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Amplitude equation with quintic nonlinearities for the generalized Swift-Hohenberg equation with additive degenerate noise

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Abstract

In this paper, we are interested in the approximation of a stochastic generalized Swift-Hohenberg equation with quadratic and cubic nonlinearity by using the natural separation of time-scales near a change of stability. The main results show that the behavior of the SPDE is well approximated by a stochastic ordinary differential equation describing the amplitude of the dominant mode. The cubic and the quadratic nonlinearities lead to cubic nonlinearities of opposite sign. Here we study the interesting case, where both contributions cancel and in the right scaling a quintic nonlinearity emerges in the amplitude equation. Also, we give a brief indication of how the effect of additive degenerate noise (*i.e.* noise that does not act directly to the dominant mode) might lead to the stabilization of the trivial solution.

MSC: 60H10; 60H15

Keywords: generalized Swift-Hohenberg equation; multi-scale analysis; amplitude equation; additive noise

1 Introduction

We consider the stochastic generalized Swift-Hohenberg equation (SGSH) in the following form:

$$du = \left[-\left(1+\partial_x^2\right)^2 u + \nu_\varepsilon u + \gamma u^2 - u^3\right] dt + \mu_\varepsilon \, dW, \tag{1}$$

where ν_ε is the control parameter, W is a finite dimensional Wiener process. The Swift-Hohenberg equation (1), which describes the temperature and fluid velocity dynamics of the thermal convection, was derived with $\gamma = \mu_\varepsilon = 0$ by Swift and Hohenberg [1] in the year 1977. Also, it plays key role in the studies of pattern formation [2]. Here the quadratic term γu^2 plays an essential role; it was first introduced into the GSH equation mathematically in [3] in order to model the threshold character of periodic pattern formation.

In [4], I derived rigorously the stochastic amplitude equation with additive noise of the SGSH equation (1) in the two cases when $\gamma^2 < \frac{27}{38}$ and $\gamma^2 = \frac{27}{38}$. Also, I supposed that the noise acts directly on the dominant modes. While in our previous papers [5, 6] (written in

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collaboration with Blömker and Klepel), we assumed that $\gamma^2 < \frac{27}{38}$ and derived rigorously the amplitude equation for the amplitude of the dominant modes $\{\cos, \sin\}$ of the SGSH equation (1) with $\nu_\varepsilon = \varepsilon^2\nu$ and $\mu_\varepsilon = \varepsilon$ in the following form:

$$db_i = \left[\left(\nu - \frac{3}{2}\rho^2 + 3\rho^2\gamma^2\right)b_i + \frac{3}{4}\left(\frac{38\gamma^2}{27} - 1\right)b_i\left(b_1^2 + b_{-1}^2\right)\right]dT + 2\gamma\rho b_i\, d\tilde{\beta} \quad \text{for } i = \pm 1, \tag{2}$$

where the noise is a constant in the space ($W(t) = \rho\beta(t)$) and $\tilde{\beta}$ is a rescaled version of a Brownian motion, and we showed that the solution of equation (1) is well approximated by

$$u(t) = \varepsilon b\left(\varepsilon^2 t\right) + \text{error}.$$

In this paper we deal with the case $\gamma^2 = \frac{27}{38}$ and the noise does not act directly to the dominant mode, which is not treated in [4–6]. In this case the amplitude equation (2) loses its cubic nonlinearity term and it becomes a linear equation only. Therefore, the scaling we considered lead to solutions that were too small to see any of the nonlinear effects. Here we will change the scaling, and go to larger time-scales (of order ε^{-4}) and closer to bifurcation (*i.e.*, ν_ε of order ε^4). But changing the scaling considered here to the time-scale of order ε^{-4} considered in [5, 6] (*i.e.* by replacing ε^2 by $\hat{\varepsilon}$) one could see that this would lead to a larger scaling (of order $\hat{\varepsilon}^{-1/2}$) of the solutions in the ansatz and a larger noise strength of order $\hat{\varepsilon}^{-1/2}$. Moreover, due to noise and nonlinear interaction, deterministic linear terms appear in the amplitude equation. Other examples of this effect are [7–13]. Related work in this direction is in [14, 15].

Our aim of this paper is to derive rigorously this amplitude equation with the quintic nonlinearity for the SGSH equation (1) with $\nu_\varepsilon = \varepsilon^4\nu$, $\mu_\varepsilon = \varepsilon^2$ and $\gamma^2 = \frac{27}{38}$. Furthermore, we discuss the stabilization, without proof, by looking at the amplitude equation with Stratonovich type. We show that degenerate additive noise (*i.e.* noise that does not act directly to the dominant mode) has the potential to stabilize or destabilize the dynamics of the dominant modes. For example, if we consider (1) with respect to periodic boundary conditions on the interval $[0, 2\pi]$, then we obtain the stochastic amplitude equation with multiplicative noise and with an additional deterministic linear term, appearing due to noise and nonlinear interaction, in the Stratonovich form:

$$db_i = \left[\left(\nu - \frac{10}{9}\rho^2\right)b_i - C_0 b_i\left(b_1^2 + b_{-1}^2\right)^2\right]dT + \rho b_i \circ d\tilde{\beta} \quad \text{for } i = \pm 1,$$

where C_0 is a positive constant. We note that if ρ is large compared with ν, then the constant in front of the linear term, $(\nu - \frac{10}{9}\rho^2)$ is negative. In this case the degenerate additive noise stabilizes the dynamics of the dominant modes.

The rest of this paper is organized as follows. In Section 2 we state our precise assumptions. In Section 3 we derive rigorously the amplitude equation with error term and state the main theorem of this paper. In Section 4 we prove the main results of this paper. Finally, we give several cases of the amplitude equation of the stochastic generalized Swift-Hohenberg depending on the type of the noise and the boundary conditions.

2 Assumptions and definitions

We work in some Hilbert space $\mathcal{H}$ equipped with scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. For a more general setting, we study the following abstract equation:

$$du = \left[\mathcal{A}u + \varepsilon^4 \nu u + \gamma B(u) - \mathcal{F}(u)\right] dt + \varepsilon^2\, dW, \tag{3}$$

where $\mathcal{A}$ is a non-positive operator with finite dimensional kernel, $\varepsilon^4 \nu u$ is a linear small deterministic perturbation, $B(u,u) = B(u)$ is a quadratic nonlinearity given by bilinear map, $\mathcal{F}(u,u,u) = \mathcal{F}(u)$ is a cubic nonlinearity given by trilinear map, and W is a finite dimensional Wiener process. To be more precise we make the following assumptions.

For the linear operator $\mathcal{A}$ in (3) we assume the following.

Assumption 1 (Linear operator $\mathcal{A}$) Suppose $\mathcal{A}$ is a non-positive self-adjoint operator on $\mathcal{H}$ with eigenvalues

$$0 = \lambda_1 = \cdots = \lambda_n < \lambda_{n+1} \leq \cdots \leq \lambda_k \leq \cdots \quad \text{and} \quad \lambda_k \geq Ck^m$$

for all sufficiently large k, for one $m > 0$, and for a constant $C > 0$. The corresponding eigenvectors $\{e_k\}_{k=0}^{\infty}$ form a complete orthonormal system in $\mathcal{H}$ such that $-\mathcal{A}e_k = \lambda_k e_k$ (*cf.* Courant and Hilbert [16]).

We use the notation $\mathcal{C} := \ker \mathcal{A}$, where $\mathcal{C}$ has the finite dimension n and orthonormal basis $(e_1,\ldots,e_n)$. Define $S = \mathcal{C}^{\perp}$ the orthogonal complement of $\mathcal{C}$ in$\mathcal{H}$, and P_c for the orthogonal projection $P_c : \mathcal{H} \to \mathcal{C}$ and define $P_s := \mathcal{I} - P_c$ where $\mathcal{I}$ is the identity operator on $\mathcal{H}$.

