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Free Bosonic Vertex Operator Algebras on Genus Two Riemann Surfaces I

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Abstract: We define the partition and n-point functions for a vertex operator algebra on a genus two Riemann surface formed by sewing two tori together. We obtain closed formulas for the genus two partition function for the Heisenberg free bosonic string and for any pair of simple Heisenberg modules. We prove that the partition function is holomorphic in the sewing parameters on a given suitable domain and describe its modular properties for the Heisenberg and lattice vertex operator algebras and a continuous orbifolding of the rank two fermion vertex operator super algebra. We compute the genus two Heisenberg vector n-point function and show that the Virasoro vector one point function satisfies a genus two Ward identity for these theories.

Contents

1. Introduction

One of the most striking features of Vertex Operator Algebras (VOAs) or chiral conformal field theory is the occurrence of *elliptic functions* and *modular forms*, manifested

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in the form of n -point correlation trace functions. This phenomenon has been present in string theory since the earliest days e.g. [GSW,P]. In mathematics it dates from the Conway-Norton conjectures [CN] proved by Borcherds ([B1,B2]), and Zhu's important paper [Z1]. Physically, we are dealing with probability amplitudes corresponding to a complex torus (compact Riemann surface of genus one) inflicted with n punctures corresponding to local fields (vertex operators). For a VOA $V = \oplus V_n$, the most familiar correlation function is the 0-point function, also called the *partition function* or *graded dimension*

$$Z_V^{(1)}(q) = q^{-c/24} \sum_n \dim V_n q^n, \tag{1}$$

(c is the central charge). An example which motivates much of the present paper is that of a lattice theory V_L associated to a positive-definite even lattice L. Then c is the rank of L and

$$Z_{V_L}^{(1)}(q) = \frac{\theta_L(q)}{\eta(q)^c}, \tag{2}$$

for the Dedekind eta function $\eta(q) = q^{1/24}\prod_n(1-q^n)$ and $\theta_L(q)$ is the usual theta function of L. Both $\theta_L(q)$ and $\eta(q)^c$ are (holomorphic) elliptic modular forms of weight $c/2$ on a certain congruence subgroup of $SL(2,\mathbb{Z})$, so that Z_{V_L} is an elliptic modular function of weight zero on the same subgroup. It is widely expected that an analogous result holds for *any rational* vertex operator algebra, namely that $Z_V^{(1)}(q)$ is a modular function of weight zero on a congruence subgroup of $SL(2,\mathbb{Z})$.

There are natural physical and mathematical reasons for wanting to extend this picture to Riemann surfaces of *higher genus*. In particular, we want to know if there are natural analogs of (1) and (2) for arbitrary rational vertex operator algebras and arbitrary genus, in which *genus g Siegel modular forms* occur. This is considerably more challenging than the case of genus one. Many, but not all, of the new difficulties that arise are already present at genus two, and it is this case that we are concerned with in the present paper and a companion paper [MT4]. Our goal, then, is this: given a vertex operator algebra V, to define the partition and n-point correlation function on a compact Riemann surface of genus two which is associated to V, and study their convergence and automorphic properties. An overview of aspects of this program is given in the Introduction to [MT2]. Brief discussions of some of our methods and results can also be found in [T,MT3 and MT6].

The study of genus two (and higher) partition functions and correlation functions has a long history in conformal field theory e.g. [EO,FS,DP,So1,So2,BK,Kn,GSW,P] and, indeed, these ideas have heavily influenced our approach. Likewise, in pure mathematics, other approaches based on algebraic geometry have been been developed to describe n-point correlation functions but *not* the partition function e.g. [TUY,KNTY,Z2,U]. Our approach is constructively based *only* on the properties of a VOA in the spirit of Zhu's genus one theory [Z1] with no a priori assumptions made about the analytic or modular properties of partition or n-point functions. Rather, in our approach, these genus two objects are formally defined and are then proved to be analytic and modular in appropriate domains for the VOAs considered.

