPIRE].

[8] M.B. Green, J.H. Schwarz and L. Brink, $N = 4$ *Yang-Mills and* $N = 8$ *supergravity as limits of string theories*, *Nucl. Phys.* **B 198** (1982) 474 [INSPIRE].

[9] A. Brandhuber, B. Spence, G. Travaglini and K. Zoubos, *One-loop MHV rules and pure Yang-Mills*, *JHEP* **07** (2007) 002 [arXiv:0704.0245] [INSPIRE].

[10] N. Berkovits and B.C. Vallilo, *One loop N point superstring amplitudes with manifest* $D = 4$ *supersymmetry*, *Nucl. Phys.* **B 624** (2002) 45 [hep-th/0110168] [INSPIRE].

[11] N. Berkovits, *Covariant quantization of the Green-Schwarz superstring in a Calabi-Yau background*, *Nucl. Phys.* **B 431** (1994) 258 [hep-th/9404162] [INSPIRE].

JHEP01(2016)043

[12] N. Berkovits, C. Vafa and E. Witten, *Conformal field theory of AdS background with Ramond-Ramond flux*, *JHEP* **03** (1999) 018 [hep-th/9902098] [INSPIRE].

[13] N. Berkovits, *Quantization of the superstring with manifest* U(5) *super-Poincaré invariance*, *Phys. Lett.* **B 457** (1999) 94 [hep-th/9902099] [INSPIRE].

[14] N. Berkovits, S. Gukov and B.C. Vallilo, *Superstrings in* 2*D backgrounds with RR flux and new extremal black holes*, *Nucl. Phys.* **B 614** (2001) 195 [hep-th/0107140] [INSPIRE].

[15] S. Kachru and E. Silverstein, 4*D conformal theories and strings on orbifolds*, *Phys. Rev. Lett.* **80** (1998) 4855 [hep-th/9802183] [INSPIRE].

[16] K. Zoubos, *Review of AdS/CFT integrability, chapter IV.2: deformations, orbifolds and open boundaries*, *Lett. Math. Phys.* **99** (2012) 375 [arXiv:1012.3998] [INSPIRE].

[17] E. Kiritsis, *String theory in a nutshell*, Princeton University Press, Princeton U.S.A. (2007).

[18] C.R. Mafra, O. Schlotterer and S. Stieberger, *Complete* N*-point superstring disk amplitude I. Pure spinor computation*, *Nucl. Phys.* **B 873** (2013) 419 [arXiv:1106.2645] [INSPIRE].

[19] C.R. Mafra, O. Schlotterer and S. Stieberger, *Complete* N*-point superstring disk amplitude II. Amplitude and hypergeometric function structure*, *Nucl. Phys.* **B 873** (2013) 461 [arXiv:1106.2646] [INSPIRE].

[20] L.A. Barreiro and R. Medina, *Revisiting the S-matrix approach to the open superstring low energy effective lagrangian*, *JHEP* **10** (2012) 108 [arXiv:1208.6066] [INSPIRE].

[21] L.A. Barreiro and R. Medina, *RNS derivation of N-point disk amplitudes from the revisited S-matrix approach*, *Nucl. Phys.* **B 886** (2014) 870 [arXiv:1310.5942] [INSPIRE].

[22] M.B. Green, J.H. Schwarz and E. Witten, *Superstring Theory 25th anniversary edition*, Cambridge University Press, Camrbidge U.K. (2012).

[23] M.B. Green and J.H. Schwarz, *The hexagon gauge anomaly in Type I superstring theory*, *Nucl. Phys.* **B 255** (1985) 93 [INSPIRE].

[24] D.J. Gross and P.F. Mende, *Modular subgroups, odd spin structures and gauge invariance in the heterotic string*, *Nucl. Phys.* **B 291** (1987) 653 [INSPIRE].

[25] D. Friedan, S.H. Shenker and E.J. Martinec, *Covariant quantization of superstrings*, *Phys. Lett.* **B 160** (1985) 55 [INSPIRE].

[26] M. Bianchi and A. Sagnotti, *The partition function of the* SO(8192) *bosonic string*, *Phys. Lett.* **B 211** (1988) 407 [INSPIRE].

[27] M. Bianchi and A. Sagnotti, *Twist symmetry and open string Wilson lines*, *Nucl. Phys.* **B 361** (1991) 519 [INSPIRE].

[28] M. Bianchi and A. Sagnotti, *On the systematics of open string theories*, *Phys. Lett.* **B 247** (1990) 517 [INSPIRE].

[29] C. Angelantonj, M. Bianchi, G. Pradisi, A. Sagnotti and Ya.S. Stanev, *Chiral asymmetry in four-dimensional open string vacua*, *Phys. Lett.* **B 385** (1996) 96 [hep-th/9606169] [INSPIRE].