Definition 2 For $\sigma \in \mathbb{R}$, we define the fractional interpolation space $\mathcal{H}^\sigma$ as

$$\mathcal{H}^\sigma = \left\{ \sum_{k=0}^{\infty} \eta_k e_k : \sum_{k=0}^{\infty} \eta_k^2 k^{2\sigma} < \infty \right\} \quad \text{with norm} \left\| \sum_{k=0}^{\infty} \eta_k e_k \right\|_\sigma^2 = \eta_0^2 + \sum_{k=1}^{\infty} \eta_k^2 k^{2\sigma}.$$

Moreover, the operator $\mathcal{A}$ given by Assumption 1 generates an analytic semigroup $\{e^{t\mathcal{A}}\}_{t\geq 0}$ (*cf.* Dan Henry [17]), on the space $\mathcal{H}^\sigma$ defined by

$$e^{\mathcal{A}t}\left(\sum_{k=0}^{\infty} \eta_k e_k \right) = \sum_{k=0}^{\infty} e^{-\lambda_k t} \eta_k e_k \quad \forall t \geq 0. \tag{4}$$

Lemma 3 *For all $t \geq 0$ and all $u \in \mathcal{H}^\sigma$, then there exists an $0 < \omega < \lambda_{n+1}$ such that*

$$\left\| e^{t\mathcal{A}} P_s u \right\|_{\mathcal{H}^\sigma} \leq e^{-\omega t} \|P_s u\|_{\mathcal{H}^\sigma}. \tag{5}$$

Proof From (4) we obtain

$$\begin{aligned}\left\| e^{t\mathcal{A}} P_s u \right\|_{\mathcal{H}^\sigma} &= \left\| \sum_{k=1}^{\infty} e^{-\lambda_k t} \eta_k e_k \right\|_{\mathcal{H}^\sigma} \leq \left\| \sum_{k=1}^{\infty} e^{-\lambda_{n+1} t} \eta_k e_k \right\|_{\mathcal{H}^\sigma} \\ &\leq e^{-\omega t} \left\| \sum_{k=1}^{\infty} \eta_k e_k \right\|_{\mathcal{H}^\sigma} \leq e^{-\omega t} \|P_s u\|_{\mathcal{H}^\sigma}.\end{aligned}$$

□

For the cubic term defined in (3), we assume the following.

Assumption 4 Assume that $\mathcal{F} : (\mathcal{H}^\sigma)^3 \to \mathcal{H}^\sigma$ is trilinear and symmetric, and it satisfies the following condition for some $C > 0$:

$$\left\|\mathcal{F}(u,v,\omega)\right\|_\sigma \le C\|u\|_\sigma\|v\|_\sigma\|w\|_\sigma \quad \forall u,v,w \in \mathcal{H}^\sigma. \tag{6}$$

Symmetry of $\mathcal{F}$ means that any permutation of the arguments yields the same result. For the quadratic nonlinearity B defined in (3), we assume the following.

Assumption 5 (Bilinear operator B) Let B be a bounded bilinear mapping from $\mathcal{H}^\sigma \times \mathcal{H}^\sigma$ to $\mathcal{H}^\sigma$. Suppose that B is symmetric and satisfies the following conditions for some $C > 0$:

$$\left\|B(u,w)\right\|_\sigma \le C\|u\|_\sigma\|w\|_\sigma \quad \forall u,w \in \mathcal{H}^\sigma \tag{7}$$

and

$$P_cB(e_k,e_k) = 0 \quad \text{for } k \in \mathbb{N}. \tag{8}$$

Assumption 6 We assume for $u \in \mathcal{C}$ that

6.1. $2\gamma^2B_c(u,\mathcal{A}_s^{-1}B_s(u)) + \mathcal{F}_c(u) = 0$,
6.2. $B_c(u,\mathcal{A}_s^{-1}\mathcal{F}_s(u)) = 0$,
6.3. $B_c(u,\mathcal{A}_s^{-1}B_s(u,\mathcal{A}_s^{-1}B_s(u))) = 0$,
6.4. $\mathcal{F}_c(u,u,\mathcal{A}_s^{-1}B_s(u)) = 0$,
6.5. $\sum_{k,\ell=n+1}^{\infty} \frac{B^k(u)B^\ell(u)}{\lambda_\ell(\lambda_k+\lambda_\ell)}B_c(e_k,e_\ell) = 0$,

where $B^\ell(u) = \langle B(u),e_\ell\rangle$.

We denote the projections by indices. This means $\mathcal{F}_c = P_c\mathcal{F}$, $\mathcal{F}_s = P_s\mathcal{F}$, $B_c = P_cB$, and $B_s = P_sB$. Moreover, we use $\mathcal{F}_s(u) = \mathcal{F}_s(u,u,u)$ and $B_s(u) = B_s(u,u)$ for short. Note that in Assumption 6, we need 6.1 in order for the cubic term to vanish in the amplitude equation and the other 6.2-6.5 for quartic terms disappear in the amplitude equation. These conditions need to be checked in examples.

For the noise W defined in (3) we have the following.

Assumption 7 We assume $\alpha_0 = \alpha_1 = \cdots = \alpha_n = \alpha_{N+1} = \cdots = 0$ and let W be a finite Wiener process on an abstract probability space $(\Omega,\mathcal{F},\mathbb{P})$. For $t \ge 0$, we can write $W(t)$ (*cf.* Da Prato and Zabczyk [18]) as

$$W(t) = \sum_{k=n+1}^{N} \alpha_k\beta_k(t)e_k \quad \text{for some } N \ge n+1,$$

where $(\beta_k)_{k\in\{n+1,\dots,N\}}$ are independent, standard Brownian motions in $\mathbb{R}$ and $(\alpha_k)_{k\in\{n+1,\dots,N\}}$ are real numbers.

Remark 8 We take $N < \infty$ in the above assumption for simplicity of presentation. For $N = \infty$, we can prove the most results by using the same method of proof. We only need to control the convergence of various infinite series, which is possible if the noise is not too irregular, which means for α_k decaying sufficiently fast for $k \to \infty$.

For the quintic nonlinearities term $\mathcal{G}$, which is defined later in (25), we assume the following.

Assumption 9 There are constants δ_1, $\delta_2 \geq 0$ such that for $u, w \in \mathcal{C}$ (identify $\mathcal{C} \cong \mathbb{R}^n$) the following inequalities are satisfied:

$$\langle \mathcal{G}(u,u,u,w,w), u \rangle \leq -\delta_1 |u|^4 |w|^2 \tag{9}$$

and

$$\langle \mathcal{G}(u,w,w,w,w), u \rangle \leq -\delta_2 |u|^2 |w|^4. \tag{10}$$

Remark 10 Setting $u = w$ in the above assumption we obtain for some $\delta \geq 0$

$$\langle \mathcal{G}(u), u \rangle \leq -\delta |u|^6. \tag{11}$$

For our result we rely on a cut off argument. We consider only solutions (a, ψ) that are not too large, as given by the next definition.

Definition 11 For the $\mathcal{C} \times S$-valued stochastic process (a, ψ) that will be defined later in (14) we define, for some $T_0 > 0$ and $\kappa \in (0, \frac{1}{16})$, the stopping time τ^* as

$$\tau^* := T_0 \wedge \inf\{T > 0 : \|a(T)\|_\sigma > \varepsilon^{-\kappa} \text{ or } \|\psi(T)\|_\sigma > \varepsilon^{-3\kappa}\}. \tag{12}$$

For a real-valued family of processes $\{X_\varepsilon(t)\}_{\geq 0}$ we say $X_\varepsilon = \mathcal{O}(f_\varepsilon)$, if for every $p \geq 1$ there exists a constant C_p such that

$$\mathbb{E} \sup_{t \in [0,\tau^*]} |X_\varepsilon(t)|^p \leq C_p f_\varepsilon^p. \tag{13}$$

We use also the analogous notation for time-independent random variables.

