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Positive solutions for a second-order differential equation with integral boundary conditions and deviating arguments

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Abstract

Using a well-known fixed point theorem on cones, we study the number of positive solutions for a second-order differential equation with integral boundary conditions and deviating arguments. We discuss our problems under two cases when the deviating arguments are delayed and advanced. Our results extend and improve those of Boucherif (Nonlinear Anal. 70:364-371, 2009) and Kong (Nonlinear Anal. 72:2628-2638, 2010) by generalizing the nonlinearity $f(t,u(t))$ to $f(t,u(\alpha(t)))$ with general $\alpha(t) \not\equiv t$. The dependence of solutions on the parameter λ is also studied.

Keywords: differential equations with advanced or delayed arguments; integral boundary conditions; number of positive solutions; parameter dependence of positive solutions

1 Introduction

Boundary value problems with integral boundary conditions arise naturally in thermal conduction problems [3], semiconductor problems [4], hydrodynamic problems [5], thermostat problems [6] and so on. It is interesting to point out that such problems include two, three, multi-point and nonlocal boundary value problems as special cases and have been extensively studied in the last ten years; see for example [1, 7–21]. Recently, Boucherif [1] applied the fixed point theorem in cones to study the existence of positive solutions for the problem given by

$$\begin{cases} x''(t) = f(t,x(t)), & 0 < t < 1, \\ x(0) - cx'(0) = \int_0^1 g_0(s)x(s)\,ds, \\ x(1) - dx'(1) = \int_0^1 g_1(s)x(s)\,ds, \end{cases}$$

where $f : [0,1] \times \mathbf{R} \to \mathbf{R}$ is continuous, $g_0, g_1 : [0,1] \to [0,+\infty)$ are continuous and positive, c and d are nonnegative real parameters. The author established some excellent results for the existence of positive solutions to the above problem by using the fixed point theorem in cones.

We notice that a type of boundary value problems with deviating arguments have received much attention. For example, in [22], Jankowski considered the following three-point boundary value problem:

$$\begin{cases} x''(t) = f(t, x(t), x(\alpha(t))), & t \in [0, T], \\ x(0) = 0, \quad x(T) = rx(\gamma), & 0 < \gamma < T. \end{cases}$$

The author obtained some solvability results by using monotone iterative technique.

In [23], Yang *et al.* studied the existence and multiplicity of positive solutions to a three-point boundary value problem with an advanced argument

$$\begin{cases} x''(t) + a(t)f(x(\alpha(t))) = 0, & t \in (0,1), \\ x(0) = 0, \quad bx(\eta) = x(1), \end{cases}$$

where $0 < \eta < 1$, $b > 0$, and $1 - b\eta > 0$. The main tool is the fixed point index theory. For some other excellent results and applications of the case that ordinary differential equation with deviating arguments to a variety of problems from Jankowski [24–27], Jiang and Wei [28], Wang [29], Wang *et al.* [30] and Hu *et al.* [31]. However, few papers have been reported on the same problems with a parameter.

At the same time, the dependence of positive solution $x_\lambda(t)$ on the parameter λ has received much attention; see [2, 20, 32–37] and the references cited therein. In particular, we would like to mention some excellent results of Kong [2] and Dai *et al.* [33]. In [2], Kong considered the existence and uniqueness of positive solutions for second-order singular boundary value problem

$$\begin{cases} u''(t) + \lambda f(u(t)) = 0, & t \in (0,1), \\ u(0) = \int_0^1 u(s)\,dA(s), \quad u(1) = \int_0^1 u(s)\,dB(s). \end{cases}$$

The author examined the uniqueness of the solution and its dependence on the parameter λ under condition

(H) $f : [0,\infty) \to (0,\infty)$ is nondecreasing, and there exists $\nu \in (0,1)$ such that

$$f(kx) \geq k^\nu f(x), \quad \text{for } k \in (0,1) \text{ and } x \in [0,+\infty).$$

In [33], Dai *et al.* investigated the existence of one-sign solutions for the following periodic p-Laplacian problem:

$$\begin{cases} -(\varphi_p(u'))' + q(t)\varphi_p(u) = \lambda\omega(t)f(u), & 0 < t < T, \\ u(0) = u(T), \quad u'(0) = u'(T). \end{cases}$$

The authors also examined the uniqueness of the solution and its dependence on the parameter λ under condition

(H*) $\frac{f(s)}{\varphi_p(s)}$ is strictly decreasing in $(0,\infty)$.

But to the best of our knowledge, there are no results for the dependence of positive solution $x_\lambda(t)$ on the parameter λ of second-order boundary value problems with deviating arguments without a condition similar to (H) or (H*). The objective of the present paper is to fill this gap.

In this paper, we consider the following second-order boundary value problem with integral boundary conditions and deviating arguments:

$$\begin{cases} (g(t)x'(t))' + \lambda\omega(t)f(t,x(\alpha(t))) = 0, \quad 0 < t < 1, \\ ax(0) - b\lim_{t\to 0^+} g(t)x'(t) = \int_0^1 h(s)x(s)\,ds, \\ ax(1) + b\lim_{t\to 1^-} g(t)x'(t) = \int_0^1 h(s)x(s)\,ds, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a parameter, $a, b > 0$, and ω may be singular at $t = 0$ and/or $t = 1$.

Throughout this paper we assume that $\alpha(t) \not\equiv t$ on $J = [0,1]$. In addition, g, ω, f, α, and h satisfy

(H_1) $g \in C^1(J, (0, +\infty))$, $\alpha \in C(J, J)$;
(H_2) $\omega \in C((0,1), [0, +\infty))$ with

$$0 < \int_0^1 \omega(s)\,ds < \infty$$

and ω does not vanish on any subinterval of $(0, 1)$;
(H_3) $f \in C([0,1] \times [0, +\infty), [0, +\infty))$ with $f(t,x) > 0$ for all t and $x > 0$;
(H_4) $h \in C[0,1]$ is nonnegative with $\nu \in [0, a)$, where

$$\nu = \int_0^1 h(t)\,dt. \tag{1.2}$$

Some special cases of (1.1) have been investigated. For example, Boucherif [1] considered problem (1.1) under the case that $\lambda = 1$, $g(t) \equiv 1$, $\omega(t) \equiv 1$, and $\alpha(t) \equiv t$ on J. By using Krasnoselskii's fixed point theorem in a cone, the author proved the existence results of positive solution for problem (1.1).

Kong [2] considered problem (1.1) under the case that $g(t) \equiv 1$, $\omega(t) \equiv 1$, and $\alpha(t) \equiv t$ on J. By using the mixed monotone operator theory, the author obtained the existence and uniqueness of positive solutions for problem (1.1).

In this paper, we shall show that the number of positive solutions of problem (1.1) can be determined by the asymptotic behaviors of the quotient of $\frac{f(t,x)}{x}$ at zero and infinity. Specifically, let

$$f^0 = \limsup_{x\to 0^+} \max_{t\in J} \frac{f(t,x)}{x}, \qquad f_0 = \liminf_{x\to 0^+} \min_{t\in J} \frac{f(t,x)}{x},$$
$$f^\infty = \limsup_{x\to \infty} \max_{t\in J} \frac{f(t,x)}{x}, \qquad f_\infty = \liminf_{x\to \infty} \min_{t\in J} \frac{f(t,x)}{x}.$$

We also define as in [38]

$$i_0 = \text{number of zeros in the set } \{f^0, f^\infty\},$$
$$i_\infty = \text{number of infinities in the set } \{f_0, f_\infty\}.$$

It is clear that $i_0, i_\infty = 0, 1$ or 2. Then we shall show that problem (1.1) has i_0 or i_∞ positive solution(s) for sufficiently large or small λ, respectively.

Moreover, being directly inspired by [20] and [37], we study the dependence of positive solution $x_\lambda(t)$ on the parameter λ for problem (1.1), *i.e.*,

$$\lim_{\lambda\to+\infty}\|x_\lambda\| = +\infty \quad \text{or} \quad \lim_{\lambda\to+\infty}\|x_\lambda\| = 0,$$

and the condition is weaker than that of Kong [2], Graef *et al.* [32], Dai *et al.* [33], Liu and Li [34], He and Su [35] and Li and Liu [36]. To our knowledge, it is the first paper when the dependence of positive solution $x_\lambda(t)$ on the parameter λ has been investigated for second-order boundary value problems with deviating arguments α, which can be both of advanced and of delayed type.