In our approach, we define the genus two partition and n-point functions in terms of genus one data coming from the VOA V. There are two rather different ways to obtain a compact Riemann surface of genus two from surfaces of genus one - one may sew two separate tori together, or self-sew a torus (i.e. attach a handle). This is discussed at length in [MT2] where we refer to these two schemes as the ϵ- and ρ -formalism respectively. In the present paper we concentrate solely on developing a theory of partition and n-point correlation functions in the ϵ-formalism. We discuss the corresponding theory in the ρ -formalism in a companion paper [MT4].

The ϵ-formalism developed in [MT2] is reviewed in Sect. 2 below. This is concerned with expressing a differential 2-form $\omega^{(2)}$ (the normalized differential of the second kind) in terms of a pair of infinite matrices A_i, whose entries are quasi-modular forms associated with the two sewn tori. This allows us to obtain explicit expressions for genus two holomorphic one forms ν_1, ν_2 and the period matrix Ω in terms of this genus one data. In particular, Ω is determined by a holomorphic map

$$\mathcal{D}^{\epsilon} \xrightarrow{F^{\epsilon}} \mathbb{H}_2, \tag{3}$$

where for $g \geq 1, \mathbb{H}_g$ denotes the genus g Siegel upper half-space. Then $\mathcal{D}^{\epsilon} \subseteq \mathbb{H}_1 \times \mathbb{H}_1 \times \mathbb{C}$ is the domain consisting of triples $(\tau_1, \tau_2, \epsilon)$ which correspond to a pair of complex tori of modulus τ_1, τ_2 sewn together by identifying two annular regions via a sewing parameter ϵ. This sewing produces a compact Riemann surface of genus two, which assigns to each point of $\mathcal{D}^{\epsilon}$ the period matrix Ω of the sewn surface via the map F^{ϵ}.

In Sect. 3 we introduce some graph-theoretic technology which provides a convenient way of describing $\omega^{(2)}$, ν_i and Ω in terms of the ϵ-formalism. Similar graphical techniques are employed later on as a means of computing the genus two partition function and n-point functions for the free bosonic Heisenberg VOA and its modules.

Section 4 is a brief review of some necessary background on VOA theory and the Li-Zamolodchikov or Li-Z metric. We assume throughout that the Li-Z metric is unique and invertible (which follows if V is simple [Li]).

Section 5 develops a theory of n-point functions for VOAs on Riemann surfaces of genus 0, 1 and 2 motivated by ideas in conformal field theory. The Zhu theory [Z1] of genus one n-point functions is reformulated in this language in terms of the self-sewing of a Riemann sphere to obtain a torus. We give a formal definition of genus two n-point functions based on the given sewing formalism. We also emphasize the interpretation of n-point functions in terms of formal differential forms.

The genus two partition function involves extending (3) to a diagram

$$\begin{array}{ccc} \mathcal{D}^{\epsilon} & \xrightarrow{F^{\epsilon}} & \mathbb{H}_2 \\ & \searrow & \downarrow \\ & & \mathbb{C} \end{array}$$

where the partition function maps $\mathcal{D}^{\epsilon} \to \mathbb{C}$, and is defined purely in terms of genus one data coming from V. Explicitly, the genus two partition function of V is *a priori* a formal power series in the variables ϵ, q_1, q_2 (where as usual, $q = e^{2\pi i \tau}$, etc.) given by

$$Z_V^{(2)}(\tau_1, \tau_2, \epsilon) = \sum_{n \geq 0} \epsilon^n \sum_{u \in V_{[n]}} Z_V^{(1)}(u, \tau_1) Z_V^{(1)}(\bar{u}, \tau_2). \tag{4}$$

Here, $Z_V^{(1)}(u, \tau)$ is a genus one 1-point function with $\bar{u}$ the Li-Z metric dual of u. The precise meaning of (4) together with similar definitions for n-point functions, is given in Sect. 5.