3 Amplitude equation

In this section we derive the amplitude equation with error term. We are interested here the studying behavior of the solutions of (3) on time-scales of order ε^{-4}. So, we split the solution u into

$$u(t) = \varepsilon a(\varepsilon^4 t) + \varepsilon^2 \psi(\varepsilon^4 t), \tag{14}$$

where $a \in \mathcal{C}$ and $\psi \in S$. After rescaling to the slow time-scale $T = \varepsilon^4 t$, we obtain the following system of equations:

$$\begin{aligned} da = \big[&\nu a + 2\gamma \varepsilon^{-2} B_c(a, \psi) + \gamma \varepsilon^{-1} B_c(\psi) - \varepsilon^{-2} \mathcal{F}_c(a) \\ &- 3\varepsilon^{-1} \mathcal{F}_c(a,a,\psi) - 3\mathcal{F}_c(a,\psi,\psi) - \varepsilon \mathcal{F}_c(\psi) \big] \, dT \end{aligned} \tag{15}$$

and

$$d\psi = \big[\varepsilon^{-4}\mathcal{A}_s\psi + \nu\psi + \gamma\varepsilon^{-4}B_s(a+\varepsilon\psi) - \varepsilon^{-3}\mathcal{F}_s(a+\varepsilon\psi)\big]\,dT + \varepsilon^{-2}\,d\tilde{W}_s, \tag{16}$$

where $\tilde{W}_s(T) := \varepsilon^2 W_s(\varepsilon^{-4}T)$ is a rescaled version of the Wiener process with $\tilde{W}_s = P_s\tilde{W}$. Equation (15) reads in integrated form

$$\begin{aligned} a(T) = a(0) + \nu\int_0^T a\,d\tau + \frac{2\gamma}{\varepsilon^2}\int_0^T B_c(a,\psi)\,d\tau + \frac{\gamma}{\varepsilon}\int_0^T B_c(\psi)\,d\tau - \frac{1}{\varepsilon^2}\int_0^T \mathcal{F}_c(a)\,d\tau \\ - 3\varepsilon^{-1}\int_0^T \mathcal{F}_c(a,a,\psi)\,d\tau - 3\int_0^T \mathcal{F}_c(a,\psi,\psi)\,d\tau - \varepsilon\int_0^T \mathcal{F}_c(\psi)\,d\tau. \end{aligned} \tag{17}$$

First, let us apply Itô's formula to $B_c(a,\mathcal{A}_s^{-1}\psi)$ in order to obtain the cubic term $B_c(a,\mathcal{A}_s^{-1}B_s(a))$. After that, we use Assumption 6(6.1) to remove the cubic term $[2\gamma^2 B_c(a,\mathcal{A}_s^{-1}B_s(a)) + \mathcal{F}_c(a)]$ from the amplitude equation. Applying Itô's formula to $B_c(a,\mathcal{A}_s^{-1}\psi)$, yields

$$\begin{aligned} &\frac{2\gamma}{\varepsilon^2}\int_0^T B_c(a,\psi)\,d\tau \\ &\quad= -\frac{2\gamma^2}{\varepsilon^2}\int_0^T B_c\big(a,\mathcal{A}_s^{-1}B_s(a)\big)\,d\tau - 4\gamma^2\int_0^T B_c\big(B_c(a,\psi),\mathcal{A}_s^{-1}\psi\big)\,d\tau \\ &\qquad+ 2\gamma\int_0^T B_c\big(\mathcal{F}_c(a),\mathcal{A}_s^{-1}\psi\big)\,d\tau - 2\gamma^2\int_0^T B_c\big(a,\mathcal{A}_s^{-1}B_s(\psi)\big)\,d\tau \\ &\qquad- \frac{4\gamma^2}{\varepsilon}\int_0^T B_c\big(a,\mathcal{A}_s^{-1}B_s(a,\psi)\big)\,d\tau + \frac{2\gamma}{\varepsilon}\int_0^T B_c\big(a,\mathcal{A}_s^{-1}\mathcal{F}_s(a)\big)\,d\tau \\ &\qquad+ 6\gamma\int_0^T B_c\big(a,\mathcal{A}_s^{-1}\mathcal{F}_s(a,a,\psi)\big)\,d\tau - 2\gamma\int_0^T B_c\big(a,\mathcal{A}_s^{-1}\,d\tilde{W}_s\big) + R_1, \end{aligned} \tag{18}$$

where the error term R_1 contains only terms that contain at least one ε, and it is given by

$$\begin{aligned} R_1(T) &= 2\varepsilon^2\gamma B_c\big(a(T),\mathcal{A}_s^{-1}\psi(T)\big) - 2\varepsilon^2\gamma B_c\big(a(0),\mathcal{A}_s^{-1}\psi(0)\big) \\ &\quad- 4\gamma\nu\varepsilon^2\int_0^T B_c\big(a,\mathcal{A}_s^{-1}\psi\big)\,d\tau - 2\gamma^2\varepsilon\int_0^T B_c\big(B_c(\psi),\mathcal{A}_s^{-1}\psi\big)\,d\tau \\ &\quad+ 6\gamma\varepsilon\int_0^T B_c\big(\mathcal{F}_c(a,a,\psi),\mathcal{A}_s^{-1}\psi\big)\,d\tau + 6\gamma\varepsilon^2\int_0^T B_c\big(\mathcal{F}_c(a,\psi,\psi),\mathcal{A}_s^{-1}\psi\big)\,d\tau \\ &\quad+ 2\gamma\varepsilon^3\int_0^T B_c\big(\mathcal{F}_c(\psi),\mathcal{A}_s^{-1}\psi\big)\,d\tau + 6\gamma\varepsilon\int_0^T B_c\big(a,\mathcal{A}_s^{-1}\mathcal{F}_s(a,\psi,\psi)\big)\,d\tau \\ &\quad+ 2\gamma\varepsilon^2\int_0^T B_c\big(a,\mathcal{A}_s^{-1}\mathcal{F}_s(\psi)\big)\,d\tau. \end{aligned} \tag{19}$$

Now, applying Itô's formula to $B_c(\psi_k e_k,\psi_\ell e_\ell)$ we obtain

$$\begin{aligned} \frac{\gamma}{\varepsilon}\int_0^T B_c(\psi,\psi)\,d\tau &= \frac{1}{\varepsilon}\sum_{k,\ell}\frac{2\gamma^2}{(\lambda_k+\lambda_\ell)}\int_0^T B_c\big(B_k(a)e_k,\psi_\ell e_\ell\big)\,d\tau \\ &\quad+ \sum_{k,\ell}\frac{4\gamma^2}{(\lambda_k+\lambda_\ell)}\int_0^T B_c\big(B_k(a,\psi)e_k,\psi_\ell e_\ell\big)\,d\tau \\ &\quad- \sum_{k,\ell}\frac{2\gamma\mathcal{F}_k(a)}{(\lambda_k+\lambda_\ell)}\int_0^T B_c(e_k,\psi_\ell e_\ell)\,d\tau + \mathcal{O}\big(\varepsilon^{1-15\kappa}\big), \end{aligned} \tag{20}$$

where we used $B_k(w) = \langle B(w), e_k\rangle$, and $\mathcal{F}_k(w) = \langle \mathcal{F}(w), e_k\rangle$ for short hand notation. Substituting equations (18) and (20) into equation (17) yields