The paper is organized in the following fashion. In Section 2, we provide some necessary background. In particular, we state some properties of the Green's function associated with problem (1.1). In Section 3, we use a well-known fixed point theorem to study the existence, multiplicity and nonexistence of positive solutions for problem (1.1) with advanced argument α. In Section 4 we discuss the dependence of solution $x_\lambda(t)$ on the parameter λ for problem (1.1) with advanced argument α and we formulate sufficient conditions under which delayed problem (1.1) has positive solutions in Section 5. Finally, an example is given to illustrate the main results in Section 6.

2 Preliminaries

Let $E = C[0,1]$. It is well known that E is a real Banach space with the norm $\|\cdot\|$ defined by $\|x\| = \max_{t\in J}|x(t)|$.

In our main results, we will make use of the following definitions and lemmas.

Definition 2.1 (see [39]) Let E be a real Banach space over $\mathbf{R}$. A nonempty closed set $P\subset E$ is said to be a cone provided that

(i) $cu + dv \in P$ for all $u, v \in P$ and all $c \geq 0$, $d \geq 0$ and
(ii) $u, -u \in P$ implies $u = 0$.

Every cone $P \subset E$ induces an ordering in E given by $x \leq y$ if and only if $y - x \in P$.

Definition 2.2 A function $x \in E \cap C^2(0,1)$ is called a solution of problem (1.1) if it satisfies (1.1). If $x(t) \geq 0$ and $x(t) \not\equiv 0$ on J, then x is called a positive solution of problem (1.1).

Lemma 2.1 *Assume that* (H_1) *and* (H_4) *hold. Then for any* $y \in E$, *boundary value problem*

$$\begin{cases} -(g(t)x'(t))' = y(t), \quad 0 < t < 1, \\ ax(0) - b\lim_{t\to 0^+} g(t)x'(t) = \int_0^1 h(s)x(s)\,ds, \\ ax(1) + b\lim_{t\to 1^-} g(t)x'(t) = \int_0^1 h(s)x(s)\,ds, \end{cases} \tag{2.1}$$

has a unique solution x given by

$$x(t) = \int_0^1 H(t,s)y(s)\,ds, \tag{2.2}$$

where

$$H(t,s) = G(t,s) + \frac{1}{a-\nu}\int_0^1 G(\tau,s)h(\tau)\,d\tau, \tag{2.3}$$

$$G(t,s) = \frac{1}{\Delta}\begin{cases}(b + a\int_0^s \frac{dr}{g(r)})(b + a\int_t^1 \frac{dr}{g(r)}), & \text{if } 0 \le s \le t \le 1,\\ (b + a\int_0^t \frac{dr}{g(r)})(b + a\int_s^1 \frac{dr}{g(r)}), & \text{if } 0 \le t \le s \le 1,\end{cases} \tag{2.4}$$

where $\Delta = 2ab + a^2\int_0^1 \frac{1}{g(r)}\,dr$, $\nu = \int_0^1 h(s)\,ds$.

Proof The proof is similar to that of Lemma 2.1 in [11]. □

Lemma 2.2 *Let G and H be given as in Lemma* 2.1. *Then we have the following results*:

$$G(t,s) \le G(s,s), \qquad H(t,s) \le H(s,s) \le \frac{a}{a-\nu}G(s,s), \quad \forall t,s \in J, \tag{2.5}$$

$$G(t,s) \ge \delta G(s,s), \qquad H(t,s) \ge \delta H(s,s) \ge \frac{\delta^2 a}{a-\nu}G(s,s), \quad \forall t,s \in J, \tag{2.6}$$

where

$$\delta = \frac{b}{b + a\int_0^1 \frac{1}{g(r)}\,dr}.$$

Proof It follows from the definition of $G(t,s)$ and $H(t,s)$ that (2.5) holds. Now, we show that (2.6) also holds.

Note that

$$\frac{G(t,s)}{G(s,s)} = \frac{b + a\int_t^1 \frac{1}{g(r)}\,dr}{b + a\int_s^1 \frac{1}{g(r)}\,dr} \ge \frac{b}{b + a\int_0^1 \frac{1}{g(r)}\,dr} \quad \text{for } s \le t,$$

$$\frac{G(t,s)}{G(s,s)} = \frac{b + a\int_0^t \frac{1}{g(r)}\,dr}{b + a\int_0^s \frac{1}{g(r)}\,dr} \ge \frac{b}{b + a\int_0^1 \frac{1}{g(r)}\,dr} \quad \text{for } t \le s,$$

for $t,s \in J$.

Similarly, we can prove that $H(t,s) \ge \delta H(s,s)$ for $t,s \in J$. Hence, it follows from $G(t,s) \ge \delta G(s,s)$ that

$$H(t,s) \ge \delta H(s,s) \ge \frac{\delta^2 a}{a-\nu}G(s,s), \quad \forall t,s \in J.$$

This gives the proof of Lemma 2.2. □

Remark 2.1 Noticing that $a,b > 0$, it follows from (2.3) and (2.4) that

$$\frac{1}{\Delta}b^2 \le G(t,s) \le G(s,s) \le \frac{1}{\Delta}D; \tag{2.7}$$

$$\frac{1}{\Delta}ab^2\gamma \le H(t,s) \le H(s,s) \le \frac{1}{\Delta}a\gamma D, \quad \text{for } t,s \in J, \tag{2.8}$$

where

$$D = \left(b + a\int_0^1 \frac{dr}{g(r)}\right)^2, \qquad \gamma = \frac{1}{a-\nu}. \tag{2.9}$$

Being directly inspired by [11], we define a cone K in E by

$$K = \left\{x \in E : x \geq 0, \min_{t\in J} x(t) \geq \delta\|x\|\right\}. \tag{2.10}$$

Also, define, for a positive number r, Ω_r by

$$\Omega_r = \left\{x \in E : \|x\| < r\right\}.$$

Note that $\partial\Omega_r = \{x \in E : \|x\| = r\}$.

Define $T : K \to K$ by

$$(Tx)(t) = \lambda \int_0^1 H(t,s)\omega(s)f\bigl(s, x\bigl(\alpha(s)\bigr)\bigr)\,ds. \tag{2.11}$$

From (2.11), we know that a function $x \in K$ is a solution of problem (1.1) if and only if x is a fixed point of operator T, and we obtain Lemma 2.3.

Lemma 2.3 *Assume that* (H_1)-(H_4) *hold. If x is a fixed point of operator T, then $x \in E \cap C^2(0,1)$, and x is a solution of problem* (1.1).

Lemma 2.4 *Suppose that* (H_1)-(H_4) *hold. Then $T(K) \subset K$ and $T : K \to K$ is completely continuous.*

Proof For all $x \in K$, from (2.11) we have $Tx \geq 0$ and

$$\begin{aligned}(Tx)(t) &= \lambda \int_0^1 H(t,s)\omega(s)f\bigl(s, x\bigl(\alpha(s)\bigr)\bigr)\,ds \\ &\leq \lambda \int_0^1 H(s,s)\omega(s)f\bigl(s, x\bigl(\alpha(s)\bigr)\bigr)\,ds, \quad t \in J.\end{aligned} \tag{2.12}$$

It follows from (2.6), (2.11), and (2.12) that

$$\begin{aligned}\min_{t\in J}(Tx)(t) &= \min_{t\in J} \lambda \int_0^1 H(t,s)\omega(s)f\bigl(s, x\bigl(\alpha(s)\bigr)\bigr)\,ds \\ &\geq \delta\lambda \int_0^1 H(s,s)\omega(s)f\bigl(s, x\bigl(\alpha(s)\bigr)\bigr)\,ds \\ &\geq \delta\|Tx\|.\end{aligned}$$

Thus, $T(K) \subset K$.

Finally, similar to the proof of Theorem 4.1 in [40], one can prove that $T : K \to K$ is completely continuous. This gives the proof of Lemma 2.4. □

In the rest of this section, we state a well-known fixed point theorem which we need later.

