

Generalised CP and $\Delta(6n^2)$ family symmetry in semi-direct models of leptons

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ABSTRACT: We perform a detailed analysis of $\Delta(6n^2)$ family symmetry combined with a generalised CP symmetry in the lepton sector, breaking to different remnant symmetries G_ν in the neutrino and G_l in the charged lepton sector, together with different remnant CP symmetries in each sector. We discuss the resulting mass and mixing predictions for $G_\nu = Z_2$ with $G_l = K_4, Z_p, p > 2$ and $G_\nu = K_4$ with $G_l = Z_2$. All cases correspond to the preserved symmetry smaller than the full Klein symmetry, as in the semi-direct approach, leading to predictions which depend on a single undetermined real parameter, which mainly determines the reactor angle. We focus on five phenomenologically allowed cases for which we present the resulting predictions for the PMNS parameters as a function of n , as well as the predictions for neutrinoless double beta decay.

KEYWORDS: Neutrino Physics, CP violation, Discrete and Finite Symmetries

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1 Introduction

Following the pioneering measurements of the reactor mixing angle θ_{13} by the Daya Bay [1, 2], RENO [3], and Double Chooz [4, 5] reactor neutrino experiments, the three lepton mixing angles θ_{12} , θ_{23} , θ_{13} and both mass-squared differences Δm_{sol}^2 and Δm_{atm}^2 have by now been measured to quite good accuracy, as recently summarised, for example, at the Neutrino Oscillation Workshop 2014 [6].

However the Dirac CP violating oscillation phase has not been measured so far and the neutrino mass squared ordering remains ambiguous. Moreover, if neutrinos are Majorana particles, there are two more unmeasured Majorana CP phases which play a role in determining the neutrinoless double-beta decay mass observable $|m_{ee}|$, which is the (1,1) element of the neutrino mass matrix in the flavour basis. Determining the neutrino mass squared ordering and measuring the Dirac CP violating phase are primary goals of the

next generation neutrino oscillation experiments. The CP violation has been firmly established in the quark sector and therefore it is natural to expect that CP violation occurs in the lepton sector as well. Indeed hints of a nonzero $\delta_{\text{CP}} \sim 3\pi/2$ have begun to show up in global analyses of neutrino oscillation data [7–9], although we emphasise that such a hint could equally well be due to an upward statistical fluctuation in the T2K electron appearance measurements [6] which are largely responsible for the effect.

In recent years, much effort has been devoted to explaining the structure of the lepton mixing angles through the introduction of a non-abelian discrete family symmetry group. For recent reviews of model building and relevant group theory aspects (see for example [10–14]). Three different model building approaches have been identified, known as “direct”, “semi-direct” and “indirect” [13, 14]. In the “direct” approach the Klein symmetry $Z_2 \times Z_2$ of the Majorana neutrino mass matrix is identified as a subgroup of the family symmetry, while in the “semi-direct” approach only part of the Klein symmetry, typically Z_2 , is contained in the family symmetry group, allowing the reactor angle to be fitted with one free parameter. In the “indirect approach” the neutrino mass matrix is constructed from “vacuum alignments” which are orthogonal to the symmetry preserving ones, allowing the Klein symmetry to arise as an indirect consequence of the family symmetry (for a recent example of this approach see e.g. [15]).

Inspired by the predictive power of discrete family symmetry, it is conceivable to extend the family symmetry approaches to include a generalised CP symmetry H_{CP} [16–20] in order to allow the prediction of not only the mixing angles but also the CP phases as well. The idea was first explored in the context of CP symmetry combined with $\mu - \tau$ reflection symmetry, resulting in the atmospheric mixing angle θ_{23} and the Dirac CP phase δ_{CP} both being determined to be maximal, as discussed in [21–27]. More recently, the phenomenological consequences of imposing both an S_4 family symmetry and a generalised CP symmetry have been investigated in a model-independent way [28–33]. Assuming the symmetry breaking of $S_4 \times H_{\text{CP}}$ to $Z_2 \times \text{CP}$ in the neutrino sector and to some abelian subgroup of S_4 in the charged lepton sector, all the three lepton mixing angles and CP phases were then given in terms of only one free parameter, where this free parameter results from the fact that only part of the Klein symmetry comes from the discrete family symmetry which is the case for the “semi-direct” approach. Concrete “semi-direct” S_4 family models with a generalised CP symmetry have been constructed in refs. [28–33] where the spontaneous breaking of the $S_4 \times H_{\text{CP}}$ down to $Z_2 \times \text{CP}$ in the neutrino sector was implemented. A similar generalised analysis has also been considered for A_4 family symmetry [34]. The typical prediction for the Dirac CP phase for the “semi-direct” approach to S_4 and A_4 is $|\sin \delta_{\text{CP}}| = 0$ or 1 in each case, although other model dependent predictions are also possible.

Other models with a family symmetry and a generalised CP symmetry can also be found in refs. [35–39]. The interplay between flavor symmetries and CP symmetries has been generally discussed in [40, 41]. In addition, there are other theoretical approaches involving both family symmetry and CP violation [42–53].

A generalised CP analysis of $\Delta(6n^2)$ has been performed recently [54] based on a “direct” approach with the full Klein symmetry $Z_2 \times Z_2$ preserved in the neutrino sector and a Z_3 preserved in the charged lepton sector. However the result of this analysis is

that $|\sin \delta_{\text{CP}}| = 0$ corresponding to the Dirac CP phase being either zero or $\pm\pi$. In fact, this result was obtained originally without imposing any CP symmetry [55], the difference being that in the earlier approach the Majorana phases were not determined. Moreover, extensive scan over discrete symmetry groups shows that the Dirac CP phase is trivial for mixing angles within 3σ ranges of the global fits [56, 57]. Clearly the result of conserved Dirac CP is rather disappointing from an experimental point of view, given the large effort going into measuring the Dirac CP phase and the difficulty of proving that $|\sin \delta_{\text{CP}}| = 0$. Therefore it is of interest to consider the analogous situation for the “semi-direct” approach. Here we shall focus on the infinite series of finite groups $\Delta(6n^2)$ but relax the requirement of having the full Klein symmetry in the neutrino sector, and in addition consider more general preserved symmetries in the charged lepton sector. In other words we follow the “semi-direct” approach for the infinite series of groups based on $\Delta(6n^2)$.

In this paper, then, we study generalised CP symmetry for all $\Delta(6n^2)$ family symmetry groups where the CP symmetry is assumed to exist at a high energy scale and the requirement of having the full Klein symmetry is relaxed, as in the so-called “semi-direct” approach. The work here follows on from a similar analysis of the semi-direct approach performed by two of us based on the group $\Delta(96)$ [58]. Here we investigate the lepton mixing parameters which can be obtained from the original symmetry $\Delta(6n^2) \times H_{\text{CP}}$ breaking to different remnant symmetries in the neutrino and charged lepton sectors, namely G_ν and G_l subgroups in the neutrino and the charged lepton sector respectively, while the remnant CP symmetries from the breaking of H_{CP} are H_{CP}^ν and H_{CP}^l , respectively. The generalised CP transformation compatible with an $\Delta(6n^2)$ family symmetry is defined, and a model-independent analysis of the lepton mixing matrix is performed by scanning all the possible remnant subgroups in the neutrino and charged lepton sectors. Relaxing the requirement of having the full Klein symmetry in the neutrino sector given by a subgroup of $\Delta(6n^2)$, as in the semi-direct approach, we are led to a large number of possibilities where the results depend on a single parameter, expressed as an angle which determines the reactor angle. We systematically discuss all such possibilities consistent with existing phenomenological data, then analyse in detail the resulting predictions for mixing parameters. Our results divide into two cases, the first in which the residual symmetry in the neutrino sector is $Z_2 \times CP$ and the second in which the same residual symmetry $Z_2 \times CP$ is preserved by the charged lepton sector. More precisely, we discuss the resulting mass and mixing predictions for all possible cases where the family symmetry $\Delta(6n^2)$ enhanced with generalised CP is broken to $G_\nu = Z_2$ with $G_l = K_4, Z_p, p > 2$ and $G_\nu = K_4$ with $G_l = Z_2$. We are led to five phenomenologically allowed mixing patterns and present the resulting predictions for PMNS parameters as a function of n , as well as the predictions for neutrinoless double beta decay.

While this paper was being prepared, a study of generalised CP within the semi-direct approach appeared based on the infinite series of finite groups $\Delta(6n^2)$ and $\Delta(3n^2)$ [59]. Where the results overlap for $\Delta(6n^2)$ they appear to be broadly in agreement, although the case that the residual symmetry $Z_2 \times CP$ is preserved by the charged lepton sector was not considered in [59]. The present paper focuses exclusively on $\Delta(6n^2)$, and, apart from considering extra cases not previously considered, presents the numerical results in a quite different and complementary way. Many of the numerical results contained in this

paper, for example, the predictions for neutrinoless double beta decay, were not previously considered at all.

The remainder of this paper is organised as follows. In section 2 we consider Generalised CP with $\Delta(6n^2)$. In section 3 we consider possible lepton mixing from “semi-direct” approach with residual symmetry $Z_2 \times CP$ in the neutrino sector. In section 4 we consider possible lepton mixing from “semi-direct” approach with residual symmetry $Z_2 \times CP$ in the charged lepton sector. The phenomenological predictions of the neutrinoless double beta decay for all the viable cases are presented in section 5, Finally section 6 concludes the paper.

2 Generalised CP with $\Delta(6n^2)$

Let us consider a theory with both family symmetry G_f and generalized CP symmetry at high energy scale. A field multiplet $\varphi_{\mathbf{r}}$ embedded into the representation \mathbf{r} of G_f transforms under the action of the family symmetry group G_f as

$$\varphi_{\mathbf{r}} \xrightarrow{g} \rho_{\mathbf{r}}(g)\varphi_{\mathbf{r}}, \quad g \in G_f, \quad (2.1)$$

where $\rho_{\mathbf{r}}(g)$ is the representation matrix of g in the representation \mathbf{r} . Furthermore, the most general CP transformations act on the field $\varphi_{\mathbf{r}}$ as:

$$\varphi_{\mathbf{r}} \xrightarrow{CP} X_{\mathbf{r}}\varphi_{\mathbf{r}}^*(x_P), \quad (2.2)$$

where $x_P = (t, -\mathbf{x})$, $X_{\mathbf{r}}$ is a unitary matrix, and it is the so-called generalized CP transformation. The generalized CP symmetry has to be consistent with the family symmetry. It has been firmly established that the generalized CP symmetry can only be compatible with the family symmetry if the following consistency equation is satisfied [16–20, 28, 40]:

$$X_{\mathbf{r}}\rho_{\mathbf{r}}^*(g)X_{\mathbf{r}}^\dagger = \rho_{\mathbf{r}}(g'), \quad g, g' \in G_f. \quad (2.3)$$

Hence the generalized CP transformation is related to an automorphism which maps g into g' . Furthermore, it was recently shown that physical CP transformations have to be given by class-inverting automorphism of G_f [41]. In this work, we shall investigate the $\Delta(6n^2)$ series as the family symmetry group. The group theory of $\Delta(6n^2)$ is presented in appendix A. With the help of the computer algebra program system GAP [60–63], we have studied the automorphism group of the $\Delta(6n^2)$ until $n = 19$.¹ The results are collected in table 1. We find that the outer automorphism groups of members of the $\Delta(6n^2)$ series are generally non-trivial except for $\Delta(6) \cong S_3$ and $\Delta(24) \cong S_4$. However, there is only one class-inverting outer automorphism for $n \neq 3\mathbb{Z}$ while no class-inverting automorphism exists for $n = 3\mathbb{Z}$. In fact, we find an outer automorphism u acting on the generators as:

$$a \xrightarrow{u} a^2, \quad b \xrightarrow{u} b, \quad c \xrightarrow{u} d, \quad d \xrightarrow{u} c. \quad (2.4)$$

¹The $\Delta(6n^2)$ group with $n > 19$ are not available in GAP so far.

It can be straightforwardly checked that this automorphism u maps each conjugacy class into the inverse one for $n \neq 3\mathbb{Z}$. In the case of $n = 3\mathbb{Z}$, we have

$$\frac{2n^2}{3}C_2^{(\tau)} \xrightarrow{u} \frac{2n^2}{3}C_2^{(-\tau)}, \quad \left(\frac{2n^2}{3}C_2^{(\tau)}\right)^{-1} = \frac{2n^2}{3}C_2^{(\tau)}, \quad \tau = 0, 1, 2. \quad (2.5)$$

Hence both $\frac{2n^2}{3}C_2^{(1)}$ and $\frac{2n^2}{3}C_2^{(2)}$ are not mapped into their inverse classes although the latter is still true for the remaining classes. As a result, we conjecture that the $\Delta(6n^2)$ group with $n \neq 3\mathbb{Z}$ admits a unique class-inverting automorphism given by eq. (2.4) although we can not prove it in a strict mathematical manner so far. In the following, we shall concentrate on the $n \neq 3\mathbb{Z}$ case without mention. The generalized CP transformation corresponding to u , which is denoted by $X_{\mathbf{r}}(u)$, would be physically well-defined, as suggested in ref. [41]. Its concrete form is fixed by the consistency equations as follows:

$$\begin{aligned} X_{\mathbf{r}}(u) \rho_{\mathbf{r}}^*(a) X_{\mathbf{r}}^\dagger(u) &= \rho_{\mathbf{r}}(u(a)) = \rho_{\mathbf{r}}(a^2), \\ X_{\mathbf{r}}(u) \rho_{\mathbf{r}}^*(b) X_{\mathbf{r}}^\dagger(u) &= \rho_{\mathbf{r}}(u(b)) = \rho_{\mathbf{r}}(b), \\ X_{\mathbf{r}}(u) \rho_{\mathbf{r}}^*(c) X_{\mathbf{r}}^\dagger(u) &= \rho_{\mathbf{r}}(u(c)) = \rho_{\mathbf{r}}(d), \\ X_{\mathbf{r}}(u) \rho_{\mathbf{r}}^*(d) X_{\mathbf{r}}^\dagger(u) &= \rho_{\mathbf{r}}(u(d)) = \rho_{\mathbf{r}}(c). \end{aligned} \quad (2.6)$$

In our basis, presented in appendix A, we can determine that

$$X_{\mathbf{r}}(u) = \rho_{\mathbf{r}}(b). \quad (2.7)$$

Furthermore, including inner automorphisms, the full set of generalized CP transformations compatible with $\Delta(6n^2)$ family symmetry is

$$X_{\mathbf{r}} = \rho_{\mathbf{r}}(g), \quad g \in \Delta(6n^2). \quad (2.8)$$

Consequently the generalized CP transformations are of the same form as the family symmetry transformations in the chosen basis. In particular, we see that the conventional CP transformation with $\rho_{\mathbf{r}}(1)=1$ is allowed. As a consequence, all coupling constants would be real in a $\Delta(6n^2)$ model with imposed CP symmetry since all the CG coefficients are real, as shown in appendix B. In the case of $n = 3\mathbb{Z}$, the consistency equations of eq. (2.6) are also satisfied except for \mathbf{r} being the doublet representations $\mathbf{2}_2$, $\mathbf{2}_3$ or $\mathbf{2}_4$. Hence the generalized CP transformations in eq. (2.7) can also be imposed on a model with $n = 3\mathbb{Z}$ if the fields transforming as $\mathbf{2}_2$, $\mathbf{2}_3$ or $\mathbf{2}_4$ are absent.

3 Possible lepton mixing from “semi-direct” approach with residual symmetry $Z_2 \times CP$ in the neutrino sector

In the following, we shall analyze all possible lepton mixing patterns in the “semi-direct” method [13, 14, 29–31, 34, 38, 39, 58]. In this context, both family symmetry $\Delta(6n^2)$ and the consistent generalized CP symmetry are imposed onto the theory at high energy scales. The full symmetry group is $\Delta(6n^2) \times H_{CP}$, which is broken down to $G_l \times H_{CP}^l$ and $Z_2 \times H_{CP}^\nu$ residual symmetries in the charged lepton and neutrino sectors respectively. G_l is usually

n	G_f	GAP-Id	$\text{Inn}(G_f)$	$\text{Out}(G_f)$	Num.
1	$\Delta(6) \equiv S_3$	[6,1]	S_3	Z_1	1
2	$\Delta(24) \equiv S_4$	[24,12]	S_4	Z_1	1
3	$\Delta(54)$	[54,8]	$(Z_3 \times Z_3) \rtimes Z_2$	S_4	0
4	$\Delta(96)$	[96,64]	$\Delta(96)$	Z_2	1
5	$\Delta(150)$	[150,5]	$\Delta(150)$	Z_4	1
6	$\Delta(216)$	[216,95]	$(Z_3 \times A_4) \rtimes Z_2$	S_3	0
7	$\Delta(294)$	[294,7]	$\Delta(294)$	Z_6	1
8	$\Delta(384)$	[384,568]	$\Delta(384)$	K_4	1
9	$\Delta(486)$	[486,61]	$((Z_9 \times Z_3) \times Z_3) \rtimes Z_2$	$Z_3 \times S_3$	0
10	$\Delta(600)$	[600,179]	$\Delta(600)$	Z_4	1
11	$\Delta(726)$	[726,5]	$\Delta(726)$	Z_{10}	1
12	$\Delta(864)$	[864,701]	$(Z_3 \times ((Z_4 \times Z_4) \times Z_3)) \rtimes Z_2$	D_{12}	0
13	$\Delta(1014)$	[1014,7]	$\Delta(1014)$	Z_{12}	1
14	$\Delta(1176)$	[1176,243]	$\Delta(1176)$	Z_6	1
15	$\Delta(1350)$	[1350,46]	$(Z_3 \times ((Z_5 \times Z_5) \times Z_3)) \rtimes Z_2$	$Z_4 \times S_3$	0
16	$\Delta(1536)$	[1536,408544632]	$\Delta(1536)$	$Z_4 \times Z_2$	1
17	$\Delta(1734)$	[1734,5]	$\Delta(1734)$	Z_{16}	1
18	$\Delta(1944)$	[1944,849]	$((Z_{18} \times Z_6) \times Z_3) \rtimes Z_2$	$Z_3 \times S_3$	0
19	$\Delta(2166)$	[2166,15]	$\Delta(2166)$	Z_{18}	1

Table 1. The automorphism groups of the $\Delta(6n^2)$ group series, where $\text{Inn}(G_f)$ and $\text{Out}(G_f)$ denote inner automorphism group and outer automorphism group of the family symmetry group G_f respectively. The last column gives the number of class-inverting outer automorphisms. Note that the inner automorphism group of $\Delta(6n^2)$ with $n = 3\mathbb{Z}$ is isomorphic to $\Delta(6n^2)/Z_3$ since its center is the Z_3 subgroup generated by $c^{\frac{n}{3}}d^{\frac{2n}{3}}$.

taken to be an abelian subgroup of $\Delta(6n^2)$ of order larger than 2 to avoid degenerate charged lepton masses. The misalignment between the two residual symmetries generates the PMNS matrix. In this approach, only remnant symmetries are considered and we do not discuss how the required symmetry breaking is dynamically achieved as there are generally more than one mechanism and many possible specific model realizations. It is remarkable that one column of the lepton mixing matrix can be fixed and the resulting lepton mixing parameters are generally constrained to depend on only one free parameter in this approach. We shall consider all the possible remnant symmetries $G_l \times H_{CP}^l$ and $Z_2 \times H_{CP}^\nu$, and predictions for the lepton mixing angles and CP phases would be investigated. As usual the three generation of the left-handed lepton doublet fields are assigned to the faithful representation $\mathbf{3}_{1,1}$ which is denoted by $\mathbf{3}$ for simplicity in the following.

3.1 Charged lepton sector

The remnant symmetry G_l implies that the charged lepton mass matrix is invariant under the transformation $\ell_L \rightarrow \rho_{\mathbf{3}}(g_l)\ell_L$, where ℓ_L stands for the three generations of left-handed lepton doublets, g_l is the generator of G_l , and $\rho_{\mathbf{3}}(g_l)$ is the representation matrix of g_l in the triplet representation $\mathbf{3}$. As a consequence, the charged lepton mass matrix satisfies

$$\rho_{\mathbf{3}}^\dagger(g_l)m_l^\dagger m_l \rho_{\mathbf{3}}(g_l) = m_l^\dagger m_l, \tag{3.1}$$

where the charged lepton mass matrix m_l is defined in the convention, $\ell^c m_l \ell_L$. Let us denote the diagonalization matrix of $m_l^\dagger m_l$ by U_l , i.e.

$$U_l^\dagger m_l^\dagger m_l U_l = \text{diag}(m_e^2, m_\mu^2, m_\tau^2) \equiv \widehat{m}_l^2. \tag{3.2}$$

where m_e , m_μ and m_τ are the electron, muon and tau masses respectively. Substituting eq. (3.2) into eq. (3.1), we obtain

$$\widehat{m}_l^2 [U_l^\dagger \rho_{\mathbf{3}}(g_l) U_l] = [U_l^\dagger \rho_{\mathbf{3}}(g_l) U_l] \widehat{m}_l^2. \tag{3.3}$$

One can see that $U_l^\dagger \rho_{\mathbf{3}}(g_l) U_l$ has to be diagonal. Therefore U_l not only diagonalizes $m_l^\dagger m_l$ but also the matrix $\rho_{\mathbf{3}}(g_l)$. As a result, the unitary diagonalization matrix U_l is completely fixed by the remnant flavor symmetry G_l once the eigenvalues of $\rho_{\mathbf{3}}(g_l)$ are non-degenerate. In the present work, we shall focus on the case that G_l is a cyclic subgroup of $\Delta(6n^2)$. Hence the generator g_l of G_l could be of the form $c^s d^t$, $bc^s d^t$, $ac^s d^t$, $a^2 c^s d^t$, $abc^s d^t$ or $a^2 bc^s d^t$ with $s, t = 0, 1, \dots, n-1$. If the eigenvalues of $\rho_{\mathbf{3}}(g_l)$ are degenerate such that its diagonalization matrix U_l can not be fixed uniquely, we could extend G_l from a single cyclic subgroup to the product of several cyclic subgroups. This scenario is beyond the scope of this work except that the simplest K_4 extension is included. Given the explicit form of the representation matrices listed in appendix A, the charged lepton diagonalization matrices U_l for different cases of G_l can be calculated straightforwardly, and the results are summarized in the following. Since the charged lepton masses can not be constrained at all in the present approach (in other word, the order of the eigenvalues of $\rho_{\mathbf{3}}(g_l)$ is indeterminate), U_l can undergo rephasing and permutations from the left.

- $G_l = \langle c^s d^t \rangle$

$$U_l = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.4}$$

Note that the parameters s and t should be subject to the following constraints

$$s + t \neq 0 \pmod n, \quad s - 2t \neq 0 \pmod n, \quad t - 2s \neq 0 \pmod n, \tag{3.5}$$

otherwise the eigenvalues of $c^s d^t$ would be degenerate and consequently U_l can not be determined uniquely. For the value of $s = t = n/2$, the remnant symmetry could be chose to be $K_4^{(c^{n/2}, d^{n/2})} \equiv \{1, c^{n/2}, d^{n/2}, c^{n/2} d^{n/2}\}$ instead, and then corresponding unitary transformation U_l is still a unit matrix. The constraints of eq. (3.5) will be assumed for the subgroup $G_l = \langle c^s d^t \rangle$ in the following.

- $G_l = \langle bc^s d^t \rangle$

$$U_l = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\pi \frac{s+t}{2n}} & 0 & e^{-i\pi \frac{s+t}{2n}} \\ 0 & \sqrt{2} & 0 \\ -e^{i\pi \frac{s+t}{2n}} & 0 & e^{i\pi \frac{s+t}{2n}} \end{pmatrix}. \quad (3.6)$$

To avoid degenerate eigenvalues, we should exclude the values

$$s - t = 0, n/3, 2n/3 \pmod n. \quad (3.7)$$

For the case of $s = t$, the order of the element $bc^s d^s$ is two and one could extend G_l from $\langle bc^s d^s \rangle = \{1, bc^s d^s\}$ to the Klein four subgroup $K_4^{(c^{n/2} d^{n/2}, bc^s d^s)} \equiv \{1, c^{n/2} d^{n/2}, bc^s d^s, bc^{s+n/2} d^{s+n/2}\}$. Then the unitary transformation U_l is still of the form in eq. (3.6) with $s = t$.

- $G_l = \langle ac^s d^t \rangle$

$$U_l = \frac{1}{\sqrt{3}} \begin{pmatrix} e^{-2i\pi \frac{s}{n}} & \omega^2 e^{-2i\pi \frac{s}{n}} & \omega e^{-2i\pi \frac{s}{n}} \\ e^{-2i\pi \frac{t}{n}} & \omega e^{-2i\pi \frac{t}{n}} & \omega^2 e^{-2i\pi \frac{t}{n}} \\ 1 & 1 & 1 \end{pmatrix}, \quad (3.8)$$

where $\omega = e^{2i\pi/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ is the third root of unity. Notice that the order of the element $ac^s d^t$ is three regardless of the values of s and t , and its eigenvalues are 1, ω and ω^2 .

- $G_l = \langle a^2 c^s d^t \rangle$

$$U_l = \frac{1}{\sqrt{3}} \begin{pmatrix} e^{-2i\pi \frac{t}{n}} & \omega^2 e^{-2i\pi \frac{t}{n}} & \omega e^{-2i\pi \frac{t}{n}} \\ e^{2i\pi \frac{s-t}{n}} & \omega e^{2i\pi \frac{s-t}{n}} & \omega^2 e^{2i\pi \frac{s-t}{n}} \\ 1 & 1 & 1 \end{pmatrix}. \quad (3.9)$$

Note that because $(ac^t d^{t-s})^2 = a^2 c^s d^t$ holds, this U_l can be obtained from the one in eq. (3.8) by the replacement $s \rightarrow t, t \rightarrow t - s$.

- $G_l = \langle abc^s d^t \rangle$

$$\bar{U}_l = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\pi \frac{t-2s}{2n}} & e^{i\pi \frac{t-2s}{2n}} & 0 \\ -e^{-i\pi \frac{t-2s}{2n}} & e^{-i\pi \frac{t-2s}{2n}} & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}. \quad (3.10)$$

Non-degeneracy of the eigenvalues of $abc^s d^t$ requires $t \neq 0, n/3, 2n/3$. In the case of $t = 0$, the degeneracy can be avoided by expanding G_l to the Klein four subgroup $K_4^{(c^{n/2}, abc^s)} \equiv \{1, c^{n/2}, abc^s, abc^{s+n/2}\}$, whose diagonalization matrix is of the same form as eq. (3.10) with $t = 0$.

- $G_l = \langle a^2 bc^s d^t \rangle$

$$U_l = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & e^{i\pi \frac{s-2t}{2n}} & e^{i\pi \frac{s-2t}{2n}} \\ 0 & -e^{-i\pi \frac{s-2t}{2n}} & e^{-i\pi \frac{s-2t}{2n}} \end{pmatrix}. \quad (3.11)$$

Here the parameter s can not be equal to 0, $n/3$ or $2n/3$, otherwise two eigenvalues of $a^2bc^s d^t$ would be identical. For the extended residual symmetry $G_l = K_4^{(d^{n/2}, a^2bd^t)} \equiv \{1, d^{n/2}, a^2bd^t, a^2bd^{t+n/2}\}$, the corresponding unitary transformation is still given by eq. (3.11) with $s = 0$.

3.2 Neutrino sector

In the present work, we assume the light neutrinos are Majorana particles. As a consequence, the remnant flavor symmetry G_ν in the neutrino sector can only be a K_4 or Z_2 subgroup. The phenomenological consequence of $G_\nu = K_4$ has been studied in refs. [54] by two of us. Here we shall concentrate on $G_\nu = Z_2$ case with generalized CP symmetry which allows us to predict CP phases. The Z_2 subgroups of $\Delta(6n^2)$ can be generated by

$$bc^x d^x, \quad abc^y, \quad a^2bd^z, \quad x, y, z = 0, 1 \dots n-1 \quad (3.12)$$

and additionally

$$c^{n/2}, \quad d^{n/2}, \quad c^{n/2}d^{n/2} \quad (3.13)$$

for $n = 2\mathbb{Z}$. It is notable that the Z_2 elements in eq. (3.12) and eq. (3.13) are conjugate to each other respectively:

$$\begin{aligned} (c^\gamma d^\delta) bc^x d^x (c^\gamma d^\delta)^{-1} &= bc^{x-\delta-\gamma} d^{x-\delta-\gamma}, & (bc^\gamma d^\delta) bc^x d^x (bc^\gamma d^\delta)^{-1} &= bc^{-x+\delta+\gamma} d^{-x+\delta+\gamma}, \\ (ac^\gamma d^\delta) bc^x d^x (ac^\gamma d^\delta)^{-1} &= a^2bd^{-x+\delta+\gamma}, & (a^2c^\gamma d^\delta) bc^x d^x (a^2c^\gamma d^\delta)^{-1} &= abc^{-x+\delta+\gamma}, \\ (abc^\gamma d^\delta) bc^x d^x (abc^\gamma d^\delta)^{-1} &= a^2bd^{x-\delta-\gamma}, & (a^2bc^\gamma d^\delta) bc^x d^x (a^2bc^\gamma d^\delta)^{-1} &= abc^{x-\delta-\gamma}. \end{aligned} \quad (3.14a)$$

$$\begin{aligned} (c^\gamma d^\delta) c^{n/2} (c^\gamma d^\delta)^{-1} &= c^{n/2}, & (bc^\gamma d^\delta) c^{n/2} (bc^\gamma d^\delta)^{-1} &= d^{n/2}, \\ (ac^\gamma d^\delta) c^{n/2} (ac^\gamma d^\delta)^{-1} &= c^{n/2}d^{n/2}, & (a^2c^\gamma d^\delta) c^{n/2} (a^2c^\gamma d^\delta)^{-1} &= d^{n/2}, \\ (abc^\gamma d^\delta) c^{n/2} (abc^\gamma d^\delta)^{-1} &= c^{n/2}, & (a^2bc^\gamma d^\delta) c^{n/2} (a^2bc^\gamma d^\delta)^{-1} &= c^{n/2}d^{n/2}. \end{aligned} \quad (3.14b)$$

The remnant generalized CP symmetry should be compatible with the residual Z_2 symmetry in the neutrino sector, and therefore the corresponding consistency equation should be satisfied, i.e.,

$$X_{\nu\mathbf{r}}\rho_{\mathbf{r}}^*(g)X_{\nu\mathbf{r}}^{-1} = \rho_{\mathbf{r}}(g), \quad g \in Z_2, \quad (3.15)$$

which means that the residual CP and residual flavor transformations are commutable with each other [28, 29] in the neutrino sector. For a given solution $X_{\nu\mathbf{r}}$ of eq. (3.15), one can check that $\rho_{\mathbf{r}}(g)X_{\nu\mathbf{r}}$ is also a solution. The remnant CP symmetries consistent with the Z_2 elements in eq. (3.12) and eq. (3.13) are summarized as follows.