$$\begin{aligned} a(T) &= a(0) + \nu\int_0^T a\,d\tau - 4\gamma^2\int_0^T B_c\big(B_c(a,\psi),\mathcal{A}_s^{-1}\psi\big)\,d\tau \\ &\quad - 2\gamma^2\int_0^T B_c\big(a,\mathcal{A}_s^{-1}B_s(\psi)\big)\,d\tau - \frac{4\gamma^2}{\varepsilon}\int_0^T B_c\big(a,\mathcal{A}_s^{-1}B_s(a,\psi)\big)\,d\tau \\ &\quad + 2\gamma\int_0^T B_c\big(\mathcal{F}_c(a),\mathcal{A}_s^{-1}\psi\big)\,d\tau - 2\gamma\int_0^T B_c\big(a,\mathcal{A}_s^{-1}\,d\tilde{W}_s\big) \\ &\quad + 6\gamma\int_0^T B_c\big(a,\mathcal{A}_s^{-1}\mathcal{F}_s(a,a,\psi)\big)\,d\tau + \frac{\gamma}{\varepsilon}\int_0^T B_c(\psi)\,d\tau \\ &\quad - \frac{3}{\varepsilon}\int_0^T \mathcal{F}_c(a,a,\psi)\,d\tau - 3\int_0^T \mathcal{F}_c(a,\psi,\psi)\,d\tau \\ &\quad + \frac{1}{\varepsilon}\sum_{k,\ell=n+1}^{\infty}\frac{2\gamma^2 B^k(a)}{(\lambda_k+\lambda_\ell)}\int_0^T B_c(e_k,\psi_\ell e_\ell)\,d\tau \\ &\quad + \sum_{k,\ell=n+1}^{\infty}\frac{4\gamma^2}{(\lambda_k+\lambda_\ell)}\int_0^T B_c\big(B^k(a,\psi)e_k,\psi_\ell e_\ell\big)\,d\tau \\ &\quad - \sum_{k,\ell=n+1}^{N}\frac{2\gamma\mathcal{F}^k(a)}{(\lambda_k+\lambda_\ell)}\int_0^T B_c(e_k,\psi_\ell e_\ell)\,d\tau + R_2(T), \end{aligned} \tag{21}$$

where we used Assumption 6(6.1) and the error term R_2 is given by

$$R_2(T) = R_1(T) - \varepsilon\int_0^T \mathcal{F}_c(\psi)\,d\tau + \mathcal{O}\big(\varepsilon^{1-15\kappa}\big). \tag{22}$$

To remove ψ from the right hand side of (21), we note that there are two kinds of terms in that equation that contains ψ. The first its kind contains only one ψ, which is $\mathcal{F}_c(a,a,\psi)$, $B_c(a,\mathcal{A}_s^{-1}B_s(a,\psi))$, $B_c(\mathcal{F}_c(a),\mathcal{A}_s^{-1}\psi)$, $B_c(a,\mathcal{A}_s^{-1}\mathcal{F}_s(a,a,\psi))$, and $B_c(e_k,\psi_\ell e_\ell)$. For these terms, let us define $\Theta_1(\cdot,\hbar\psi)$ as one of them, where $\hbar$ is an operator (such as $\hbar = \mathcal{I}$ or $\hbar = \mathcal{A}_s^{-1}$). Now, to get rid of ψ from $\Theta_1(\cdot,\hbar\psi)$, we apply the Itô formula to $\Theta_1(\cdot,\hbar\mathcal{A}_s^{-1}\psi)$ and subsequently the following two cases arise.

First case: if there is no ε^{-1} in front of $\Theta_1(\cdot,\hbar\psi)$, then we obtain

$$\Theta_1(\cdot,\hbar\psi) = -\gamma\Theta_1\big(\cdot,\hbar\mathcal{A}_s^{-1}B_s(a)\big) + \mathcal{O}\big(\varepsilon^{1-14\kappa}\big).$$

Second case: if there is ε^{-1} in front of $\Theta_1(\cdot,\hbar\psi)$, then, by using Assumption 6, we obtain the following formula:

$$\varepsilon^{-1}\Theta_1(\cdot,\hbar\psi) = \Theta_1\big(\cdot,\hbar\mathcal{A}_s^{-1}\mathcal{F}_s(a)\big) + 2\gamma^2\Theta_1\big(\cdot,\hbar\mathcal{A}_s^{-1}B_s\big(a,\mathcal{A}_s^{-1}B_s(a)\big)\big) + \mathcal{O}\big(\varepsilon^{1-14\kappa}\big).$$

The second of its kind contains ψ^2, which is $B_c(B_c(a,\psi),\mathcal{A}_s^{-1}\psi)$, $B_c(a,\mathcal{A}_s^{-1}B_s(\psi))$, $B_c(\psi)$, $\mathcal{F}_c(a,\psi,\psi)$ and $B_c(B^k(a,\psi)e_k,\psi_\ell e_\ell)$. Let us define $\Theta_2(\cdot,\hbar_1\psi,\hbar_2\psi)$ as one of the previous terms, where $\hbar_i$ is an operator for $i = 1,2$ ($\hbar_i = \mathcal{I}$ or $\hbar_i = \mathcal{A}_s^{-1}$, for $i = 1,2$). To remove ψ from

$\Theta_2(\cdot,\hbar_1\psi,\hbar_2\psi)$, we apply the Itô formula to $\Theta_2(\cdot,\hbar_1\psi_k e_k,\hbar_2\psi_\ell e_\ell)$ and therefore we get

$$\begin{aligned}\Theta_2(\cdot,\hbar_1\psi,\hbar_2\psi) &= \sum_{k,\ell=n+1}^{\infty}\frac{2\gamma^2 B^k(a)B^\ell(a)}{\lambda_\ell(\lambda_k+\lambda_\ell)}\Theta_2(\cdot,\hbar_1 e_k,\hbar_2 e_\ell)\\ &\quad+\sum_{k=n+1}^{N}\frac{\alpha_k^2}{4\lambda_k}\Theta_2(\cdot,\hbar_1 e_k,\hbar_2 e_k)+\mathcal{O}\bigl(\varepsilon^{1-15\kappa}\bigr).\end{aligned}$$

Thus, we can obtain the following amplitude equation with error terms:

$$a(T)=a(0)+\int_0^T\bigl[\mathcal{L}(a)+\mathcal{G}(a)\bigr](\tau)\,d\tau+\sum_{k=n+1}^{N}\frac{\gamma\alpha_k}{\lambda_k}\int_0^T B_c(a,e_k)\,d\tilde{\beta}_k(\tau)+R_2(T), \tag{23}$$

where the linear term $\mathcal{L}(a)$ and the quintic term $\mathcal{G}(a)$ are defined as, respectively,

$$\begin{aligned}\mathcal{L}(a) &= \nu a+\sum_{k=n+1}^{N}\frac{\gamma^2\alpha_k^2}{\lambda_k^2}B_c\bigl(B_c(a,e_k),e_k\bigr)\\ &\quad-\sum_{k=n+1}^{N}\frac{\gamma^2\alpha_k^2}{2\lambda_k}B_c\bigl(a,\mathcal{A}_s^{-1}B_s(e_k)\bigr)-\sum_{k=n+1}^{N}\frac{3\alpha_k^2}{2\lambda_k}\mathcal{F}_c(a,e_k,e_k)\end{aligned}\tag{24}$$