Lemma 2.5 (see [39]) *Let P be a cone in a real Banach space E. Assume Ω_1, Ω_2 are bounded open sets in E with $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$. If*

$$A : P \cap (\bar{\Omega}_2 \backslash \Omega_1) \to P$$

is completely continuous such that either

(i) $\|Ax\| \leq \|x\|$, $\forall x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \geq \|x\|$, $\forall x \in P \cap \partial\Omega_2$, *or*

(ii) $\|Ax\| \geq \|x\|$, $\forall x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \leq \|x\|$, $\forall x \in P \cap \partial\Omega_2$,

then A *has at least one fixed point in* $P \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

The fixed point theorem in a cone has often been used to study the existence and multiplicity of positive solutions of boundary value problems over the last several years. As recent example, we mention the paper of Baleanu *et al.* [41].

3 Existence of positive solutions for problem (1.1) under $\alpha(t) \geq t$ on J

In this section, we show that problem (1.1) has i_0 or i_∞ positive solution(s) for sufficiently large or small λ under $\alpha(t) \geq t$ on J.

For convenience we introduce the following notation:

$$\beta = \int_0^1 \omega(s)\,ds.$$

Theorem 3.1 *Assume* (H_1)-(H_4) *hold and* $\alpha(t) \geq t$ *on* J.

(i) *If* $i_0 = 1$ *or* 2, *then there exists* $\lambda_0 > 0$ *such that problem* (1.1) *has* i_0 *positive solution(s) for* $\lambda > \lambda_0$.

(ii) *If* $i_\infty = 1$ *or* 2, *then there exists* $\lambda_0 > 0$ *such that problem* (1.1) *has* i_∞ *positive solution(s) for* $0 < \lambda < \lambda_0$.

(iii) *If* $i_0 = 0$ *or* $i_\infty = 0$, *then problem* (1.1) *has no positive solution for sufficiently large or small* λ, *respectively.*

Proof Part (i). Noticing that $f(t,x) > 0$ for all t and $x > 0$, we can define

$$m_r = \min_{t\in J, \delta r \leq x \leq r} \bigl\{f(t,x)\bigr\} > 0,$$

where $r > 0$.

Since $0 \leq t \leq \alpha(t) \leq 1$ on J, for a function $x \in K$ with $\|x\| = r$, it follows from $\delta r \leq x(t) \leq r$ on J that

$$\delta r \leq x\bigl(\alpha(t)\bigr) \leq r \quad \text{for } t \in J.$$

Let $\lambda_0 = \frac{\Delta r}{m_r \beta a b^2 \gamma}$. Then, for $x \in K \cap \partial\Omega_r$ and $\lambda > \lambda_0$, we have

$$\begin{aligned}
(Tx)(t) &= \lambda \int_0^1 H(t,s)\omega(s) f\bigl(s, x\bigl(\alpha(s)\bigr)\bigr)\,ds \\
&\geq \frac{ab^2\gamma}{\Delta}\lambda \int_0^1 \omega(s) f\bigl(s, x\bigl(\alpha(s)\bigr)\bigr)\,ds \\
&\geq \frac{ab^2\gamma}{\Delta}\lambda m_r \int_0^1 \omega(s)\,ds \\
&\geq \frac{ab^2\gamma}{\Delta}\lambda m_r \beta \\
&> \frac{ab^2\gamma}{\Delta}\lambda_0 m_r \beta \\
&= r = \|x\|,
\end{aligned}$$

which implies that

$$\|Tx\| > \|x\|, \quad \forall x \in K \cap \partial\Omega_r, \lambda > \lambda_0. \tag{3.1}$$

If $f^0 = 0$, we can choose $0 < r_1 < r$ such that

$$f(t,x) \leq \frac{\Delta}{a\lambda\beta\gamma D} x, \quad \forall t \in J, 0 \leq x \leq r_1.$$

Since $0 \leq t \leq \alpha(t) \leq 1$ on J, for a function $x \in K$ with $\|x\| = r$, it follows from $0 \leq x(t) \leq r_1$ on J that

$$0 \leq x(\alpha(t)) \leq r_1 \quad \text{for } t \in J.$$

Consequently, for any $t \in J$ and $x \in K \cap \partial\Omega_{r_1}$, (2.8) and (2.11) imply

$$\begin{aligned}
(Tx)(t) &= \lambda \int_0^1 H(t,s)\omega(s)f\bigl(s,x(\alpha(s))\bigr)\,ds \\
&\leq \frac{a\gamma D}{\Delta}\lambda \int_0^1 \omega(s)f\bigl(s,x(\alpha(s))\bigr)\,ds \\
&\leq \frac{a\gamma D}{\Delta}\lambda \int_0^1 \omega(s)\frac{\Delta}{a\lambda\beta\gamma D}x(\alpha(s))\,ds \\
&\leq \int_0^1 \omega(s)\frac{1}{\beta}\|x\|\,ds \\
&\leq \frac{1}{\beta}\|x\| \int_0^1 \omega(s)\,ds \\
&= \|x\|,
\end{aligned}$$

which implies

$$\|Tx\| \leq \|x\|, \quad \forall x \in K \cap \partial\Omega_{r_1}. \tag{3.2}$$

Thus by (i) of Lemma 2.5, it follows from (3.1) and (3.2) that T has a fixed point x in $K \cap (\bar{\Omega}_r \backslash \Omega_{r_1})$ with $r_1 \leq \|x\| \leq r$. Lemma 2.3 implies that problem (1.1) has at least one positive solution x with $r_1 \leq \|x\| \leq r$.

If $f^\infty = 0$, we can choose $0 < \varepsilon < \frac{\Delta}{a\gamma D\lambda\beta\varepsilon}$ and $l > 0$ such that

$$f(t,x) \leq \varepsilon x \quad \text{for } t \in J \text{ and } x \geq l.$$

Letting $\zeta = \max_{t\in J, x\in[0,l]} f(t,x)$, then

$$0 \leq f(t,x) \leq \varepsilon x + \zeta \quad \text{for } t \in J \text{ and } x \in [0,\infty).$$

Since $0 \leq t \leq \alpha(t) \leq 1$ on J, for a function $x \in K$ with $\|x\| = r$, it follows from $x(t) \geq l$ or $0 \leq x(t) \leq l$ on J that

$$x(\alpha(t)) \geq l \quad \text{or} \quad 0 \leq x(\alpha(t)) \leq l \quad \text{for } t \in J.$$

Let $r_2 > \max\{2r, \frac{a\gamma D\lambda\zeta\beta}{\Delta - a\gamma D\lambda\beta\varepsilon}\}$. Then for $t \in J$ and $x \in K \cap \partial\Omega_{r_2}$, (2.8) and (2.11) imply

$$\begin{aligned}
(Tx)(t) &= \lambda \int_0^1 H(t,s)\omega(s)f\bigl(s,x\bigl(\alpha(s)\bigr)\bigr)\,ds \\
&\leq \frac{a\gamma D}{\Delta}\lambda \int_0^1 \omega(s)f\bigl(s,x\bigl(\alpha(s)\bigr)\bigr)\,ds \\
&\leq \frac{a\gamma D}{\Delta}\lambda \int_0^1 \omega(s)\bigl(\varepsilon x\bigl(\alpha(s)\bigr) + \zeta\bigr)\,ds \\
&\leq \frac{a\gamma D}{\Delta}\lambda \int_0^1 \omega(s)\bigl(\varepsilon\|x\| + \zeta\bigr)\,ds \\
&\leq \frac{a\gamma D}{\Delta}\lambda\beta(\varepsilon r_2 + \zeta) \\
&< r_2,
\end{aligned}$$

which implies

$$\|Tx\| \leq \|x\|, \quad \forall x \in K \cap \partial\Omega_{r_2}. \tag{3.3}$$

Thus by (ii) of Lemma 2.5, it follows from (3.1) and (3.3) that T has a fixed point x in $K \cap (\bar{\Omega}_{r_2} \backslash \Omega_r)$ with $r \leq \|x\| \leq r_2$. Lemma 2.3 implies that problem (1.1) has at least one positive solution x with $r \leq \|x\| \leq r_2$.