- $g = bc^x d^x, \quad x = 0, 1, 2 \dots n-1$

$$X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(c^\gamma d^{-2x-\gamma}), \quad \rho_{\mathbf{r}}(bc^\gamma d^{-\gamma}), \quad \gamma = 0, 1, 2 \dots n-1. \quad (3.16)$$

- $g = abc^y, \quad y = 0, 1, 2 \dots n-1$

$$X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(c^\gamma d^{2y+2\gamma}), \quad \rho_{\mathbf{r}}(abc^\gamma d^{2\gamma}), \quad \gamma = 0, 1, 2 \dots n-1. \quad (3.17)$$

- $g = a^2bd^z$, $z = 0, 1, 2 \dots n-1$

$$X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(c^{2z+2\delta}d^\delta), \quad \rho_{\mathbf{r}}(a^2bc^{2\delta}d^\delta), \quad \delta = 0, 1, 2 \dots n-1. \quad (3.18)$$

- $g = c^{n/2}$

$$X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(c^\gamma d^\delta), \quad \rho_{\mathbf{r}}(abc^\gamma d^\delta), \quad \gamma, \delta = 0, 1, 2 \dots n-1. \quad (3.19)$$

- $g = d^{n/2}$

$$X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(c^\gamma d^\delta), \quad \rho_{\mathbf{r}}(a^2bc^\gamma d^\delta), \quad \gamma, \delta = 0, 1, 2 \dots n-1. \quad (3.20)$$

- $g = c^{n/2}d^{n/2}$

$$X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(c^\gamma d^\delta), \quad \rho_{\mathbf{r}}(bc^\gamma d^\delta), \quad \gamma, \delta = 0, 1, 2 \dots n-1. \quad (3.21)$$

As we shall demonstrate in the following, the remnant CP symmetry should be symmetric to avoid degenerate lepton masses. Then the viable CP transformations would be constrained to be $\rho_{\mathbf{r}}(abc^\gamma d^{2\gamma})$, $\rho_{\mathbf{r}}(a^2bc^{2\delta}d^\delta)$ and $\rho_{\mathbf{r}}(bc^\gamma d^{-\gamma})$ together with $\rho_{\mathbf{r}}(c^\gamma d^\delta)$ for $g = c^{n/2}$, $d^{n/2}$ and $c^{n/2}d^{n/2}$ respectively. The full symmetry $\Delta(6n^2) \times H_{CP}$ is broken down to $Z_2 \times H_{CP}^\nu$ in the neutrino sector. The invariance of the light neutrino mass matrix m_ν under the remnant family symmetry $G_\nu = Z_2$ and the remnant CP symmetry H_{CP}^ν leads to

$$\begin{aligned} \rho_{\mathbf{3}}^T(g_\nu)m_\nu\rho_{\mathbf{3}}(g_\nu) &= m_\nu, & g_\nu &\in Z_2^\nu, \\ X_{\nu\mathbf{3}}^T m_\nu X_{\nu\mathbf{3}} &= m_\nu^*, & X_\nu &\in H_{CP}^\nu, \end{aligned} \quad (3.22)$$

from which we can construct the explicit form of m_ν and then diagonalize it.

(i) $G_\nu = Z_2^{bc^x d^x} \equiv \{1, bc^x d^x\}$, $X_{\nu\mathbf{r}} = \{\rho_{\mathbf{r}}(c^\gamma d^{-2x-\gamma}), \rho_{\mathbf{r}}(bc^{x+\gamma} d^{-x-\gamma})\}$ The light neutrino mass matrix satisfying eq. (3.22) is of the following form

$$m_\nu = \begin{pmatrix} m_{11}e^{-2i\pi\frac{\gamma}{n}} & m_{12}e^{i\pi\frac{2x+\gamma}{n}} & m_{13}e^{-2i\pi\frac{x+\gamma}{n}} \\ m_{12}e^{i\pi\frac{2x+\gamma}{n}} & m_{22}e^{4i\pi\frac{x+\gamma}{n}} & m_{12}e^{i\pi\frac{\gamma}{n}} \\ m_{13}e^{-2i\pi\frac{x+\gamma}{n}} & m_{12}e^{i\pi\frac{\gamma}{n}} & m_{11}e^{-2i\pi\frac{2x+\gamma}{n}} \end{pmatrix}, \quad (3.23)$$

where m_{11} , m_{12} , m_{13} and m_{22} are real parameters. This neutrino mass matrix is diagonalized by the unitary transformation U_ν via

$$U_\nu^T m_\nu U_\nu = \text{diag}(m_1, m_2, m_3), \quad (3.24)$$

where U_ν is

$$U_\nu = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\pi\frac{\gamma}{n}} & -e^{i\pi\frac{\gamma}{n}} \sin \theta & e^{i\pi\frac{\gamma}{n}} \cos \theta \\ 0 & e^{-2i\pi\frac{x+\gamma}{n}} \sqrt{2} \cos \theta & e^{-2i\pi\frac{x+\gamma}{n}} \sqrt{2} \sin \theta \\ -e^{i\pi\frac{2x+\gamma}{n}} & -e^{i\pi\frac{2x+\gamma}{n}} \sin \theta & e^{i\pi\frac{2x+\gamma}{n}} \cos \theta \end{pmatrix} K_\nu, \quad (3.25)$$

where K_ν is a diagonal unitary matrix with entries ± 1 and $\pm i$ which encode the CP parity of the neutrino states and renders the light neutrino masses positive. We shall omit the factor K_ν in the following cases for simplicity of notation. The angle θ is given by

$$\tan 2\theta = \frac{2\sqrt{2}m_{12}}{m_{11} + m_{13} - m_{22}}. \quad (3.26)$$

The light neutrino masses are

$$\begin{aligned} m_1 &= |m_{11} - m_{13}|, \\ m_2 &= \frac{1}{2} \left| m_{11} + m_{13} + m_{22} - \text{sign}((m_{11} + m_{13} - m_{22}) \cos 2\theta) \sqrt{(m_{11} + m_{13} - m_{22})^2 + 8m_{12}^2} \right|, \\ m_3 &= \frac{1}{2} \left| m_{11} + m_{13} + m_{22} + \text{sign}((m_{11} + m_{13} - m_{22}) \cos 2\theta) \sqrt{(m_{11} + m_{13} - m_{22})^2 + 8m_{12}^2} \right|. \end{aligned}$$

Here the order of the three eigenvalues m_1 , m_2 and m_3 can not be pinned down, consequently the unitary matrix U_ν is determined up to permutations of the columns (the same turns out to be true in the following cases), and the neutrino mass spectrum can be either normal ordering or inverted ordering. Moreover, as four parameters m_{11} , m_{12} , m_{13} and m_{22} are involved in the neutrino masses, the measured mass squared splitting can be accounted for easily.

(ii) $G_\nu = Z_2^{abc^y} \equiv \{1, abc^y\}$, $X_{\nu\mathbf{r}} = \{\rho_{\mathbf{r}}(c^\gamma d^{2y+2\gamma}), \rho_{\mathbf{r}}(abc^{y+\gamma} d^{2y+2\gamma})\}$ In this case, the light neutrino mass matrix takes the form:

$$m_\nu = \begin{pmatrix} m_{11}e^{-2i\pi\frac{\gamma}{n}} & m_{12}e^{-2i\pi\frac{y+\gamma}{n}} & m_{13}e^{i\pi\frac{2y+\gamma}{n}} \\ m_{12}e^{-2i\pi\frac{y+\gamma}{n}} & m_{11}e^{-2i\pi\frac{2y+\gamma}{n}} & m_{13}e^{i\pi\frac{\gamma}{n}} \\ m_{13}e^{i\pi\frac{2y+\gamma}{n}} & m_{13}e^{i\pi\frac{\gamma}{n}} & m_{33}e^{4i\pi\frac{y+\gamma}{n}} \end{pmatrix}, \quad (3.27)$$

where m_{11} , m_{12} , m_{13} and m_{33} are real. The unitary matrix U_ν which diagonalizes the above neutrino mass matrix is given by

$$U_\nu = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\pi\frac{\gamma}{n}} & e^{i\pi\frac{\gamma}{n}} \cos \theta & e^{i\pi\frac{\gamma}{n}} \sin \theta \\ -e^{i\pi\frac{2y+\gamma}{n}} & e^{i\pi\frac{2y+\gamma}{n}} \cos \theta & e^{i\pi\frac{2y+\gamma}{n}} \sin \theta \\ 0 & -e^{-2i\pi\frac{y+\gamma}{n}} \sqrt{2} \sin \theta & e^{-2i\pi\frac{y+\gamma}{n}} \sqrt{2} \cos \theta \end{pmatrix}, \quad (3.28)$$

with

$$\tan 2\theta = \frac{2\sqrt{2}m_{13}}{m_{33} - m_{11} - m_{12}}. \quad (3.29)$$

The light neutrino mass eigenvalues are determined to be

$$\begin{aligned} m_1 &= |m_{11} - m_{12}|, \\ m_2 &= \frac{1}{2} \left| m_{11} + m_{12} + m_{33} + \text{sign}((m_{11} + m_{12} - m_{33}) \cos 2\theta) \sqrt{(m_{11} + m_{12} - m_{33})^2 + 8m_{13}^2} \right|, \\ m_3 &= \frac{1}{2} \left| m_{11} + m_{12} + m_{33} - \text{sign}((m_{11} + m_{12} - m_{33}) \cos 2\theta) \sqrt{(m_{11} + m_{12} - m_{33})^2 + 8m_{13}^2} \right|. \end{aligned}$$

(iii) $G_\nu = Z_2^{a^2bd^z} \equiv \{1, a^2bd^z\}$, $X_{\nu\mathbf{r}} = \{\rho_{\mathbf{r}}(c^{2z+2\delta}d^\delta), \rho_{\mathbf{r}}(a^2bc^{2z+2\delta}d^{z+\delta})\}$ The light neutrino mass matrix, which is invariant under both remnant family symmetry and remnant CP symmetry, is of the form:

$$m_\nu = \begin{pmatrix} m_{11}e^{-4i\pi\frac{z+\delta}{n}} & m_{12}e^{-i\pi\frac{\delta}{n}} & m_{12}e^{-i\pi\frac{2z+\delta}{n}} \\ m_{12}e^{-i\pi\frac{\delta}{n}} & m_{22}e^{2i\pi\frac{2z+\delta}{n}} & m_{23}e^{2i\pi\frac{z+\delta}{n}} \\ m_{12}e^{-i\pi\frac{2z+\delta}{n}} & m_{23}e^{2i\pi\frac{z+\delta}{n}} & m_{22}e^{2i\pi\frac{\delta}{n}} \end{pmatrix}, \quad (3.30)$$

where m_{11} , m_{12} , m_{22} and m_{23} are real parameters. The neutrino diagonalization matrix U_ν is given by

$$U_\nu = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -e^{2i\pi\frac{z+\delta}{n}}\sqrt{2}\sin\theta & e^{2i\pi\frac{z+\delta}{n}}\sqrt{2}\cos\theta \\ e^{-i\pi\frac{2z+\delta}{n}} & e^{-i\pi\frac{2z+\delta}{n}}\cos\theta & e^{-i\pi\frac{2z+\delta}{n}}\sin\theta \\ -e^{-i\pi\frac{\delta}{n}} & e^{-i\pi\frac{\delta}{n}}\cos\theta & e^{-i\pi\frac{\delta}{n}}\sin\theta \end{pmatrix}, \quad (3.31)$$

where the angle θ fulfills

$$\tan 2\theta = \frac{2\sqrt{2}m_{12}}{m_{11} - m_{22} - m_{23}}. \quad (3.32)$$

Finally the light neutrino masses are

$$\begin{aligned} m_1 &= |m_{22} - m_{23}|, \\ m_2 &= \frac{1}{2} \left| m_{11} + m_{22} + m_{23} - \text{sign}((m_{11} - m_{22} - m_{23}) \cos 2\theta) \sqrt{(m_{11} - m_{22} - m_{23})^2 + 8m_{12}^2} \right|, \\ m_3 &= \frac{1}{2} \left| m_{11} + m_{22} + m_{23} + \text{sign}((m_{11} - m_{22} - m_{23}) \cos 2\theta) \sqrt{(m_{11} - m_{22} - m_{23})^2 + 8m_{12}^2} \right|. \end{aligned}$$

(iv) $G_\nu = Z_2^{c^{n/2}} \equiv \{1, c^{n/2}\}$, $X_{\nu\mathbf{r}} = \{\rho_{\mathbf{r}}(c^\gamma d^\delta), \rho_{\mathbf{r}}(abc^\gamma d^\delta)\}$

- $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(c^\gamma d^\delta)$ The light neutrino mass matrix is constrained to be of the following form

$$m_\nu = \begin{pmatrix} m_{11}e^{-2i\pi\frac{\gamma}{n}} & m_{12}e^{-i\pi\frac{\delta}{n}} & 0 \\ m_{12}e^{-i\pi\frac{\delta}{n}} & m_{22}e^{-2i\pi\frac{\delta-\gamma}{n}} & 0 \\ 0 & 0 & m_{33}e^{2i\pi\frac{\delta}{n}} \end{pmatrix}, \quad (3.33)$$

where m_{11} , m_{12} , m_{22} and m_{33} are real. The unitary transformation U_ν is

$$U_\nu = \begin{pmatrix} e^{i\pi\frac{\gamma}{n}}\cos\theta & e^{i\pi\frac{\gamma}{n}}\sin\theta & 0 \\ -e^{i\pi\frac{\delta-\gamma}{n}}\sin\theta & e^{i\pi\frac{\delta-\gamma}{n}}\cos\theta & 0 \\ 0 & 0 & e^{-i\pi\frac{\delta}{n}} \end{pmatrix}, \quad (3.34)$$

where

$$\tan 2\theta = \frac{2m_{12}}{m_{22} - m_{11}}. \quad (3.35)$$

The light neutrino masses are determined to be

$$m_1 = \frac{1}{2} \left| m_{11} + m_{22} - \text{sign}((m_{22} - m_{11}) \cos 2\theta) \sqrt{(m_{22} - m_{11})^2 + 4m_{12}^2} \right|,$$

$$\begin{aligned}
 m_2 &= \frac{1}{2} \left| m_{11} + m_{22} + \text{sign}((m_{22} - m_{11}) \cos 2\theta) \sqrt{(m_{22} - m_{11})^2 + 4m_{12}^2} \right|, \\
 m_3 &= |m_{33}|.
 \end{aligned}
 \tag{3.36}$$

- $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(abc^\gamma d^\delta)$ For the case of $\delta \neq 2\gamma \pmod n$, the light neutrino masses would be partially degenerate. This is obviously unviable. The reason is that the corresponding generalized CP transformation matrix is not symmetric.² Therefore we shall concentrate on the case of $\delta = 2\gamma \pmod n$ in the following. The neutrino mass matrix is given by

$$m_\nu = \begin{pmatrix} m_{11}e^{i\phi} & m_{12}e^{-2i\pi\frac{\gamma}{n}} & 0 \\ m_{12}e^{-2i\pi\frac{\gamma}{n}} & m_{11}e^{-i(4\pi\frac{\gamma}{n}+\phi)} & 0 \\ 0 & 0 & m_{33}e^{4i\pi\frac{\gamma}{n}} \end{pmatrix}, \tag{3.37}$$

where m_{11} , m_{12} , m_{33} and ϕ are real free parameters. The neutrino diagonalization matrix is

$$U_\nu = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\frac{\phi}{2}} & e^{-i\frac{\phi}{2}} & 0 \\ -e^{i(\frac{\phi}{2}+2\pi\frac{\gamma}{n})} & e^{i(\frac{\phi}{2}+2\pi\frac{\gamma}{n})} & 0 \\ 0 & 0 & \sqrt{2}e^{-2i\pi\frac{\gamma}{n}} \end{pmatrix}. \tag{3.38}$$

The light neutrino mass eigenvalues are

$$m_1 = |m_{11} - m_{12}|, \quad m_2 = |m_{11} + m_{12}|, \quad m_3 = |m_{33}|. \tag{3.39}$$

The ordering of the neutrino masses can not be determined as well.

(v) $G_\nu = Z_2^{d^{n/2}} \equiv \{1, d^{n/2}\}$, $X_{\nu\mathbf{r}} = \{\rho_{\mathbf{r}}(c^\gamma d^\delta), \rho_{\mathbf{r}}(a^2bc^\gamma d^\delta)\}$

- $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(c^\gamma d^\delta)$ The light neutrino mass matrix is constrained by residual family and residual CP symmetries to be

$$m_\nu = \begin{pmatrix} m_{11}e^{-2i\pi\frac{\gamma}{n}} & 0 & 0 \\ 0 & m_{22}e^{-2i\pi\frac{\delta-\gamma}{n}} & m_{23}e^{i\pi\frac{\gamma}{n}} \\ 0 & m_{23}e^{i\pi\frac{\gamma}{n}} & m_{33}e^{2i\pi\frac{\delta}{n}} \end{pmatrix}, \tag{3.40}$$

where m_{11} , m_{22} , m_{23} and m_{33} are real. The neutrino diagonalization matrix is

$$U_\nu = \begin{pmatrix} e^{i\pi\frac{\gamma}{n}} & 0 & 0 \\ 0 & e^{i\pi\frac{\delta-\gamma}{n}} \cos \theta & e^{i\pi\frac{\delta-\gamma}{n}} \sin \theta \\ 0 & -e^{-i\pi\frac{\delta}{n}} \sin \theta & e^{-i\pi\frac{\delta}{n}} \cos \theta \end{pmatrix}, \tag{3.41}$$

²In the basis in which the neutrino mass matrix is diagonal with $m_\nu = \text{diag}(m_1, m_2, m_3)$, the general CP transformation \widehat{X} which leaves m_ν invariant: $\widehat{X}^T m_{mu} \widehat{X} = m_\nu^*$, should be of the form $\widehat{X} = \text{diag}(\pm 1, \pm 1, \pm 1)$. One can go to an arbitrary basis and define the corresponding CP symmetry transformation $X = V^\dagger \widehat{X} V^*$ as a symmetry of the general neutrino mass matrix, where V is the basis transformation. As a result, the remnant CP symmetry X in the neutrino sector should be symmetric. The same conclusion has been obtained in ref. [28].

with

$$\tan 2\theta = \frac{2m_{23}}{m_{33} - m_{22}}. \quad (3.42)$$

The light neutrino masses take the form

$$\begin{aligned} m_1 &= |m_{11}|, \\ m_2 &= \frac{1}{2} \left| m_{22} + m_{33} - \text{sign}((m_{33} - m_{22}) \cos 2\theta) \sqrt{(m_{33} - m_{22})^2 + 4m_{23}^2} \right|, \\ m_3 &= \frac{1}{2} \left| m_{22} + m_{33} + \text{sign}((m_{33} - m_{22}) \cos 2\theta) \sqrt{(m_{33} - m_{22})^2 + 4m_{23}^2} \right|. \end{aligned} \quad (3.43)$$

- $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(a^2 b c^\gamma d^\delta)$ As has been shown above, $X_{\nu\mathbf{r}}$ has to be symmetric. Then the requirement $\gamma = 2\delta \pmod n$ follows immediately, otherwise the light neutrino masses would be partially degenerate. In this case, the neutrino mass matrix takes the form:

$$m_\nu = \begin{pmatrix} m_{11}e^{-4i\pi\frac{\delta}{n}} & 0 & 0 \\ 0 & m_{22}e^{i\phi} & m_{23}e^{2i\pi\frac{\delta}{n}} \\ 0 & m_{23}e^{2i\pi\frac{\delta}{n}} & m_{22}e^{i(4\pi\frac{\delta}{n}-\phi)} \end{pmatrix}, \quad (3.44)$$

where m_{11} , m_{22} , m_{23} and ϕ are real. It is diagonalized by the unitary matrix

$$U_\nu = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}e^{2i\pi\frac{\delta}{n}} & 0 & 0 \\ 0 & e^{-i\frac{\phi}{2}} & e^{-i\frac{\phi}{2}} \\ 0 & -e^{i(\frac{\phi}{2}-2\pi\frac{\delta}{n})} & e^{i(\frac{\phi}{2}-2\pi\frac{\delta}{n})} \end{pmatrix}. \quad (3.45)$$

The light neutrino masses are

$$m_1 = |m_{11}|, \quad m_2 = |m_{22} - m_{23}|, \quad m_3 = |m_{22} + m_{23}|. \quad (3.46)$$

(vi) $G_\nu = Z_2^{c^{n/2}d^{n/2}} \equiv \{1, c^{n/2}d^{n/2}\}$, $X_{\nu\mathbf{r}} = \{\rho_{\mathbf{r}}(c^\gamma d^\delta), \rho_{\mathbf{r}}(bc^\gamma d^\delta)\}$

- $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(c^\gamma d^\delta)$ The light neutrino mass matrix invariant under both the residual family and residual CP symmetries is

$$m_\nu = \begin{pmatrix} m_{11}e^{-2i\pi\frac{\gamma}{n}} & 0 & m_{13}e^{-i\pi\frac{\gamma-\delta}{n}} \\ 0 & m_{22}e^{-2i\pi\frac{\delta-\gamma}{n}} & 0 \\ m_{13}e^{-i\pi\frac{\gamma-\delta}{n}} & 0 & m_{33}e^{2i\pi\frac{\delta}{n}} \end{pmatrix}, \quad (3.47)$$

where m_{11} , m_{13} , m_{22} and m_{33} are real parameters. The unitary transformation U_ν is given by

$$U_\nu = \begin{pmatrix} e^{i\pi\frac{\gamma}{n}} \cos \theta & 0 & e^{i\pi\frac{\gamma}{n}} \sin \theta \\ 0 & e^{i\pi\frac{\delta-\gamma}{n}} & 0 \\ -e^{-i\pi\frac{\delta}{n}} \sin \theta & 0 & e^{-i\pi\frac{\delta}{n}} \cos \theta \end{pmatrix}, \quad (3.48)$$

with

$$\tan 2\theta = \frac{2m_{13}}{m_{33} - m_{11}}. \quad (3.49)$$

The light neutrino mass eigenvalues are

$$\begin{aligned}
 m_1 &= \frac{1}{2} \left| m_{11} + m_{33} - \text{sign}((m_{33} - m_{11}) \cos 2\theta) \sqrt{(m_{33} - m_{11})^2 + 4m_{13}^2} \right|, \\
 m_2 &= |m_{22}|, \\
 m_3 &= \frac{1}{2} \left| m_{11} + m_{33} + \text{sign}((m_{33} - m_{11}) \cos 2\theta) \sqrt{(m_{33} - m_{11})^2 + 4m_{13}^2} \right|. \quad (3.50)
 \end{aligned}$$

- $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(bc^\gamma d^\delta)$ In the case of $\gamma + \delta \neq 0 \pmod n$, the generalized CP transformation $\rho_{\mathbf{r}}(bc^\gamma d^\delta)$ is not symmetric. As a consequence, the light neutrino masses are partially degenerate. In the following, we shall focus on the case of $\gamma + \delta = 0 \pmod n$. The neutrino mass matrix is determined to be of the following form:

$$m_\nu = \begin{pmatrix} m_{11}e^{i\phi} & 0 & m_{13}e^{-2i\pi\frac{\gamma}{n}} \\ 0 & m_{22}e^{4i\pi\frac{\gamma}{n}} & 0 \\ m_{13}e^{-2i\pi\frac{\gamma}{n}} & 0 & m_{11}e^{-i(\phi+4\pi\frac{\gamma}{n})} \end{pmatrix}, \quad (3.51)$$

where m_{11} , m_{13} , m_{22} and ϕ are real. The neutrino diagonalization matrix is

$$U_\nu = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\frac{\phi}{2}} & 0 & e^{-i\frac{\phi}{2}} \\ 0 & \sqrt{2}e^{-2i\pi\frac{\gamma}{n}} & 0 \\ -e^{i(\frac{\phi}{2}+2\pi\frac{\gamma}{n})} & 0 & e^{i(\frac{\phi}{2}+2\pi\frac{\gamma}{n})} \end{pmatrix}. \quad (3.52)$$

Finally the light neutrino masses are given by

$$m_1 = |m_{11} - m_{13}|, \quad m_2 = |m_{22}|, \quad m_3 = |m_{11} + m_{13}|. \quad (3.53)$$

3.3 Predictions for lepton flavor mixing

Since the possible forms of the neutrino and charged lepton mass matrices and their diagonalization matrices have been worked out in previous sections, the lepton flavor mixing matrix can be pinned down immediately

$$U_{PMNS} = U_l^\dagger U_\nu. \quad (3.54)$$

Because the order of the charged-lepton and neutrino masses is indeterminate in the present framework, the PMNS matrix U_{PMNS} is determined up to independent permutations of rows and columns. From eqs. (3.14a), (3.14b), we know that the remnant Z_2 symmetries generated by $bc^x d^x$, abc^y , a^2bd^z are conjugate to each other, and the same is true for the Z_2 symmetry generated by $c^{n/2}$, $d^{n/2}$ and $c^{n/2}d^{n/2}$. If a pair of residual flavor symmetries (G'_ν, G'_l) is conjugated to the pair of groups (G_ν, G_l) under the group element $g \in \Delta(6n^2)$, it has been established that both pairs lead to the same result for U_{PMNS} even after the generalized CP symmetry is included [34]. As a consequence, we only need to consider the representative residual symmetry $G_\nu = Z_2^{bc^x d^x}$, $Z_2^{c^{n/2}}$ and $G_l = \langle c^s d^t \rangle$, $\langle bc^s d^t \rangle$, $\langle ac^s d^t \rangle$, $\langle abc^s d^t \rangle$ and $\langle a^2bc^s d^t \rangle$. Because the remnant flavor symmetry in the neutrino sector is taken to be Z_2 instead of K_4 subgroup, only one column of the PMNS matrix can

be fixed up to permutation and rephasing of the elements in this scenario. The concrete form of the determined columns for different choices of the remnant flavor symmetry is summarized in table 2. The present 3σ confidence level ranges for the magnitude of the elements of the leptonic mixing matrix are fitted to be [8]:

$$\|U_{PMNS}\|_{3\sigma} = \begin{pmatrix} 0.789 \rightarrow 0.853 & 0.501 \rightarrow 0.594 & 0.133 \rightarrow 0.172 \\ 0.194 \rightarrow 0.558 & 0.408 \rightarrow 0.735 & 0.602 \rightarrow 0.784 \\ 0.194 \rightarrow 0.558 & 0.408 \rightarrow 0.735 & 0.602 \rightarrow 0.784 \end{pmatrix}, \quad (3.55)$$

for normal ordering neutrino mass spectrum, and a very similar result is obtained for inverted ordering spectrum. We see that no entry of the PMNS matrix can be zero. As a result, the mixing patterns with a zero element have been ruled out by experimental data of neutrino mixing. In the following, we shall concentrate on the viable case in which no element of the fixed column is zero, and the predictions for the lepton flavor mixing parameters will be investigated for various remnant CP symmetries compatible with remnant flavor symmetry.

$$\text{(I)} \quad G_l = \langle ac^s d^t \rangle, \quad G_\nu = Z_2^{bc^x d^x}, \quad X_{\nu\mathbf{r}} = \{ \rho_{\mathbf{r}}(c^\gamma d^{-2x-\gamma}), \rho_{\mathbf{r}}(bc^{x+\gamma} d^{-x-\gamma}) \}$$

The PMNS matrix is determined to be

$$U_{PMNS}^I = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2} \sin \varphi_1 & e^{i\varphi_2} \cos \theta - \sqrt{2} \sin \theta \cos \varphi_1 & e^{i\varphi_2} \sin \theta + \sqrt{2} \cos \theta \cos \varphi_1 \\ \sqrt{2} \cos(\frac{\pi}{6} - \varphi_1) & -e^{i\varphi_2} \cos \theta - \sqrt{2} \sin \theta \sin(\frac{\pi}{6} - \varphi_1) & -e^{i\varphi_2} \sin \theta + \sqrt{2} \cos \theta \sin(\frac{\pi}{6} - \varphi_1) \\ \sqrt{2} \cos(\frac{\pi}{6} + \varphi_1) & e^{i\varphi_2} \cos \theta + \sqrt{2} \sin \theta \sin(\frac{\pi}{6} + \varphi_1) & e^{i\varphi_2} \sin \theta - \sqrt{2} \cos \theta \sin(\frac{\pi}{6} + \varphi_1) \end{pmatrix},$$

where

$$\varphi_1 = \frac{s-x}{n}\pi, \quad \varphi_2 = \frac{2t-s-3(\gamma+x)}{n}\pi. \quad (3.56)$$

These two parameters φ_1 and φ_2 are interdependent of each other, and they can take the discrete values

$$\begin{aligned} \varphi_1 &= 0, \pm \frac{1}{n}\pi, \pm \frac{2}{n}\pi, \dots, \pm \frac{n-1}{n}\pi, \\ \varphi_2 \bmod 2\pi &= 0, \frac{1}{n}\pi, \frac{2}{n}\pi, \dots, \frac{2n-1}{n}\pi. \end{aligned} \quad (3.57)$$

Obviously one column of the PMNS matrix is fixed to be

$$\sqrt{\frac{2}{3}} \begin{pmatrix} \sin \varphi_1 \\ \cos(\pi/6 - \varphi_1) \\ \cos(\pi/6 + \varphi_1) \end{pmatrix} \equiv \mathcal{C}. \quad (3.58)$$

For the group $\Delta(24) \equiv S_4$, we have

$$\varphi_1 = 0 : \mathcal{C} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \varphi_1 = \pm \frac{\pi}{2} : \mathcal{C} = \pm \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}. \quad (3.59)$$

	$G_\nu = Z_2^{bc^x d^x}$	$G_\nu = Z_2^{c^{n/2}}$
$G_l = \langle c^s d^t \rangle$	$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \boldsymbol{\times}$	$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \boldsymbol{\times}$
$G_l = \langle bc^s d^t \rangle$	$\begin{pmatrix} 0 \\ \cos\left(\frac{s+t-2x}{2n}\pi\right) \\ \sin\left(\frac{s+t-2x}{2n}\pi\right) \end{pmatrix} \boldsymbol{\times}$	$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \boldsymbol{\times}$
$G_l = \langle ac^s d^t \rangle$	$\sqrt{\frac{2}{3}} \begin{pmatrix} \sin\left(\frac{s-x}{n}\pi\right) \\ \cos\left(\frac{\pi}{6} - \frac{s-x}{n}\pi\right) \\ \cos\left(\frac{\pi}{6} + \frac{s-x}{n}\pi\right) \end{pmatrix} \boldsymbol{\checkmark}$	$\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \boldsymbol{\checkmark}$
$G_l = \langle abc^s d^t \rangle$	$\frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -\sqrt{2} \end{pmatrix} \boldsymbol{\checkmark}$	$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \boldsymbol{\times}$
$G_l = \langle a^2 bc^s d^t \rangle$	$\frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -\sqrt{2} \end{pmatrix} \boldsymbol{\checkmark}$	$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \boldsymbol{\times}$

Table 2. The determined form of one column of the PMNS matrix for different remnant symmetries G_ν and G_l which are Z_2 and abelian subgroups of $\Delta(6n^2)$ family symmetry groups respectively. The symbol “ $\boldsymbol{\times}$ ” denotes that the resulting lepton mixing is ruled out since there is at least one zero element in the fixed column, and the symbol “ $\boldsymbol{\checkmark}$ ” denotes that the resulting mixing is viable. Note that for $G_\nu = Z_2^{bc^x d^x}$, the cases of $G_l = \langle abc^s d^t \rangle$ and $G_l = \langle a^2 bc^s d^t \rangle$ are not independent as we have $b(abc^s d^t) b = a^2 bc^{-t} d^{-s}$ and $b(bc^x d^x) b = bc^{-x} d^{-x}$.