and

$$\begin{aligned}\mathcal{G}(a) &= \sum_{k,\ell=n+1}^{\infty}\frac{8\gamma^4 B^k(a)B^\ell(a)}{\lambda_\ell^2(\lambda_k+\lambda_\ell)}B_c\bigl(B_c(a,e_k),e_\ell\bigr)-2\gamma^2 B_c\bigl(\mathcal{F}_c(a),\mathcal{A}_s^{-1}\mathcal{A}_s^{-1}B_s(a)\bigr)\\ &\quad-8\gamma^4 B_c\bigl(a,\mathcal{A}_s^{-1}B_s\bigl(a,\mathcal{A}_s^{-1}B_s\bigl(a,\mathcal{A}_s^{-1}B_s(a)\bigr)\bigr)\bigr)-4\gamma^2 B_c\bigl(a,\mathcal{A}_s^{-1}B_s\bigl(a,\mathcal{A}_s^{-1}\mathcal{F}_s(a)\bigr)\bigr)\\ &\quad-\sum_{k,\ell=n+1}^{\infty}\frac{4\gamma^4 B^k(a)B^\ell(a)}{\lambda_\ell(\lambda_k+\lambda_\ell)}B_c\bigl(a,\mathcal{A}_s^{-1}B_s(e_k,e_\ell)\bigr)-6\gamma^2 B_c\bigl(a,\mathcal{A}_s^{-1}\mathcal{F}_s\bigl(a,a,\mathcal{A}_s^{-1}B_s(a)\bigr)\bigr)\\ &\quad-\sum_{k,\ell=n+1}^{\infty}\frac{4\gamma^4 B^k(a)}{\lambda_\ell(\lambda_k+\lambda_\ell)}B_c\bigl(e_k,B^\ell\bigl(a,\mathcal{A}_s^{-1}B_s(a)\bigr)e_\ell\bigr)-\sum_{k,\ell=n+1}^{\infty}\frac{2\gamma^2 B^k(a)\mathcal{F}^\ell(a)}{\lambda_\ell\lambda_k}B_c(e_k,e_\ell)\\ &\quad-\sum_{k,\ell=n+1}^{\infty}\frac{6\gamma^2 B^k(a)B^\ell(a)}{\lambda_\ell(\lambda_k+\lambda_\ell)}\mathcal{F}_c(a,e_k,e_\ell)-6\gamma^2\mathcal{F}_c\bigl(a,a,\mathcal{A}_s^{-1}B_s\bigl(a,\mathcal{A}_s^{-1}B_s(a)\bigr)\bigr)\\ &\quad+\sum_{k,\ell,j=n+1}^{\infty}\frac{8\gamma^4 B^j(a)B^\ell(a)}{\lambda_\ell(\lambda_k+\lambda_\ell)(\lambda_j+\lambda_\ell)}B_c\bigl(B^k(a,e_j)e_k,e_\ell\bigr)-3\mathcal{F}_c\bigl(a,a,\mathcal{A}_s^{-1}\mathcal{F}_s(a)\bigr).\end{aligned}\tag{25}$$

The main result of this paper is that near a change of stability on a time-scale of order ε^{-4} the solution of (3) is of the type

$$u(t)=\varepsilon b\bigl(\varepsilon^4 t\bigr)+\text{error}, \tag{26}$$

where b is the solution of the amplitude equation on the slow time-scale

$$db=\bigl[\mathcal{L}(b)+\mathcal{G}(b)\bigr]\,dT+\sum_{k=n+1}^{N}\frac{\gamma\alpha_k}{\lambda_k}B_c(b,e_k)\,d\tilde{\beta}_k, \tag{27}$$

where the linear term $\mathcal{L}(b)$ and the quintic term $\mathcal{G}(b)$ are defined in (24) and (25), respectively.

The main result of this paper is the following.

Theorem 12 (Approximation) *Under Assumptions* 1, 4, 5, 6, 7, *and* 9 *let u be a solution of* (3) *defined in* (14) *with the initial condition* $u(0) = \varepsilon a(0) + \varepsilon^2\psi(0)$ *with* $a(0) \in \mathcal{C}$ *and* $\psi(0) \in S$ *where* $a(0)$ *and* $\psi(0)$ *are of order one, and b is a solution of* (27) *with* $b(0) = a(0)$. *Then, for all* $p \geq 1$ *and* $T_0 > 0$ *and all* $\kappa \in (0, \frac{1}{16})$, *there exists* $C > 0$ *such that*

$$\mathbb{P}\Big(\sup_{t\in[0,\varepsilon^{-4}T_0]} \big\|u(t) - \varepsilon b\big(\varepsilon^4 t\big)\big\|_\sigma > \varepsilon^{2-32\kappa}\Big) \leq C\varepsilon^p, \tag{28}$$

for all sufficiently small $\varepsilon > 0$.

4 Proof of the main result

Lemma 13 *Under Assumptions* 1, 4, *and* 5, *there is a constant* $C > 0$ *such that, for* κ *from the definition of* τ^* *and* $p \geq 1$,

$$\mathbb{E}\sup_{T\in[0,\tau^*]} \big\|\psi(T) - \mathcal{Q}(T)\big\|_\sigma^p \leq C\varepsilon^{p-9p\kappa}, \tag{29}$$

where

$$\mathcal{Q}(T) = e^{\varepsilon^{-4}T\mathcal{A}_s}\psi(0) + \gamma\varepsilon^{-4}\int_0^T e^{\varepsilon^{-4}\mathcal{A}_s(T-\tau)}B_s(a)\,d\tau + \mathcal{Z}(T), \tag{30}$$

with

$$\mathcal{Z}(T) = \varepsilon^{-2}\int_0^T e^{\varepsilon^{-4}\mathcal{A}_s(T-\tau)}\,d\tilde{W}_s(\tau). \tag{31}$$

Proof The mild formulation of (16) is

$$\begin{aligned}\psi(T) = {}& e^{\varepsilon^{-4}T\mathcal{A}_s}\psi(0) + \nu\int_0^T e^{\varepsilon^{-4}\mathcal{A}_s(T-\tau)}\psi\,d\tau + \frac{\gamma}{\varepsilon^4}\int_0^T e^{\varepsilon^{-4}\mathcal{A}_s(T-\tau)}B_s(a)\,d\tau \\ & + \frac{2\gamma}{\varepsilon^3}\int_0^T e^{\varepsilon^{-4}\mathcal{A}_s(T-\tau)}B_s(a,\psi)\,d\tau + \frac{\gamma}{\varepsilon^2}\int_0^T e^{\varepsilon^{-4}\mathcal{A}_s(T-\tau)}B_s(\psi)\,d\tau \\ & - \frac{1}{\varepsilon^3}\int_0^T e^{\varepsilon^{-4}\mathcal{A}_s(T-\tau)}\mathcal{F}_s(a+\varepsilon\psi)\,d\tau + \mathcal{Z}(T).\end{aligned} \tag{32}$$

Using the triangle inequality

$$\begin{aligned}\big\|\psi(T) - \mathcal{Q}(T)\big\|_\alpha \leq {}& C\bigg\|\int_0^T e^{\varepsilon^{-4}\mathcal{A}_s(T-\tau)}\psi\,d\tau\bigg\|_\sigma \\ & + C\varepsilon^{-3}\bigg\|\int_0^T e^{\varepsilon^{-4}\mathcal{A}_s(T-\tau)}B_s(a,\psi)\,d\tau\bigg\|_\sigma \\ & + \varepsilon^{-2}C\bigg\|\int_0^T e^{\varepsilon^{-4}\mathcal{A}_s(T-\tau)}B_s(\psi)\,d\tau\bigg\|_\sigma\end{aligned}$$

$$+ C\varepsilon^{-3}\left\| \int_0^T e^{\varepsilon^{-4}\mathcal{A}_s(T-\tau)} \mathcal{F}_s(a+\varepsilon\psi)\, d\tau \right\|_\sigma$$
$$=: I_1 + I_2 + I_3 + I_4.$$

We now bound all four terms separately. For the first term, using (5) we obtain for all $T \le \tau^*$

$$I_1 \le C \int_0^T e^{-\varepsilon^{-4}w(T-\tau)} \left\|\psi(\tau)\right\|_\sigma d\tau \le C \sup_{\tau\in[0,\tau^*]} \left\|\psi(\tau)\right\|_\sigma \int_0^{\varepsilon^{-4}wT} e^{-\eta}\, d\eta \le C\varepsilon^{4-3\kappa},$$

where we used the definition of τ^*. For the second term, we obtain by using (5) and Assumption 5

$$\begin{aligned} I_2 &\le C\varepsilon^{-3} \int_0^T e^{-\varepsilon^{-4}w(T-\tau)} \left\|B_s(a,\psi)(\tau)\right\|_\sigma d\tau \\ &\le C\varepsilon \sup_{[0,\tau^*]} \left\{ \|a\|_\sigma \|\psi\|_\sigma \right\} \cdot \int_0^{\varepsilon^{-4}wT} e^{-\eta}\, d\eta \le C\varepsilon^{1-4\kappa}, \end{aligned}$$

where we used again the definition of τ^*. Analogously, for the third term we obtain

$$\begin{aligned} I_3 &\le \varepsilon^{-2} \int_0^T e^{-\varepsilon^{-4}w(T-\tau)} \left\|B_s(\psi)(\tau)\right\|_\sigma d\tau \\ &\le C\varepsilon^2 \sup_{[0,\tau^*]} \|\psi\|_\sigma^2 \int_0^{\varepsilon^{-4}wT} e^{-\eta}\, d\eta \le C\varepsilon^{2-6\kappa}. \end{aligned}$$