Turning to $f^0 = f^\infty = 0$. Choose two numbers r_3 and r_4 satisfying

$$0 < r_1 < r_3 < \delta r_4 < r_4 < \delta r_2 < r_2 < +\infty. \tag{3.4}$$

Similar to the proof of (3.1), there exists $\lambda_0 > 0$ such that for $\lambda > \lambda_0$

$$\|Tx\| > \|x\|, \quad \forall x \in K \cap \partial\Omega_{r_i}, i = 3, 4, \tag{3.5}$$

which together with (3.2) and (3.3) shows that T has a fixed point x_1 in $K \cap (\Omega_{r_3} \backslash \Omega_{r_1})$ and a fixed point x_2 in $K \cap (\bar{\Omega}_{r_2} \backslash \Omega_{r_4})$ with

$$r_1 \leq \|x_1\| \leq r_3 < r_4 \leq \|x_2\| \leq r_2.$$

Consequently, it follows from Lemma 2.3 that problem (1.1) has two positive solutions for $\lambda > \lambda_0$ if $f^0 = f^\infty = 0$.

Part (ii). Noticing that $f(t,x) > 0$ for all t and $x > 0$, we can define

$$M_r = \max_{t \in J, 0 \leq x \leq r} \bigl\{f(t,x)\bigr\} > 0,$$

where $r > 0$.

Since $0 \leq t \leq \alpha(t) \leq 1$ on J, for a function $x \in K$ with $\|x\| = r$, it follows from $0 \leq x(t) \leq r$ on J that

$$0 \leq x\bigl(\alpha(t)\bigr) \leq r \quad \text{for } t \in J.$$

Let $\lambda_0 \le \frac{\Delta r}{M_r \beta a D \gamma}$. Then, for $x \in K \cap \partial\Omega_r$ and $0 < \lambda < \lambda_0$, we have

$$\begin{aligned}(Tx)(t) &= \lambda \int_0^1 H(t,s)\omega(s) f\bigl(s, x\bigl(\alpha(s)\bigr)\bigr)\, ds \\ &\le \frac{a\gamma D}{\Delta}\lambda \int_0^1 \omega(s) f\bigl(s, x\bigl(\alpha(s)\bigr)\bigr)\, ds \\ &\le \frac{a\gamma D}{\Delta} M_r \lambda \int_0^1 \omega(s)\, ds \\ &= \frac{a\gamma D}{\Delta} M_r \lambda \beta \\ &< \frac{a\gamma D}{\Delta} M_r \lambda_0 \beta \\ &\le r,\end{aligned}$$

which implies that

$$\|Tx\| < \|x\|, \quad \forall x \in K \cap \partial\Omega_r, 0 < \lambda < \lambda_0. \tag{3.6}$$

If $f_0 = \infty$, we can choose $0 < r_1 < r$ such that

$$f(t,x) \ge \frac{\Delta}{ab^2\delta\lambda\beta\gamma} x, \quad \forall t \in J, 0 \le x \le r_1.$$

Since $0 \le t \le \alpha(t) \le 1$ on J, for a function $x \in K$ with $\|x\| = r$, it follows from $0 \le x(t) \le r_1$ on J that

$$0 \le x\bigl(\alpha(t)\bigr) \le r_1 \quad \text{for } t \in J.$$

Consequently, for any $t \in J$ and $x \in K \cap \partial\Omega_{r_1}$, (2.8) and (2.11) imply

$$\begin{aligned}(Tx)(t) &= \lambda \int_0^1 H(t,s)\omega(s) f\bigl(s, x\bigl(\alpha(s)\bigr)\bigr)\, ds \\ &\ge \frac{ab^2\gamma}{\Delta}\lambda \int_0^1 \omega(s) f\bigl(s, x\bigl(\alpha(s)\bigr)\bigr)\, ds \\ &\ge \frac{ab^2\gamma}{\Delta}\lambda \int_0^1 \omega(s) \frac{\Delta}{ab^2\delta\lambda\beta\gamma} x\bigl(\alpha(s)\bigr)\, ds \\ &\ge \frac{ab^2\gamma}{\Delta}\lambda \int_0^1 \omega(s) \frac{\Delta}{ab^2\delta\lambda\beta\gamma} \delta\|x\|\, ds \\ &\ge \|x\|,\end{aligned}$$

which implies

$$\|Tx\| \ge \|x\|, \quad \forall x \in K \cap \partial\Omega_{r_1}. \tag{3.7}$$

Thus by (ii) of Lemma 2.5, it follows from (3.6) and (3.7) that T has a fixed point x in $K \cap (\bar{\Omega}_r \backslash \Omega_{r_1})$ with $r_1 \le \|x\| \le r$. Lemma 2.3 shows that problem (1.1) has at least one positive solution x with $r_1 \le \|x\| \le r$.

If $f_\infty = \infty$, we can choose sufficiently large $\varepsilon > 0$ and $l > 0$ such that

$$f(t,x) \geq \varepsilon x \quad \text{for } t \in J \text{ and } x \geq l,$$

where ε satisfies

$$\frac{ab^2\gamma}{\Delta}\lambda\varepsilon\delta\beta \geq 1.$$

Since $0 \leq t \leq \alpha(t) \leq 1$ on J, it follows from $x(t) \geq l$ on J that

$$x(\alpha(t)) \geq l \quad \text{for } t \in J.$$

Let $r_2 > \max\{2r, \frac{l}{\delta}\}$. Then for $t \in J$ and $x \in K \cap \partial\Omega_{r_2}$ we have

$$x(t) \geq \delta\|x\| > l.$$

Hence, for $t \in J$ and $x \in K \cap \partial\Omega_{r_2}$, it follows from (2.8) and (2.11) that

$$\begin{aligned}(Tx)(t) &= \lambda \int_0^1 H(t,s)\omega(s)f(s,x(\alpha(s)))\,ds \\ &\geq \frac{ab^2\gamma}{\Delta}\lambda \int_0^1 \omega(s)f(s,x(\alpha(s)))\,ds \\ &\geq \frac{ab^2\gamma}{\Delta}\lambda \int_0^1 \omega(s)\varepsilon x(\alpha(s))\,ds \\ &\geq \frac{ab^2\gamma}{\Delta}\lambda \int_0^1 \omega(s)\varepsilon\delta\|x\|\,ds \\ &= \frac{ab^2\gamma}{\Delta}\lambda\varepsilon\delta\|x\|\beta \\ &\geq \|x\|,\end{aligned}$$

which implies

$$\|Tx\| \geq \|x\|, \quad \forall x \in K \cap \partial\Omega_{r_2}. \tag{3.8}$$

Thus by (i) of Lemma 2.5, it follows from (3.6) and (3.8) that T has a fixed point x in $K \cap (\bar{\Omega}_{r_2} \backslash \Omega_r)$ with $r \leq \|x\| \leq r_2$. Lemma 2.3 shows that problem (1.1) has at least one positive solution x with $r \leq \|x\| \leq r_2$.

Turning to $f_0 = f_\infty = \infty$. Choose two numbers r_3 and r_4 satisfying (3.4). Similar to the proof of (3.6), there exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$

$$\|Tx\| < \|x\|, \quad \forall x \in K \cap \partial\Omega_{r_i}, i = 3,4, \tag{3.9}$$

which together with (3.7) and (3.8) shows that T has a fixed point x_1 in $K \cap (\Omega_{r_3} \backslash \Omega_{r_1})$ and a fixed point x_2 in $K \cap (\bar{\Omega}_{r_2} \backslash \Omega_{r_4})$ with

$$r_1 \leq \|x_1\| \leq r_3 < r_4 \leq \|x_2\| \leq r_2.$$

Consequently, it follows from Lemma 2.3 that problem (1.1) has two positive solutions for $0 < \lambda < \lambda_0$ if $f_0 = f_\infty = \infty$.