[30] C. Angelantonj and A. Sagnotti, *Open strings*, *Phys. Rept.* **371** (2002) 1 [hep-th/0204089] [INSPIRE].

[31] E. Kiritsis and N.A. Obers, *Heterotic type-I duality in* $D < 10$*-dimensions, threshold corrections and D instantons*, *JHEP* **10** (1997) 004 [hep-th/9709058] [INSPIRE].

[32] P. Anastasopoulos, M. Bianchi, G. Sarkissian and Y.S. Stanev, *On gauge couplings and thresholds in type I gepner models and otherwise*, *JHEP* **03** (2007) 059 [hep-th/0612234] [INSPIRE].

JHEP01(2016)043

[33] J.A. Minahan, *One loop amplitudes on orbifolds and the renormalization of coupling constants*, *Nucl. Phys.* **B 298** (1988) 36 [INSPIRE].

[34] M. Berg, M. Haack and J.U. Kang, *One-loop Kähler metric of D-branes at angles*, *JHEP* **11** (2012) 091 [arXiv:1112.5156] [INSPIRE].

[35] M. Berg, M. Haack, J.U. Kang and S. Sjörs, *Towards the one-loop Kähler metric of Calabi-Yau orientifolds*, *JHEP* **12** (2014) 077 [arXiv:1407.0027] [INSPIRE].

[36] H. Elvang and Y.-t. Huang, *Scattering amplitudes*, arXiv:1308.1697 [INSPIRE].

[37] A. Sagnotti, *Open Strings and their Symmetry Groups*, in the proceedings of the *NATO Advanced Summer Institute on Nonperturbative Quantum Field Theory*, July 16–30, Cargese, France (1987), hep-th/0208020 [INSPIRE].

[38] G. Pradisi and A. Sagnotti, *Open string orbifolds*, *Phys. Lett.* **B 216** (1989) 59 [INSPIRE].

[39] M. Bianchi and A. Sagnotti, *The geometry of open string partition functions*, in the proceedings of the *Theoretical Advanced Study Institute in Elementary Particle Physics: Particles, Strings and Supernovae (TASI 88)*, June 5–July 1, Providence, U.S.A. (1988).

[40] M. Bianchi and A. Sagnotti, *Open strings and the relative modular group*, *Phys. Lett.* **B 231** (1989) 389 [INSPIRE].

[41] J. Broedel, C.R. Mafra, N. Matthes and O. Schlotterer, *Elliptic multiple zeta values and one-loop superstring amplitudes*, *JHEP* **07** (2015) 112 [arXiv:1412.5535] [INSPIRE].

[42] J.D. Fay, *Theta functions on riemann surfaces*, Lecture Notes in Mathematics volume 352, Springer, Berlin, Germany (1973).

[43] M. Cvetič and P. Langacker, *Implications of Abelian extended gauge structures from string models*, *Phys. Rev.* **D 54** (1996) 3570 [hep-ph/9511378] [INSPIRE].

[44] L.E. Ibáñez, R. Rabadán and A.M. Uranga, *Anomalous* U(1)*'s in type-I and type IIB* $D = 4$, $N = 1$ *string vacua*, *Nucl. Phys.* **B 542** (1999) 112 [hep-th/9808139] [INSPIRE].

[45] G. Aldazabal, A. Font, L.E. Ibáñez and G. Violero, $D = 4$, $N = 1$, *type IIB orientifolds*, *Nucl. Phys.* **B 536** (1998) 29 [hep-th/9804026] [INSPIRE].

[46] I. Antoniadis, E. Kiritsis and J. Rizos, *Anomalous* U(1)*s in type* 1 *superstring vacua*, *Nucl. Phys.* **B 637** (2002) 92 [hep-th/0204153] [INSPIRE].

[47] P. Anastasopoulos, 4*D anomalous* U(1)*'s, their masses and their relation to* 6*D anomalies*, *JHEP* **08** (2003) 005 [hep-th/0306042] [INSPIRE].

[48] P. Anastasopoulos, M. Bianchi, E. Dudas and E. Kiritsis, *Anomalies, anomalous* U(1)*'s and generalized Chern-Simons terms*, *JHEP* **11** (2006) 057 [hep-th/0605225] [INSPIRE].

[49] M. Bianchi, G. Pradisi and A. Sagnotti, *Toroidal compactification and symmetry breaking in open string theories*, *Nucl. Phys.* **B 376** (1992) 365 [INSPIRE].

[50] M. Bianchi, *A note on toroidal compactifications of the type-I superstring and other superstring vacuum configurations with sixteen supercharges*, *Nucl. Phys.* **B 528** (1998) 73 [hep-th/9711201] [INSPIRE].