For the fourth term, we obtain by using (5) and Assumption 4

$$\begin{aligned} I_4 &\le C\varepsilon^{-3} \int_0^T e^{-\varepsilon^{-4}w(T-\tau)} \left\|\mathcal{F}_s(a+\varepsilon\psi)\right\|_\sigma d\tau \\ &\le C\varepsilon\Big(\sup_{[0,\tau^*]} \|a\|_\sigma^3 + \varepsilon^3 \sup_{[0,\tau^*]} \|\psi\|_\sigma^3 \Big) \int_0^{\varepsilon^{-4}wT} e^{-\eta}\, d\eta \le C\varepsilon^{1-9\kappa}, \end{aligned}$$

where we used the definition of τ^*. Combining all results yields (29). □

The next lemma provides bounds for the stochastic convolution $\mathcal{Z}(T)$ defined in (31).

Lemma 14 *Under Assumption* 7, *for every* $\kappa_0 > 0$ *and* $p \ge 1$, *there is a constant* $C > 0$, *depending on* p, α_k, λ_k, κ_0, *and* T_0, *such that*

$$\mathbb{E} \sup_{T\in[0,T_0]} \left\|\mathcal{Z}(T)\right\|_\sigma^p \le C\varepsilon^{-\kappa_0}.$$

Proof See the proof of Lemma 20 in [19]. □

We now need the following simple estimate.

Lemma 15 *Using* τ^* *defined in Definition* 11, *then*

$$\mathbb{E} \sup_{T\in[0,\tau^*]} \left\| \int_0^T e^{\varepsilon^{-4}\mathcal{A}_s(T-\tau)} B_s(a,a)\, d\tau \right\|_\sigma^p \le C\varepsilon^{4p-2p\kappa}, \tag{33}$$

for all $\varepsilon \in (0,1)$.

Proof Using (5) we obtain for $T < \tau^*$

$$\begin{aligned}\left\|\int_0^T e^{\varepsilon^{-4}\mathcal{A}_s(T-\tau)}B_s(a)\,d\tau\right\|_\sigma &\leq C\int_0^T e^{-\varepsilon^{-4}\omega(T-\tau)}\left\|B_s(a)\right\|_\sigma\,d\tau \\ &\leq C\varepsilon^4\sup_{\tau\in[0,\tau^*]}\left\|a(\tau)\right\|_\sigma^2\int_0^{\varepsilon^{-4}\omega T}e^{-\eta}\,d\eta \\ &\leq C\varepsilon^{4-2\kappa}.\end{aligned}$$

□

The following corollary states that $\psi(T)$ is with high probability much smaller than $\varepsilon^{-3\kappa}$ as asserted by the Definition 11 for $T \leq \tau^*$. We will show later $\tau^* \geq T_0$ with high probability (*cf.* the proof of Theorem 12).

Corollary 16 *Under the assumptions of Lemmas* 13 *and* 14, *if* $\psi(0) = \mathcal{O}(1)$, *then for all* $p \geq 1$ *and for all* $\kappa_0 > 0$ *there exist a constant* $C > 0$ *such that*

$$\mathbb{E}\Big(\sup_{T\in[0,\tau^*]}\left\|\psi(T)\right\|_\sigma^p\Big) \leq C\varepsilon^{-2\kappa}. \tag{34}$$

Proof From (32), by the triangle inequality and Lemmas 14 and 15 we obtain

$$\mathbb{E}\sup_{T\in[0,\tau^*]}\left\|\psi(T)\right\|_\sigma^p \leq C + C\varepsilon^{-2p\kappa} + C\varepsilon^{p-9p\kappa},$$

for $\kappa < \frac{1}{9}$ and $\kappa_0 \leq \kappa$, which yields (34). □

Now the next step is to bound the remainder R_2 defined in (22).

Lemma 17 *If Assumption* 7 *holds, then for all* $p \geq 1$ *there exists a constant* $C > 0$ *such that*

$$\mathbb{E}\Big(\sup_{T\in[0,\tau^*]}\left\|R_2(T)\right\|_\sigma^p\Big) \leq C\varepsilon^{p-15p\kappa}. \tag{35}$$

Proof We follow the proof of Lemma 13 to obtain (35). □

We need the following *a priori* estimate for solutions of the amplitude equation (27).

Lemma 18 *Let Assumptions* 5, 7, *and* 9 *hold. Define* $b(t)$ *in* $\mathcal{C}$ *as the solution of* (27). *If the initial condition satisfies* $\mathbb{E}|b(0)|^p \leq C$ *for some* $p \geq 1$, *then there exists another constant* C *such that*

$$\mathbb{E}\sup_{T\in[0,\tau^*]}\left|b(T)\right|^p \leq C. \tag{36}$$

Proof See the proof of Lemma 23 in [6]. □

Definition 19 Define the set $\Omega^* \subset \Omega$ such that all these estimates

$$\sup_{[0,\tau^*]}\|\psi\|_\sigma < C\varepsilon^{-\frac{5}{2}\kappa}, \tag{37}$$

$$\sup_{[0,\tau^*]}\|R_2\|_\sigma < C\varepsilon^{1-16\kappa}, \tag{38}$$

and

$$\sup_{[0,\tau^*]} |b| < C\varepsilon^{-\frac{1}{2}\kappa}, \tag{39}$$

hold on Ω^*.

In the following we show that the set Ω^* has approximately probability 1.

Proposition 20 *For any $p \geq 1$ there is a constant $C > 0$ such that*

$$\mathbb{P}(\Omega^*) \geq 1 - C\varepsilon^p,$$

for all ε sufficiently small.

Proof Ω^* has probability

$$\mathbb{P}(\Omega^*) \geq 1 - \mathbb{P}\Big(\sup_{[0,\tau^*]} \|\psi\|_\sigma \geq C\varepsilon^{-\frac{5}{2}\kappa}\Big) - \mathbb{P}\Big(\sup_{[0,\tau^*]} \|R_2\|_\sigma \geq C\varepsilon^{1-16\kappa}\Big)$$
$$- \mathbb{P}\Big(\sup_{[0,\tau^*]} |b| \geq C\varepsilon^{-\frac{1}{2}\kappa}\Big).$$

Using the Chebychev inequality, Corollary 16, and Lemmas 17, 18, we obtain for sufficiently large $q > \frac{2p}{\kappa}$ for any $p \geq 1$

$$\mathbb{P}(\Omega^*) \geq 1 - C\big[\varepsilon^{\frac{1}{2}q\kappa} + \varepsilon^{q\kappa} + \varepsilon^{\frac{1}{2}q\kappa}\big] \geq 1 - C\varepsilon^{\frac{1}{2}q\kappa} \geq 1 - C\varepsilon^p. \tag{40}$$

□

Theorem 21 *Assume that Assumption 9 holds and suppose $a(0) = \mathcal{O}(1)$ and $\psi(0) = \mathcal{O}(1)$. Let b be a solution of (27) and a is defined in (23). If the initial conditions satisfy $a(0) = b(0)$, then for $\kappa < \frac{1}{16}$, we obtain*

$$\sup_{T\in[0,\tau^*]} \big|a(T) - b(T)\big| \leq C\varepsilon^{2-32\kappa} \quad \textit{on } \Omega^* \tag{41}$$

and

$$\sup_{T\in[0,\tau^*]} \big|a(T)\big| \leq C\varepsilon^{-\frac{1}{2}\kappa} \quad \textit{on } \Omega^*. \tag{42}$$

Proof See the proof of Theorem 24 in [6]. □

Now, we can use the above results to prove the main result of Theorem 12 for the approximation of the solution (27) of the SPDE (3).