Note that the first column of the tri-bimaximal mixing matrix is reproduced for $\varphi_1 = \pm\pi/2$. For the group $\Delta(96)$, we have

$$\begin{aligned}
 \varphi_1 = \frac{3\pi}{4}, -\frac{\pi}{4} : \mathcal{C} = \pm \frac{1}{6} \begin{pmatrix} 2\sqrt{3} \\ -3 + \sqrt{3} \\ -3 - \sqrt{3} \end{pmatrix}, & \quad \varphi_1 = \pm \frac{\pi}{2} : \mathcal{C} = \pm \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \\
 \varphi_1 = \frac{\pi}{4}, -\frac{3\pi}{4} : \mathcal{C} = \pm \frac{1}{6} \begin{pmatrix} 2\sqrt{3} \\ 3 + \sqrt{3} \\ 3 - \sqrt{3} \end{pmatrix}, & \quad \varphi_1 = 0 : \mathcal{C} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}. \quad (3.60)
 \end{aligned}$$

It is interesting that the first and the third columns (up to permutations) of the Toorop-

Feruglio-Hagedorn mixing [64–67] can be obtained in the case of $\varphi_1 = \pm\pi/4, \pm 3\pi/4$. Now we consider the permutations of the rows and the columns, i.e. the PMNS matrix can be multiplied by a 3×3 permutation matrix from both the left-hand side and the right-hand side. There are six permutation matrices corresponding to six possible orderings of rows (or columns):

$$\begin{aligned}
 P_{123} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & P_{132} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & P_{213} &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
 P_{231} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, & P_{312} &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, & P_{321} &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
 \end{aligned} \tag{3.61}$$

It is well-known that the atmospheric mixing angle θ_{23} becomes $\pi/2 - \theta_{23}$, the Dirac CP phases δ_{CP} becomes $\pi + \delta_{CP}$ and the other mixing parameters are kept intact if the second and third row of the PMNS matrix are exchanged. Hence the permutation of the second and third row will not be explored explicitly in the following. First we study the situation that the constant vector $\sqrt{2/3}(\sin \varphi_1, \cos(\pi/6 - \varphi_1), \cos(\pi/6 + \varphi_1))^T$ is in the first column. Then the PMNS matrix can be arranged as follows:

$$U_{PMNS}^{I,1st} = U_{PMNS}^I, \quad U_{PMNS}^{I,2nd} = P_{231} U_{PMNS}^I, \quad U_{PMNS}^{I,3rd} = P_{312} U_{PMNS}^I. \tag{3.62}$$

Note that exchanging its second and third row does not lead to new mixing patterns. Moreover, we see that the above three arrangements are related with each other:

$$\begin{aligned}
 U_{PMNS}^{I,2nd}(\theta, \varphi_1, \varphi_2) &= \text{diag}(1, 1, -1) U_{PMNS}^{I,1st} \left(\pi - \theta, \frac{\pi}{3} + \varphi_1, \varphi_2 \right) \text{diag}(1, 1, -1), \\
 U_{PMNS}^{I,3rd}(\theta, \varphi_1, \varphi_2) &= \text{diag}(-1, 1, 1) U_{PMNS}^{I,1st} \left(-\theta, -\frac{\pi}{3} + \varphi_1, \varphi_2 \right) \text{diag}(1, -1, 1),
 \end{aligned} \tag{3.63}$$

where the phase factor $\text{diag}(\pm 1, \pm 1, \pm 1)$ can be absorbed by the lepton fields. Hence it is sufficient to only discuss the first PMNS matrix $U_{PMNS}^{I,1st}$ in detail, the phenomenological predictions for the other two can be easily obtained by variable substitution. In this case, the lepton mixing parameters are predicted to be

$$\begin{aligned}
 \sin^2 \theta_{13} &= \frac{1}{3} \left(1 + \cos^2 \theta \cos 2\varphi_1 + \sqrt{2} \sin 2\theta \cos \varphi_2 \cos \varphi_1 \right), \\
 \sin^2 \theta_{12} &= \frac{1 + \sin^2 \theta \cos 2\varphi_1 - \sqrt{2} \sin 2\theta \cos \varphi_2 \cos \varphi_1}{2 - \cos^2 \theta \cos 2\varphi_1 - \sqrt{2} \sin 2\theta \cos \varphi_2 \cos \varphi_1}, \\
 \sin^2 \theta_{23} &= \frac{1 - \cos^2 \theta \sin(\pi/6 + 2\varphi_1) - \sqrt{2} \sin 2\theta \cos \varphi_2 \sin(\pi/6 - \varphi_1)}{2 - \cos^2 \theta \cos 2\varphi_1 - \sqrt{2} \sin 2\theta \cos \varphi_2 \cos \varphi_1}, \\
 |\tan \delta_{CP}| &= \left| 2\sqrt{2} \sin 2\theta \sin \varphi_2 (1 + 2 \cos 2\varphi_1) \left(2 - \cos^2 \theta \cos 2\varphi_1 - \sqrt{2} \sin 2\theta \cos \varphi_2 \cos \varphi_1 \right) \right. \\
 &\quad \left. \left\{ 2 \sin^2 2\theta \cos 2\varphi_2 (\cos 3\varphi_1 - 2 \cos \varphi_1) + \cos \varphi_1 (9 - 4 \cos 2\theta + 3 \cos 4\theta - 16 \cos^2 \theta \cos 2\varphi_1) \right. \right. \\
 &\quad \left. \left. - 2\sqrt{2} \sin 2\theta \cos \varphi_2 [2 - \cos^2 \theta (5 + \cos 2\varphi_1 + \cos 4\varphi_1)] \right\} \right|
 \end{aligned}$$

$$\begin{aligned}
 |J_{CP}| &= \frac{1}{6\sqrt{6}} |\sin 2\theta \sin \varphi_2 \sin 3\varphi_1|, \\
 |\tan \alpha_{21}| &= \left| \frac{2 \sin \varphi_2 (\cos \varphi_2 - \sqrt{2} \cos \varphi_1 \tan \theta)}{\cos 2\varphi_2 - 2 \cos \varphi_1 \tan \theta (\sqrt{2} \cos \varphi_2 - \cos \varphi_1 \tan \theta)} \right|, \\
 |\tan \alpha'_{31}| &= \left| \frac{2 \sin \varphi_2 (\cos \varphi_2 + \sqrt{2} \cos \varphi_1 \cot \theta)}{\cos 2\varphi_2 + 2 \cos \varphi_1 \cot \theta (\sqrt{2} \cos \varphi_2 + \cos \varphi_1 \cot \theta)} \right|,
 \end{aligned} \tag{3.64}$$

where $\alpha'_{31} = \alpha_{31} - 2\delta_{CP}$, δ_{CP} is the Dirac CP phase, α_{21} and α_{31} are the Majorana CP phases in the standard parameterization [68]. If we embed the three generations of left-handed lepton doublets into the triplet $\mathbf{3}_{1,n-1}$ which is the complex conjugate representation of $\mathbf{3}_{1,1}$, all three CP phases δ_{CP} , α_{21} and α_{31} would become their opposite numbers modulo 2π . Furthermore, the overall sign of $\tan \alpha_{21}$ and $\tan \alpha'_{31}$ depends on the CP parity of the neutrino states which is encoded in the matrix K_ν (please see eq. (3.25)), and the sign of the Jarlskog invariant J_{CP} depends on the ordering of rows and columns. As a result, all these quantities are presented in terms of absolute values here.

It is notable that all three CP phases depend on both the free continuous parameter θ and the discrete parameters φ_1 and φ_2 associated with flavor and CP symmetries. Obviously both Dirac CP and Majorana CP are conserved for $\varphi_2 = 0$. Furthermore, the solar mixing angle θ_{12} and reactor angle θ_{13} are related by

$$3 \cos^2 \theta_{12} \cos^2 \theta_{13} = 2 \sin^2 \varphi_1, \tag{3.65}$$

which is independent of the free parameter θ . Taking into account measured values of θ_{12} and θ_{13} [8], we obtain the constraint on φ_1 as

$$0.417\pi \leq \varphi_1 \leq 0.583\pi, \quad \text{or} \quad -0.583\pi \leq \varphi_1 \leq -0.417\pi, \tag{3.66}$$

which indicates that φ_1 is around $\pm\pi/2$. This mixing pattern is very interesting, and it can accommodate the present neutrino oscillation data very well. After the measured 3σ ranges of the three lepton mixing angles are imposed, the allowed values of the lepton mixing parameters for $n = 2, 3, \dots, 100$ are displayed in figure 1 and figure 2. In the case that n is divisible by 3, the doublet representations $\mathbf{2}_2$, $\mathbf{2}_3$ and $\mathbf{2}_4$ are assumed to be absent such that generalized CP symmetry in eq. (2.8) is consistently defined. Notice that here n should start from $n = 2$ since $\Delta(6) \equiv S_3$ does not have three dimensional irreducible representations. If n is divisible by 3, the three permutations $U_{PMNS}^{I,1st}$, $U_{PMNS}^{I,2nd}$ and $U_{PMNS}^{I,3rd}$ give rise to the same predictions for the mixing parameters. The observed values of the three lepton mixing angles can not be achieved simultaneously for $n = 3$. In case of $n = 2$ and $n = 4$, both the atmospheric mixing angle θ_{23} and the Dirac CP phase δ_{CP} are maximal while the Majorana phases are zero. It is remarkable that the three CP phases can take any values for sufficiently large n , while θ_{12} is always constrained to be in the range of $0.313 \leq \sin^2 \theta_{12} \leq 0.344$. Hence this mixing pattern can be tested by precisely measuring the solar mixing angle θ_{12} . Notice that θ_{12} can be measured with rather good accuracy by JUNO experiment [69]. In the end, the correlations between mixing parameters for $n \rightarrow \infty$ and $n = 8$ are displayed in figure 3 where we only show the

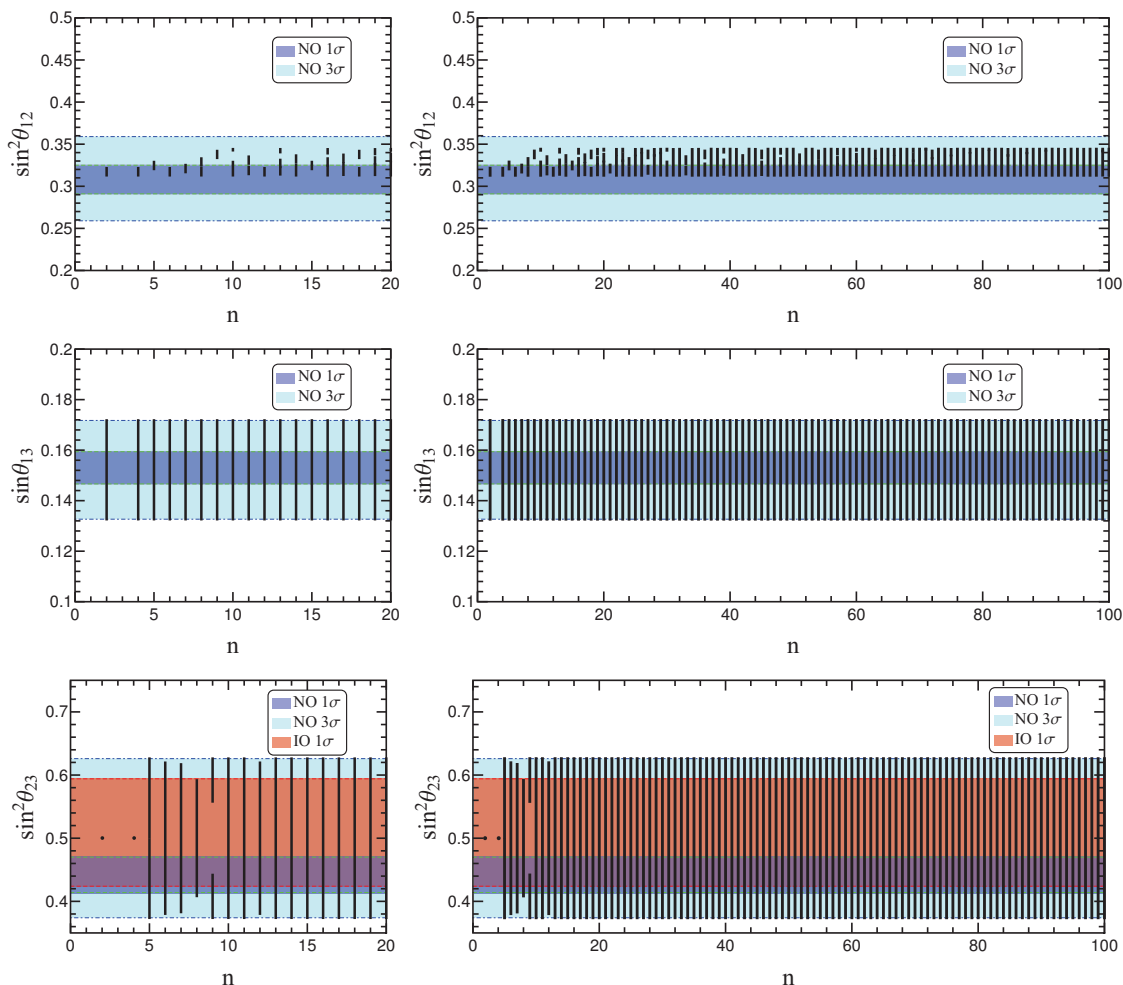


Figure 1. Numerical results in case I for the 1st-3rd ordering with the PMNS matrices given in eq. (3.62): the allowed values of $\sin^2 \theta_{12}$, $\sin \theta_{13}$ and $\sin^2 \theta_{23}$ for different n , where the three lepton mixing angles are required to lie in their 3σ ranges. The 1σ and 3σ bounds of the mixing parameters are taken from ref. [8].

phenomenologically viable cases for which the measured values of θ_{12} , θ_{13} and θ_{23} can be accommodated for certain values of the parameter θ .

The vector $\sqrt{2/3}(\sin \varphi_1, \cos(\pi/6 - \varphi_1), \cos(\pi/6 + \varphi_1))^T$ enforced by the remnant Z_2 symmetry can also be the second column of the PMNS matrix. Disregarding the exchange of the second and the third rows, three arrangements are possible as well,

$$U_{PMNS}^{I,4th} = U_{PMNS}^I P_{213}, \quad U_{PMNS}^{I,5th} = P_{231} U_{PMNS}^I P_{213}, \quad U_{PMNS}^{I,6th} = P_{312} U_{PMNS}^I P_{213}. \quad (3.67)$$

Analogous to that in eq. (3.63), these three forms of the PMNS matrix are related by parameter redefinition as follows

$$U_{PMNS}^{I,5th}(\theta, \varphi_1, \varphi_2) = \text{diag}(1, 1, -1) U_{PMNS}^{I,4th} \left(\pi - \theta, \frac{\pi}{3} + \varphi_1, \varphi_2 \right) \text{diag}(1, 1, -1),$$

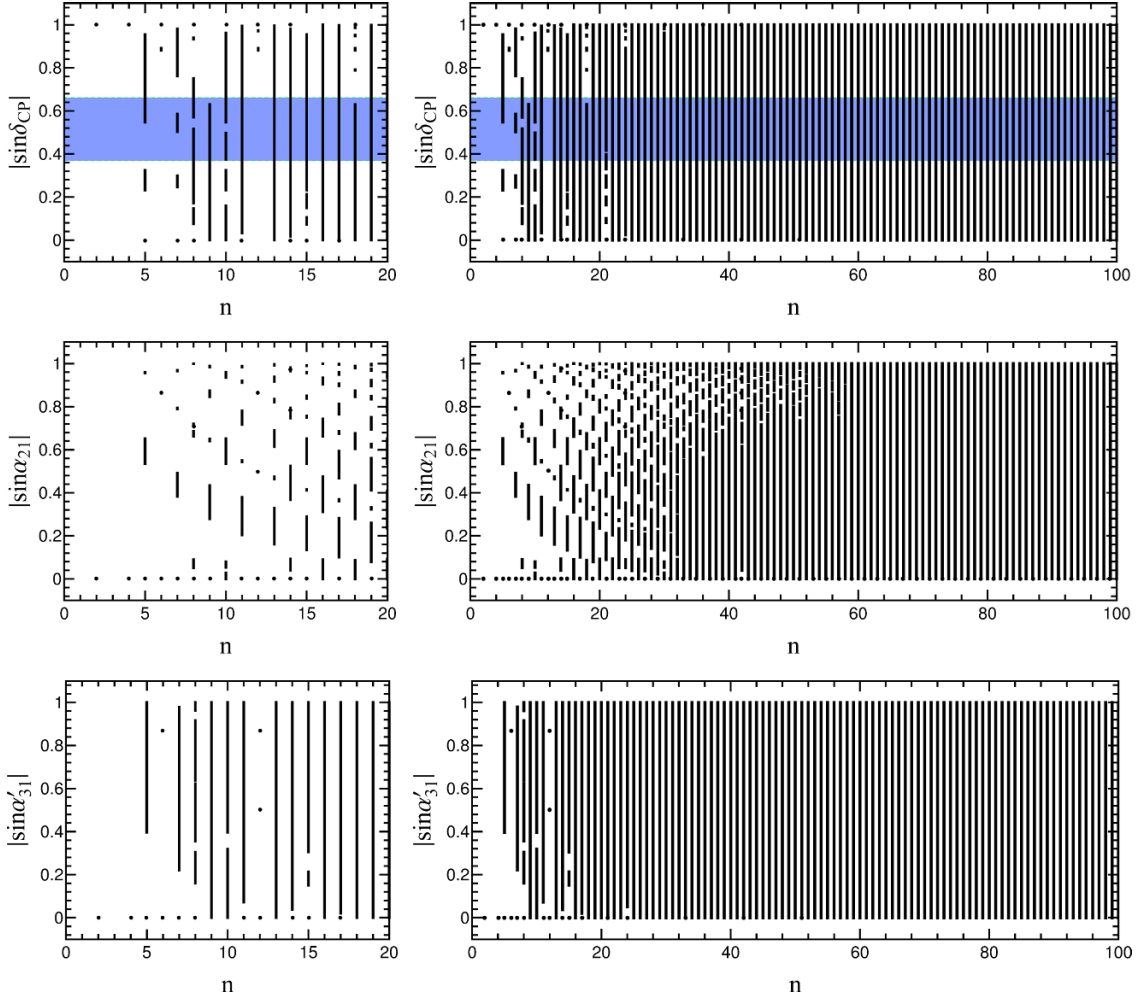


Figure 2. Numerical results in case I for the 1st-3rd ordering with the PMNS matrices given in eq. (3.62): the possible values of $|\sin \delta_{CP}|$, $|\sin \alpha_{21}|$ and $|\sin \alpha'_{31}|$ for different n , where the three lepton mixing angles are required to lie in the 3σ ranges. The 1σ and 3σ bounds of the mixing parameters are taken from ref. [8].

$$U_{PMNS}^{6th}(\theta, \varphi_1, \varphi_2) = \text{diag}(-1, 1, 1) U_{PMNS}^{I, 4th} \left(-\theta, -\frac{\pi}{3} + \varphi_1, \varphi_2 \right) \text{diag}(-1, 1, 1). \quad (3.68)$$

For the lepton flavor mixing matrix U_{PMNS}^{4th} , one can straightforwardly extract the flavor mixing parameters:

$$\begin{aligned} \sin^2 \theta_{13} &= \frac{1}{3} \left(1 + \cos^2 \theta \cos 2\varphi_1 + \sqrt{2} \sin 2\theta \cos \varphi_2 \cos \varphi_1 \right), \\ \sin^2 \theta_{12} &= \frac{2 \sin^2 \varphi_1}{2 - \cos^2 \theta \cos 2\varphi_1 - \sqrt{2} \sin 2\theta \cos \varphi_2 \cos \varphi_1}, \\ \sin^2 \theta_{23} &= \frac{1 - \cos^2 \theta \sin(\pi/6 + 2\varphi_1) - \sqrt{2} \sin 2\theta \cos \varphi_2 \sin(\pi/6 - \varphi_1)}{2 - \cos^2 \theta \cos 2\varphi_1 - \sqrt{2} \sin 2\theta \cos \varphi_2 \cos \varphi_1}, \\ |J_{CP}| &= \frac{1}{6\sqrt{6}} |\sin 2\theta \sin \varphi_2 \sin 3\varphi_1|, \\ |\tan \delta_{CP}| &= \left| 4\sqrt{2} \sin 2\theta \sin \varphi_2 \sin 3\varphi_1 \csc \varphi_1 \left(2 - \cos 2\varphi_1 \cos^2 \theta - \sqrt{2} \cos \varphi_2 \cos \varphi_1 \sin 2\theta \right) \right| / \end{aligned}$$

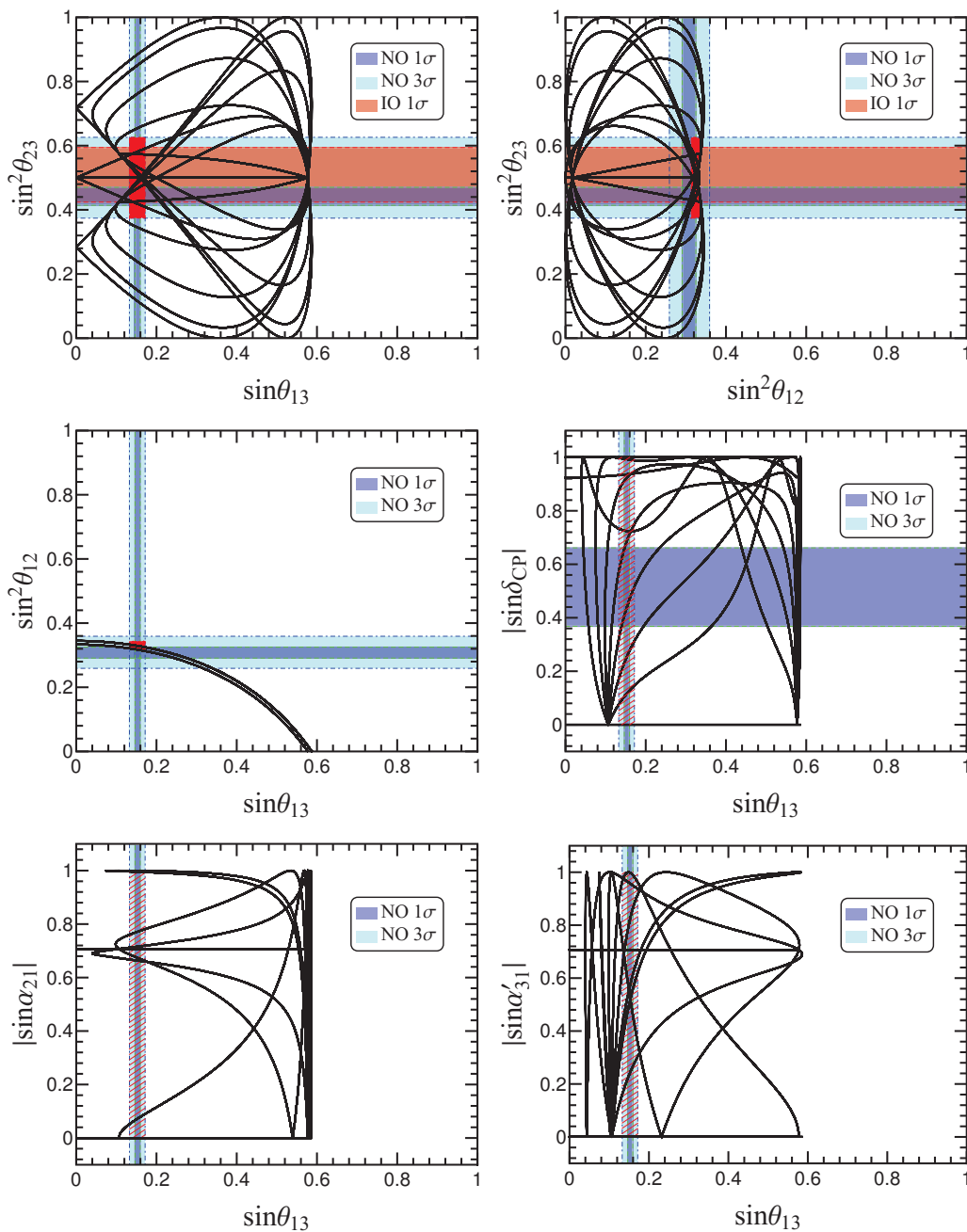


Figure 3. The correlations among mixing parameters in case I for the 1st-3rd ordering with the PMNS matrices given in eq. (3.62). The red filled regions denote the allowed values of the mixing parameters if we take the parameters φ_1 and φ_2 to be continuous (which is equivalent to taking the limit $n \rightarrow \infty$) and the three mixing angles are required to lie in their 3σ regions. Note that the three CP phases δ_{CP} , α_{21} and α'_{31} are not constrained in this limit. The black curves represent the phenomenologically viable correlations for $n = 8$. The 1σ and 3σ bounds of the mixing parameters are taken from ref. [8].

$$\begin{aligned}
 & \left\{ -16 \cos 3\varphi_1 \cos^2 \theta + 8(1 - 3 \cos 2\theta) \cos \varphi_1 \sin^2 \theta + 4 \cos 2\varphi_2 (\cos 3\varphi_1 - 2 \cos \varphi_1) \sin^2 2\theta \right. \\
 & \quad \left. + \sqrt{2} \cos \varphi_2 [8(\cos 2\varphi_1 + \cos 4\varphi_1) \sin \theta \cos^3 \theta + 2 \sin 2\theta + 5 \sin 4\theta] \right\}, \\
 |\tan \alpha_{21}| &= \left| \frac{2 \sin \varphi_2 (\cos \varphi_2 - \sqrt{2} \cos \varphi_1 \tan \theta)}{\cos 2\varphi_2 - 2 \cos \varphi_1 \tan \theta (\sqrt{2} \cos \varphi_2 - \cos \varphi_1 \tan \theta)} \right|, \\
 |\tan \alpha'_{31}| &= \left| 8 \cos \varphi_1 \left(\sqrt{2} \cos 2\varphi_1 \sin 2\theta \sin \varphi_2 - 2 \cos 2\theta \cos \varphi_1 \sin 2\varphi_2 \right) / \left\{ 4(3 + \cos 4\theta) \right. \right. \\
 & \quad \left. \left. \times \cos 2\varphi_2 \cos^2 \varphi_1 - 4\sqrt{2} \cos \varphi_2 \cos \varphi_1 \cos 2\varphi_1 \sin 4\theta - (3 - \cos 4\varphi_1 + 4 \cos 2\varphi_1) \sin^2 2\theta \right\} \right|. \quad (3.69)
 \end{aligned}$$

In this case, we have the following relation

$$3 \sin^2 \theta_{12} \cos^2 \theta_{13} = 2 \sin^2 \varphi_1, \quad (3.70)$$

which yields $0.614 \leq |\sin \varphi_1| \leq 0.727$ at 3σ confidence level, and therefore the parameter φ_1 is determined to be in the range

$$\varphi_1 \in \pm ([0.210\pi, 0.259\pi]) \cup [0.741\pi, 0.790\pi]. \quad (3.71)$$

For the representative values $\pm\pi/4$ and $\pm 3\pi/4$ of φ_1 , the relatively small θ_{13} leads to

$$(\varphi_1, \varphi_2, \theta) \simeq \left(\pm \frac{\pi}{4}, 0, \frac{3\pi}{4} \right), \quad \left(\pm \frac{\pi}{4}, \pi, \frac{\pi}{4} \right), \quad \left(\pm \frac{3\pi}{4}, 0, \frac{\pi}{4} \right), \quad \left(\pm \frac{3\pi}{4}, \pi, \frac{3\pi}{4} \right). \quad (3.72)$$

Accordingly the atmospheric mixing angle θ_{23} would be

$$\sin^2 \theta_{23} \simeq \frac{1}{4}(2 - \sqrt{3}) \simeq 0.067, \quad \text{or} \quad \sin^2 \theta_{23} \simeq \frac{1}{4}(2 + \sqrt{3}) \simeq 0.933, \quad (3.73)$$

which is clearly not compatible with the global analysis of neutrino oscillation data [8]. As a result, the three lepton mixing angles can not be accommodated simultaneously in this case, and this mixing pattern is not viable. The detailed numerical results are presented in figure 4. We see that the correct values of the atmospheric mixing angle really can not be achieved for realistic θ_{12} and θ_{13} .