In Sects. 6 and 7 we investigate the case of the free bosonic Heisenberg VOA M and the expression corresponding to (4) for a pair of simple M-modules. This later case is used to analyze lattice VOAs and the bosonized version of the rank two fermion Vertex Operator Super Algebra. We find in all these cases that (4) is a holomorphic function on $\mathcal{D}^{\epsilon}$. It is natural to expect that this result holds in much wider generality. Section 6 is devoted to the Heisenberg VOA M. In this case, holomorphy depends on an interesting new formula for the genus two partition function. Namely, we prove (Theorem 5) by reinterpreting (4) in terms of certain graphical expansion, that

$$Z_M^{(2)}(\tau_1, \tau_2, \epsilon) = \frac{Z_M^{(1)}(\tau_1)Z_M^{(1)}(\tau_2)}{\det(I - A_1A_2)^{1/2}}. \tag{5}$$

Here, the A_i are the infinite matrices of Sect. 2 and $Z_M^{(1)}(\tau_i) = 1/\eta(q_i)$. The infinite determinant that occurs in (5) was introduced and discussed at length in [MT2]. The results obtained there are important here, as are the explicit computations of genus one 1-point functions obtained in [MT1]. We also give in Sect. 6 a product formula for the infinite determinant (Theorem 6) which depends on the graphical interpretation of the entries of the A_i.

The domain $\mathcal{D}^{\epsilon}$ admits the group $G_0 = SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$ as automorphisms (in fact, there is a larger automorphism group G that contains G_0 with index 2). We show (cf. Theorem 8) that the partition function $Z_M^{(2)}(\tau_1, \tau_2, \epsilon)$ is an automorphic form of weight $-1/2$ on G. This is a bit imprecise in several ways: we have not explained here what the automorphy factor is, and in fact this is an interesting point because it depends on the map F^{ϵ}. Similarly to the eta-function, there is a 24$^{\text{th}}$ root of unity, corresponding to a character of G, that intervenes in the functional equation. These properties of $Z_M(\tau_1, \tau_2, \epsilon)$ justify the idea that it should be thought of as the genus two analog of $\eta(q)^{-1}$ in the ϵ-formalism.

We conclude Sect. 6 by computing, by means of the graphical technique, the genus two n-point function for n Heisenberg vectors in terms of symmetric tensor products of the differential 2-form $\omega^{(2)}$ in Theorem 10. This allows us to also find the Virasoro vector 1-point function in terms of the genus two projective connection.

Section 7 is concerned with the genus two n-point function associated with a pair of Heisenberg simple modules. We obtain a closed formula for the partition function in Theorem 11 and the Heisenberg vector n-point function in terms of symmetric tensor products of $\omega^{(2)}$ and ν_i in Theorem 13. We also derive a genus two Ward identity for the Virasoro vector 1-point function in Proposition 10. We apply these results in Theorem 14 to the case of a lattice VOA V_L to find a natural genus two generalization of (2), namely

$$\frac{Z_{V_L}^{(2)}(\tau_1, \tau_2, \epsilon)}{Z_M^{(2)}(\tau_1, \tau_2, \epsilon)} = \theta_L^{(2)}(\Omega), \tag{6}$$

where $\theta_L^{(2)}(\Omega)$ is the genus two Siegel theta function of the lattice L. Similarly, the Virasoro 1-point function obeys a Ward identity. Finally, we consider the bosonized version of a continuous orbifolding of the rank two fermion vertex super algebra to find the partition function is expressed in terms of the genus two Riemann theta series.

2. Genus Two Riemann Surface from Two Sewn Tori

In this section we review some of the main results of [MT2] relevant to the present work. We review one of the two separate constructions of a genus two Riemann surface discussed there based on a general sewing formalism due to Yamada [Y]. In this construction, which we refer to as the ϵ-formalism, we parameterize a genus two Riemann surface by sewing together two once-punctured tori. Then various genus two structures such as the period matrix Ω can be determined in terms of genus one data. In particular, Ω is described by an explicit formula which defines a holomorphic map from a specified domain $\mathcal{D}^{\epsilon}$ into the genus two Siegel upper half plane $\mathbb{H}_2$. This map is equivariant under a suitable subgroup of $Sp(4, \mathbb{Z})$. We also review the convergence and holomorphy of an infinite determinant that naturally arises and which plays a dominant rôle later on.