Part (iii). If $i_0 = 0$, then $f_0 > 0$ and $f_\infty > 0$. It follows that there exist positive numbers $\eta_1 > 0$, $\eta_2 > 0$, $h_1 > 0$, and $h_2 > 0$ such that $h_1 < h_2$ and, for $t \in J$, $0 < x \leq h_1$, we have

$$f(t,x) \geq \eta_1 x, \tag{3.10}$$

and, for $t \in J$, $x \geq h_2$, we have

$$f(t,x) \geq \eta_2 x. \tag{3.11}$$

Let

$$\eta = \min\left\{\eta_1, \eta_2, \min\left\{\frac{f(t,x)}{x} : t \in J, \delta h_1 \leq x \leq h_2\right\}\right\} > 0.$$

Thus, for $t \in J$, $x \geq \delta h_1$, we have

$$f(t,x) \geq \eta x, \tag{3.12}$$

and, for $t \in J$, $x \leq h_1$, we have

$$f(t,x) \geq \eta x. \tag{3.13}$$

Assume $y \in K$ is a positive solution of problem (1.1). We will show that this leads to a contradiction for $\lambda > \lambda_0 = [ab^2\gamma\eta\delta\beta]^{-1}\Delta$.

In fact, if $\|y\| \leq h_1$, (3.13) shows that

$$f(t,y) \geq \eta y, \quad \text{for } t \in J.$$

On the other hand, if $\|y\| > h_1$, then

$$\min_{t\in J} y(t) \geq \delta\|y\| > \delta h_1.$$

Since $0 \leq t \leq \alpha(t) \leq 1$, it follows from $y(t) > \delta h_1$ on $t \in J$ that $y(\alpha(t)) > \delta h_1$ on $t \in J$, which, together with (3.12), shows that

$$f(t, y(\alpha(t))) \geq \eta y(\alpha(t)), \quad t \in J.$$

Since $(Ty)(t) = y(t)$, for $\lambda > \lambda_0$, it follows from (2.8) and (2.11) that

$$\begin{aligned}
\|y\| &= \|Ty\| \\
&= \max_{t\in J} \lambda \int_0^1 H(t,s)w(s)f\bigl(s, y(\alpha(s))\bigr)\,ds \\
&\geq \frac{ab^2\gamma}{\Delta}\lambda \int_0^1 \omega(s)f\bigl(s, y(\alpha(s))\bigr)\,ds
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{ab^2\gamma}{\Delta}\lambda \int_0^1 \omega(s)\eta y\bigl(\alpha(s)\bigr)\,ds \\
&\geq \frac{ab^2\gamma}{\Delta}\lambda \int_0^1 \omega(s)\eta\delta\|y\|\,ds \\
&= \frac{ab^2\gamma}{\Delta}\lambda\eta\delta\|y\|\beta \\
&> \frac{ab^2\gamma}{\Delta}\lambda_0\eta\delta\|y\|\beta \\
&= \|y\|,
\end{aligned}$$

which is a contradiction.

If $i_\infty = 0$, then $f^0 < \infty$ and $f^\infty < \infty$. It follows that there exist positive numbers $\eta_3 > 0$, $\eta_4 > 0$, $h_3 > 0$, and $h_4 > 0$ such that $h_3 < h_4$ and, for $t \in J$, $0 < x \leq h_3$, we have

$$f(t,x) \leq \eta_3 x, \tag{3.14}$$

and, for $t \in J$, $x \geq h_4$, we have

$$f(t,x) \leq \eta_4 x. \tag{3.15}$$

Let

$$\eta^* = \max\left\{\eta_3, \eta_4, \max\left\{\frac{f(t,x)}{x} : t \in J, h_3 \leq x \leq h_4\right\}\right\} > 0.$$

Thus, we have

$$f(t,x) \leq \eta^* x, \quad t \in J, x \in [0,\infty). \tag{3.16}$$

Since $0 \leq t \leq \alpha(t) \leq 1$ on J, it follows from $0 \leq x(t) \leq h_3$, $x(t) \geq h_4$, and $h_3 \leq x(t) \leq h_4$ on J that $0 \leq x(\alpha(t)) \leq h_3$, $x(\alpha(t)) \geq h_4$, and $h_3 \leq x(\alpha(t)) \leq h_4$ on J, respectively.

Assume $y \in K$ is a positive solution of problem (1.1). We will show that this leads to a contradiction for $0 < \lambda < \lambda_0 = [a\gamma D\eta^*\beta]^{-1}\Delta$.

Since $(Ty)(t) = y(t)$, for $0 < \lambda < \lambda_0$, it follows from (2.8) and (2.11) that

$$\begin{aligned}
\|y\| &= \|Ty\| \\
&= \max_{t\in J} \lambda \int_0^1 H(t,s)w(s)f\bigl(s, y\bigl(\alpha(s)\bigr)\bigr)\,ds \\
&\leq \frac{a\gamma D}{\Delta}\lambda \int_0^1 \omega(s)f\bigl(s, y\bigl(\alpha(s)\bigr)\bigr)\,ds \\
&\leq \frac{a\gamma D}{\Delta}\lambda \int_0^1 \omega(s)\eta^* y\bigl(\alpha(s)\bigr)\,ds \\
&\leq \frac{a\gamma D}{\Delta}\lambda\eta^*\|y\| \int_0^1 \omega(s)\,ds \\
&= \frac{a\gamma D}{\Delta}\lambda\eta^*\|y\|\beta
\end{aligned}$$

$$< \frac{a\gamma D}{\Delta}\lambda_0\eta^*\|y\|\beta$$
$$= \|y\|,$$

which is a contradiction. □

Theorem 3.2 is a direct consequence of the proof of Theorem 3.1(iii). Under the conditions of Theorem 3.2 we are able to give explicit intervals of λ such that (1.1) has no positive solution.

Theorem 3.2 *Assume* (H_1)-(H_4) *hold and* $\alpha(t) \geq t$ *on* J.

(i) *If there exists* $l > 0$ *such that* $f(t,x) \geq lx$ *for* $t \in J$ *and* $x \in [0,\infty)$, *then there exists* $\lambda_0 > 0$ *such that problem* (1.1) *has no positive solution for* $\lambda > \lambda_0$.

(ii) *If there exists* $L > 0$ *such that* $f(t,x) \leq Lx$ *for* $t \in J$ *and* $x \in [0,\infty)$, *then there exists* $\lambda_0 > 0$ *such that problem* (1.1) *has no positive solution for* $0 < \lambda < \lambda_0$.

Theorem 3.3 *Assume* (H_1)-(H_4) *hold*, $\alpha(t) \geq t$ *on* J *and* $i_0 = i_\infty = 0$. *Then problem* (1.1) *has at least one positive solution in* K *provided*

$$\frac{\Delta}{ab^2\gamma\beta\delta\max\{f_\infty,f^0\}} < \lambda < \frac{\Delta}{aD\gamma\beta\min\{f_\infty,f^0\}}. \tag{3.17}$$

Proof We give the proof under two cases of $f_\infty > f^0$ and $f_\infty < f^0$.

If $f_\infty > f^0$, then (3.17) implies that

$$\frac{\Delta}{ab^2\gamma\beta\delta f_\infty} < \lambda < \frac{\Delta}{aD\gamma\beta f^0}.$$

It is easy to see that there exists $\varepsilon > 0$ such that

$$\frac{\Delta}{ab^2\gamma\beta\delta(f_\infty - \varepsilon)} \leq \lambda \leq \frac{\Delta}{aD\gamma\beta(f^0 + \varepsilon)}.$$

Now, considering f^0 and f_∞, there exists $r_1 > 0$ such that $f(t,x) \leq (f^0 + \varepsilon)x$ for $t \in J$ and $0 \leq x \leq r_1$.

Since $0 \leq t \leq \alpha(t) \leq 1$, it follows from $0 \leq x(t) \leq r_1$ on J that $0 \leq x(\alpha(t)) \leq r_1$. Hence, similar to the proof of (3.2), for $x \in K \cap \partial\Omega_{r_1}$ we have

$$\|Tx\| \leq \frac{a\gamma D}{\Delta}\lambda\int_0^1 \omega(s)f\big(s,x\big(\alpha(s)\big)\big)\,ds \leq \frac{a\gamma D}{\Delta}\lambda\big(f^0 + \varepsilon\big)\beta\|x\| \leq \|x\|.$$

On the other hand, there exists $L > 0$ with $L > r_1$ such that $f(t,x) \geq (f_\infty - \varepsilon)x$ for $t \in J$ and $x \geq L$.

Since $0 \leq t \leq \alpha(t) \leq 1$, it follows from $0 \leq x(t) \leq r_1$ on J that $x(\alpha(t)) \geq L$.