Finally the predicted vector $\sqrt{2/3}(\sin \varphi_1, \cos(\pi/6 - \varphi_1), \cos(\pi/6 + \varphi_1))^T$ can be placed in the third column. Using the freedom of exchanging the rows of the PMNS matrix, three configurations are found as well,

$$U_{PMNS}^{I,7th} = U_{PMNS}^I P_{321}, \quad U_{PMNS}^{I,8th} = P_{231} U_{PMNS}^I P_{321}, \quad U_{PMNS}^{I,9th} = P_{312} U_{PMNS}^I P_{321}, \quad (3.74)$$

which are related by

$$\begin{aligned}
 U_{PMNS}^{I,8th}(\theta, \varphi_1, \varphi_2) &= \text{diag}(1, 1, -1) U_{PMNS}^{I,7th}(\pi - \theta, \frac{\pi}{3} + \varphi_1, \varphi_2) \text{diag}(-1, 1, 1), \\
 U_{PMNS}^{I,9th}(\theta, \varphi_1, \varphi_2) &= \text{diag}(-1, 1, 1) U_{PMNS}^{I,7th}(-\theta, -\frac{\pi}{3} + \varphi_1, \varphi_2) \text{diag}(1, -1, 1). \quad (3.75)
 \end{aligned}$$

The lepton mixing parameters for $U_{PMNS}^{I,7th}$ are determined to be

$$\sin^2 \theta_{13} = \frac{2}{3} \sin^2 \varphi_1, \quad \sin^2 \theta_{12} = \frac{1 + \sin^2 \theta \cos 2\varphi_1 - \sqrt{2} \sin 2\theta \cos \varphi_2 \cos \varphi_1}{2 + \cos 2\varphi_1},$$

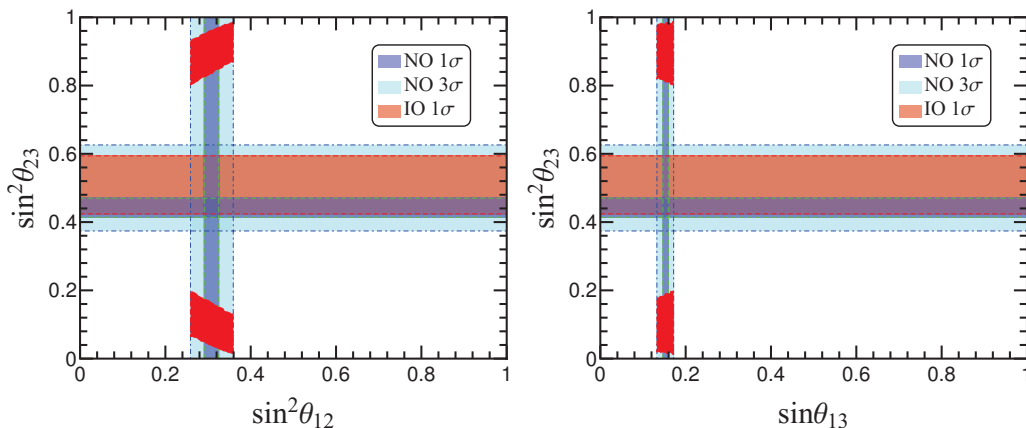


Figure 4. Numerical results in case I for the 4th-6th ordering with the PMNS matrices given in eq. (3.67). The red filled regions denote the allowed values of the mixing parameters if we take the parameters φ_1 and φ_2 to be continuous (which is equivalent to taking the limit $n \rightarrow \infty$), where θ_{12} and θ_{13} are required to lie in their 3σ ranges. Obviously the resulting predictions for θ_{23} are far beyond its 3σ range. The 1σ and 3σ bounds of the mixing parameters are taken from ref. [8].

$$\begin{aligned}
 \sin^2 \theta_{23} &= \frac{1 + \sin(\pi/6 + 2\varphi_1)}{2 + \cos 2\varphi_1}, & |J_{CP}| &= \frac{1}{6\sqrt{6}} |\sin 2\theta \sin \varphi_2 \sin 3\varphi_1|, \\
 |\tan \delta_{CP}| &= \left| \frac{\sin \varphi_2 (2 + \cos 2\varphi_1)}{\cos \varphi_2 \cos 2\varphi_1 - 2\sqrt{2} \cot 2\theta \cos \varphi_1} \right|, \\
 |\tan \alpha_{21}| &= \left| 4\sqrt{2} \cos \varphi_1 \left(\cos 2\varphi_1 \sin 2\theta \sin \varphi_2 - \sqrt{2} \cos 2\theta \cos \varphi_1 \sin 2\varphi_2 \right) / \left\{ 2(\cos 4\theta + 3) \right. \right. \\
 &\quad \left. \left. \times \cos 2\varphi_2 \cos^2 \varphi_1 - 2\sqrt{2} \cos \varphi_2 \cos \varphi_1 \cos 2\varphi_1 \sin 4\theta + (\cos^2 2\varphi_1 - 4 \cos^2 \varphi_1) \sin^2 2\theta \right\} \right|, \\
 |\tan \alpha'_{31}| &= \left| \frac{2 \sin \varphi_2 \sin \theta (\sqrt{2} \cos \theta \cos \varphi_1 + \cos \varphi_2 \sin \theta)}{2 \cos^2 \theta \cos^2 \varphi_1 + \sqrt{2} \cos \varphi_2 \cos \varphi_1 \sin 2\theta + \cos 2\varphi_2 \sin^2 \theta} \right|. \tag{3.76}
 \end{aligned}$$

The lepton mixing parameters for $U_{PMNS}^{I,8th}$ and $U_{PMNS}^{I,9th}$ can be obtained from eq. (3.76) by the replacement $\theta \rightarrow \pi - \theta$, $\varphi_1 \rightarrow \frac{\pi}{3} + \varphi_1$ and $\theta \rightarrow -\theta$, $\varphi_1 \rightarrow -\frac{\pi}{3} + \varphi_1$ respectively. We see that both θ_{13} and θ_{23} are only determined by the discrete group parameter φ_1 , and they are related by

$$\sin^2 \theta_{23} = \frac{1}{2} \pm \frac{1}{2} \tan \theta_{13} \sqrt{2 - \tan^2 \theta_{13}}, \tag{3.77}$$

which yields

$$\theta_{23} \simeq \frac{\pi}{4} \pm \frac{\theta_{13}}{\sqrt{2}}. \tag{3.78}$$

For the 3σ interval $1.76 \times 10^{-2} \leq \sin^2 \theta_{13} \leq 2.95 \times 10^{-2}$ [8], we have

$$0.378 \leq \sin^2 \theta_{23} \leq 0.406, \quad \text{or} \quad 0.594 \leq \sin^2 \theta_{23} \leq 0.622. \tag{3.79}$$

This mixing pattern can be directly tested by future atmospheric neutrino oscillation experiments or long baseline neutrino oscillation experiments. If θ_{23} is found to be nearly maximal, this mixing would be ruled out. Furthermore, the precisely measured θ_{13} leads

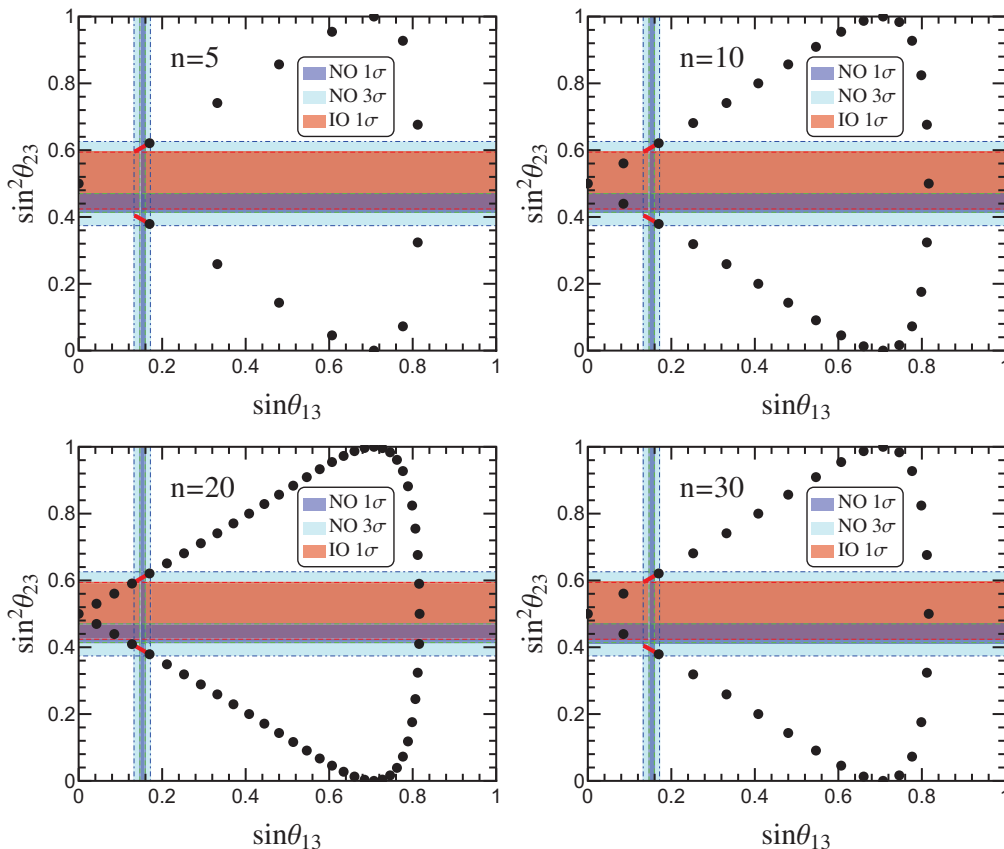


Figure 5. The possible values of $\sin^2 \theta_{23}$ and $\sin \theta_{13}$ for the 7th-9th ordering with the PMNS matrices shown in eq. (3.74) in case I. The 1σ and 3σ bounds of the mixing angles are taken from ref. [8].

to $0.162 \leq |\sin \varphi_1| \leq 0.210$, and therefore φ_1 has to be in the following range

$$\varphi_1 \in \pm ([0.0519\pi, 0.0675\pi] \cup [0.933\pi, 0.948\pi]) , \quad (3.80)$$

which implies that φ_1 should be rather close to 0 or π . To reproduce the observed value of the reactor mixing angle, the two smallest values for n are 5 and 10, i.e. at least $\Delta(150)$ or $\Delta(600)$ is needed to produce viable mixing in this case. The admissible values of $\sin^2 \theta_{23}$ and $\sin \theta_{13}$ for $n = 5, 10, 20$ and 30 are plotted in figure 5. Furthermore, the variation of the allowed values of the lepton mixing parameters with respect to n are shown in figure 6 and figure 7. Compared with previous cases, both θ_{23} and θ_{13} are predicted to take several discrete values until $n = 100$ in this case. It is interesting that the Majorana phase α'_{31} is constrained to be in the range of $0 \leq |\sin \alpha'_{31}| \leq 0.91$ while both δ_{CP} and α_{21} can take any values between 0 and 2π for large n .

(II) $G_l = \langle abc^s d^t \rangle, G_\nu = Z_2^{bc^x d^x}, X_{\nu\mathbf{r}} = \{ \rho_{\mathbf{r}}(c^\gamma d^{-2x-\gamma}), \rho_{\mathbf{r}}(bc^{x+\gamma} d^{-x-\gamma}) \}$

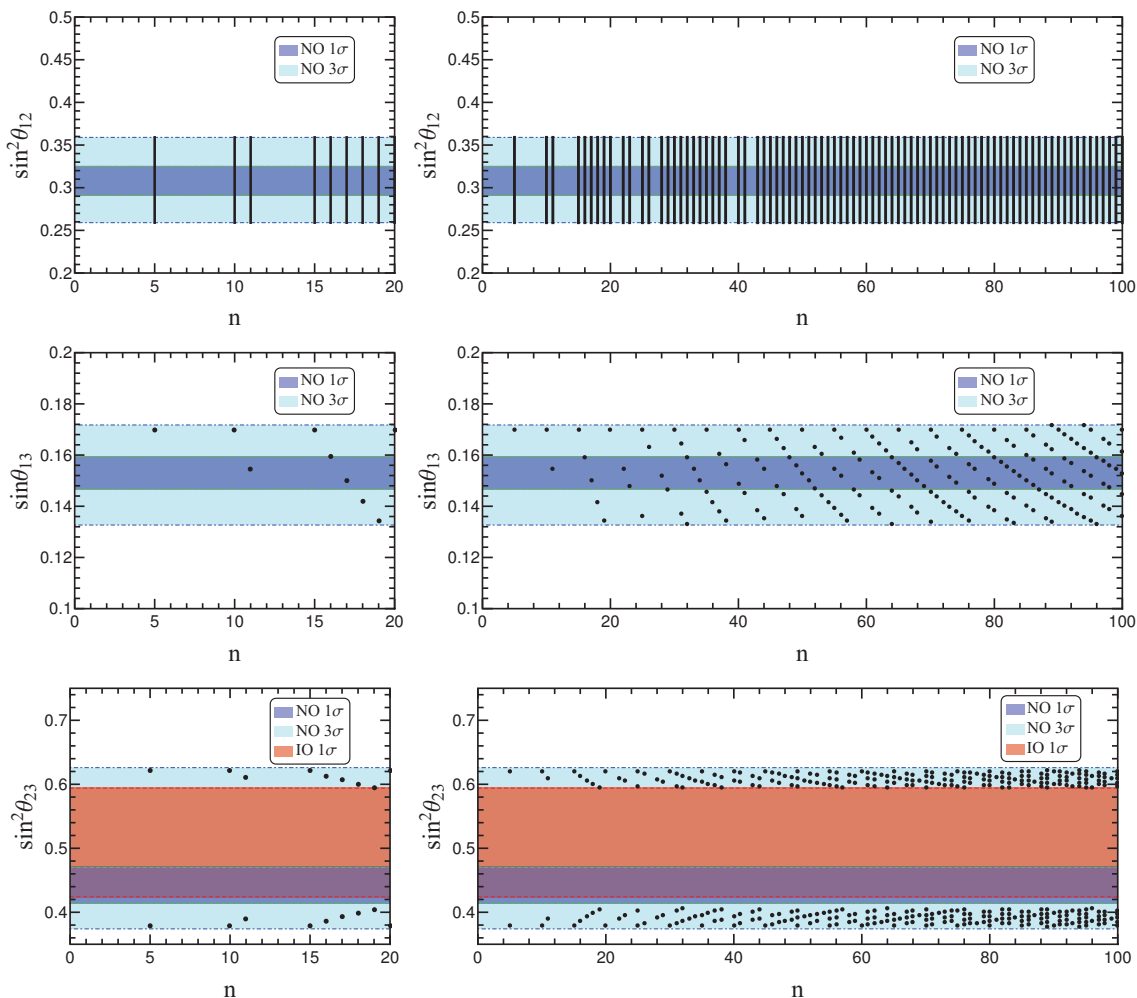


Figure 6. Numerical results in case I for the 7th-9th ordering with the PMNS matrices given in eq. (3.74): the allowed values of $\sin^2 \theta_{12}$, $\sin \theta_{13}$ and $\sin^2 \theta_{23}$ for different n , where the three lepton mixing angles are required to lie in their 3σ ranges. The 1σ and 3σ bounds of the mixing parameters are taken from ref. [8].

In this case, the PMNS matrix is determined to be

$$U_{PMNS}^I = \frac{1}{2} \begin{pmatrix} -\sin \theta - \sqrt{2}e^{i\varphi_3} \cos \theta & 1 & \cos \theta - \sqrt{2}e^{i\varphi_3} \sin \theta \\ -\sin \theta + \sqrt{2}e^{i\varphi_3} \cos \theta & 1 & \cos \theta + \sqrt{2}e^{i\varphi_3} \sin \theta \\ -\sqrt{2} \sin \theta & -\sqrt{2} & \sqrt{2} \cos \theta \end{pmatrix}, \quad (3.81)$$

or the one obtained by exchanging the second and the third rows, where the parameter φ_3 is

$$\varphi_3 = -\frac{3\gamma + 2s - t + 2x}{n} \pi. \quad (3.82)$$

It can take $2n$ discrete values:

$$\varphi_3 \bmod 2\pi = 0, \frac{1}{n}\pi, \frac{2}{n}\pi, \dots, \frac{2n-1}{n}\pi. \quad (3.83)$$

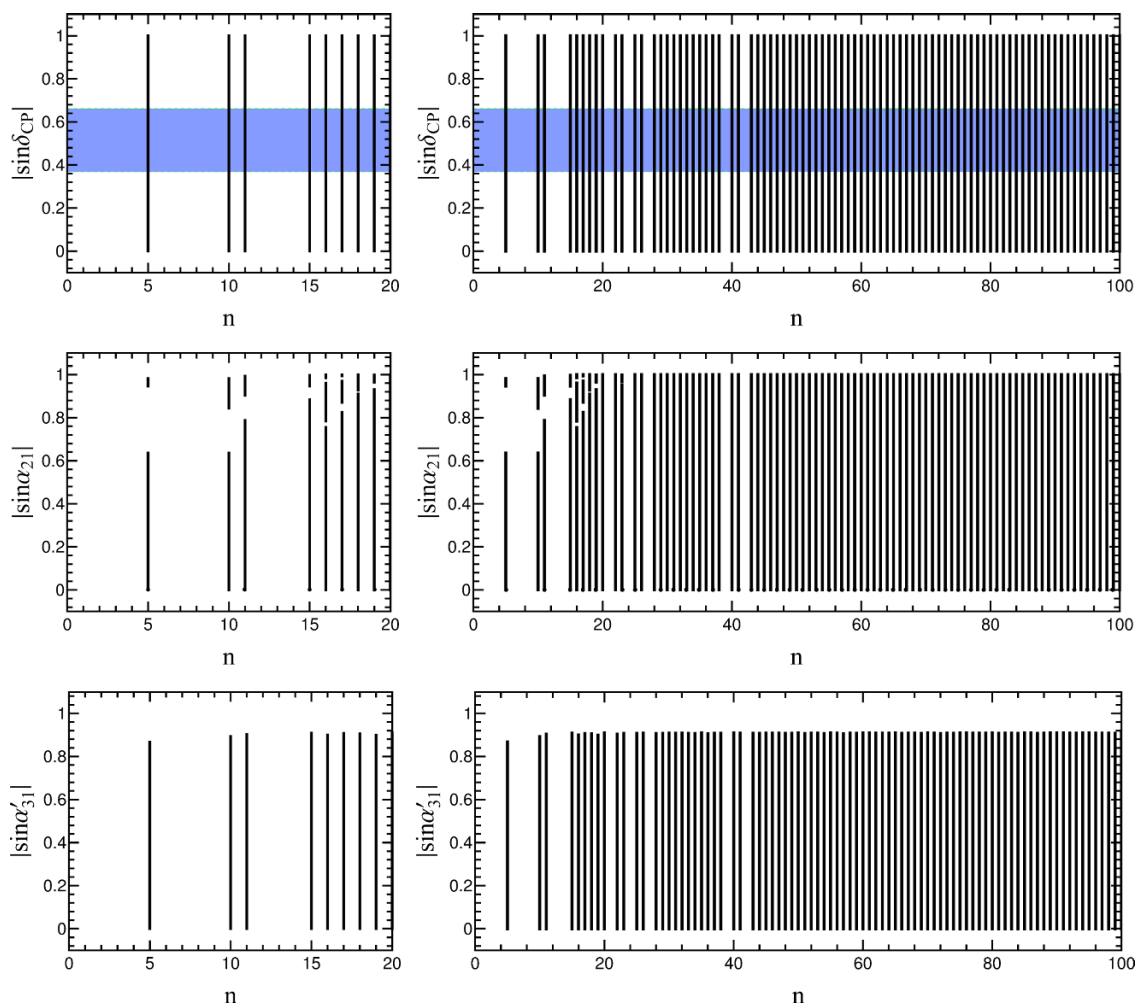


Figure 7. Numerical results in case I for the 7th-9th ordering with the PMNS matrices given in eq. (3.74): the allowed values of $|\sin \delta_{CP}|$, $|\sin \alpha_{21}|$ and $|\sin \alpha'_{31}|$ for different n , where the three lepton mixing angles are required to lie in their 3σ ranges. The 1σ and 3σ bounds of the mixing angles are taken from ref. [8].

The eigenvalues of $abc^s d^t$ would be degenerate for $t = 0$ such that the unitary transformation U_l can be pinned down uniquely. If that is the case, we could choose the residual symmetry to be $G_l = K_4^{(c^{n/2}, abc^s)}$ which leads to same PMNS matrix shown in eq. (3.81) with $t = 0$. Obviously this mixing pattern has one column $(1/2, 1/2, -1/\sqrt{2})^T$ which is the same as the first (second) column of the Bimaximal mixing up to permutations. In order to be in accordance with the experimental data, the fixed vector $(1/2, 1/2, -1/\sqrt{2})^T$ can only be the second column of the PMNS matrix. We can straightforwardly read out the predictions for the lepton mixing angles:

$$\sin^2 \theta_{13} = \frac{1}{8} \left(3 - \cos 2\theta - 2\sqrt{2} \sin 2\theta \cos \varphi_3 \right), \quad \sin^2 \theta_{12} = \frac{2}{5 + \cos 2\theta + 2\sqrt{2} \sin 2\theta \cos \varphi_3},$$

$$\begin{aligned}
 \sin^2 \theta_{23} &= \frac{3 - \cos 2\theta + 2\sqrt{2} \sin 2\theta \cos \varphi_3}{5 + \cos 2\theta + 2\sqrt{2} \sin 2\theta \cos \varphi_3}, & |J_{CP}| &= \frac{1}{8\sqrt{2}} |\sin 2\theta \sin \varphi_3|, \\
 |\tan \delta_{CP}| &= \left| \frac{8 \cos \theta \sin^2 \theta \sin 2\varphi_3 + \sqrt{2}(9 \sin \theta + \sin 3\theta) \sin \varphi_3}{4 \cos 3\theta + \cos \theta (4 - 8 \sin^2 \theta \cos 2\varphi_3) + \sqrt{2}(3 \sin 3\theta - 5 \sin \theta) \cos \varphi_3} \right|, \\
 |\tan \alpha_{21}| &= \left| \frac{2 \cos^2 \theta \sin 2\varphi_3 + \sqrt{2} \sin 2\theta \sin \varphi_3}{\sin^2 \theta + 2 \cos^2 \theta \cos 2\varphi_3 + \sqrt{2} \sin 2\theta \cos \varphi_3} \right|, \\
 |\tan \alpha'_{31}| &= \left| \frac{16 \cos 2\theta \sin 2\varphi_3 - 8\sqrt{2} \sin 2\theta \sin \varphi_3}{6 \sin^2 2\theta + 4\sqrt{2} \sin 4\theta \cos \varphi_3 - 4(3 + \cos 4\theta) \cos 2\varphi_3} \right|. \tag{3.84}
 \end{aligned}$$

The following correlation is satisfied:

$$4 \sin^2 \theta_{12} \cos^2 \theta_{13} = 1, \tag{3.85}$$

which leads to $0.254 \leq \sin^2 \theta_{12} \leq 0.258$ for the measured value of the reactor mixing angle [8]. Therefore $\sin^2 \theta_{12}$ is predicted to be very close to its 3σ lower bound 0.259 [8] in this case. As a consequence, we suggest that this mixing pattern is a good leading order approximation to the present neutrino mixing data. The reason is that the subleading contributions could easily pull θ_{12} into a experimentally more favored range. Furthermore, the expression for $\sin^2 \theta_{13}$ in eq. (3.84) yields

$$\frac{1}{8} \left(3 - \sqrt{1 + 8 \cos^2 \varphi_3} \right) \leq \sin^2 \theta_{13} \leq \frac{1}{8} \left(3 + \sqrt{1 + 8 \cos^2 \varphi_3} \right). \tag{3.86}$$

In order to be in accordance with experimental data, the parameter φ_3 has to be in the range

$$\varphi_3 \in [0, 0.135\pi] \cup [0.865\pi, 1.135\pi] \cup [1.865\pi, 2\pi]. \tag{3.87}$$

The allowed values of the mixing parameters with respect to n are shown in figure 8 and figure 9, and the correlations between them are plotted in figure 10, where the 3σ lower bound of $\sin^2 \theta_{12}$ is chosen to be 0.254 instead of 0.259 given in ref. [8]. The values of $\varphi_3 = 0, \pi$ are always acceptable, and the corresponding Dirac and Majorana CP phases are conserved. Note that only the CP conserved cases are viable for $n = 2, 3, \dots, 7$. Moreover the CP violating phases δ_{CP} and α_{21} are predicted to fulfill $|\sin \delta_{CP}| \leq 0.895$ and $|\sin \alpha_{21}| \leq 0.545$ while α'_{31} is not constrained at all for large n .

(III) $G_l = \langle ac^s d^t \rangle$, $G_\nu = Z_2^{c^{n/2}}$, $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(c^\gamma d^\delta)$

This case can be realizable if n is divisible by 2, and the PMNS matrix takes the form

$$U_{PMNS}^{III} = \frac{1}{\sqrt{3}} \begin{pmatrix} e^{i\varphi_4} \cos \theta - e^{i\varphi_5} \sin \theta & 1 & e^{i\varphi_4} \sin \theta + e^{i\varphi_5} \cos \theta \\ \omega e^{i\varphi_4} \cos \theta - \omega^2 e^{i\varphi_5} \sin \theta & 1 & \omega e^{i\varphi_4} \sin \theta + \omega^2 e^{i\varphi_5} \cos \theta \\ \omega^2 e^{i\varphi_4} \cos \theta - \omega e^{i\varphi_5} \sin \theta & 1 & \omega^2 e^{i\varphi_4} \sin \theta + \omega e^{i\varphi_5} \cos \theta \end{pmatrix}, \tag{3.88}$$

where

$$\varphi_4 = \frac{\gamma + \delta + 2s}{n} \pi, \quad \varphi_5 = \frac{2\delta - \gamma + 2t}{n} \pi, \tag{3.89}$$

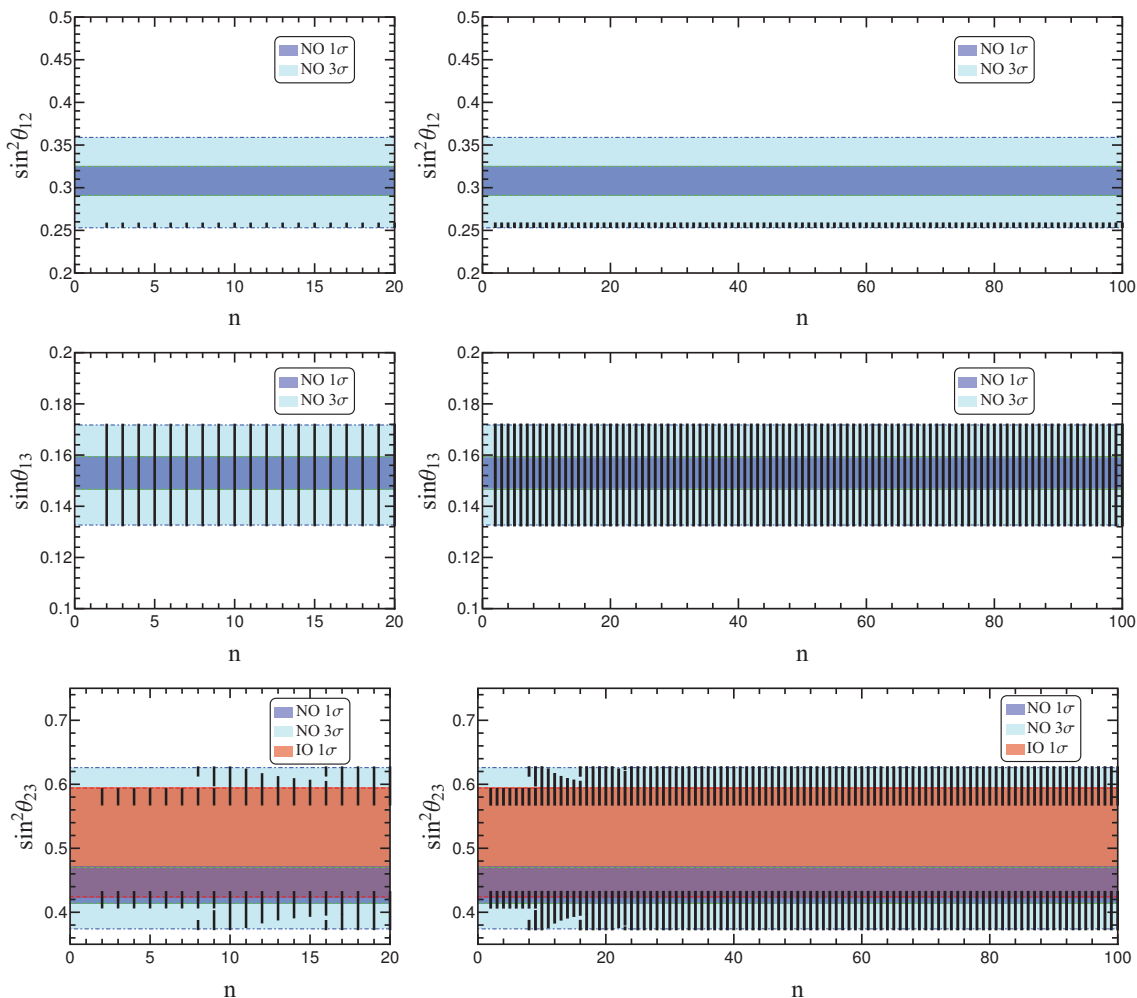


Figure 8. Numerical results in case II: the allowed values of $\sin^2 \theta_{12}$, $\sin \theta_{13}$ and $\sin^2 \theta_{23}$ for different n , where the three lepton mixing angles are required to lie in their 3σ ranges (the 3σ lower bound of $\sin^2 \theta_{12}$ is chosen to be 0.254 instead of 0.259 given in ref. [8]). The 1σ and 3σ bounds of the mixing parameters are taken from ref. [8].

which can take the values

$$\varphi_4, \varphi_5 \bmod 2\pi = 0, \frac{1}{n}\pi, \frac{2}{n}\pi, \dots, \frac{2n-1}{n}\pi. \quad (3.90)$$

Agreement with experimental data can be achieved only if the vector $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})^T$ is placed in the second column. It is the so-called TM_2 mixing [70]. There are three independent arrangements up to the exchange of the second and the third row,

$$U_{PMNS}^{III,1st} = U_{PMNS}^{III}, \quad U_{PMNS}^{III,2nd} = P_{231} U_{PMNS}^{III}, \quad U_{PMNS}^{III,3rd} = P_{312} U_{PMNS}^{III}. \quad (3.91)$$

Once can check that they are related as follows,

$$U_{PMNS}^{III,2nd}(\theta, \varphi_4, \varphi_5) = U_{PMNS}^{III,1st}\left(\theta, \varphi_4 + \frac{2\pi}{3}, \varphi_5 - \frac{2\pi}{3}\right),$$

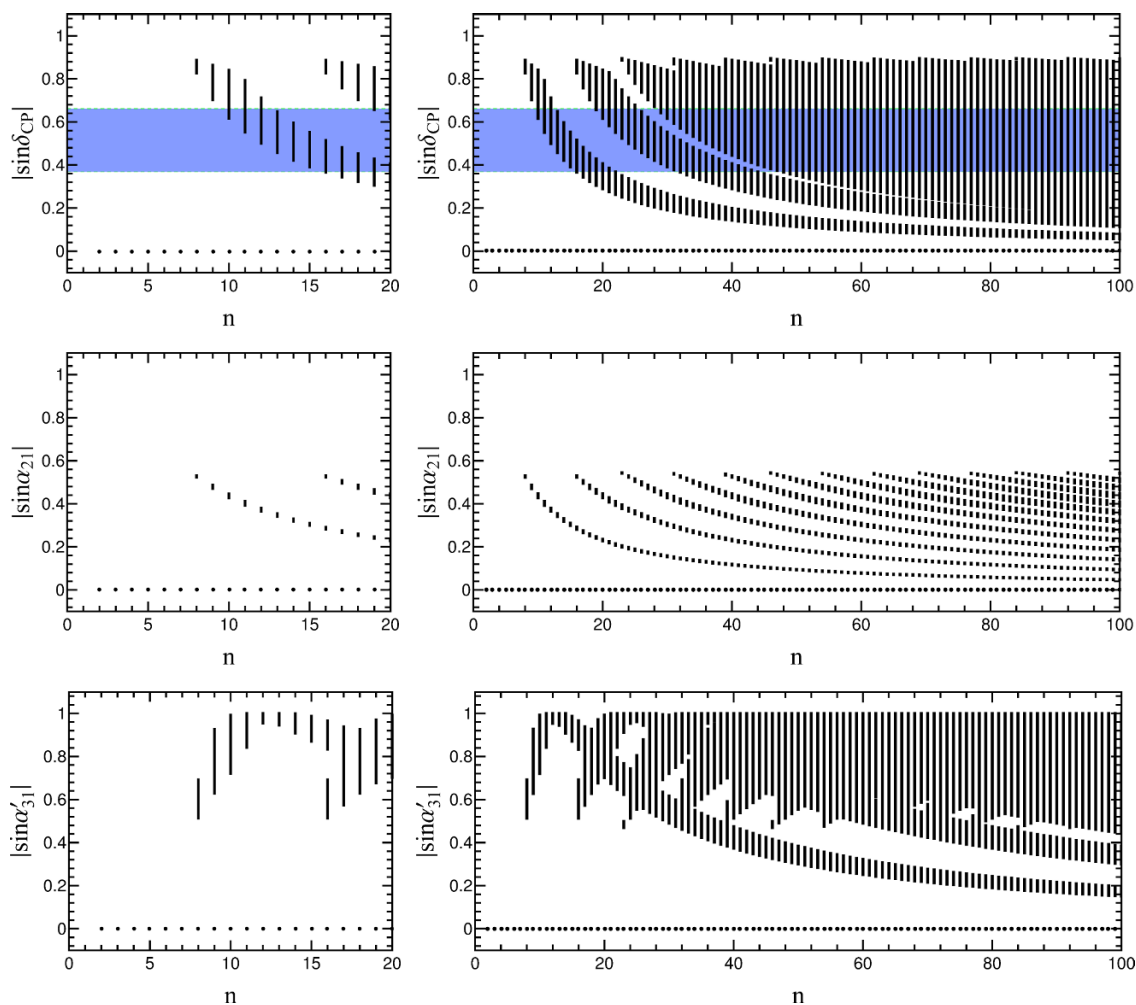


Figure 9. Numerical results in case II: the allowed values of $|\sin \delta_{CP}|$, $|\sin \alpha_{21}|$ and $|\sin \alpha'_{31}|$ for different n , where the three lepton mixing angles are required to lie in their 3σ ranges (the 3σ lower bound of $\sin^2 \theta_{12}$ is chosen to be 0.254 instead of 0.259 given in ref. [8]). The 1σ and 3σ bounds of the mixing parameters are taken from ref. [8].