Proof of Theorem 12 For the stopping time, we note that

$$\Omega \supset \{\tau^* = T_0\} \supseteq \Big\{\sup_{T\in[0,T_0]} \big\|a(T)\big\|_\sigma < \varepsilon^{-\kappa}, \sup_{T\in[0,T_0]} \big\|\psi(T)\big\|_\sigma < \varepsilon^{-3\kappa}\Big\} \supseteq \Omega^*.$$

Hence

$$\mathbb{P}\{\tau^* < T_0\} \leq \mathbb{P}\Big\{\sup_{[0,\tau^*]} \|a\|_\sigma > \varepsilon^{-\kappa}, \sup_{[0,\tau^*]} \|\psi\|_\sigma > \varepsilon^{-3\kappa}\Big\} \leq C\varepsilon^{q\kappa}, \tag{43}$$

where we used Chebychev's inequality and (34). Now let us turn to the approximation result. Using (14) and the triangle inequality yields

$$\sup_{T\in[0,\tau^*]} \big\|u(\varepsilon^{-4}T) - \varepsilon b(T)\big\|_\sigma \leq \varepsilon \sup_{[0,\tau^*]} \|a-b\|_\sigma + \varepsilon^2 \sup_{[0,\tau^*]} \|\psi\|_\sigma.$$

From (37) and (41), we obtain

$$\begin{aligned}\sup_{t\in[0,\varepsilon^{-4}T_0]} \big\|u(t) - \varepsilon b(\varepsilon^4 t)\big\|_\sigma &= \sup_{t\in[0,\varepsilon^{-4}\tau^*]} \big\|u(t) - \varepsilon b(\varepsilon^4 t)\big\|_\sigma \\ &\leq C\varepsilon^{2-32\kappa} \quad \text{on } \Omega^*.\end{aligned}$$

Thus

$$\mathbb{P}\Big(\sup_{t\in[0,\varepsilon^{-4}T_0]} \big\|u(t) - \varepsilon b(\varepsilon^4 t)\big\|_\sigma > \varepsilon^{2-32\kappa}\Big) \leq 1 - \mathbb{P}(\Omega^*).$$

Using (40), the above estimate yields (28). □

5 Stochastic generalized Swift-Hohenberg equation

We consider the SGSH equation (1) with $\gamma^2 = \frac{27}{38}$. The Swift-Hohenberg equation was first used as a toy model for the convective instability in Rayleigh-Bénard problem (see [2] or [20]). Today it is one of the most popular equations for the examination of the dynamics of pattern formation.

For this model (1), we note that

$$\mathcal{A} = -(1+\partial_x^2)^2, \qquad B(u) = u^2, \quad \text{and} \quad \mathcal{F}(u) = u^3.$$

In the following we derive the amplitude equation of (1) with respect to Neumann boundary condition on the interval $[0,\pi]$ and with respect to periodic boundary conditions on $[0,2\pi]$.

5.1 Neumann boundary condition

Define

$$e_k(x) = \begin{cases} \frac{1}{\sqrt{\pi}} & \text{if } k = 0, \\ \sqrt{\frac{2}{\pi}}\cos(kx) & \text{if } k > 0, \end{cases}$$

and

$$\mathcal{H} = L^2([0,\pi]) \quad \text{and} \quad \mathcal{C} = \operatorname{span}\{\cos\}.$$

Then the eigenvalues of $-\mathcal{A} = (1 + \partial_x^2)^2$ are $\lambda_k = (1 - k^2)^2$ for $k \in \mathbb{N}_0$ with $m = 4$, $\lambda_0 = 1 > 0$ and $\lim_{k\to\infty} \lambda_k = \infty$. Moreover, with $\sigma = 1$, it is easy to check that for $u, v, w \in \mathcal{H}^1$

$$\|\mathcal{F}(u,v,w)\|_{\mathcal{H}^1} = \|uvw\|_{\mathcal{H}^1} \leq C\|u\|_{\mathcal{H}^1}\|v\|_{\mathcal{H}^1}\|w\|_{\mathcal{H}^1}$$

and

$$\|B(u,w)\|_{\mathcal{H}^1} = \|uw\|_{\mathcal{H}^1} \leq C\|u\|_{\mathcal{H}^1}\|w\|_{\mathcal{H}^1}.$$

Moreover, Assumption 6 is satisfied as follows:

$$\begin{aligned} B_c\bigl(a\cos(kx), a\cos(kx)\bigr) &= P_c\bigl[a^2\cos^2(kx)\bigr] = \frac{a^2}{2}P_c\bigl[1 + \cos(2kx)\bigr] \\ &= 0 \quad \text{for } u = a\cos(kx) \in \mathcal{H}. \end{aligned}$$

If $\gamma^2 = \frac{27}{38}$ and $u = a\cos(x) \in \mathcal{C}$, then we have

$$2\gamma^2 B_c\bigl(u, \mathcal{A}_s^{-1}B_s(u)\bigr) + \mathcal{F}_c(u) = \frac{3}{4}\left(\frac{38\gamma^2}{27} - 1\right)a^3\cos(x) = 0$$

and

$$\begin{aligned} B_c\bigl(u, \mathcal{A}_s^{-1}\mathcal{F}_s(u)\bigr) &= B_c\left(a\cos(x), \frac{-a^3}{4\lambda_3}\cos(3x)\right) = \frac{-a^4}{4\lambda_3}P_c\bigl[\cos(x)\cos(3x)\bigr] \\ &= \frac{-a^4}{8\lambda_3}P_c\bigl[\cos(2x) + \cos(4x)\bigr] = 0, \\ B_c\bigl(u, \mathcal{A}_s^{-1}B_s\bigl(u, \mathcal{A}_s^{-1}B_s(u)\bigr)\bigr) &= B_c\left(u, \mathcal{A}_s^{-1}B_s\left(u, \frac{-a^2}{2}\left(1 + \frac{1}{\lambda_2}\cos(2x)\right)\right)\right) \\ &= \frac{a^4}{4\lambda_2\lambda_3}P_c\bigl[\cos(x)\cos(3x)\bigr] = 0, \\ \mathcal{F}_c\bigl(u, u, \mathcal{A}_s^{-1}B_s(u)\bigr) &= \mathcal{F}_c\left(u, u, \frac{-a^2}{2}\left(1 + \frac{1}{\lambda_2}\cos(2x)\right)\right) \\ &= \frac{-a^4}{4}P_c\left[\bigl(1 + \cos(2x)\bigr)\left(1 + \frac{1}{\lambda_2}\cos(2x)\right)\right] \\ &= \frac{-a^4}{4}P_c\left[\frac{19}{18} + \frac{10}{9}\cos(2x) + \frac{1}{18}\cos(4x)\right] = 0, \end{aligned}$$

and

$$\begin{aligned} \sum_{k,\ell}\frac{B^k(u)B^\ell(u)}{\lambda_\ell(\lambda_k + \lambda_\ell)}B_c(e_k, e_\ell) &= \sum_{k\neq\ell}\frac{B^k(u)B^\ell(u)}{\lambda_\ell(\lambda_k + \lambda_\ell)}B_c(e_k, e_\ell) \\ &= \frac{B^1(u)B^2(u)}{90}B_c(e_1, e_2) + \frac{B^2(u)B^1(u)}{10}B_c(e_1, e_2) \\ &= \frac{a^4}{36}P_c\bigl[\cos(2x)\bigr] = 0. \end{aligned}$$

After a straightforward calculation of (25) we derive

$$\mathcal{G}(b) := -C_0 b^5 \cos(x) \quad \text{with } C_0 \simeq 0.6.$$

The function $\mathcal{G}$ satisfies the condition (9), for $u = \sqrt{\frac{2}{\pi}}\gamma_1 \cos(x) \in \mathcal{C}$ and $w = \sqrt{\frac{2}{\pi}}\gamma_2 \cos(x) \in \mathcal{C}$ ($\mathcal{C} \cong \mathbb{R}^n$) as follows:

$$\langle \mathcal{G}(u,u,u,w,w), u\rangle = -\sqrt{\frac{\pi}{2}} C_0 \gamma_1^4 \gamma_2^2 \cos^2(x) \geq -\delta_1 \gamma_1^4 \gamma_2^2 \geq -\delta_1 |u|^4 |w|^2,$$

where $\delta_1 = \sqrt{\frac{\pi}{2}} C_0$. We argue analogously for the conditions (10) and (11).