Let $r_2 = \max\{2r_1, \frac{L}{\delta}\}$ and it follows that $x(t) \geq \delta\|x\| \geq L$ for $t \in J$ and $x \in K \cap \partial\Omega_{r_2}$. Similar to the proof of (3.8), for $t \in J$ and $x \in K \cap \partial\Omega_{r_2}$ we have

$$\|Tx\| \geq \frac{ab^2\gamma}{\Delta}\lambda\int_0^1 \omega(s)f\big(s,x\big(\alpha(s)\big)\big)\,ds \geq \frac{ab^2\gamma}{\Delta}\lambda(f_\infty - \varepsilon)\beta\delta\|x\| \geq \|x\|.$$

It follows from Lemma 2.5 that T has a fixed point in $K \cap (\bar{\Omega}_{r_2} \backslash \Omega_{r_1})$. Consequently, problem (1.1) has a positive solution.

If $f_\infty < f^0$, then (3.17) shows that

$$\frac{\Delta}{ab^2\gamma\beta\delta f^0} < \lambda < \frac{\Delta}{aD\gamma\beta f_\infty}.$$

It is easy to see that there exists $\varepsilon > 0$ such that

$$\frac{\Delta}{ab^2\gamma\beta\delta(f^0 - \varepsilon)} \le \lambda \le \frac{\Delta}{aD\gamma\beta(f_\infty + \varepsilon)}.$$

Now, turning to f^0 and f_∞, there exists $r_1 > 0$ such that $f(t,x) \ge (f^0 - \varepsilon)x$ for $t \in J$ and $0 \le x \le r_1$.

Since $0 \le t \le \alpha(t) \le 1$, it follows from $0 \le x(t) \le r_1$ that $0 \le x(\alpha(t)) \le r_1$ on J. Hence, similar to the proof of (3.8), for $x \in K \cap \partial\Omega_{r_1}$ we have

$$\|Tx\| \ge \frac{ab^2\gamma}{\Delta}\lambda \int_0^1 \omega(s) f\big(s, x\big(\alpha(s)\big)\big)\, ds \ge \frac{ab^2\gamma}{\Delta}\lambda\big(f^0 - \varepsilon\big)\beta\delta\|x\| \ge \|x\|.$$

On the other hand, there exists $L > 0$ with $L > r_1$ such that $f(t,x) \le (f_\infty + \varepsilon)x$ for $t \in J$ and $x \ge L$.

Letting $\zeta = \max_{t\in J, x\in[0,L]} f(t,x)$, then

$$0 \le f(t,x) \le (f_\infty + \varepsilon)x + \zeta \quad \text{for } t \in J \text{ and } x \in [0,\infty).$$

Since $0 \le t \le \alpha(t) \le 1$ on J, it follows from $x(t) \ge L$ or $0 \le x(t) \le L$ on J that

$$x\big(\alpha(t)\big) \ge L \quad \text{or} \quad 0 \le x\big(\alpha(t)\big) \le L \quad \text{for } t \in J.$$

Let $r_2 > \max\{2r, \frac{a\gamma D\lambda\zeta\beta}{\Delta - a\gamma D\lambda\beta(f_\infty+\varepsilon)}\}$. Then, for $t \in J$ and $x \in K \cap \partial\Omega_{r_2}$, similar to the proof of (3.3) we get

$$\|Tx\| \le \frac{a\gamma D}{\Delta}\lambda \int_0^1 \omega(s) f\big(s, x\big(\alpha(s)\big)\big)\, ds \le \frac{a\gamma D}{\Delta}\lambda\beta\big((f_\infty + \varepsilon)r_2 + \zeta\big) < r_2 = \|x\|.$$

It follows from Lemma 2.5 that T has a fixed point in $K \cap (\bar{\Omega}_{r_2} \backslash \Omega_{r_1})$. Consequently, problem (1.1) has a positive solution. □

Corollary 3.1 *Assume* (H_1)-(H_4) *hold,* $\alpha(t) \ge t$ *on* J *and* $i_0 = i_\infty = 0$. *Then problem* (1.1) *has at least one positive solution in* K *provided*

$$\frac{\Delta}{ab^2\gamma\beta\delta \max\{f^\infty, f_0\}} < \lambda < \frac{\Delta}{aD\gamma\beta \min\{f^\infty, f_0\}}.$$

Proof The proof is similar to that of Theorem 3.3. □

4 The dependence of solution $x_\lambda(t)$ on the parameter λ for problem (1.1) under $\alpha(t) \geq t$ on J

In this section, we consider the dependence of positive solution $x_\lambda(t)$ on the parameter λ under a weaker condition $(H_3)^*$ than (H_3),

$(H_3)^*$ $f \in C([0,1] \times [0,+\infty),[0,+\infty))$.

For convenience we introduce the following notation:

$$\mathbf{M} = \max\Big\{\max_{t\in J} f(t,x), \|x\| \leq \varsigma\Big\},$$

where $\varsigma > 0$.

Theorem 4.1 *Assume* (H_1), (H_2), $(H_3)^*$, (H_4) *hold*, $\alpha(t) \geq t$ *on* J *and* $i_0 = i_\infty = 1$. *Then the following two conclusions hold.*

(i) *If* $f^0 = 0$ *and* $f_\infty = \infty$, *then for every* $\lambda > 0$ *problem* (1.1) *has a positive solution* $x_\lambda(t)$ *satisfying* $\lim_{\lambda\to 0^+} \|x_\lambda\| = \infty$.

(ii) *If* $f_0 = \infty$ *and* $f^\infty = 0$, *then for every* $\lambda > 0$ *problem* (1.1) *has a positive solution* $x_\lambda(t)$ *satisfying* $\lim_{\lambda\to 0^+} \|x_\lambda\| = 0$.

Proof By definition, $i_0 = i_\infty = 1$ implies that $f^0 = 0$ and $f_\infty = \infty$ or $f_0 = \infty$ and $f^\infty = 0$. We need only prove this theorem under the condition $f^0 = 0$ and $f_\infty = \infty$ since the proof is similar to that of $f_0 = \infty$ and $f^\infty = 0$.

Let $\lambda > 0$. Considering $f^0 = 0$, then similar to the proof of (3.2), there exists $r > 0$ such that

$$\|Tx\| \leq \|x\|, \quad \forall x \in K \cap \partial\Omega_r. \tag{4.1}$$

On the other hand, turning to $f_\infty = \infty$, similar to the proof of (3.8), there exists $R > 0$ satisfying $R > r$ such that

$$\|Tx\| \geq \|x\|, \quad \forall x \in K \cap \partial\Omega_R. \tag{4.2}$$

Applying (i) of Lemma 2.5 to (4.1) and (4.2) shows that the operator T has a fixed point $x_\lambda \in K \cap (\bar{\Omega}_R \backslash \Omega_r)$. Consequently, it follows from Lemma 2.3 that problem (1.1) has a positive solution $x_\lambda \in K \cap (\bar{\Omega}_R \backslash \Omega_r)$ with $r \leq \|x_\lambda\| \leq R$.

Next we prove that $\|x_\lambda\| = +\infty$ as $\lambda \to 0^+$. In fact, if not, there exist a number $\varsigma > 0$ and a sequence $\lambda_n \to 0^+$ such that

$$\|x_{\lambda_n}\| \leq \varsigma \quad (n = 1,2,3,\ldots).$$

Furthermore, the sequence $\{\|x_{\lambda_n}\|\}$ contains a subsequence that converges to a number η $(0 \leq \eta \leq \varsigma)$. For simplicity, suppose that $\{\|x_{\lambda_n}\|\}$ itself converges to η.

If $\eta > 0$, then $\|x_{\lambda_n}\| > \frac{\eta}{2}$ for sufficiently large n $(n > \mathbf{N})$. Since $0 \leq t \leq \alpha(t) \leq 1$, it follows from $0 \leq x(t) \leq \varsigma$ that

$$0 \leq x\big(\alpha(t)\big) \leq \varsigma.$$

Hence, it follows from the definition of $\mathbf{M}$, (2.8), and (2.11) that

$$\begin{aligned}\frac{1}{\lambda_n} &= \frac{\|\int_0^1 H(t,s)\omega(s)f(s,x_{\lambda_n}(\alpha(s)))\,ds\|}{\|x_{\lambda_n}\|} \\ &\leq \frac{a\gamma D\Delta^{-1}\int_0^1 \omega(s)f(s,x_{\lambda_n}(\alpha(s)))\,ds}{\|x_{\lambda_n}\|} \\ &\leq \frac{a\gamma D\Delta^{-1}\beta\mathbf{M}}{\|x_{\lambda_n}\|} \\ &< \frac{2a\gamma D\Delta^{-1}\beta\mathbf{M}}{\eta} \quad (n > \mathbf{N}),\end{aligned}$$

which contradicts $\lambda_n \to 0^+$.