$$U_{PMNS}^{III,3rd}(\theta, \varphi_4, \varphi_5) = U_{PMNS}^{III,1st}\left(\theta, \varphi_4 - \frac{2\pi}{3}, \varphi_5 + \frac{2\pi}{3}\right). \quad (3.92)$$

It is enough to study the phenomenological predictions of $U_{PMNS}^{III,1st}$. The lepton mixing parameters are given by

$$\begin{aligned} \sin^2 \theta_{13} &= \frac{1}{3} [1 + \sin 2\theta \cos(\varphi_5 - \varphi_4)], & \sin^2 \theta_{12} &= \frac{1}{2 - \sin 2\theta \cos(\varphi_5 - \varphi_4)}, \\ \sin^2 \theta_{23} &= \frac{1 - \sin 2\theta \sin(\varphi_5 - \varphi_4 + \pi/6)}{2 - \sin 2\theta \cos(\varphi_5 - \varphi_4)}, & |J_{CP}| &= \frac{1}{6\sqrt{3}} |\cos 2\theta|, \\ |\tan \delta_{CP}| &= \left| \frac{\cot 2\theta [2 - \sin 2\theta \cos(\varphi_5 - \varphi_4)]}{\sin(\varphi_5 - \varphi_4) - \sin 2\theta \sin(2\varphi_5 - 2\varphi_4)} \right|, \end{aligned}$$

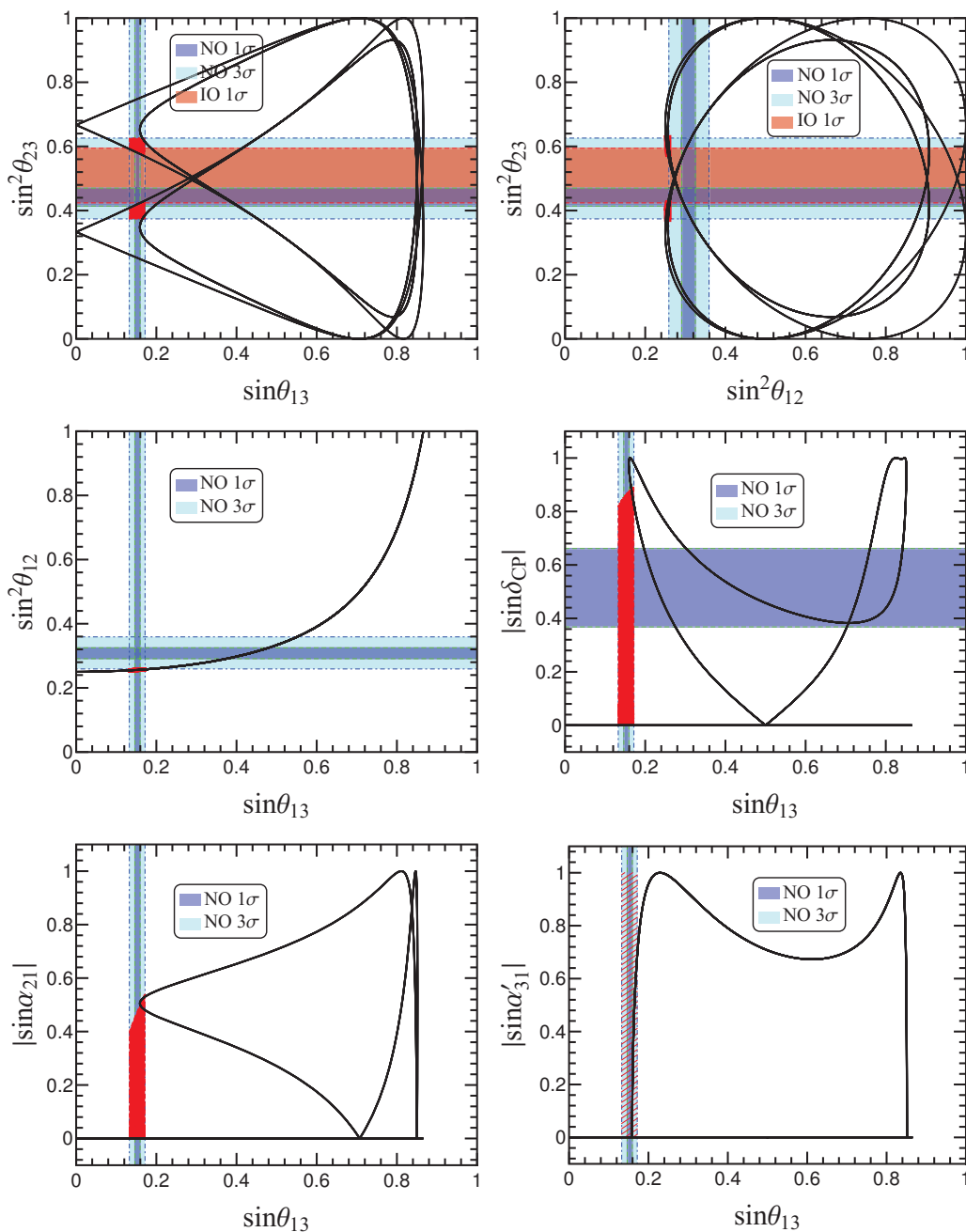


Figure 10. The correlations among mixing parameters in case II. The red filled regions denote the allowed values of the mixing parameters if we take the parameter φ_3 to be continuous (which is equivalent to taking the limit $n \rightarrow \infty$) and the three mixing angles are required to lie in their 3σ ranges (the 3σ lower bound of $\sin^2 \theta_{12}$ is chosen to be 0.254 instead of 0.259 given in ref. [8]). Note that the Majorana phase α'_{31} is not constrained in this limit. The black curves represent the phenomenologically viable correlations for $n = 8$. The 1σ and 3σ bounds of the mixing parameters are taken from ref. [8].

$$\begin{aligned}
 |\tan \alpha_{21}| &= \left| \frac{\cos^2 \theta \sin 2\varphi_4 + \sin^2 \theta \sin 2\varphi_5 - \sin 2\theta \sin(\varphi_5 + \varphi_4)}{\cos^2 \theta \cos 2\varphi_4 + \sin^2 \theta \cos 2\varphi_5 - \sin 2\theta \cos(\varphi_5 + \varphi_4)} \right|, \\
 |\tan \alpha'_{31}| &= \left| \frac{4 \cos 2\theta \sin(2\varphi_5 - 2\varphi_4)}{1 - 3 \cos(2\varphi_5 - 2\varphi_4) - 2 \cos 4\theta \cos^2(\varphi_5 - \varphi_4)} \right|. \tag{3.93}
 \end{aligned}$$

We see that all the mixing parameters depend on the combination $\varphi_5 - \varphi_4$ except $|\tan \alpha_{21}|$. Common to all TM_2 mixing, θ_{13} and θ_{12} are related with each other via:

$$3 \cos^2 \theta_{13} \sin^2 \theta_{12} = 1. \tag{3.94}$$

Therefore θ_{12} admits a lower bound $\sin^2 \theta_{12} > 1/3$. Given the 3σ interval of θ_{13} [8], we find $0.339 \leq \sin^2 \theta_{12} \leq 0.343$. This prediction can be tested at JUNO in near future. In addition, θ_{13} and θ_{23} are correlated as follows

$$\frac{3 \cos^2 \theta_{13} \sin^2 \theta_{23} - 1}{1 - 3 \sin^2 \theta_{13}} = \frac{1}{2} + \frac{\sqrt{3}}{2} \tan(\varphi_5 - \varphi_4). \tag{3.95}$$

The expression for θ_{13} in eq. (3.93) implies that

$$\begin{aligned}
 \frac{1}{3}(1 - |\sin 2\theta|) &\leq \sin^2 \theta_{13} \leq \frac{1}{3}(1 + |\sin 2\theta|), \\
 \frac{1}{3}(1 - |\cos(\varphi_5 - \varphi_4)|) &\leq \sin^2 \theta_{13} \leq \frac{1}{3}(1 + |\cos(\varphi_5 - \varphi_4)|), \tag{3.96}
 \end{aligned}$$

which yields

$$\begin{aligned}
 \theta &\in [0.183\pi, 0.317\pi] \cup [0.683\pi, 0.817\pi], \\
 \varphi_5 - \varphi_4 &\in [-0.135\pi, 0.135\pi] \cup [0.865\pi, 1.135\pi]. \tag{3.97}
 \end{aligned}$$

The allowed values of the mixing parameters for different n are shown in figure 11 and figure 12. The case of $\varphi_4 = \varphi_5$ is always viable for any n , and the resulting θ_{23} and δ_{CP} are maximal while the Majorana phase α'_{31} is trivial. Correlations among the mixing parameters are plotted in figure 13. The three CP phases can take any values for large n .

(IV) $G_l = \langle ac^s d^t \rangle$, $G_\nu = Z_2^{c^{n/2}}$, $X_{\nu r} = \rho_r(abc^\gamma d^\delta)$

In this case, the PMNS matrix is of the form

$$U_{PMNS}^{IV} = \frac{1}{\sqrt{3}} \begin{pmatrix} i\sqrt{2}e^{i\varphi_7} \sin\left(\varphi_6 - \frac{\phi}{2}\right) & 1 & \sqrt{2}e^{i\varphi_7} \cos\left(\varphi_6 - \frac{\phi}{2}\right) \\ i\sqrt{2}e^{i\varphi_7} \cos\left(\varphi_6 - \frac{\phi}{2} + \frac{\pi}{6}\right) & 1 & -\sqrt{2}e^{i\varphi_7} \sin\left(\varphi_6 - \frac{\phi}{2} + \frac{\pi}{6}\right) \\ -i\sqrt{2}e^{i\varphi_7} \cos\left(\varphi_6 - \frac{\phi}{2} - \frac{\pi}{6}\right) & 1 & \sqrt{2}e^{i\varphi_7} \sin\left(\varphi_6 - \frac{\phi}{2} - \frac{\pi}{6}\right) \end{pmatrix}, \tag{3.98}$$

with

$$\varphi_6 = \frac{s - t - \gamma}{n}\pi, \quad \varphi_7 = \frac{s + t + 3\gamma}{n}\pi. \tag{3.99}$$

The constant vector $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})^T$ must be the second column to account for the measured values of the lepton mixing angles. The PMNS matrices corresponding to other ordering of rows and columns are related to the above one through redefinition of the

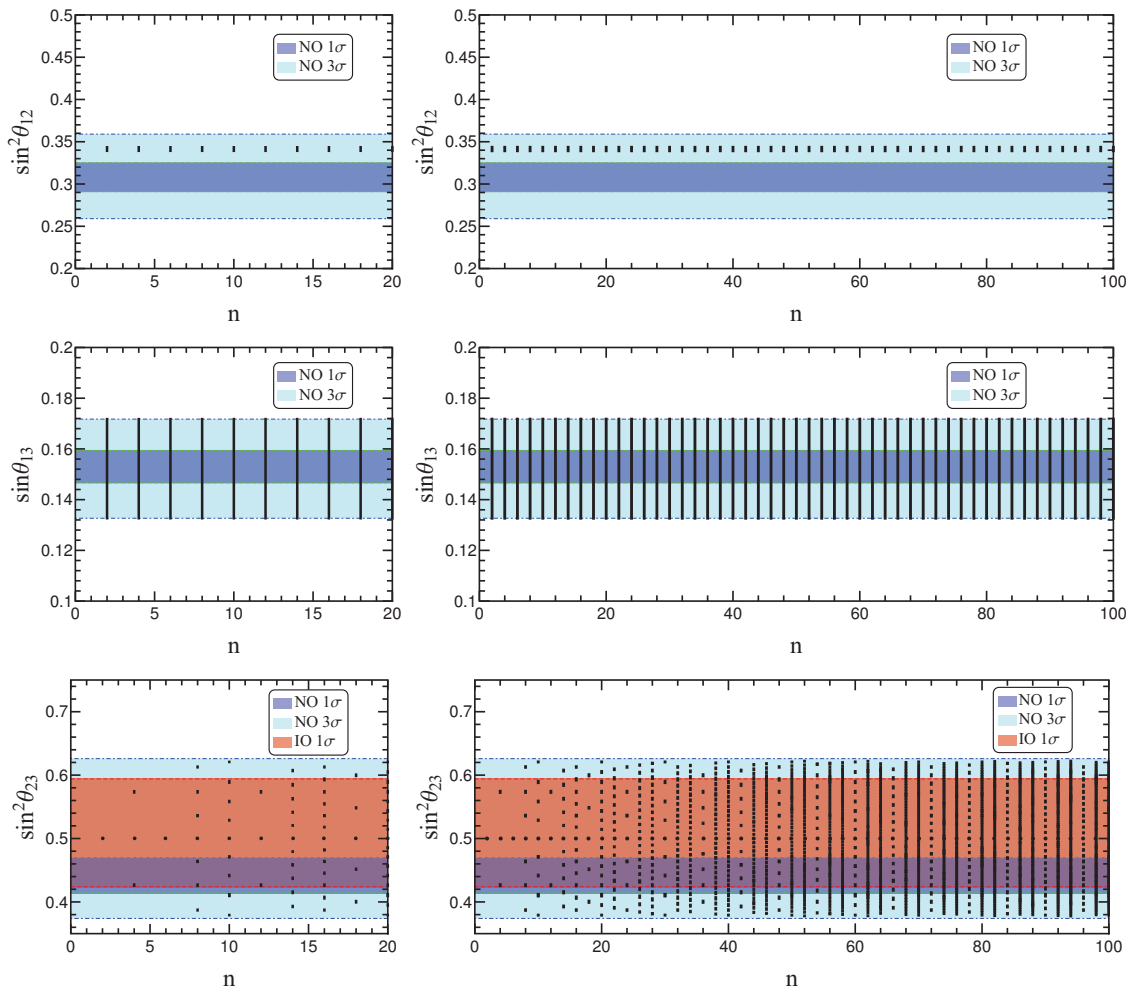


Figure 11. Numerical results in case III: the allowed values of $\sin^2 \theta_{12}$, $\sin \theta_{13}$ and $\sin^2 \theta_{23}$ for different n , where the three lepton mixing angles are required to lie in their 3σ regions. The 1σ and 3σ bounds of the mixing parameters are taken from ref. [8]. Note that n should be even in this case.

free parameter ϕ . This case differs from case III in the remnant CP symmetry, and the resulting PMNS matrix in eq. (3.98) is still of TM_2 form. The associated lepton mixing parameters read as:

$$\begin{aligned}
 \sin^2 \theta_{13} &= \frac{1}{3} [1 + \cos(\phi - 2\varphi_6)], & \sin^2 \theta_{12} &= \frac{1}{2 - \cos(\phi - 2\varphi_6)}, \\
 \sin^2 \theta_{23} &= \frac{1 - \sin(\phi - 2\varphi_6 + \pi/6)}{2 - \cos(\phi - 2\varphi_6)}, \\
 \tan \delta_{CP} = \tan \alpha'_{31} = J_{CP} &= 0, & |\tan \alpha_{21}| &= |\tan(2\varphi_7)|.
 \end{aligned}
 \tag{3.100}$$

It is remarkable that the contribution of φ_6 can be absorbed into the free parameter ϕ via redefinition $\phi \rightarrow \phi + 2\varphi_6$, the reason is that the PMNS matrix in eq. (3.98) and the resulting mixing parameters in eq. (3.100) depend on the combination $\phi - 2\varphi_6$. Regarding to the CP violating phases, both δ_{CP} and α'_{31} are always conserved while α_{21} can be any

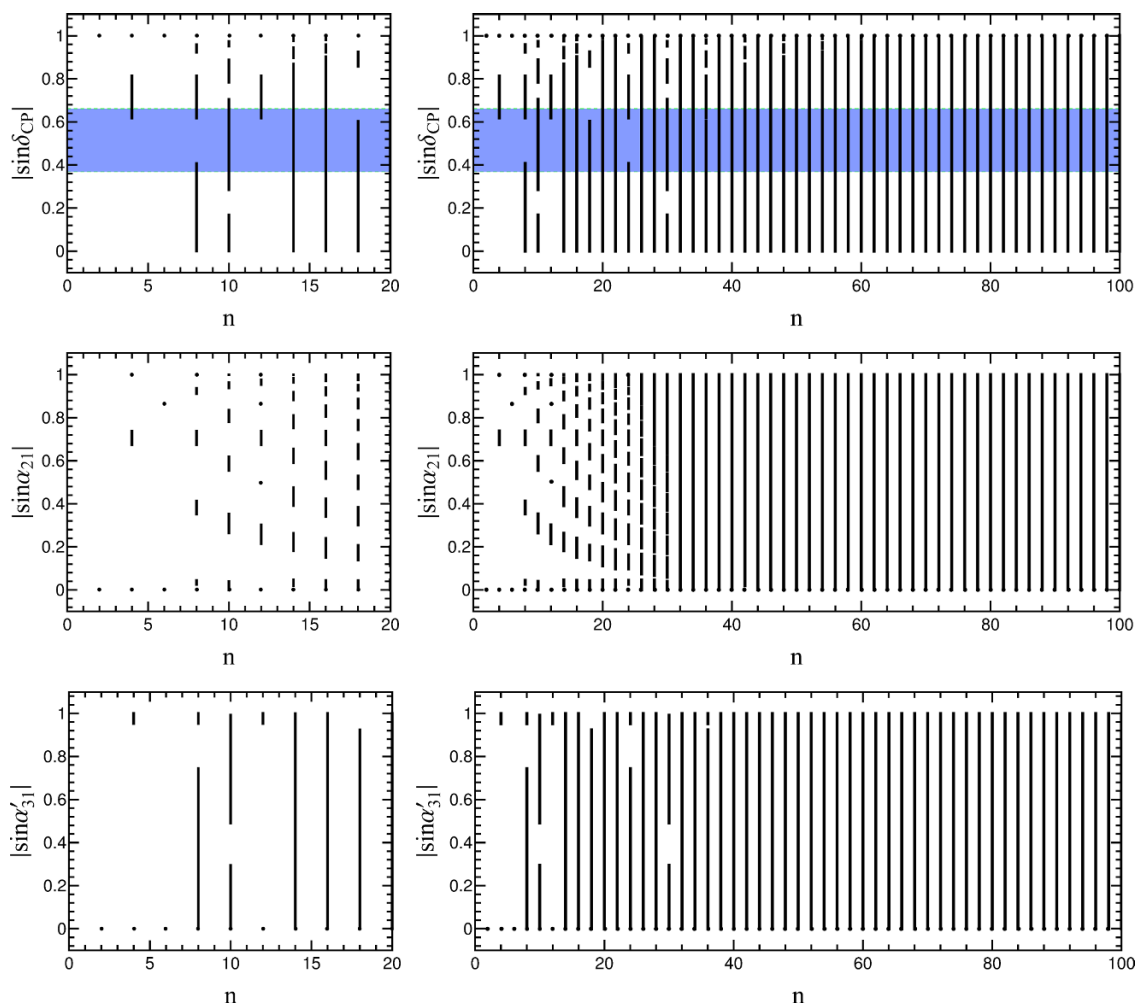


Figure 12. Numerical results in case III: the allowed values of $|\sin \delta_{CP}|$, $|\sin \alpha_{21}|$ and $|\sin \alpha'_{31}|$ for different n , where the three lepton mixing angles are required to lie in their 3σ ranges. The 1σ and 3σ bounds of the mixing angles are taken from ref. [8]. Note that n should be even in this case.

value of $0, \frac{1}{n}\pi, \frac{2}{n}\pi, \dots, \frac{2n-1}{n}\pi$ in this scenario. Furthermore, the three mixing angles are strongly related with each other as follows:

$$3 \cos^2 \theta_{13} \sin^2 \theta_{12} = 1, \quad \sin^2 \theta_{23} = \frac{1}{2} \pm \frac{1}{2} \tan \theta_{13} \sqrt{2 - \tan^2 \theta_{13}}. \quad (3.101)$$

For the best fitting value of $\sin^2 \theta_{13} = 0.0234$ [8], the solar and atmospheric angles are determined to be

$$\sin^2 \theta_{12} \simeq 0.341, \quad \sin^2 \theta_{23} \simeq 0.391 \text{ or } 0.609, \quad (3.102)$$

which are compatible with the experimentally allowed regions. These correlations between the three mixing angles are shown in figure 14. We see that both θ_{12} and θ_{23} are constrained to be in a narrow range. The deviation of θ_{23} from maximal mixing is somewhat

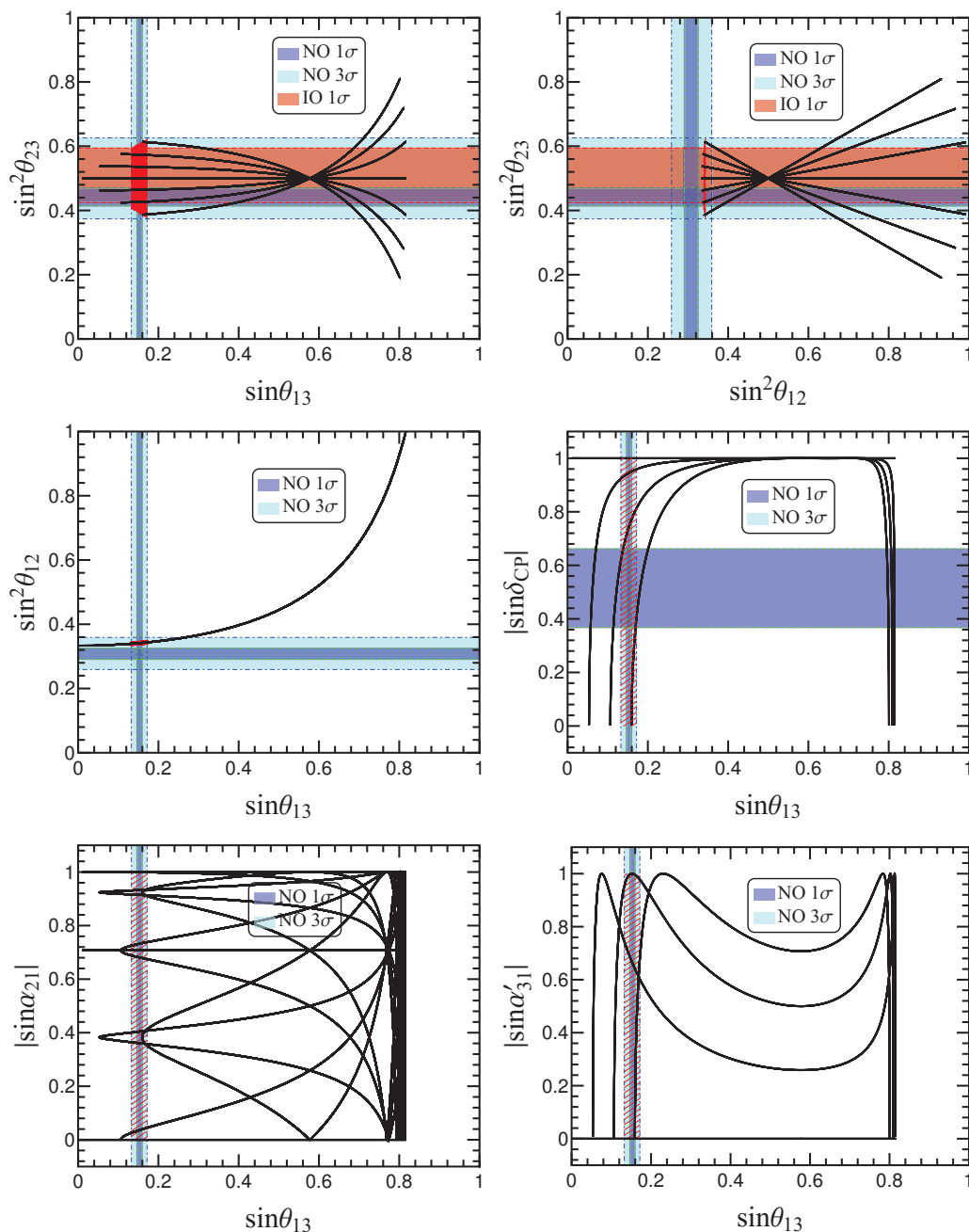


Figure 13. The correlations among mixing parameters in case III. The red filled regions denote the allowed values of the mixing parameters if we take the parameters φ_4 and φ_5 to be continuous (which is equivalent to taking the limit $n \rightarrow \infty$) and the three mixing angles are required to lie in their 3σ ranges. Note that the three CP phases δ_{CP} , α_{21} and α'_{31} are not constrained in this limit. The black curves represent the phenomenologically viable correlations for $n = 8$. The 1σ and 3σ bounds of the mixing parameters are taken from ref. [8].

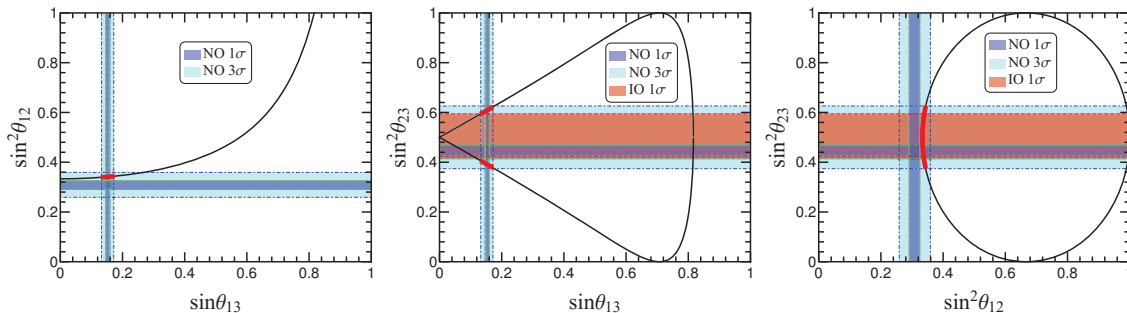


Figure 14. The correlations among mixing parameters in case IV. The red filled regions denote the allowed values of the mixing parameters if we take the parameters φ_6 and φ_7 to be continuous (which is equivalent to taking the limit $n \rightarrow \infty$), where θ_{12} and θ_{13} are required to lie in their 3σ ranges. The 1σ and 3σ bounds of the mixing parameters are taken from ref. [8].

large. Hence this mixing pattern can be checked or ruled out by precisely measuring θ_{12} and θ_{23} in next generation neutrino oscillation experiments.

4 Lepton mixing from “semi-direct” approach with residual symmetry $Z_2 \times CP$ in the charged lepton sector

In the previous section, we assumed a $Z_2 \times CP$ remnant symmetry in the neutrino sector and that an abelian subgroup of $\Delta(6n^2)$ is preserved in the charged lepton sector. In this section, we shall investigate another scenario: the remnant symmetry $Z_2 \times CP$ is preserved in the charged lepton sector and the full symmetry $\Delta(6n^2) \times H_{CP}$ is broken down to $K_4 \times H_{CP}^\nu$ in the neutrino sector. The phenomenological consequences of this scenario have been analyzed for the simple flavor symmetry group $\Delta(24) = S_4$ in ref. [31]. All the Z_2 subgroups of $\Delta(6n^2)$ have been listed in eq. (3.12) and eq. (3.13). The K_4 subgroups of $\Delta(6n^2)$ are classified as follows:

$$\begin{aligned}
 K_4^{(c^{n/2}, d^{n/2})} &\equiv \left\{ 1, c^{n/2}, d^{n/2}, c^{n/2}d^{n/2} \right\}, \\
 K_4^{(c^{n/2}, abc^y)} &\equiv \left\{ 1, c^{n/2}, abc^y, abc^{y+n/2} \right\}, \\
 K_4^{(d^{n/2}, a^2bd^z)} &\equiv \left\{ 1, d^{n/2}, a^2bd^z, a^2bd^{z+n/2} \right\}, \\
 K_4^{(c^{n/2}d^{n/2}, bc^x d^x)} &\equiv \left\{ 1, c^{n/2}d^{n/2}, bc^x d^x, bc^{x+n/2}d^{x+n/2} \right\},
 \end{aligned} \tag{4.1}$$

where $K_4^{(c^{n/2}, d^{n/2})}$ is a normal subgroup of $\Delta(6n^2)$, and the remaining three K_4 subgroups are conjugate to each other. Obviously this scenario is only possible if n is divisible by 2. Because of the conjugate relations shown in eq. (3.14a) and eq. (3.14b), we only need to consider the representative cases of $G_l = Z_2^{bc^x d^x}, Z_2^{c^{n/2}}$ and $G_\nu = K_4^{(c^{n/2}, d^{n/2})}, K_4^{(c^{n/2}, abc^y)}, K_4^{(d^{n/2}, a^2bd^z)}$ and $K_4^{(c^{n/2}d^{n/2}, bc^x d^x)}$. Other possible choices of G_l and G_ν are related to these representative residual symmetry by similarity transformations, and therefore the same lepton mixing matrices are generated.

Following the same procedure demonstrated in section 3, the hermitian combination $m_l^\dagger m_l$ of the charged lepton mass matrix and its diagonalization matrix can be straightforwardly calculated from the invariance under the remnant symmetry although the involved algebraic calculations are somewhat lengthy and tedious. To avoid distractions by too many details, we have left the calculations for this to appendix C. Comparing with the scenario of $Z_2 \times H_{CP}'$ preserved in the neutrino sector which has been studied in section 3.2, we find that the unitary transformation U_l is of the same form as U_ν listed in section 3.2 if both residual flavor symmetry and residual CP symmetry in the two occasions are identical. For the sake of presentation in the following, we briefly recapitulate the predicted form of U_l for $G_l = Z_2^{bc^x d^x}$ and $Z_2^{c^{n/2}}$ here.