[51] E. Witten, *Toroidal compactification without vector structure*, *JHEP* **02** (1998) 006 [hep-th/9712028] [INSPIRE].

[52] M. Bianchi, G. Inverso, J.F. Morales and D.R. Pacifici, *Unoriented quivers with flavour*, *JHEP* **01** (2014) 128 [arXiv:1307.0466] [INSPIRE].

[53] O. Aharony, A. Fayyazuddin and J.M. Maldacena, *The large-N limit of* $N = 2$, $N = 1$ *field theories from three-branes in F-theory*, *JHEP* **07** (1998) 013 [hep-th/9806159] [INSPIRE].

JHEP01(2016)043

[54] M. Bianchi et al., *Mass-deformed brane tilings*, *JHEP* **10** (2014) 027 [arXiv:1408.1957] [INSPIRE].

[55] E. Dudas and J. Mourad, *String theory predictions for future accelerators*, *Nucl. Phys.* **B 575** (2000) 3 [hep-th/9911019] [INSPIRE].

[56] N. Berkovits and D.Z. Marchioro, *Relating the Green-Schwarz and pure spinor formalisms for the superstring*, *JHEP* **01** (2005) 018 [hep-th/0412198] [INSPIRE].

[57] O.A. Bedoya and N. Berkovits, *GGI lectures on the pure spinor formalism of the superstring*, arXiv:0910.2254 [INSPIRE].

[58] N. Berkovits and E. Witten, *Supersymmetry breaking effects using the pure spinor formalism of the superstring*, *JHEP* **06** (2014) 127 [arXiv:1404.5346] [INSPIRE].

[59] N. Berkovits, *Origin of the pure spinor and Green-Schwarz formalisms*, *JHEP* **07** (2015) 091 [arXiv:1503.03080] [INSPIRE].

[60] M. Bianchi, S. He, Y.-t. Huang and C. Wen, *More on soft theorems: trees, loops and strings*, *Phys. Rev.* **D 92** (2015) 065022 [arXiv:1406.5155] [INSPIRE].

[61] M. Bianchi and A.L. Guerrieri, *On the soft limit of open string disk amplitudes with massive states*, *JHEP* **09** (2015) 164 [arXiv:1505.05854] [INSPIRE].

[62] Z. Bern, S. Davies, P. Di Vecchia and J. Nohle, *Low-energy behavior of gluons and gravitons from gauge invariance*, *Phys. Rev.* **D 90** (2014) 084035 [arXiv:1406.6987] [INSPIRE].

[63] P. Di Vecchia, R. Marotta and M. Mojaza, *Double-soft behavior for scalars and gluons from string theory*, *JHEP* **12** (2015) 150 [arXiv:1507.00938] [INSPIRE].

[64] P. Di Vecchia, R. Marotta and M. Mojaza, *Soft theorem for the graviton, dilaton and the Kalb-Ramond field in the bosonic string*, *JHEP* **05** (2015) 137 [arXiv:1502.05258] [INSPIRE].

[65] M. Ademollo et al., *Soft dilations and scale renormalization in dual theories*, *Nucl. Phys.* **B 94** (1975) 221 [INSPIRE].

[66] R. Donagi and E. Witten, *Supermoduli space is not projected*, arXiv:1304.7798 [INSPIRE].

[67] A. Sen and E. Witten, *Filling the gaps with PCO's*, *JHEP* **09** (2015) 004 [arXiv:1504.00609] [INSPIRE].

[68] R. Blumenhagen, B. Körs, D. Lüst and S. Stieberger, *Four-dimensional string compactifications with D-branes, orientifolds and fluxes*, *Phys. Rept.* **445** (2007) 1 [hep-th/0610327] [INSPIRE].

[69] D. Lüst, S. Stieberger and T.R. Taylor, *The LHC string hunter's companion*, *Nucl. Phys.* **B 808** (2009) 1 [arXiv:0807.3333] [INSPIRE].

[70] L.A. Anchordoqui et al., *LHC phenomenology for string hunters*, *Nucl. Phys.* **B 821** (2009) 181 [arXiv:0904.3547] [INSPIRE].

[71] D. Lüst, O. Schlotterer, S. Stieberger and T.R. Taylor, *The LHC string Hunter's companion (II): five-particle amplitudes and universal properties*, *Nucl. Phys.* **B 828** (2010) 139 [arXiv:0908.0409] [INSPIRE].

[72] L.E. Ibáñez and A.M. Uranga, *String theory and particle physics*, Cambridge University Press, Cambrdige U.K. (2012).

JHEP01(2016)043