If $\eta = 0$, then $\|x_{\lambda_n}\| \to 0$ $(n \to +\infty)$, and therefore it follows from $f^0 = 0$ that for any $\varepsilon > 0$ there exists $r^* > 0$ such that

$$f(t,x_{\lambda_n}) \leq \varepsilon x_{\lambda_n}, \quad \forall t \in J, 0 \leq x_{\lambda_n} \leq r^*.$$

Since $0 \leq t \leq \alpha(t) \leq 1$, it follows from $0 \leq x_{\lambda_n}(t) \leq r^*$ that

$$0 \leq x_{\lambda_n}\bigl(\alpha(t)\bigr) \leq r^*.$$

Therefore, $x_{\lambda_n} \in K \cap \partial\Omega_{r^*}$ and $\|x_{\lambda_n}\| = r^*$ imply that

$$\begin{aligned}\frac{1}{\lambda_n} &= \frac{\|\int_0^1 H(t,s)\omega(s)f(s,x_{\lambda_n}(\alpha(s)))\,ds\|}{\|x_{\lambda_n}\|} \\ &\leq \frac{a\gamma D\Delta^{-1}\int_0^1 \omega(s)f(s,x_{\lambda_n}(\alpha(s)))\,ds}{\|x_{\lambda_n}\|} \\ &\leq \frac{a\gamma D\Delta^{-1}\int_0^1 \omega(s)\varepsilon x_{\lambda_n}(\alpha(s))}{\|x_{\lambda_n}\|} \\ &\leq \frac{\beta a\gamma D\Delta^{-1}\varepsilon\|x_{\lambda_n}\|}{\|x_{\lambda_n}\|} \\ &= \beta a\gamma D\Delta^{-1}\varepsilon.\end{aligned}$$

Since ε is arbitrary, we have $\lambda_n \to \infty$ $(n \to +\infty)$ in contradiction with $\lambda_n \to 0^+$. Therefore, $\|x_\lambda\| \to +\infty$ as $\lambda \to 0^+$. This finishes the proof of Theorem 4.1. □

Remark 4.1 In contrast to [20] and [37], the behavior of the solution as $\lambda \to 0^+$ is investigated.

5 Positive solutions of problem (1.1) for the case of $\alpha(t) \leq t$ on J

Now we deal with problem (1.1) for the case of $\alpha(t) \leq t$ on J. Let E, K, and T be as defined in Section 2. Similarly as Lemmas 2.2-2.4, we can prove the following results.

Lemma 5.1 *Let G and H be given as in Lemma* 2.1. *Then we have the following results*:

$$G(t,s) \geq \delta G(s,s), \qquad H(t,s) \geq \delta H(s,s) \geq \frac{\delta^2 a}{a-\nu}G(s,s), \quad \forall t,s \in J, \tag{5.1}$$

where

$$\delta = \frac{b}{b + a\int_0^1 \frac{1}{g(r)}\,dr}.$$

Lemma 5.2 *Assume that* (H_1)-(H_4) *hold. If* x *is a fixed point of the operator* T, *then* $x \in E \cap C^2(0,1)$, *and* x *is a solution of problem* (1.1).

Lemma 5.3 *Assume that* (H_1)-(H_4) *hold. Then* $T(K) \subset K$ *and* $T: K \to K$ *is completely continuous.*

By analogous methods, we have the following results.

Theorem 5.1 *Assume* (H_1)-(H_4) *hold and* $\alpha(t) \le t$ *on* J.

(i) *If* $i_0 = 1$ *or* 2, *then there exists* $\lambda_0 > 0$ *such that problem* (1.1) *has* i_0 *positive solution(s) for* $\lambda > \lambda_0$.

(ii) *If* $i_\infty = 1$ *or* 2, *then there exists* $\lambda_0 > 0$ *such that problem* (1.1) *has* i_∞ *positive solution(s) for* $0 < \lambda < \lambda_0$.

(iii) *If* $i_0 = 0$ *or* $i_\infty = 0$, *then problem* (1.1) *has no positive solution for sufficiently large or small* λ, *respectively.*

Theorem 5.2 *Assume* (H_1)-(H_4) *hold and* $\alpha(t) \le t$ *on* J.

(i) *If there exists* $l > 0$ *such that* $f(t,x) \ge lx$ *for* $t \in J$ *and* $x \in [0,\infty)$, *then there exists* $\lambda_0 > 0$ *such that problem* (1.1) *has no positive solution for* $\lambda > \lambda_0$.

(ii) *If there exists* $L > 0$ *such that* $f(t,x) \le lx$ *for* $t \in J$ *and* $x \in [0,\infty)$, *then there exists* $\lambda_0 > 0$ *such that problem* (1.1) *has no positive solution for* $0 < \lambda < \lambda_0$.

Theorem 5.3 *Assume* (H_1)-(H_4) *hold,* $\alpha(t) \le t$ *on* J, *and* $i_0 = i_\infty = 0$. *Then problem* (1.1) *has at least one positive solution in* K *provided*

$$\frac{\Delta}{ab^2\gamma\beta_1\delta\max\{f_\infty, f^0\}} < \lambda < \frac{\Delta}{aD\gamma\beta\min\{f_\infty, f^0\}}.$$

Corollary 5.1 *Assume* (H_1)-(H_4) *hold,* $\alpha(t) \le t$ *on* J *and* $i_0 = i_\infty = 0$. *Then problem* (1.1) *has at least one positive solution in* K *provided*

$$\frac{\Delta}{ab^2\gamma\beta_1\delta\max\{f^\infty, f_0\}} < \lambda < \frac{\Delta}{aD\gamma\beta\min\{f^\infty, f_0\}}.$$

Theorem 5.4 *Assume* (H_1), (H_2), $(\mathrm{H}_3)^*$, (H_4) *hold,* $\alpha(t) \le t$ *on* J *and* $i_0 = i_\infty = 1$. *Then the following two conclusions hold.*

(i) *If* $f^0 = 0$ *and* $f_\infty = \infty$, *then for every* $\lambda > 0$ *problem* (1.1) *has a positive solution* $x_\lambda(t)$ *satisfying* $\lim_{\lambda\to 0^+}\|x_\lambda\| = \infty$.

(ii) *If* $f_0 = \infty$ *and* $f^\infty = 0$, *then for every* $\lambda > 0$ *problem* (1.1) *has a positive solution* $x_\lambda(t)$ *satisfying* $\lim_{\lambda\to 0^+}\|x_\lambda\| = 0$.

6 An example

To illustrate how our main results can be used in practice we present an example.

Example 6.1 Consider the following boundary value problem:

$$\begin{cases} -(\frac{1}{e^t}x'(t))' = \lambda \frac{1}{\sqrt{t}}(1+t^2)x^n(\alpha(t)), & t \in J, \\ x(0) - x'(0) = \int_0^1 \frac{1}{2}x(t)\,dt, & x(1) + x'(1) = \int_0^1 \frac{1}{2}x(t)\,dt, \end{cases} \tag{6.1}$$

where $\alpha \in C(J,J)$, $\alpha(t) \geq t$ on J, and

$$\omega(t) = \frac{1}{\sqrt{t}}, \qquad f(t,x) = \bigl(1+t^2\bigr)x^n,$$

here $n \geq 2$ is a positive integral.

This means that problem (6.1) involves the advanced argument α. For example, we can take $\alpha(t) = \sqrt[3]{t}$. It is clear that ω is singular at $t = 0$ and f is both nonnegative and continuous.

We claim that problem (6.1) has at least one positive solution for any $\lambda > \frac{e^2(e+1)}{4}$.