In case of $G_l = Z_2^{bc^x d^x}$, $X_{l\mathbf{r}} = \{\rho_{\mathbf{r}}(c^\gamma d^{-2x-\gamma}), \rho_{\mathbf{r}}(bc^{x+\gamma} d^{-x-\gamma})\}$, as shown in appendix C, the charged lepton mass matrix is diagonalized by

$$U_l = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\pi\frac{\gamma}{n}} & -e^{i\pi\frac{\gamma}{n}} \sin \theta & e^{i\pi\frac{\gamma}{n}} \cos \theta \\ 0 & e^{-2i\pi\frac{x+\gamma}{n}} \sqrt{2} \cos \theta & e^{-2i\pi\frac{x+\gamma}{n}} \sqrt{2} \sin \theta \\ -e^{i\pi\frac{2x+\gamma}{n}} & -e^{i\pi\frac{2x+\gamma}{n}} \sin \theta & e^{i\pi\frac{2x+\gamma}{n}} \cos \theta \end{pmatrix}. \quad (4.2)$$

For the residual symmetry $G_l = Z_2^{c^{n/2}}$, $X_{l\mathbf{r}} = \rho_{\mathbf{r}}(c^\gamma d^\delta)$, the unitary matrix U_l is of the form

$$U_l = \begin{pmatrix} e^{i\pi\frac{\gamma}{n}} \cos \theta & e^{i\pi\frac{\gamma}{n}} \sin \theta & 0 \\ -e^{i\pi\frac{\delta-\gamma}{n}} \sin \theta & e^{i\pi\frac{\delta-\gamma}{n}} \cos \theta & 0 \\ 0 & 0 & e^{-i\pi\frac{\delta}{n}} \end{pmatrix}. \quad (4.3)$$

The remnant symmetry could also be $G_l = Z_2^{c^{n/2}}$, $X_{l\mathbf{r}} = \rho_{\mathbf{r}}(abc^\gamma d^\delta)$. The non-degeneracy of the charged lepton masses requires $\delta = 2\gamma \pmod n$, and then the diagonalization matrix U_l is determined to be

$$U_l = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi} & e^{i\phi} & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}, \quad (4.4)$$

where ϕ is a free real parameter.

4.1 Neutrino sector

In this section, we shall assume that the $\Delta(6n^2)$ flavor symmetry is broken down to K_4 in the neutrino sector. Hence the neutrino diagonalization matrix U_ν is entirely fixed by the remnant K_4 , and the residual CP symmetry allows us to further determine the three leptonic CP violating phases up to π . The residual CP symmetry H_{CP}' in the neutrino sector must be compatible with the remnant K_4 symmetry, and the consistency condition should be satisfied,

$$X_{\nu\mathbf{r}} \rho_{\mathbf{r}}^*(g) X_{\nu\mathbf{r}}^{-1} = \rho_{\mathbf{r}}(g'), \quad g, g' \in K_4. \quad (4.5)$$

Solving this equation, we can find the consistent remnant CP symmetries for different K_4 subgroups are as follows:

- $K_4^{(c^{n/2}, d^{n/2})}$

$$X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(h), \quad h \in \Delta(6n^2). \quad (4.6)$$

- $K_4^{(c^{n/2}, abc^y)}$, $y = 0, 1, \dots, n-1$

$$X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(c^\gamma d^{2y+2\gamma}), \rho_{\mathbf{r}}(c^\gamma d^{2y+2\gamma+n/2}), \rho_{\mathbf{r}}(abc^\gamma d^{2\gamma}), \rho_{\mathbf{r}}(abc^\gamma d^{2\gamma+n/2}), \quad (4.7)$$

with $\gamma = 0, 1, \dots, n-1$.

- $K_4^{(d^{n/2}, a^2bd^z)}$, $z = 0, 1, \dots, n-1$

$$X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(c^{2z+2\delta} d^\delta), \rho_{\mathbf{r}}(c^{2z+2\delta+n/2} d^\delta), \rho_{\mathbf{r}}(a^2bc^{2\delta} d^\delta), \rho_{\mathbf{r}}(a^2bc^{2\delta+n/2} d^\delta), \quad (4.8)$$

where $\delta = 0, 1, \dots, n-1$.

- $K_4^{(c^{n/2}d^{n/2}, bc^x d^x)}$, $x = 0, 1, \dots, n-1$

$$X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(c^\gamma d^{-2x-\gamma}), \rho_{\mathbf{r}}(c^\gamma d^{-2x-\gamma+n/2}), \rho_{\mathbf{r}}(bc^\gamma d^{-\gamma}), \rho_{\mathbf{r}}(bc^\gamma d^{-\gamma+n/2}), \quad (4.9)$$

with $\gamma = 0, 1, \dots, n-1$.

In this case, the light neutrino mass matrix must be subject to the constraints from the remnant family symmetry K_4 and the residual CP symmetry H_{CP}^ν :

$$\begin{aligned} \rho_{\mathbf{3}}^T(g_\nu) m_\nu \rho_{\mathbf{3}}(g_\nu) &= m_\nu, & g_\nu &\in K_4, \\ X_{\nu\mathbf{3}}^T m_\nu X_{\nu\mathbf{3}} &= m_\nu^*, & X_\nu &\in H_{CP}^\nu. \end{aligned} \quad (4.10)$$

(i) $G_\nu = K_4^{(c^{n/2}, d^{n/2})}$, $X_{\nu\mathbf{r}} = \{\rho_{\mathbf{r}}(c^\gamma d^\delta)\}$ Since the representation matrices of both $c^{n/2}$ and $d^{n/2}$ are diagonal, the light neutrino mass matrix is constrained to be diagonal as well. Including the remnant CP symmetry, we find

$$m_\nu = \begin{pmatrix} m_{11} e^{-2i\pi \frac{\gamma}{n}} & 0 & 0 \\ 0 & m_{22} e^{2i\pi \frac{\gamma-\delta}{n}} & 0 \\ 0 & 0 & m_{33} e^{2i\pi \frac{\delta}{n}} \end{pmatrix}, \quad (4.11)$$

where m_{11} , m_{22} and m_{33} are real parameters. The neutrino diagonalization matrix can be easily read out

$$U_\nu = \text{diag} \left(e^{i\pi \frac{\gamma}{n}}, e^{-i\pi \frac{\gamma-\delta}{n}}, e^{-i\pi \frac{\delta}{n}} \right) K_\nu, \quad (4.12)$$

where K_ν is a diagonal matrix with element ± 1 or $\pm i$ to set the light neutrino masses being positive. The light neutrino masses are

$$m_1 = |m_{11}|, \quad m_2 = |m_{22}|, \quad m_3 = |m_{33}|. \quad (4.13)$$

We see that the light neutrino masses depend on only three real parameters, and we would like to stress again that the order of the light neutrino masses can not be fixed here, and therefore U_ν here and henceforth is determined up to column permutations. For other residual CP symmetries $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(bc^\gamma d^\delta)$, $\rho_{\mathbf{r}}(ac^\gamma d^\delta)$, $\rho_{\mathbf{r}}(a^2c^\gamma d^\delta)$, $\rho_{\mathbf{r}}(abc^\gamma d^\delta)$ and $\rho_{\mathbf{r}}(a^2bc^\gamma d^\delta)$ with $\gamma, \delta = 0, 1, \dots, n-1$, the light neutrino masses are partially degenerate such that they are not viable.

(ii) $G_\nu = K_4^{(c^{n/2}, abc^y)}$, $X_{\nu\mathbf{r}} = \{\rho_{\mathbf{r}}(c^\gamma d^{2y+2\gamma}), \rho_{\mathbf{r}}(abc^{y+\gamma} d^{2y+2\gamma})\}$ In this case, the light neutrino mass matrix takes the form

$$m_\nu = \begin{pmatrix} m_{11}e^{-2i\pi\frac{\gamma}{n}} & m_{12}e^{-2i\pi\frac{y+\gamma}{n}} & 0 \\ m_{12}e^{-2i\pi\frac{y+\gamma}{n}} & m_{11}e^{-2i\pi\frac{2y+\gamma}{n}} & 0 \\ 0 & 0 & m_{33}e^{4i\pi\frac{y+\gamma}{n}} \end{pmatrix}, \quad (4.14)$$

where m_{11} , m_{12} and m_{33} are real. It is diagonalized by the unitary matrix U_ν with

$$U_\nu = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\pi\frac{\gamma}{n}} & e^{i\pi\frac{\gamma}{n}} & 0 \\ -e^{i\pi\frac{2y+\gamma}{n}} & e^{i\pi\frac{2y+\gamma}{n}} & 0 \\ 0 & 0 & \sqrt{2}e^{-2i\pi\frac{y+\gamma}{n}} \end{pmatrix}. \quad (4.15)$$

The light neutrino masses are given by

$$m_1 = |m_{11} - m_{12}|, \quad m_2 = |m_{11} + m_{12}|, \quad m_3 = |m_{33}|. \quad (4.16)$$

For the case of $X_{\nu\mathbf{r}} = \{\rho_{\mathbf{r}}(c^\gamma d^{2y+2\gamma+n/2}), \rho_{\mathbf{r}}(abc^{y+\gamma} d^{2y+2\gamma+n/2})\}$, the light neutrino masses are degenerate, and therefore are not discussed here.

(iii) $G_\nu = K_4^{(d^{n/2}, a^2bd^z)}$, $X_{\nu\mathbf{r}} = \{\rho_{\mathbf{r}}(c^{2z+2\delta} d^\delta), \rho_{\mathbf{r}}(a^2bc^{2z+2\delta} d^{z+\delta})\}$ The light neutrino mass matrix, which is invariant under both residual flavor and residual CP symmetry, is determined to be

$$m_\nu = \begin{pmatrix} m_{11}e^{-4i\pi\frac{z+\delta}{n}} & 0 & 0 \\ 0 & m_{22}e^{2i\pi\frac{2z+\delta}{n}} & m_{23}e^{2i\pi\frac{z+\delta}{n}} \\ 0 & m_{23}e^{2i\pi\frac{z+\delta}{n}} & m_{22}e^{2i\pi\frac{\delta}{n}} \end{pmatrix}, \quad (4.17)$$

where m_{11} , m_{22} and m_{23} are real. The unitary matrix U_ν is

$$U_\nu = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}e^{2i\pi\frac{z+\delta}{n}} & 0 & 0 \\ 0 & e^{-i\pi\frac{2z+\delta}{n}} & e^{-i\pi\frac{2z+\delta}{n}} \\ 0 & -e^{-i\pi\frac{\delta}{n}} & e^{-i\pi\frac{\delta}{n}} \end{pmatrix}. \quad (4.18)$$

The light neutrino mass eigenvalues are given by

$$m_1 = |m_{11}|, \quad m_2 = |m_{22} - m_{23}|, \quad m_3 = |m_{22} + m_{23}|. \quad (4.19)$$

For the value of $X_{\nu\mathbf{r}} = \{\rho_{\mathbf{r}}(c^{2z+2\delta+n/2} d^\delta), \rho_{\mathbf{r}}(a^2bc^{2z+2\delta+n/2} d^{z+\delta})\}$, the neutrino masses are degenerate.

(iv) $G_\nu = K_4^{(c^{n/2}d^{n/2}, bc^x d^x)}$, $X_{\nu\mathbf{r}} = \{\rho_{\mathbf{r}}(c^\gamma d^{-2x-\gamma}), \rho_{\mathbf{r}}(bc^{x+\gamma} d^{-x-\gamma})\}$ In this case, we find the light neutrino mass matrix is of the form

$$m_\nu = \begin{pmatrix} m_{11}e^{-2i\pi\frac{\gamma}{n}} & 0 & m_{13}e^{-2i\pi\frac{x+\gamma}{n}} \\ 0 & m_{22}e^{4i\pi\frac{x+\gamma}{n}} & 0 \\ m_{13}e^{-2i\pi\frac{x+\gamma}{n}} & 0 & m_{11}e^{-2i\pi\frac{2x+\gamma}{n}} \end{pmatrix}, \quad (4.20)$$

	$G_l = Z_2^{bc^{x'} d^{x'}}$	$G_l = Z_2^{c^{n/2}}$
$G_\nu = K_4^{(c^{n/2}, d^{n/2})}$	$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}^T \quad \boldsymbol{\times}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^T \quad \boldsymbol{\times}$
$G_\nu = K_4^{(c^{n/2}, abc^y)}$	$\frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -\sqrt{2} \end{pmatrix}^T \quad \checkmark$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^T \quad \boldsymbol{\times}$
$G_\nu = K_4^{(d^{n/2}, a^2 bd^z)}$	$\frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -\sqrt{2} \end{pmatrix}^T \quad \checkmark$	$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}^T \quad \boldsymbol{\times}$
$G_\nu = K_4^{(c^{n/2} d^{n/2}, bc^x d^x)}$	$\begin{pmatrix} \cos\left(\frac{x-x'}{n}\pi\right) \\ -i \sin\left(\frac{x-x'}{n}\pi\right) \\ 0 \end{pmatrix}^T \quad \boldsymbol{\times}$	$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}^T \quad \boldsymbol{\times}$

Table 3. The determined form of one row of the PMNS matrix for different remnant symmetries G_ν and G_l which are K_4 and Z_2 subgroups of $\Delta(6n^2)$ family symmetry group respectively. The superscript “ T ” means transpose. The symbol “ $\boldsymbol{\times}$ ” denotes that the resulting lepton mixing is ruled out since there is at least one zero element in the fixed row, and the symbol “ \checkmark ” denote that the resulting mixing is viable. Note that for $G_l = Z_2^{bc^{x'} d^{x'}}$, the cases of $G_\nu = K_4^{(c^{n/2}, abc^y)}$ and $G_\nu = K_4^{(d^{n/2}, a^2 bd^z)}$ are equivalent because the remnant symmetries are related by group conjugation as $b(bc^{x'} d^{x'})b = bc^{-x'} d^{-x'}$, $bd^{n/2}b = c^{n/2}$ and $b(a^2 bd^z)b = abc^{-z}$.

where m_{11} , m_{13} and m_{22} are real. The unitary matrix U_ν diagonalizing this neutrino mass matrix is

$$U_\nu = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\pi\frac{\gamma}{n}} & 0 & e^{i\pi\frac{\gamma}{n}} \\ 0 & \sqrt{2}e^{-2i\pi\frac{x+\gamma}{n}} & 0 \\ -e^{i\pi\frac{2x+\gamma}{n}} & 0 & e^{i\pi\frac{2x+\gamma}{n}} \end{pmatrix}. \quad (4.21)$$

Finally the neutrino masses are

$$m_1 = |m_{11} - m_{13}|, \quad m_2 = |m_{22}|, \quad m_3 = |m_{11} + m_{13}|. \quad (4.22)$$

For the remaining value of $X_{\nu\mathbf{r}} = \{\rho_{\mathbf{r}}(c^\gamma d^{-2x-\gamma+n/2}), \rho_{\mathbf{r}}(bc^{x+\gamma} d^{-x-\gamma+n/2})\}$, the light neutrino masses are degenerate.

4.2 Predictions for lepton flavor mixing

As the different residual symmetries related by group conjugation lead to the same predictions for the lepton mixing matrix, we only need to consider the cases of $G_l = Z_2^{bc^x d^x}, Z_2^{c^{n/2}}$

and $G_\nu = K_4^{(c^{n/2}, d^{n/2})}$, $K_4^{(c^{n/2}, abc^y)}$, $K_4^{(d^{n/2}, a^2bd^z)}$ and $K_4^{(c^{n/2}d^{n/2}, bc^x d^x)}$. Compared with section 3, one row instead of one column of the PMNS matrix is fixed by remnant flavor symmetry in this scenario. The explicit form of this row vector for different remnant symmetry is summarized in table 3. We see that only one independent case is viable. Taking into account the compatible remnant CP symmetry, we can predict both mixing angles and CP phases in terms of one free parameter.

$$(\mathbf{V}) \quad G_l = \left\{ 1, bc^{x'} d^{x'} \right\}, \quad X_{l\mathbf{r}} = \left\{ \rho_{\mathbf{r}}(c^{\gamma'} d^{-2x' - \gamma'}), \rho_{\mathbf{r}}(bc^{x' + \gamma'} d^{-x' - \gamma'}) \right\}, \quad G_\nu = K_4^{(c^{n/2}, abc^y)} \text{ and} \\ X_{\nu\mathbf{r}} = \left\{ \rho_{\mathbf{r}}(c^\gamma d^{2y+2\gamma}), \rho_{\mathbf{r}}(abc^{y+\gamma} d^{2y+2\gamma}) \right\}$$

Combining the unitary transformation U_l in eq. (4.2) and U_ν in eq. (4.15), we can pin down the lepton flavor mixing matrix as follows:

$$U_{PMNS}^V = \frac{1}{2} \begin{pmatrix} \sin \theta + \sqrt{2} e^{i\varphi_8} \cos \theta & \sin \theta - \sqrt{2} e^{i\varphi_8} \cos \theta & \sqrt{2} e^{i\varphi_9} \sin \theta \\ 1 & 1 & -\sqrt{2} e^{i\varphi_9} \\ \cos \theta - \sqrt{2} e^{i\varphi_8} \sin \theta & \cos \theta + \sqrt{2} e^{i\varphi_8} \sin \theta & \sqrt{2} e^{i\varphi_9} \cos \theta \end{pmatrix}, \quad (4.23)$$

with

$$\varphi_8 = \frac{3\gamma' + 2x' + 2y}{n} \pi, \quad \varphi_9 = -\frac{3\gamma + 2x' + 2y}{n} \pi. \quad (4.24)$$

Here φ_8 and φ_9 are independent, they are determined by the remnant symmetry, and they can take the values,

$$\varphi_8, \varphi_9 \pmod{2\pi} = 0, \frac{1}{n}\pi, \frac{2}{n}\pi, \dots, \frac{2n-1}{n}\pi. \quad (4.25)$$

In order to be in accordance with the present neutrino oscillation data, $(1/2, 1/2, -e^{i\varphi_9}/\sqrt{2})$ can only be the second or the third row. Note that as usual permutation of the second and the third rows of U_{PMNS}^V is also viable. We can read out the lepton mixing parameters,

$$\begin{aligned} \sin^2 \theta_{13} &= \frac{1}{2} \sin^2 \theta, & \sin^2 \theta_{12} &= \frac{1}{2} - \frac{\sqrt{2} \sin 2\theta \cos \varphi_8}{3 + \cos 2\theta}, & \sin^2 \theta_{23} &= \frac{2}{3 + \cos 2\theta}, \\ |\tan \delta_{CP}| &= \left| \frac{(3 + \cos 2\theta) \tan \varphi_8}{1 + 3 \cos 2\theta} \right|, & |J_{CP}| &= \frac{1}{8\sqrt{2}} |\sin 2\theta \sin \varphi_8|, \\ |\tan \alpha_{21}| &= \left| \frac{8\sqrt{2}(1 + 3 \cos 2\theta) \sin 2\theta \sin \varphi_8}{7 + 12 \cos 2\theta + 13 \cos 4\theta + 8 \sin^2 2\theta \cos 2\varphi_8} \right|, \\ |\tan \alpha'_{31}| &= \left| \frac{\sin^2 \theta \sin 2\varphi_9 + \sqrt{2} \sin 2\theta \sin(2\varphi_9 - \varphi_8) + 2 \cos^2 \theta \sin(2\varphi_9 - 2\varphi_8)}{\sin^2 \theta \cos 2\varphi_9 + \sqrt{2} \sin 2\theta \cos(2\varphi_9 - \varphi_8) + 2 \cos^2 \theta \cos(2\varphi_9 - 2\varphi_8)} \right|. \end{aligned} \quad (4.26)$$

We see that all mixing parameters depend on θ and φ_8 except $|\tan \alpha'_{31}|$ which involves φ_9 additionally. It is interesting that the mixing angles θ_{13} and θ_{23} are related as follows

$$2 \cos^2 \theta_{13} \sin^2 \theta_{23} = 1, \quad \text{or} \quad 2 \cos^2 \theta_{13} \sin^2 \theta_{23} = 1 - 2 \sin^2 \theta_{13}, \quad (4.27)$$

where the second relation is for the PMNS matrix obtained by exchanging the second and the third rows of U_{PMNS}^V . Moreover, θ_{12} and θ_{13} are related by

$$\cos^2 \theta_{13} \cos 2\theta_{12} = \pm 2 \sin \theta_{13} \sqrt{\cos 2\theta_{13}} \cos \varphi_8, \quad (4.28)$$

which is relevant to the parameter φ_8 . The 3σ bound of $\sin^2 \theta_{13}$ gives the limit on θ :

$$\theta \in [0.060\pi, 0.078\pi] \cup [0.922\pi, 0.940\pi] . \quad (4.29)$$

The equation for $\sin^2 \theta_{12}$ in eq. (4.26) leads to

$$\frac{1}{2} (1 - |\cos \varphi_8|) \leq \sin^2 \theta_{12} \leq \frac{1}{2} (1 + |\cos \varphi_8|) , \quad (4.30)$$

Hence φ_8 is constrained to lie in the region

$$\varphi_8 \in [0, 0.409\pi] \cup [0.591\pi, 1.409\pi] \cup [1.591\pi, 2\pi] . \quad (4.31)$$

The numerical results are displayed in figure 15 and figure 16. Note that conserved CP corresponding to $\varphi_8 = 0, \pi$ is always viable. If we require that all three mixing angles are in their 3σ intervals, we find that $0.141 \leq \sin \theta_{13} \leq 0.172$, $0.328 \leq \sin^2 \theta_{12} \leq 0.359$, and $\sin^2 \theta_{23}$ is around 0.488 and 0.512 due to the correlation shown in eq. (4.27). Note that θ_{23} is very close to maximal mixing. Therefore precisely measuring the lepton mixing angles at JUNO or long baseline neutrino experiments can test this mixing pattern directly. For the CP phases, δ_{CP} and α_{21} are predicted to be in the intervals of $|\sin \delta_{CP}| \leq 0.586$ and $|\sin \alpha_{21}| \leq 0.396$ while α'_{31} can be any value for sufficient large n . The correlations between different mixing parameters are shown in figure 17.

5 Neutrinoless Double-beta decay

The rare process, neutrinoless double-beta decay ($0\nu 2\beta$), is an important probe for the Majorana nature of neutrino and lepton number violation, a sizable number of new experiments are currently running, under construction, or in the planning phase. The effective mass of neutrinoless double-beta decay is [68]

$$|m_{ee}| = \left| (m_1 c_{12}^2 + m_2 s_{12}^2 e^{i\alpha_{21}}) c_{13}^2 + m_3 s_{13}^2 e^{i\alpha'_{31}} \right| \quad (5.1)$$

For normal hierarchy, the masses are

$$m_1 = m_l, \quad m_2 = \sqrt{m_l^2 + \delta m^2}, \quad m_3 = \sqrt{m_l^2 + \Delta m^2 + \delta m^2/2}, \quad (5.2)$$

and for inverted hierarchy

$$m_1 = \sqrt{m_l^2 - \Delta m^2 - \delta m^2/2}, \quad m_2 = \sqrt{m_l^2 - \Delta m^2 + \delta m^2/2}, \quad m_3 = m_l, \quad (5.3)$$

where m_l denotes the lightest neutrino masses, and $\delta m^2 \equiv m_2^2 - m_1^2$ and $\Delta m^2 \equiv m_3^2 - (m_1^2 + m_2^2)/2$ defined in ref. [8]. The experimental error on the neutrino mass splitting is not taken into account during the analysis, instead the best fit values from [8] are used:

$$\delta m^2 = 7.54 \times 10^{-5} \text{eV}^2, \quad \Delta m^2 = 2.43 \times 10^{-3} (-2.38 \times 10^{-3}) \text{eV}^2, \quad (5.4)$$

for normal (inverted) hierarchy. In the following, the properties of the effective mass are examined for all viable cases of lepton mixing discussed in this paper. In figure 18 the

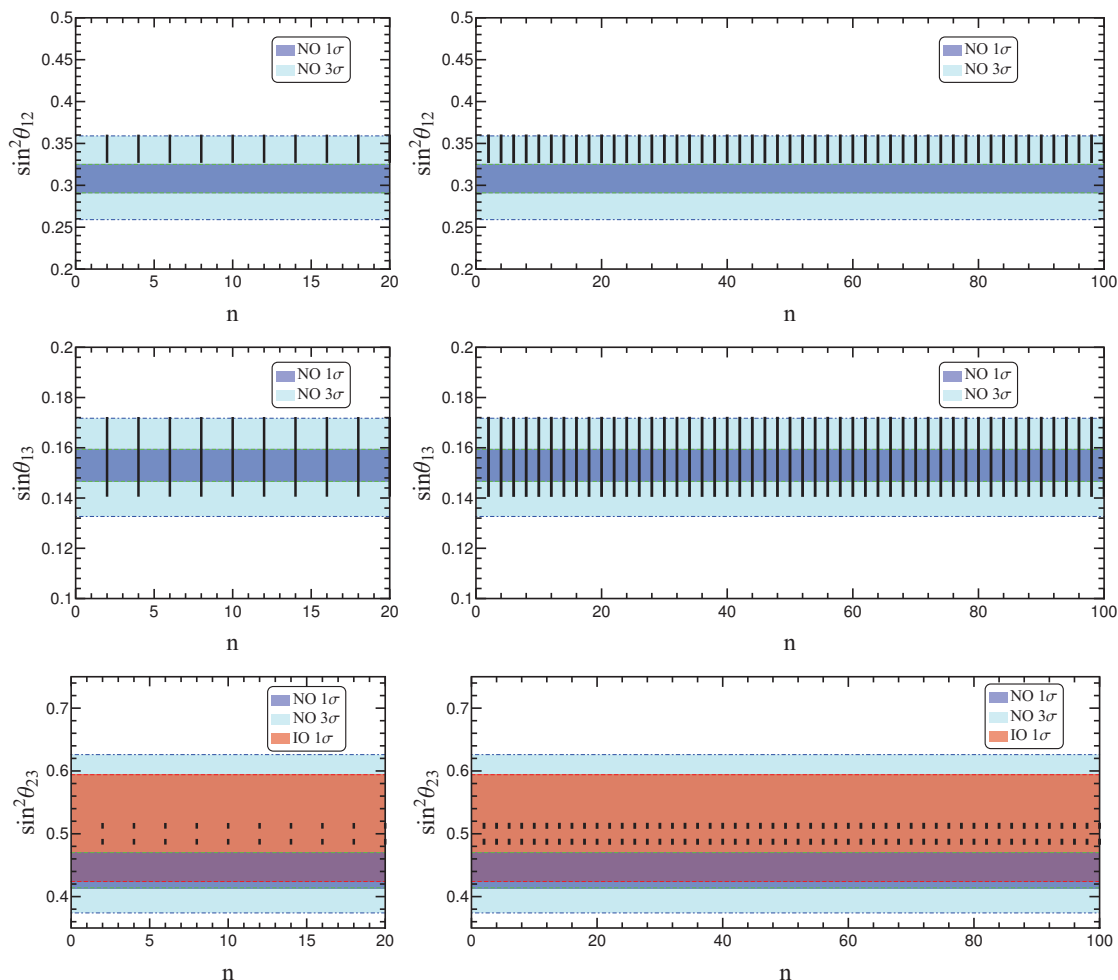


Figure 15. Numerical results in case V: the allowed ranges of $\sin^2 \theta_{12}$, $\sin \theta_{13}$ and $\sin^2 \theta_{23}$ for different n , where the three lepton mixing angles are required to lie in the 3σ regions. The 1σ and 3σ bounds of the mixing angles are taken from ref. [8]. Note that n should be divisible by 2 in this case.

allowed ranges of the effective mass are shown for each case in the limit of $n \rightarrow \infty$, where the three mixing angles are required to lie in the measured 3σ intervals [8] (As previously mentioned, the 3σ lower bound of $\sin^2 \theta_{12}$ is chosen to be 0.254 instead of 0.259 in case II). Furthermore, the predictions for the representative value $n = 8$ ($n = 5$ in case I for the 7th-9th ordering) are plotted in figure 19 in order to be read easily. The results for any finite value of n must be part of the shown one corresponding to $n \rightarrow \infty$. Moreover, the plotting would change only a little bit if the experimental errors on δm^2 and Δm^2 are taken into account. Note that only one distinct prediction for the effective mass arises except in case I. One reason for this is that, as discussed before, many of the possible permutations of the mixing matrix can be identified with each other. Furthermore, permuting the second and third row has no effect on the effective mass as θ_{23} does not appear in eq. (5.1).

As shown in figure 18, for inverted hierarchy neutrino mass spectrum, almost all the 3σ range values of the effective masses $|m_{ee}|$ can be reproduced in the limit $n \rightarrow \infty$ in case

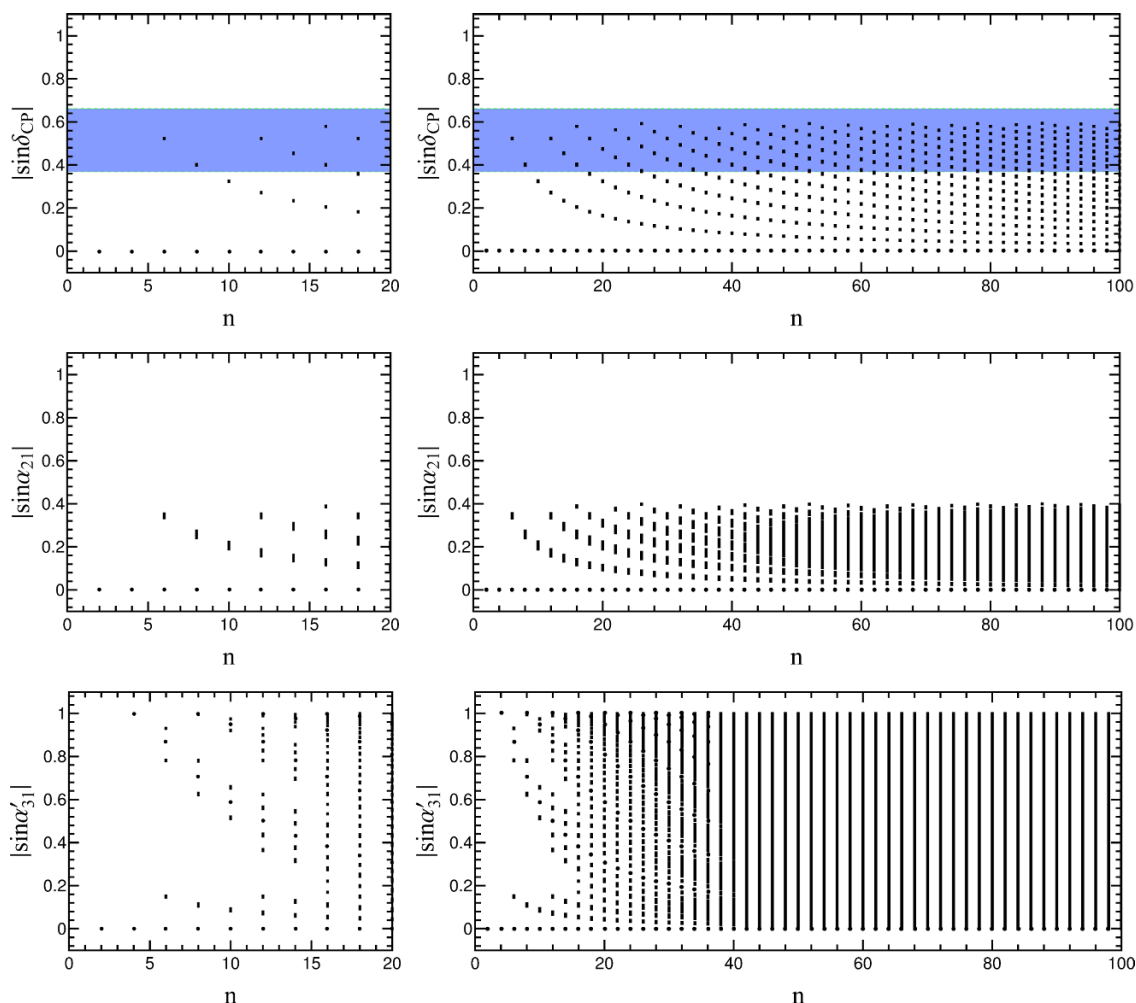


Figure 16. Numerical results in case V: the allowed ranges of $|\sin \delta_{CP}|$, $|\sin \alpha_{21}|$ and $|\sin \alpha'_{31}|$ for different n , where the three lepton mixing angles are required to lie in the 3σ regions. The 1σ and 3σ bounds of the mixing angles are taken from ref. [8]. Note that n should be divisible by 2 in this case.