For Assumption 7 we consider two cases.

First case: the noise is a constant in the space (*i.e.* $W(t) = \frac{\alpha_0}{\sqrt{\pi}}\beta_0(t)$).

In this case our main theorem states that the solution of (1) is of the type

$$u(t,x) = \varepsilon v(\varepsilon^4 t, x)$$

and

$$v(T,x) = b(T)\cos(x) + \mathcal{O}(\varepsilon^{1-32\kappa}),$$

where b is the solution of the amplitude equation of Itô type

$$db = \left[\left(\nu - \frac{11}{18}\rho_0^2\right) b - C_0 b^5\right] dT + \rho_0 b \, d\tilde{\beta}_0, \tag{44}$$

where $\rho_0 = \sqrt{\frac{27}{38\pi}}\alpha_0$.

The Stratonovich version of (44) is obtained:

$$db = \left[\left(\nu - \frac{10}{9}\rho_0^2\right) b - C_0 b^5\right] dT + \rho_0 b \circ d\tilde{\beta}_0. \tag{45}$$

Now, let us show the influence of the additive degenerate noise on the stabilization of the solution of the amplitude equation (45) by looking at the sign of the linear drift term. The constant solution 0 is locally stable if $(\nu - \frac{10}{9}\rho_0^2) < 0$ and unstable if $(\nu - \frac{10}{9}\rho_0^2) > 0$. This is well known in the literature (see for instance Arnold [21], Arnold *et al.* [22], Mao [23]). We use the Euler Maruyama method stated in [24] to simulate equation (45).

From Figure 1 we can deduce that if the noise intensity ρ_0 increases, then the solution of the amplitude tends to zero.

Second case: the noise acts on the second mode, *i.e.* the noise takes the form

$$W(t,x) = \alpha_2 \beta_2(t) e_2(x).$$

The amplitude equation of Itô type in this case takes the form

$$db = \left[\left(\nu - \frac{7\rho_2^2}{114}\right) b - C_0 b^5\right] dT + \frac{\rho_2\sqrt{27}}{18\sqrt{38}} b \, d\tilde{\beta}_2, \tag{46}$$

where $\rho_2 = \alpha_2\sqrt{\frac{2}{\pi}}$.

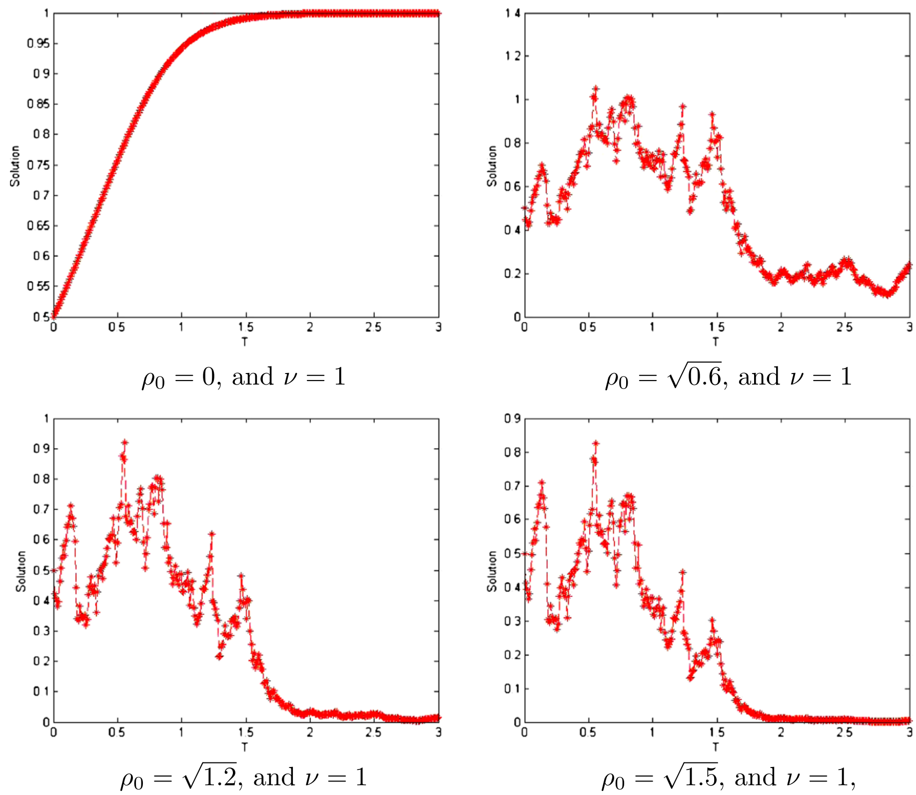

Figure 1 The solution of amplitude equation (45) with different values of the noise intensity ρ_0.

5.2 Periodic boundary conditions

In this case

$$e_k(x) = \begin{cases} \frac{1}{\sqrt{\pi}}\sin(kx) & \text{if } k > 0, \\ \frac{1}{\sqrt{2\pi}} & \text{if } k = 0, \\ \frac{1}{\sqrt{\pi}}\cos(kx) & \text{if } k < 0, \end{cases} \quad \text{and} \quad \ker\mathcal{A} = \operatorname{span}\{\cos x, \sin x\},$$

then we have

$$\mathcal{G}(b) := -C_0\bigl(b_1^2 + b_{-1}^2\bigr)^2\bigl[b_1\sin(x) + b_{-1}\cos(x)\bigr].$$

Here we consider two cases depending on the type of the noise.

First case: the noise is constant in the space ($W(t) = \frac{\alpha_0}{\sqrt{\pi}}\beta_0(t)$). In this case the amplitude equation of Itô type is a system of two equations where the dimension of $\ker\mathcal{A}$ equals two (*i.e.* for $b \in \ker\mathcal{A}$, we can write b as $b = b_1(T)\sin(x) + b_{-1}(T)\cos(x)$). Hence, the amplitude equation takes the form

$$db_i = \left[\left(\nu - \frac{11}{18}\rho_0^2\right)b_i - C_0|b|^4 b_i\right]dT + \rho_0 b_i\, d\tilde{\beta}_0 \quad \text{for } i = \pm 1. \tag{47}$$

Our main theorem in this case states that the rescaled solution of (1),

$$u(t,x) = \varepsilon v\bigl(\varepsilon^4 t, x\bigr),$$

takes the form

$$v(T,x) = b_1(T)\sin(x) + b_{-1}(T)\cos(x) + \mathcal{O}\big(\varepsilon^{1-32\kappa}\big),$$

where b_1 and b_2 are the solution of the system of the amplitude equations (47).

Second case: the noise takes the form

$$W(t,x) = \alpha_2\beta(t)\cos(2x).$$

In this case the amplitude equation of Itô type takes the form

$$db_i = \left[\left(\nu - \frac{7\rho_2^2}{114}\right)b_i - C_0|b|^4 b_i\right]dT + \frac{i\rho_2\sqrt{27}}{18\sqrt{38}} b_i\,d\tilde{\beta}_2 \quad \text{for } i = \pm 1. \tag{48}$$

Competing interests

The author declares to have no competing interests.

Author's contributions

The author read and approved the final manuscript.

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