Proof Problem (6.1) can be regarded as a problem of the form (1.1), where

$$g(t) = \frac{1}{e^t}, \qquad a = b = 1, \qquad h(t) = \frac{1}{2}.$$

Letting $n = 2$ and $r = 1$, then, by a simple computation, we have

$$\nu = \frac{1}{2}, \qquad \gamma = 2, \qquad \Delta = e+1, \qquad \beta = 2,$$
$$\delta = \frac{1}{e}, \qquad m_r = \frac{1}{e^2}, \qquad \lambda_0 = \frac{e^2(e+1)}{4}.$$

It follows from the definition of g, ω, f, α, and h that (H_1)-(H_4) hold, and $f^0 = 0$.

Therefore, for any $\lambda > \lambda_0 = \frac{e^2(e+1)}{4}$, it follows from Theorem 3.1(i) that problem (6.1) has a positive solution. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
XZ completed the main study and carried out the results of this article. MF checked the proofs and verified the calculation. All authors read and approved the final manuscript.

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References

1. Boucherif, A: Second-order boundary value problems with integral boundary conditions. Nonlinear Anal. TMA **70**, 364-371 (2009)

2. Kong, LJ: Second order singular boundary value problems with integral boundary conditions. Nonlinear Anal. TMA **72**, 2628-2638 (2010)
3. Cannon, JR: The solution of the heat equation subject to the specification of energy. Q. Appl. Math. **21**, 155-160 (1963)
4. Ionkin, NI: Solution of a boundary value problem in heat conduction theory with nonlocal boundary conditions. Differ. Equ. **13**, 294-304 (1977)
5. Chegis, RY: Numerical solution of a heat conduction problem with an integral boundary condition. Liet. Mat. Rink. **24**, 209-215 (1984)
6. Cabada, A, Infante, G, Tojo, FAF: Nonzero solutions of perturbed Hammerstein integral equations with deviated arguments and applications. arXiv preprint arXiv:1306.6560 (2013)
7. Ahmad, B, Alsaedi, A, Alghamdi, BS: Analytic approximation of solutions of the forced Duffing equation with integral boundary conditions. Nonlinear Anal., Real World Appl. **9**, 1727-1740 (2008)
8. Infante, G, Webb, JRL: Nonlinear non-local boundary-value problems and perturbed Hammerstein integral equations. Proc. Edinb. Math. Soc. **49**, 637-656 (2006)
9. Kang, P, Wei, Z, Xu, J: Positive solutions to fourth-order singular boundary value problems with integral boundary conditions in abstract spaces. Appl. Math. Comput. **206**, 245-256 (2008)
10. Ma, H: Symmetric positive solutions for nonlocal boundary value problems of fourth order. Nonlinear Anal. TMA **68**, 645-651 (2008)
11. Feng, M: Existence of symmetric positive solutions for a boundary value problem with integral boundary conditions. Appl. Math. Lett. **24**, 1419-1427 (2011)
12. Webb, JRL, Infante, G: Positive solutions of nonlocal boundary value problems: a unified approach. J. Lond. Math. Soc. **74**, 673-693 (2006)
13. Yang, Z: Positive solutions to a system of second-order nonlocal boundary value problems. Nonlinear Anal. TMA **62**, 1251-1265 (2005)
14. Hao, X, Liu, L, Wu, Y: Positive solutions for second order impulsive differential equations with integral boundary conditions. Commun. Nonlinear Sci. Numer. Simul. **16**, 101-111 (2011)
15. Tian, Y, Ji, D, Ge, W: Existence and nonexistence results of impulsive first-order problem with integral boundary condition. Nonlinear Anal. TMA **71**, 1250-1262 (2009)
16. Kong, LJ: Second order singular boundary value problems with integral boundary conditions. Nonlinear Anal. TMA **72**, 2628-2638 (2010)
17. Zhang, X, Feng, M, Ge, W: Existence results for nonlinear boundary-value problems with integral boundary conditions in Banach spaces. Nonlinear Anal. TMA **69**, 3310-3321 (2008)
18. Zhang, X, Feng, M, Ge, W: Existence of solutions of boundary value problems with integral boundary conditions for second-order impulsive integro-differential equations in Banach spaces. J. Comput. Appl. Math. **233**, 1915-1926 (2010)
19. Zhang, X, Ge, W: Symmetric positive solutions of boundary value problems with integral boundary conditions. Appl. Math. Comput. **219**, 3553-3564 (2012)
20. Zhang, X, Feng, M: Existence and dependence of positive solution on parameter for the one-dimensional singular *p*-Laplacian. J. Math. Anal. Appl. **413**, 566-582 (2014)
21. Feng, M, Du, B, Ge, W: Impulsive boundary value problems with integral boundary conditions and one-dimensional *p*-Laplacian. Nonlinear Anal. TMA **70**, 3119-3126 (2009)
22. Jankowski, T: Solvability of three point boundary value problems for second order differential equations with deviating arguments. J. Math. Anal. Appl. **312**, 620-636 (2005)
23. Yang, C, Zhai, C, Yan, J: Positive solutions of the three-point boundary value problem for second order differential equations with an advanced argument. Nonlinear Anal. TMA **65**, 2013-2023 (2006)
24. Jankowski, T: Existence of solutions of boundary value problems for differential equations with delayed arguments. J. Comput. Appl. Math. **156**, 239-252 (2003)
25. Jankowski, T: Advanced differential equations with nonlinear boundary conditions. J. Math. Anal. Appl. **304**, 490-503 (2005)
26. Jankowski, T: Nonnegative solutions to nonlocal boundary value problems for systems of second-order differential equations dependent on the first-order derivatives. Nonlinear Anal. **87**, 83-101 (2013)
27. Jankowski, T: Positive solutions to second-order differential equations with dependence on the first-order derivative and nonlocal boundary conditions. Bound. Value Probl. **2013**, 8 (2013)
28. Jiang, D, Wei, J: Monotone method for first- and second-order periodic boundary value problems and periodic solutions of functional differential equations. Nonlinear Anal. TMA **50**, 885-898 (2002)
29. Wang, G: Monotone iterative technique for boundary value problems of a nonlinear fractional differential equation with deviating arguments. J. Comput. Appl. Math. **236**, 2425-2430 (2012)
30. Wang, G, Zhang, L, Song, G: Systems of first order impulsive functional differential equations with deviating arguments and nonlinear boundary conditions. Nonlinear Anal. TMA **74**, 974-982 (2011)
31. Hu, C, Liu, B, Xie, S: Monotone iterative solutions for nonlinear boundary value problems of fractional differential equation with deviating arguments. Appl. Math. Comput. **222**, 72-81 (2013)
32. Graef, JR, Kong, L, Wang, H: Existence, multiplicity, and dependence on a parameter for a periodic boundary value problem. J. Differ. Equ. **245**, 1185-1197 (2008)
33. Dai, G, Ma, R, Wang, H: Eigenvalues, bifurcation and one-sign solutions for the periodic *p*-Laplacian. Commun. Pure Appl. Anal. **12**, 2839-2872 (2013)
34. Liu, X, Li, W: Existence and uniqueness of positive periodic solutions of functional differential equations. J. Math. Anal. Appl. **293**, 28-39 (2004)
35. He, T, Su, Y: On discrete fourth-order boundary value problems with three parameters. J. Comput. Appl. Math. **233**, 2506-2520 (2010)
36. Li, W, Liu, X: Eigenvalue problems for second-order nonlinear dynamic equations on time scales. J. Math. Anal. Appl. **318**, 578-592 (2005)
37. Zhang, X, Feng, M: Transformation techniques and fixed point theories to establish the positive solutions of second order impulsive differential equations. J. Comput. Appl. Math. **271**, 117-129 (2014)

38. Wang, H: Positive periodic solutions of functional differential equations. J. Differ. Equ. **202**, 354-366 (2004)
39. Guo, D, Lakshmikantham, V: Nonlinear Problems in Abstract Cones. Academic Press, New York (1988)
40. Jankowski, T: Positive solutions to third-order impulsive Sturm-Liouville boundary value problems with deviated arguments and one-dimensional p-Laplacian. Dyn. Syst. Appl. **20**, 575-586 (2011)
41. Baleanu, D, Mohammadi, H, Rezapour, S: Positive solutions of a boundary value problem for nonlinear fractional differential equations. Abstr. Appl. Anal. **2012**, 837437 (2012)