I, case III and case IV. However, the predictions for $|m_{ee}|$ are around the upper bound (about 0.05eV) or lower bound (about 0.013 eV) in case V. The reason is that the solar mixing angle is in a narrow region $0.328 \leq \sin^2 \theta_{12} \leq 0.359$ and the Majorana phase α_{21} is constrained to be $|\sin \alpha_{21}| \leq 0.586$ in this case, as displayed in figure 15 and figure 16. Similarly $|m_{ee}|$ is near the upper bound and 0.025 eV in case II. Therefore if the effective mass is measured to be far from 0.013 eV, 0.025 eV and 0.05 eV for inverted hierarchy by future experiments, the mixing patterns in cases II and V could be ruled out.

For normal hierarchy neutrino mass spectrum, a sizable part of the experimentally allowed 3σ region of $|m_{ee}|$ can be generated in all cases, and the effective mass could be rather small. In particular, the prediction in case I with 7th to 9th ordering approximately coincides with the present 3σ region. Unfortunately the predictions for normal hierarchy are still out of reach of projected experiments known to the author. As a result, it is should be gen-

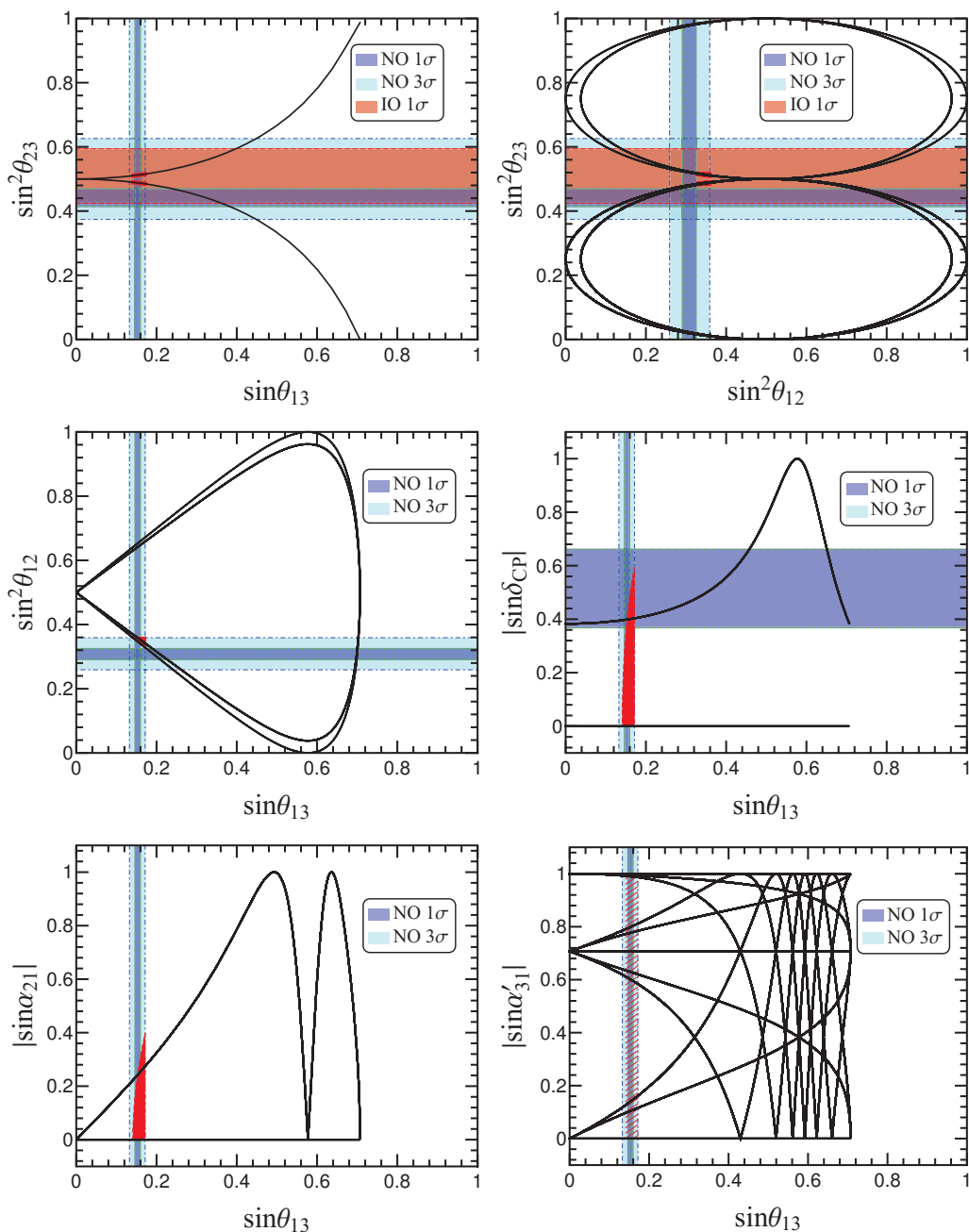


Figure 17. The correlations among mixing parameters in case V. The red filled regions denote the allowed values of the mixing parameters if we take the parameters φ_8 and φ_9 to be continuous (which is equivalent to taking the limit $n \rightarrow \infty$) and the three mixing angles are required to lie in their 3σ ranges. Note that the Majorana phase α'_{31} is not constrained in this limit. The black curves represent the phenomenologically viable correlations for $n = 8$. The 1σ and 3σ bounds of the mixing parameters are taken from ref. [8].

erally difficult to test the $\Delta(6n^2)$ family symmetry and generalized CP symmetry through neutrinoless double beta decay experiments in the case of normal ordering spectrum.

6 Conclusions

We have performed a detailed analysis of $\Delta(6n^2)$ family symmetry combined with the generalised CP symmetry H_{CP} in the lepton sector. We have investigated the lepton mixing parameters which can be obtained from the original symmetry $\Delta(6n^2) \times H_{\text{CP}}$ breaking to different remnant symmetries in the neutrino and charged lepton sectors, namely G_ν and G_l subgroups in the neutrino and the charged lepton sector respectively, while the remnant CP symmetries from the breaking of H_{CP} are H_{CP}^ν and H_{CP}^l , respectively.

We have assumed a preserved symmetry smaller than the full Klein symmetry, as in the semi-direct approach, leading to predictions which depend on a single undetermined real parameter, which mainly controls the reactor angle. We have discussed the resulting mass and mixing predictions for all possible cases where the $\Delta(6n^2)$ family symmetry with generalised CP is broken to $G_\nu = Z_2$ with $G_l = K_4, Z_p, p > 2$ and $G_\nu = K_4$ with $G_l = Z_2$. We have focused on five phenomenologically allowed cases and have presented the resulting predictions for the PMNS parameters as a function of n , as well as the predictions for neutrinoless double beta decay.

It is remarkable that the CP phases are predicted to take irregular values rather than $0, \pi$ or $\pm\pi/2$. In particular, compared to $\Delta(6n^2)$ in the direct approach where the full Klein symmetry is identified as a subgroup, the result $|\sin \delta_{\text{CP}}| = 0$ corresponding to the Dirac CP phase being either zero or $\pm\pi$ is relaxed in the indirect approach followed here. Compared to the indirect approach to S_4 (which corresponds in our notation to $n = 2$) where $|\sin \delta_{\text{CP}}| = 1$ is the only available possibility for non-zero CP violation, for larger values of n we find many different possibilities for the oscillation phase, where the number of possible predictions grows rapidly as n is increased. However it is interesting that, even for very large n , the predictions for the PMNS parameters is bounded within certain ranges, providing non-trivial tests of the semi-direct approach followed here. It remains to be seen if any of these possibilities will closely correspond to the observed future precise determination of leptonic mixing angles and CP violating parameters in the future.

Acknowledgments

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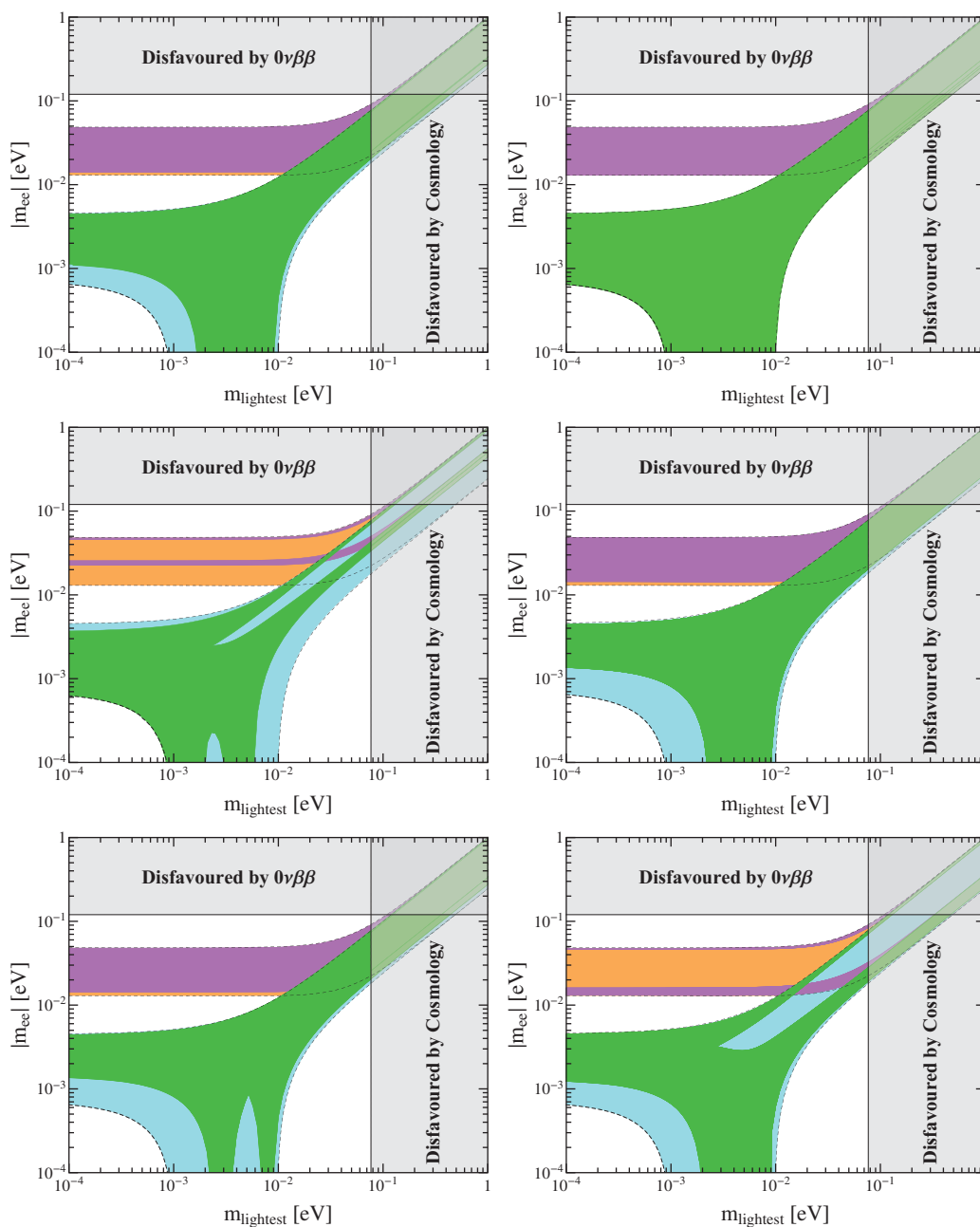


Figure 18. The allowed ranges of the effective mass for neutrinoless double-beta decay for all viable cases of lepton mixing in semidirect models with a $\Delta(6n^2)$ flavour group in the limit of $n \rightarrow \infty$. The top row corresponds to case I, with 1st-3rd ordering on the left and 7th to 9th ordering on the right, the middle row contains case II and III, and the bottom row case IV and V. Light blue and yellow areas indicate the currently allowed three sigma region for normal and inverted hierarchy, respectively. Purple regions correspond to predictions assuming inverted hierarchy, green regions to normal hierarchy. The upper bound $|m_{ee}| < 0.120$ eV is given by measurements by the EXO-200 [72, 73] and KamLAND-ZEN experiments [74]. Planck data in combination with other CMB and BAO measurements [75] provides a limit on the sum of neutrino masses of $m_1 + m_2 + m_3 < 0.230$ eV from which the upper limit on the mass of the lightest neutrino can be derived.

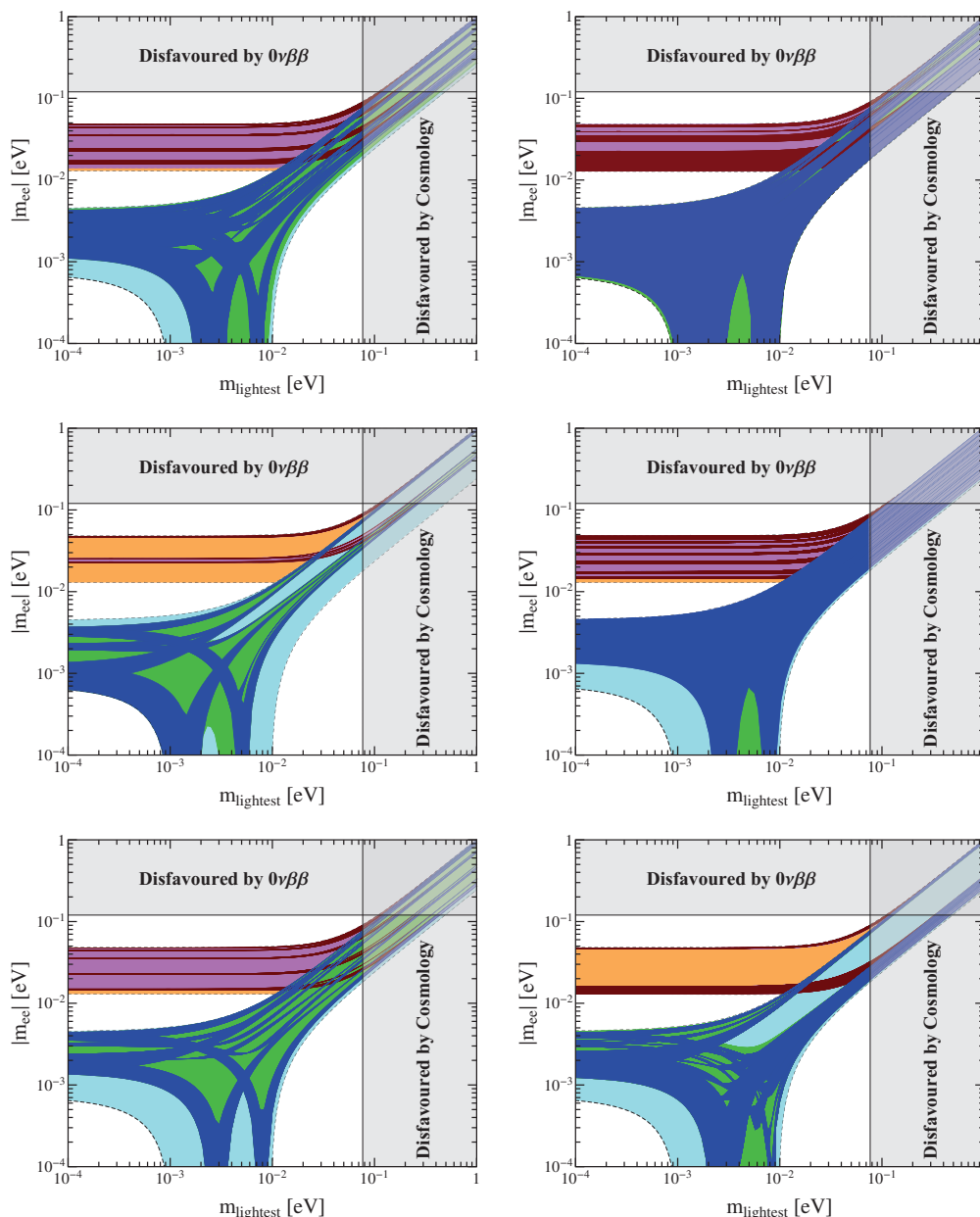


Figure 19. The allowed ranges of the effective mass for neutrinoless double-beta decay for all viable cases of lepton mixing in semidirect models with a $\Delta(6n^2)$ flavour group. The top row corresponds to case I, with 1st-3rd ordering on the left and 7th to 9th ordering on the right, the middle row contains case II and III, and the bottom row case IV and V. Light blue and yellow areas indicate the currently allowed three sigma region for normal and inverted hierarchy, respectively. Purple regions correspond to predictions assuming inverted hierarchy, green regions to normal hierarchy in the limit of $n \rightarrow \infty$. Blue and red regions represent predictions for normal and inverted hierarchy for the value $n = 8$ (in the top-right panel, we choose $n = 5$ which is the smallest viable value of n in that case). The upper bound $|m_{ee}| < 0.120$ eV is given by measurements by the EXO-200 [72, 73] and KamLAND-ZEN experiments [74]. Planck data in combination with other CMB and BAO measurements [75] provides a limit on the sum of neutrino masses of $m_1 + m_2 + m_3 < 0.230$ eV from which the upper limit on the mass of the lightest neutrino can be derived.

A The group theory of $\Delta(6n^2)$

$\Delta(6n^2)$ is non-abelian finite subgroup of $SU(3)$. The $\Delta(6n^2)$ is isomorphic to $(Z_n \times Z_n) \rtimes S_3$, where S_3 is isomorphic to $Z_3 \rtimes Z_2$, and it can be conveniently defined by four generators a, b, c and d obeying the relations [76]:

$$\begin{aligned} a^3 &= b^2 = (ab)^2 = 1, \\ c^n &= d^n = 1, \quad cd = dc, \\ aca^{-1} &= c^{-1}d^{-1}, \quad ada^{-1} = c, \quad bcb^{-1} = d^{-1}, \quad bdb^{-1} = c^{-1}. \end{aligned} \tag{A.1}$$

The elements a and b are the generators of S_3 while c and d generate $Z_n \times Z_n$, and the last line defines the semidirect product “ \rtimes ”. Note that the generator $d = bc^{-1}b^{-1}$ is not independent. All the group elements can be written into the form

$$g = a^\alpha b^\beta c^\gamma d^\delta, \tag{A.2}$$

where $\alpha = 0, 1, 2, \beta = 0, 1, \gamma, \delta = 0, 1, 2, \dots, n - 1$. In the following we list the elements of $\Delta(6n^2)$ by order of the generated cyclic subgroup.

- Elements of order 2, if n even:

$$c^{n/2}, d^{n/2}, c^{n/2}d^{n/2} \tag{A.3}$$

- Elements of order 2, always:

$$bc^\epsilon d^\epsilon, abc^\gamma, a^2bd^\delta \tag{A.4}$$

with $\epsilon, \gamma, \delta = 0, \dots, n - 1$.

- Elements of order 3, if 3 divides n :

$$c^{n/3}, d^{n/3}, \dots \tag{A.5}$$

where the dots indicate all possible products and powers of the two first elements.

- Elements of order 3, always:

$$ac^\gamma d^\delta, a^2c^\gamma d^\delta \tag{A.6}$$

with $\gamma, \delta = 0, \dots, n - 1$

- Elements of order m where m divides n , if m and n are even:

$$bc^{\delta+2kn/m} d^\delta, abc^\gamma d^{2kn/m}, a^2bc^{2kn/m} d^\delta \tag{A.7}$$

with $\gamma, \delta = 0, \dots, n - 1$ and $0 \leq k \leq m/2$

- Elements of order m where m divides n , always:

$$c^{kn/m} d^{ln/m} \tag{A.8}$$

with $k, l = 0, \dots, n - 1$.

The $\Delta(6n^2)$ group have been thoroughly studied in ref. [76]. In the following, we shall review the basic aspects, which is relevant to our present work. The conjugacy classes of $\Delta(6n^2)$ group are of the following forms:

- $n \neq 3\mathbb{Z}$

$$1 : 1C_1 = \{1\}, \tag{A.9a}$$

$$n-1 : 3C_1^{(\rho)} = \{c^\rho d^{-\rho}, c^{-2\rho} d^{-\rho}, c^\rho d^{2\rho}\}, \quad \rho = 1, 2, \dots, n-1, \tag{A.9b}$$

$$\frac{n^2 - 3n + 2}{6} : 6C_1^{(\rho, \sigma)} = \{c^\rho d^\sigma, c^{\sigma-\rho} d^{-\rho}, c^{-\sigma} d^{\rho-\sigma}, c^{-\sigma} d^{-\rho}, c^{\sigma-\rho} d^\sigma, c^\rho d^{\rho-\sigma}\}, \tag{A.9c}$$

$$1 : 2n^2 C_2 = \{ac^z d^y, a^2 c^{-y} d^{-z} | z, y = 0, 1, \dots, n-1\}, \tag{A.9d}$$

$$n : 3nC_3^{(\rho)} = \{bc^{\rho+x} d^x, a^2 bc^{-\rho} d^{-x-\rho}, abc^{-x} d^\rho | x=0, 1, \dots, n-1\}, \rho=0, 1, \dots, n-1. \tag{A.9e}$$

The convention used here is that the quantity left of the colon is the number of classes of the kind on the right of the colon. In eq. (A.9c), the parameter $\rho, \sigma = 0, 1, \dots, n-1$, but excluding possibilities given by

$$\rho + \sigma = 0 \pmod n, \quad 2\rho - \sigma = 0 \pmod n, \quad \rho - 2\sigma = 0 \pmod n. \tag{A.10}$$

- $n = 3\mathbb{Z}$

$$1 : 1C_1 = \{1\}, \tag{A.11a}$$

$$2 : 1C_1^{(\nu)} = \{c^\nu d^{2\nu}\}, \quad \nu = \frac{n}{3}, \frac{2n}{3}, \tag{A.11b}$$

$$n-3 : 3C_1^{(\rho)} = \{c^\rho d^{-\rho}, c^{-2\rho} d^{-\rho}, c^\rho d^{2\rho}\}, \quad \rho \neq \frac{n}{3}, \frac{2n}{3}, \tag{A.11c}$$

$$\frac{n^2 - 3n + 6}{6} : 6C_1^{(\rho, \sigma)} = \{c^\rho d^\sigma, c^{\sigma-\rho} d^{-\rho}, c^{-\sigma} d^{\rho-\sigma}, c^{-\sigma} d^{-\rho}, c^{\sigma-\rho} d^\sigma, c^\rho d^{\rho-\sigma}\}, \tag{A.11d}$$

$$3 : \frac{2n^2}{3} C_2^{(\tau)} = \{ac^{\tau-y-3x} d^y, a^2 c^{-y} d^{y+3x-\tau} | y = 0, 1, \dots, n-1, x = 0, 1, \dots, \frac{n-3}{3}\}, \tau = 0, 1, 2,$$

$$n : 3nC_3^{(\rho)} = \{bc^{\rho+x} d^x, a^2 bc^{-\rho} d^{-x-\rho}, abc^{-x} d^\rho | x=0, 1, \dots, n-1\}, \rho=0, 1, \dots, n-1. \tag{A.11e}$$

In eq. (A.11d), $\rho, \sigma = 0, 1, \dots, n-1$, again excluding possibilities given by eq. (A.10).

The irreducible representations and their representation matrices of the $\Delta(6n^2)$ group are as follows [76]:

- (i) $n \neq 3\mathbb{Z}$

- One-dimensional representations

$$\mathbf{1}_1 : a = b = c = d = 1, \tag{A.12a}$$

$$\mathbf{1}_2 : a = c = d = 1, \quad b = -1, \tag{A.12b}$$

- Two-dimensional representation

$$\mathbf{2} : a = \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad c = d = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{A.13}$$

which is related to the basis chosen in ref. [76] by a unitary transformation U with

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}. \quad (\text{A.14})$$

In our new basis, all the Clebsch-Gordan (CG) coefficients are real, as is shown in the appendix B. Hence our basis is the so-called the ‘‘CP’’ basis. The conventional CP transformation $\varphi \rightarrow \varphi^*$ can be consistently imposed onto the theory in our basis, and all the coupling constant would be constrained to be real.

- Three-dimensional representations

$$\mathbf{3}_{1,k} : a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} \eta^k & 0 & 0 \\ 0 & \eta^{-k} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta^k & 0 \\ 0 & 0 & \eta^{-k} \end{pmatrix}, \quad (\text{A.15a})$$

$$\mathbf{3}_{2,k} : a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad b = - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} \eta^k & 0 & 0 \\ 0 & \eta^{-k} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta^k & 0 \\ 0 & 0 & \eta^{-k} \end{pmatrix}, \quad (\text{A.15b})$$

where $\eta \equiv e^{2\pi i/n}$ and $k = 1, 2, \dots, n-1$.

- Six-dimensional representations

$$\mathbf{6}_{(k,l)} : a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & \mathbb{1}_3 \\ \mathbb{1}_3 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}, \quad d = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad (\text{A.16})$$

with

$$a_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad (\text{A.17})$$

$$c_1 = d_2^{-1} = \begin{pmatrix} \eta^l & 0 & 0 \\ 0 & \eta^k & 0 \\ 0 & 0 & \eta^{-l-k} \end{pmatrix}, \quad c_2 = d_1^{-1} = \begin{pmatrix} \eta^{l+k} & 0 & 0 \\ 0 & \eta^{-l} & 0 \\ 0 & 0 & \eta^{-k} \end{pmatrix}. \quad (\text{A.18})$$

Here $\widetilde{}$ denotes the mapping

$$\widetilde{\begin{pmatrix} k \\ l \end{pmatrix}} \mapsto \text{either } \begin{pmatrix} k \\ l \end{pmatrix}, \begin{pmatrix} -k-l \\ k \end{pmatrix}, \begin{pmatrix} l \\ -k-l \end{pmatrix}, \begin{pmatrix} -l \\ -k \end{pmatrix}, \begin{pmatrix} k+l \\ -l \end{pmatrix}, \text{ or } \begin{pmatrix} -k \\ k+l \end{pmatrix}, \quad (\text{A.19})$$

$k, l = 0, 1, \dots, n-1$, and the following cases are forbidden.

$$l = 0, \quad k = 0, \quad k + l = 0 \pmod{n}. \quad (\text{A.20})$$

(ii) $n = 3\mathbb{Z}$

- One-dimensional representations

$$\mathbf{1}_1 : a = b = c = d = 1, \quad (\text{A.21a})$$

$$\mathbf{1}_2 : a = c = d = 1, \quad b = -1, \quad (\text{A.21b})$$

- Two-dimensional representation

$$\mathbf{2}_1 : a = \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad c = d = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (\text{A.22a})$$

$$\mathbf{2}_2 : a = \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad c = d = \frac{1}{2} \begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}, \quad (\text{A.22b})$$

$$\mathbf{2}_3 : a = \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad c = d = \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \quad (\text{A.22c})$$

$$\mathbf{2}_4 : a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad c = d = \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}. \quad (\text{A.22d})$$

They are related to the representation matrices of ref. [76] by the unitary transformation U in eq. (A.14).

- Three-dimensional representations

$$\mathbf{3}_{1,k} : a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} \eta^k & 0 & 0 \\ 0 & \eta^{-k} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta^k & 0 \\ 0 & 0 & \eta^{-k} \end{pmatrix}, \quad (\text{A.23a})$$

$$\mathbf{3}_{2,k} : a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad b = - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} \eta^k & 0 & 0 \\ 0 & \eta^{-k} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta^k & 0 \\ 0 & 0 & \eta^{-k} \end{pmatrix}, \quad (\text{A.23b})$$

where $k = 1, 2, \dots, n-1$.

- Six-dimensional representations

$$\mathbf{6}_{(k,l)} : a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & \mathbb{1}_3 \\ \mathbb{1}_3 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}, \quad d = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}. \quad (\text{A.24})$$

The 3×3 unitary matrices $a_{1,2}$, $c_{1,2}$ and $d_{1,2}$ are given in eq. (A.17). Again the following values are prohibited:

$$l = 0, \quad k = 0, \quad k = l = n/3, \quad k = l = 2n/3, \quad k + l = 0 \pmod{n}. \quad (\text{A.25})$$

B Clebsch-Gordan coefficients for $\Delta(6n^2)$ group with $n \neq 3\mathbb{Z}$

In the following, we shall present all the CG coefficients in the form of $x \otimes y$ in our chosen basis, x_i denotes the element of the left base vectors x , and y_i is the element of the right base vectors y . We shall see explicitly that all the CG coefficients are real.

- $\mathbf{2} \otimes \mathbf{2} = \mathbf{1}_1 \oplus \mathbf{1}_2 \oplus \mathbf{2}$

$$\mathbf{2} \sim \begin{pmatrix} x_2 y_2 - x_1 y_1 \\ x_1 y_2 + x_2 y_1 \end{pmatrix}, \quad \mathbf{1}_1 \sim x_1 y_1 + x_2 y_2, \quad \mathbf{1}_2 \sim x_1 y_2 - x_2 y_1. \quad (\text{B.1})$$

- $\mathbf{2} \otimes \mathbf{3}_{1,k} = \mathbf{3}_{1,k} \oplus \mathbf{3}_{2,k}$

$$\mathbf{3}_{1,k} \sim \begin{pmatrix} (x_1 - \sqrt{3}x_2) y_1 \\ -2x_1 y_2 \\ (x_1 + \sqrt{3}x_2) y_3 \end{pmatrix}, \quad \mathbf{3}_{2,k} \sim \begin{pmatrix} (\sqrt{3}x_1 + x_2) y_1 \\ -2x_2 y_2 \\ (-\sqrt{3}x_1 + x_2) y_3 \end{pmatrix}. \quad (\text{B.2})$$

- $\mathbf{2} \otimes \mathbf{3}_{2,k} = \mathbf{3}_{1,k} \oplus \mathbf{3}_{2,k}$

$$\mathbf{3}_{1,k} \sim \begin{pmatrix} (\sqrt{3}x_1 + x_2) y_1 \\ -2x_2 y_2 \\ (-\sqrt{3}x_1 + x_2) y_3 \end{pmatrix}, \quad \mathbf{3}_{2,k} \sim \begin{pmatrix} (x_1 - \sqrt{3}x_2) y_1 \\ -2x_1 y_2 \\ (x_1 + \sqrt{3}x_2) y_3 \end{pmatrix}. \quad (\text{B.3})$$

- $\mathbf{2} \otimes \mathbf{6}_{(k,l)} = \mathbf{6}_{(k,l)} \oplus \mathbf{6}_{(k,l)}$

$$\mathbf{6}_{(k,l)} \sim \begin{pmatrix} (\sqrt{3}x_1 + x_2) y_1 \\ -2x_2 y_2 \\ (-\sqrt{3}x_1 + x_2) y_3 \\ (\sqrt{3}x_1 - x_2) y_4 \\ 2x_2 y_5 \\ -(\sqrt{3}x_1 + x_2) y_6 \end{pmatrix}, \quad \mathbf{6}_{(k,l)} \sim \begin{pmatrix} 2x_2 y_1 \\ (\sqrt{3}x_1 - x_2) y_2 \\ -(\sqrt{3}x_1 + x_2) y_3 \\ -2x_2 y_4 \\ (\sqrt{3}x_1 + x_2) y_5 \\ (-\sqrt{3}x_1 + x_2) y_6 \end{pmatrix}. \quad (\text{B.4})$$

- $\mathbf{3}_{1,l} \otimes \mathbf{3}_{1,l'} = \mathbf{3}_{1,l+l'} \oplus \mathbf{6}_{(l,-l')}$

$$\mathbf{3}_{1,l+l'} \sim \begin{pmatrix} x_1 y_1 \\ x_2 y_2 \\ x_3 y_3 \end{pmatrix}, \quad \mathbf{6}_{(-l,l-l')} \sim \begin{pmatrix} x_1 y_2 \\ x_2 y_3 \\ x_3 y_1 \\ x_3 y_2 \\ x_2 y_1 \\ x_1 y_3 \end{pmatrix}, \quad (\text{B.5})$$

- $\mathbf{3}_{1,l} \otimes \mathbf{3}_{2,l'} = \mathbf{3}_{2,l+l'} \oplus \mathbf{6}_{(l,-l')}$

$$\mathbf{3}_{2,l+l'} \sim \begin{pmatrix} x_1y_1 \\ x_2y_2 \\ x_3y_3 \end{pmatrix}, \quad \mathbf{6}_{(-l,l-l')} \sim \begin{pmatrix} x_1y_2 \\ x_2y_3 \\ x_3y_1 \\ -x_3y_2 \\ -x_2y_1 \\ -x_1y_3 \end{pmatrix}, \quad (\text{B.6})$$

- $\mathbf{3}_{1,l} \otimes \mathbf{6}_{(k',l')} = \mathbf{6}_{\begin{pmatrix} k' \\ l'-l \end{pmatrix}} \oplus \mathbf{6}_{\begin{pmatrix} k'-l \\ l'+l \end{pmatrix}} \oplus \mathbf{6}_{\begin{pmatrix} l+k' \\ l' \end{pmatrix}}$

$$\mathbf{6}_{\begin{pmatrix} l'-l \\ l-k'-l' \end{pmatrix}} \sim \begin{pmatrix} x_1y_3 \\ x_2y_1 \\ x_3y_2 \\ x_3y_6 \\ x_2y_4 \\ x_1y_5 \end{pmatrix}, \quad \mathbf{6}_{\begin{pmatrix} k'-l \\ l'+l \end{pmatrix}} \sim \begin{pmatrix} x_1y_1 \\ x_2y_2 \\ x_3y_3 \\ x_3y_4 \\ x_2y_5 \\ x_1y_6 \end{pmatrix}, \quad \mathbf{6}_{\begin{pmatrix} -l-k'-l' \\ l+k' \end{pmatrix}} \sim \begin{pmatrix} x_1y_2 \\ x_2y_3 \\ x_3y_1 \\ x_3y_5 \\ x_2y_6 \\ x_1y_4 \end{pmatrix}. \quad (\text{B.7})$$

- $\mathbf{3}_{2,l} \otimes \mathbf{3}_{2,l'} = \mathbf{3}_{1,l+l'} \oplus \mathbf{6}_{(l,-l')}$

$$\mathbf{3}_{1,l+l'} \sim \begin{pmatrix} x_1y_1 \\ x_2y_2 \\ x_3y_3 \end{pmatrix}, \quad \mathbf{6}_{(-l,l-l')} \sim \begin{pmatrix} x_1y_2 \\ x_2y_3 \\ x_3y_1 \\ x_3y_2 \\ x_2y_1 \\ x_1y_3 \end{pmatrix}. \quad (\text{B.8})$$

- $\mathbf{3}_{2,l} \otimes \mathbf{6}_{(k',l')} = \mathbf{6}_{\begin{pmatrix} k' \\ l'-l \end{pmatrix}} \oplus \mathbf{6}_{\begin{pmatrix} k'-l \\ l'+l \end{pmatrix}} \oplus \mathbf{6}_{\begin{pmatrix} l+k' \\ l' \end{pmatrix}}$

$$\mathbf{6}_{\begin{pmatrix} l'-l \\ l-k'-l' \end{pmatrix}} \sim \begin{pmatrix} x_1y_3 \\ x_2y_1 \\ x_3y_2 \\ -x_3y_6 \\ -x_2y_4 \\ -x_1y_5 \end{pmatrix}, \quad \mathbf{6}_{\begin{pmatrix} k'-l \\ l'+l \end{pmatrix}} \sim \begin{pmatrix} x_1y_1 \\ x_2y_2 \\ x_3y_3 \\ -x_3y_4 \\ -x_2y_5 \\ -x_1y_6 \end{pmatrix}, \quad \mathbf{6}_{\begin{pmatrix} -l-k'-l' \\ l+k' \end{pmatrix}} \sim \begin{pmatrix} x_1y_2 \\ x_2y_3 \\ x_3y_1 \\ -x_3y_5 \\ -x_2y_6 \\ -x_1y_4 \end{pmatrix}. \quad (\text{B.9})$$

$$\bullet \mathbf{6}_{(k,l)} \otimes \mathbf{6}_{(k',l')} = \sum_{p,s} \mathbf{6}_{\left(\binom{k}{l} + M_s^p \binom{k'}{l'} \right)}$$

$$\mathbf{6}_{\binom{k+k'}{l+l'}} \sim \begin{pmatrix} x_1y_1 \\ x_2y_2 \\ x_3y_3 \\ x_4y_4 \\ x_5y_5 \\ x_6y_6 \end{pmatrix}, \quad \mathbf{6}_{\binom{k-k'-l'}{l+k'}} \sim \begin{pmatrix} x_1y_2 \\ x_2y_3 \\ x_3y_1 \\ x_4y_5 \\ x_5y_6 \\ x_6y_4 \end{pmatrix}, \quad \mathbf{6}_{\binom{k+l'}{l-l'-k'}} \sim \begin{pmatrix} x_1y_3 \\ x_2y_1 \\ x_3y_2 \\ x_4y_6 \\ x_5y_4 \\ x_6y_5 \end{pmatrix},$$

$$\mathbf{6}_{\binom{k-k'}{l+k'+l'}} \sim \begin{pmatrix} x_1y_4 \\ x_2y_6 \\ x_3y_5 \\ x_4y_1 \\ x_5y_3 \\ x_6y_2 \end{pmatrix}, \quad \mathbf{6}_{\binom{k+k'+l'}{l-l'}} \sim \begin{pmatrix} x_1y_5 \\ x_2y_4 \\ x_3y_6 \\ x_4y_2 \\ x_5y_1 \\ x_6y_3 \end{pmatrix}, \quad \mathbf{6}_{\binom{k-l'}{l-k'}} \sim \begin{pmatrix} x_1y_6 \\ x_2y_5 \\ x_3y_4 \\ x_4y_3 \\ x_5y_2 \\ x_6y_1 \end{pmatrix}. \quad (\text{B.10})$$

For the case of $n = 3\mathbb{Z}$, the CG-coefficients can be straightforwardly calculated although it is somewhat lengthy. Part of the CG coefficients are complex numbers in our chosen basis, the explicit form would not be reported here since generalized CP transformations can not be consistently defined in generic settings based on such groups unless the doublet representations $\mathbf{2}_2$, $\mathbf{2}_3$ and $\mathbf{2}_4$ are not introduced in a specific model.

C Phenomenological implication of $Z_2 \times CP$ in the charged lepton sector

The full symmetry $\Delta(6n^2) \rtimes H_{CP}$ is broken down to $Z_2 \times H_{CP}^l$ in the charged lepton sector. Similar to section 3, the hermitian combination $m_l^\dagger m_l$ of the charged lepton mass matrix can be constructed from its invariance under the remnant family symmetry Z_2 and the remnant CP symmetry H_{CP}^l ,

$$\begin{aligned}
 \rho_3^\dagger(g_l) m_l^\dagger m_l \rho_3(g_l) &= m_l^\dagger m_l, & g_l &\in Z_2, \\
 X_{l3}^\dagger m_l^\dagger m_l X_{l3} &= (m_l^\dagger m_l)^*, & X_l &\in H_{CP}^l.
 \end{aligned} \quad (\text{C.1})$$

(i) $G_l = Z_2^{bc^x d^x}$, $X_{lr} = \{\rho_r(c^\gamma d^{-2x-\gamma}), \rho_r(bc^{x+\gamma} d^{-x-\gamma})\}$

In this case, $m_l^\dagger m_l$ is determined to be of the form

$$m_l^\dagger m_l = \begin{pmatrix} \tilde{m}_{11} & \tilde{m}_{12} e^{i\pi \frac{2x+3\gamma}{n}} & \tilde{m}_{13} e^{-2i\pi \frac{x}{n}} \\ \tilde{m}_{12} e^{-i\pi \frac{2x+3\gamma}{n}} & \tilde{m}_{22} & \tilde{m}_{12} e^{-i\pi \frac{4x+3\gamma}{n}} \\ \tilde{m}_{13} e^{2i\pi \frac{x}{n}} & \tilde{m}_{12} e^{i\pi \frac{4x+3\gamma}{n}} & \tilde{m}_{11} \end{pmatrix}, \quad (\text{C.2})$$

where \tilde{m}_{11} , \tilde{m}_{12} , \tilde{m}_{13} and \tilde{m}_{22} are real parameters, and they have mass dimension of 2. This charged lepton mass matrix is diagonalized by a unitary transformation U_l via

$$U_l^\dagger m_l^\dagger m_l U_l = \text{diag}(m_{l_1}^2, m_{l_2}^2, m_{l_3}^2), \quad (\text{C.3})$$

with

$$U_l = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\pi\frac{\gamma}{n}} & -e^{i\pi\frac{\gamma}{n}} \sin \theta & e^{i\pi\frac{\gamma}{n}} \cos \theta \\ 0 & e^{-2i\pi\frac{x+\gamma}{n}} \sqrt{2} \cos \theta & e^{-2i\pi\frac{x+\gamma}{n}} \sqrt{2} \sin \theta \\ -e^{i\pi\frac{2x+\gamma}{n}} & -e^{i\pi\frac{2x+\gamma}{n}} \sin \theta & e^{i\pi\frac{2x+\gamma}{n}} \cos \theta \end{pmatrix}, \quad (\text{C.4})$$

where the angle θ is

$$\tan 2\theta = \frac{2\sqrt{2}\tilde{m}_{12}}{\tilde{m}_{11} + \tilde{m}_{13} - \tilde{m}_{22}}. \quad (\text{C.5})$$

It is remarkable that the unitary transformation U_l in eq. (C.4) coincides with U_ν in eq. (3.25). The reason is that the two cases share the same residual symmetry. The charged lepton masses are given by

$$\begin{aligned} m_{l_1}^2 &= \tilde{m}_{11} - \tilde{m}_{13}, \\ m_{l_2}^2 &= \frac{1}{2} \left[\tilde{m}_{11} + \tilde{m}_{13} + \tilde{m}_{22} - \text{sign}((\tilde{m}_{11} + \tilde{m}_{13} - \tilde{m}_{22}) \cos 2\theta) \sqrt{(\tilde{m}_{11} + \tilde{m}_{13} - \tilde{m}_{22})^2 + 8\tilde{m}_{12}^2} \right], \\ m_{l_3}^2 &= \frac{1}{2} \left[\tilde{m}_{11} + \tilde{m}_{13} + \tilde{m}_{22} + \text{sign}((\tilde{m}_{11} + \tilde{m}_{13} - \tilde{m}_{22}) \cos 2\theta) \sqrt{(\tilde{m}_{11} + \tilde{m}_{13} - \tilde{m}_{22})^2 + 8\tilde{m}_{12}^2} \right]. \end{aligned}$$

In the present framework, we can not determine the order of $m_{l_1}^2$, $m_{l_2}^2$ and $m_{l_3}^2$, i.e. we don't know which one of $m_{l_1}^2$, $m_{l_2}^2$, $m_{l_3}^2$ is electron (muon or tau) mass squared. As a result, the diagonalization matrix U_l in eq. (C.4) is also determined up to rephasing and permutations of its column vectors. The same holds true for the following cases.

(ii) $G_l = Z_2^{abc^y}$, $X_{l\mathbf{r}} = \{\rho_{\mathbf{r}}(c^\gamma d^{2y+2\gamma}), \rho_{\mathbf{r}}(abc^{y+\gamma} d^{2y+2\gamma})\}$ The charged lepton mass matrix satisfying the invariant conditions of eq. (C.1) takes the form

$$m_l^\dagger m_l = \begin{pmatrix} \tilde{m}_{11} & \tilde{m}_{12} e^{-2i\pi\frac{y}{n}} & \tilde{m}_{13} e^{i\pi\frac{2y+3\gamma}{n}} \\ \tilde{m}_{12} e^{2i\pi\frac{y}{n}} & \tilde{m}_{11} & \tilde{m}_{13} e^{i\pi\frac{4y+3\gamma}{n}} \\ \tilde{m}_{13} e^{-i\pi\frac{2y+3\gamma}{n}} & \tilde{m}_{13} e^{-i\pi\frac{4y+3\gamma}{n}} & \tilde{m}_{33} \end{pmatrix}, \quad (\text{C.6})$$

where \tilde{m}_{11} , \tilde{m}_{12} , \tilde{m}_{13} and \tilde{m}_{33} are real. The charged lepton diagonalization matrix U_l is given by

$$U_l = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\pi\frac{\gamma}{n}} & e^{i\pi\frac{\gamma}{n}} \cos \theta & e^{i\pi\frac{\gamma}{n}} \sin \theta \\ -e^{i\pi\frac{2y+\gamma}{n}} & e^{i\pi\frac{2y+\gamma}{n}} \cos \theta & e^{i\pi\frac{2y+\gamma}{n}} \sin \theta \\ 0 & -e^{-2i\pi\frac{y+\gamma}{n}} \sqrt{2} \sin \theta & e^{-2i\pi\frac{y+\gamma}{n}} \sqrt{2} \cos \theta \end{pmatrix}, \quad (\text{C.7})$$

with

$$\tan 2\theta = \frac{2\sqrt{2}\tilde{m}_{13}}{\tilde{m}_{33} - \tilde{m}_{11} - \tilde{m}_{12}}. \quad (\text{C.8})$$

The charged lepton masses are determined to be

$$\begin{aligned} m_{l_1}^2 &= \tilde{m}_{11} - \tilde{m}_{12}, \\ m_{l_2}^2 &= \frac{1}{2} \left[\tilde{m}_{11} + \tilde{m}_{12} + \tilde{m}_{33} + \text{sign}((\tilde{m}_{11} + \tilde{m}_{12} - \tilde{m}_{33}) \cos 2\theta) \sqrt{(\tilde{m}_{11} + \tilde{m}_{12} - \tilde{m}_{33})^2 + 8\tilde{m}_{13}^2} \right], \\ m_{l_3}^2 &= \frac{1}{2} \left[\tilde{m}_{11} + \tilde{m}_{12} + \tilde{m}_{33} - \text{sign}((\tilde{m}_{11} + \tilde{m}_{12} - \tilde{m}_{33}) \cos 2\theta) \sqrt{(\tilde{m}_{11} + \tilde{m}_{12} - \tilde{m}_{33})^2 + 8\tilde{m}_{13}^2} \right]. \end{aligned}$$

(iii) $G_l = Z_2^{a^2bd^z}$, $X_{lr} = \{\rho_r(c^{2z+2\delta}d^\delta), \rho_r(a^2bc^{2z+2\delta}d^{z+\delta})\}$ The charged lepton mass matrix invariant under both residual flavor and residual CP symmetries is

$$m_l^\dagger m_l = \begin{pmatrix} \tilde{m}_{11} & \tilde{m}_{12}e^{i\pi\frac{4z+3\delta}{n}} & \tilde{m}_{12}e^{i\pi\frac{2z+3\delta}{n}} \\ \tilde{m}_{12}e^{-i\pi\frac{4z+3\delta}{n}} & \tilde{m}_{22} & \tilde{m}_{23}e^{-2i\pi\frac{z}{n}} \\ \tilde{m}_{12}e^{-i\pi\frac{2z+3\delta}{n}} & \tilde{m}_{23}e^{2i\pi\frac{z}{n}} & \tilde{m}_{22} \end{pmatrix}, \quad (\text{C.9})$$

where \tilde{m}_{11} , \tilde{m}_{12} , \tilde{m}_{22} and \tilde{m}_{23} are real. The unitary transformation U_l follows immediately,

$$U_l = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -e^{2i\pi\frac{z+\delta}{n}}\sqrt{2}\sin\theta & e^{2i\pi\frac{z+\delta}{n}}\sqrt{2}\cos\theta \\ e^{-i\pi\frac{2z+\delta}{n}} & e^{-i\pi\frac{2z+\delta}{n}}\cos\theta & e^{-i\pi\frac{2z+\delta}{n}}\sin\theta \\ -e^{-i\pi\frac{\delta}{n}} & e^{-i\pi\frac{\delta}{n}}\cos\theta & e^{-i\pi\frac{\delta}{n}}\sin\theta \end{pmatrix}, \quad (\text{C.10})$$

with the angle θ specified by

$$\tan 2\theta = \frac{2\sqrt{2}\tilde{m}_{12}}{\tilde{m}_{11} - \tilde{m}_{22} - \tilde{m}_{23}}. \quad (\text{C.11})$$

Finally the charged lepton mass eigenvalues are

$$\begin{aligned} m_{l_1}^2 &= \tilde{m}_{22} - \tilde{m}_{23}, \\ m_{l_2}^2 &= \frac{1}{2} \left[\tilde{m}_{11} + \tilde{m}_{22} + \tilde{m}_{23} - \text{sign}((\tilde{m}_{11} - \tilde{m}_{22} - \tilde{m}_{23}) \cos 2\theta) \sqrt{(\tilde{m}_{11} - \tilde{m}_{22} - \tilde{m}_{23})^2 + 8m_{12}^2} \right], \\ m_{l_3}^2 &= \frac{1}{2} \left[\tilde{m}_{11} + \tilde{m}_{22} + \tilde{m}_{23} + \text{sign}((\tilde{m}_{11} - \tilde{m}_{22} - \tilde{m}_{23}) \cos 2\theta) \sqrt{(\tilde{m}_{11} - \tilde{m}_{22} - \tilde{m}_{23})^2 + 8m_{12}^2} \right]. \end{aligned}$$

(iv) $G_l = Z_2^{c^{n/2}} = \{1, c^{n/2}\}$, $X_{lr} = \{\rho_r(c^\gamma d^\delta), \rho_r(abc^\gamma d^\delta)\}$

- $X_{lr} = \rho_r(c^\gamma d^\delta)$ The charged lepton mass matrix is constrained to be of the following form

$$m_l^\dagger m_l = \begin{pmatrix} \tilde{m}_{11} & \tilde{m}_{12}e^{i\pi\frac{2\gamma-\delta}{n}} & 0 \\ \tilde{m}_{12}e^{-i\pi\frac{2\gamma-\delta}{n}} & \tilde{m}_{22} & 0 \\ 0 & 0 & \tilde{m}_{33} \end{pmatrix}, \quad (\text{C.12})$$

where \tilde{m}_{11} , \tilde{m}_{12} , \tilde{m}_{22} and \tilde{m}_{33} are real. It is diagonalized by the unitary matrix U_l with

$$U_l = \begin{pmatrix} e^{i\pi\frac{\gamma}{n}}\cos\theta & e^{i\pi\frac{\gamma}{n}}\sin\theta & 0 \\ -e^{i\pi\frac{\delta-\gamma}{n}}\sin\theta & e^{i\pi\frac{\delta-\gamma}{n}}\cos\theta & 0 \\ 0 & 0 & e^{-i\pi\frac{\delta}{n}} \end{pmatrix}, \quad (\text{C.13})$$

where

$$\tan 2\theta = \frac{2\tilde{m}_{12}}{\tilde{m}_{22} - \tilde{m}_{11}}. \quad (\text{C.14})$$

The charged lepton masses are determined to be

$$m_{l_1}^2 = \frac{1}{2} \left[\tilde{m}_{11} + \tilde{m}_{22} - \text{sign}((\tilde{m}_{22} - \tilde{m}_{11}) \cos 2\theta) \sqrt{(\tilde{m}_{22} - \tilde{m}_{11})^2 + 4m_{12}^2} \right],$$

$$\begin{aligned} m_{l_2}^2 &= \frac{1}{2} \left[\tilde{m}_{11} + \tilde{m}_{22} + \text{sign}((\tilde{m}_{22} - \tilde{m}_{11}) \cos 2\theta) \sqrt{(\tilde{m}_{22} - \tilde{m}_{11})^2 + 4\tilde{m}_{12}^2} \right], \\ m_{l_3}^2 &= \tilde{m}_{33}. \end{aligned} \tag{C.15}$$

- $X_{l\mathbf{r}} = \rho_{\mathbf{r}}(abc^\gamma d^\delta)$ Similar to the discussed situation that $Z_2 \times CP$ is preserved in the neutrino sector, the CP transformation should be symmetric as well otherwise the charged lepton masses would be at least partially degenerate.³ Therefore we shall focus on the case of $\delta = 2\gamma \bmod n$ in the following. Then the charged lepton mass matrix is fixed to be

$$m_l^\dagger m_l = \begin{pmatrix} \tilde{m}_{11} & \tilde{m}_{12}e^{i\phi} & 0 \\ \tilde{m}_{12}e^{-i\phi} & \tilde{m}_{11} & 0 \\ 0 & 0 & \tilde{m}_{33} \end{pmatrix}, \tag{C.16}$$

where \tilde{m}_{11} , \tilde{m}_{12} , \tilde{m}_{33} and ϕ are free real parameters. Notice that $m_l^\dagger m_l$ is independent of the parameter γ . The unitary matrix U_l is of the form

$$U_l = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi} & e^{i\phi} & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}. \tag{C.17}$$

The charged lepton masses are given by

$$\begin{aligned} m_{l_1}^2 &= \tilde{m}_{11} - \tilde{m}_{12}, \\ m_{l_2}^2 &= \tilde{m}_{11} + \tilde{m}_{12}, \\ m_{l_3}^2 &= \tilde{m}_{33}. \end{aligned} \tag{C.18}$$

(v) $G_l = Z_2^{d^{n/2}} = \{1, d^{n/2}\}$, $X_{l\mathbf{r}} = \{\rho_{\mathbf{r}}(c^\gamma d^\delta), \rho_{\mathbf{r}}(a^2bc^\gamma d^\delta)\}$

- $X_{l\mathbf{r}} = \rho_{\mathbf{r}}(c^\gamma d^\delta)$ In this case, the charged lepton mass matrix takes the form

$$m_l^\dagger m_l = \begin{pmatrix} \tilde{m}_{11} & 0 & 0 \\ 0 & \tilde{m}_{22} & \tilde{m}_{23}e^{-i\pi\frac{\gamma-2\delta}{n}} \\ 0 & \tilde{m}_{23}e^{i\pi\frac{\gamma-2\delta}{n}} & \tilde{m}_{33} \end{pmatrix}, \tag{C.19}$$

where \tilde{m}_{11} , \tilde{m}_{22} , \tilde{m}_{23} and \tilde{m}_{33} are real. The charged lepton diagonalization matrix is

$$U_l = \begin{pmatrix} e^{i\pi\frac{\gamma}{n}} & 0 & 0 \\ 0 & e^{i\pi\frac{\delta-\gamma}{n}} \cos \theta & e^{i\pi\frac{\delta-\gamma}{n}} \sin \theta \\ 0 & -e^{-i\pi\frac{\delta}{n}} \sin \theta & e^{-i\pi\frac{\delta}{n}} \cos \theta \end{pmatrix}, \tag{C.20}$$

with

$$\tan 2\theta = \frac{2\tilde{m}_{23}}{\tilde{m}_{33} - \tilde{m}_{22}}. \tag{C.21}$$

³From the remnant symmetry invariant conditions in eq. (C.1), we can derive that $U_l^\dagger X_{l3} U_l^*$ should be a diagonal matrix. As a consequence, the CP transformation X_{l3} is symmetric.

The mass eigenvalues of the charged lepton are found to be

$$\begin{aligned}
 m_{l_1}^2 &= \tilde{m}_{11}, \\
 m_{l_2}^2 &= \frac{1}{2} \left[\tilde{m}_{22} + \tilde{m}_{33} - \text{sign}((\tilde{m}_{33} - \tilde{m}_{22}) \cos 2\theta) \sqrt{(\tilde{m}_{33} - \tilde{m}_{22})^2 + 4\tilde{m}_{23}^2} \right], \\
 m_{l_3}^2 &= \frac{1}{2} \left[\tilde{m}_{22} + \tilde{m}_{33} + \text{sign}((\tilde{m}_{33} - \tilde{m}_{22}) \cos 2\theta) \sqrt{(\tilde{m}_{33} - \tilde{m}_{22})^2 + 4\tilde{m}_{23}^2} \right]. \quad (\text{C.22})
 \end{aligned}$$

- $X_{l\mathbf{r}} = \rho_{\mathbf{r}}(a^2bc^\gamma d^\delta)$ This generalized CP transformation is symmetric only if $\gamma = 2\delta \pmod n$. One can easily find that the charged lepton mass matrix is constrained to be of the form

$$m_l^\dagger m_l = \begin{pmatrix} \tilde{m}_{11} & 0 & 0 \\ 0 & \tilde{m}_{22} & \tilde{m}_{23}e^{i\phi} \\ 0 & \tilde{m}_{23}e^{-i\phi} & \tilde{m}_{22} \end{pmatrix}, \quad (\text{C.23})$$

where \tilde{m}_{11} , \tilde{m}_{22} , \tilde{m}_{23} and ϕ are real. It is diagonalized by the unitary matrix

$$U_l = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & e^{i\phi} & e^{i\phi} \\ 0 & -1 & 1 \end{pmatrix}. \quad (\text{C.24})$$

The charged lepton masses are

$$\begin{aligned}
 m_{l_1}^2 &= \tilde{m}_{11}, \\
 m_{l_2}^2 &= \tilde{m}_{22} - \tilde{m}_{23}, \\
 m_{l_3}^2 &= \tilde{m}_{22} + \tilde{m}_{23}. \quad (\text{C.25})
 \end{aligned}$$

(vi) $G_l = Z_2^{c^{n/2}d^{n/2}} = \{1, c^{n/2}d^{n/2}\}$, $X_{l\mathbf{r}} = \{\rho_{\mathbf{r}}(c^\gamma d^\delta), \rho_{\mathbf{r}}(bc^\gamma d^\delta)\}$

- $X_{l\mathbf{r}} = \rho_{\mathbf{r}}(c^\gamma d^\delta)$ Remnant symmetry leads to the following charged lepton mass matrix

$$m_l^\dagger m_l = \begin{pmatrix} \tilde{m}_{11} & 0 & \tilde{m}_{13}e^{i\pi\frac{\gamma+\delta}{n}} \\ 0 & \tilde{m}_{22} & 0 \\ \tilde{m}_{13}e^{-i\pi\frac{\gamma+\delta}{n}} & 0 & \tilde{m}_{33} \end{pmatrix}, \quad (\text{C.26})$$

where \tilde{m}_{11} , \tilde{m}_{13} , \tilde{m}_{22} and m_{33} are real parameters. The unitary transformation U_l is of the form

$$U_l = \begin{pmatrix} e^{i\pi\frac{\gamma}{n}} \cos \theta & 0 & e^{i\pi\frac{\gamma}{n}} \sin \theta \\ 0 & e^{i\pi\frac{\delta-\gamma}{n}} & 0 \\ -e^{-i\pi\frac{\delta}{n}} \sin \theta & 0 & e^{-i\pi\frac{\delta}{n}} \cos \theta \end{pmatrix}, \quad (\text{C.27})$$

with

$$\tan 2\theta = \frac{2\tilde{m}_{13}}{\tilde{m}_{33} - \tilde{m}_{11}}. \quad (\text{C.28})$$

The charged lepton mass eigenvalues are given by

$$\begin{aligned}
 m_{l_1}^2 &= \frac{1}{2} \left[\tilde{m}_{11} + \tilde{m}_{33} - \text{sign}((\tilde{m}_{33} - \tilde{m}_{11}) \cos 2\theta) \sqrt{(\tilde{m}_{33} - \tilde{m}_{11})^2 + 4\tilde{m}_{13}^2} \right], \\
 m_{l_2}^2 &= \tilde{m}_{22}, \\
 m_{l_3}^2 &= \frac{1}{2} \left[\tilde{m}_{11} + \tilde{m}_{33} + \text{sign}((\tilde{m}_{33} - \tilde{m}_{11}) \cos 2\theta) \sqrt{(\tilde{m}_{33} - \tilde{m}_{11})^2 + 4\tilde{m}_{13}^2} \right]. \quad (\text{C.29})
 \end{aligned}$$

- $X_{l\mathbf{r}} = \rho_{\mathbf{r}}(bc^\gamma d^\delta)$ The non-degeneracy of the charged lepton masses requires $\gamma + \delta = 0 \pmod n$ for which the generalized CP transformation matrix $\rho_{\mathbf{r}}(bc^\gamma d^\delta)$ is symmetric. The charged lepton mass matrix fulfilling the invariant condition in eq. (C.1) is of the form

$$m_l^\dagger m_l = \begin{pmatrix} \tilde{m}_{11} & 0 & \tilde{m}_{13}e^{i\phi} \\ 0 & \tilde{m}_{22} & 0 \\ \tilde{m}_{13}e^{-i\phi} & 0 & \tilde{m}_{11} \end{pmatrix}, \quad (\text{C.30})$$

where \tilde{m}_{11} , \tilde{m}_{13} , \tilde{m}_{22} and ϕ are real. The charged lepton diagonalization matrix is

$$U_l = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi} & 0 & e^{i\phi} \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{pmatrix}. \quad (\text{C.31})$$

Finally the charged lepton masses are

$$\begin{aligned}
 m_{l_1}^2 &= \tilde{m}_{11} - \tilde{m}_{13}, \\
 m_{l_2}^2 &= \tilde{m}_{22}, \\
 m_{l_3}^2 &= \tilde{m}_{11} + \tilde{m}_{13}. \quad (\text{C.32})
 \end{aligned}$$

Comparing with phenomenological predictions of $Z_2 \times CP$ in the neutrino sector analyzed in section 3.2, we see that the diagonalization matrix U_l is of the same form as U_ν provided the remnant flavor and remnant CP symmetries are the same in the two occasions.

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