2.1. Some elliptic function theory. We begin with the definition of various modular and elliptic functions that permeate this work [MT1,MT2]. We define

$$P_2(\tau, z) = \wp(\tau, z) + E_2(\tau)$$
$$= \frac{1}{z^2} + \sum_{k=2}^{\infty}(k-1)E_k(\tau)z^{k-2}, \quad (7)$$

where $\tau \in \mathbb{H}_1$, the complex upper half-plane, and where $\wp(\tau, z)$ is the Weierstrass function and $E_k(\tau)$ is equal to 0 for k odd, and for k even is the Eisenstein series

$$E_k(\tau) = E_k(q) = -\frac{B_k}{k!} + \frac{2}{(k-1)!}\sum_{n\geq 1}\sigma_{k-1}(n)q^n. \quad (8)$$

Here and below, we take $q = \exp(2\pi i\tau)$; $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$, and B_k is a k^{th} Bernoulli number e.g. [Se]. If $k \geq 4$ then $E_k(\tau)$ is a holomorphic modular form of weight k on $SL(2, \mathbb{Z})$, whereas $E_2(\tau)$ is a quasi-modular form [KZ,MT2]. We define $P_1(\tau, z)$ by

$$P_1(\tau, z) = \frac{1}{z} - \sum_{k\geq 2} E_k(\tau)z^{k-1}. \quad (9)$$

Noting $P_2 = -\frac{d}{dz}P_1$ we define elliptic functions $P_k(\tau, z)$ for $k \geq 3$,

$$P_k(\tau, z) = \frac{(-1)^{k-1}}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}P_1(\tau, z). \quad (10)$$

Define for $k, l \geq 1$,

$$C(k, l) = C(k, l, \tau) = (-1)^{k+1}\frac{(k+l-1)!}{(k-1)!(l-1)!}E_{k+l}(\tau), \quad (11)$$

$$D(k, l, z) = D(k, l, \tau, z) = (-1)^{k+1}\frac{(k+l-1)!}{(k-1)!(l-1)!}P_{k+l}(\tau, z). \quad (12)$$

The Dedekind eta-function is defined by

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n). \quad (13)$$

2.2. The ϵ-formalism for sewing two tori. Consider a compact Riemann surface $\mathcal{S}$ of genus 2 with canonical homology basis a_1, a_2, b_1, b_2. There exist two holomorphic 1-forms ν_i, $i = 1, 2$ which we may normalize by [FK]

$$\oint_{a_i} \nu_j = 2\pi i \delta_{ij}. \tag{14}$$

These forms can also be defined via the unique singular bilinear two form $\omega^{(2)}$, known as the *normalized differential of the second kind*. It is defined by the following properties [FK,Y]:

$$\omega^{(2)}(x, y) = (\frac{1}{(x-y)^2} + \text{regular terms})dxdy \tag{15}$$

for any local coordinates x, y, with normalization

$$\int_{a_i} \omega^{(2)}(x, \cdot) = 0, \tag{16}$$

for $i = 1, 2$. Using the Riemann bilinear relations, one finds that

$$\nu_i(x) = \oint_{b_i} \omega^{(2)}(x, \cdot), \tag{17}$$

with ν_i normalized as in (14). The genus 2 period matrix Ω is then defined by

$$\Omega_{ij} = \frac{1}{2\pi i} \oint_{b_i} \nu_j \tag{18}$$

for $i, j = 1, 2$. One further finds that $\Omega \in \mathbb{H}_2$, the Siegel upper half plane (Fig. 1).

We now review a general method due to Yamada [Y] and discussed at length in [MT2] for calculating $\omega^{(2)}(x, y)$, $\nu_i(x)$ and Ω_{ij} on the genus two Riemann surface formed by sewing together two tori $\mathcal{S}_a$ for $a = 1, 2$. We shall sometimes refer to $\mathcal{S}_1$ and $\mathcal{S}_2$ as the left and right torus respectively. Consider an oriented torus $\mathcal{S}_a = \mathbb{C}/\Lambda_a$ with lattice $\Lambda_a = 2\pi i(\mathbb{Z}\tau_a \oplus \mathbb{Z})$ for $\tau_a \in \mathbb{H}_1$. For local coordinate $z_a \in \mathbb{C}/\Lambda_a$ consider the closed disk $|z_a| \le r_a$ which is contained in $\mathcal{S}_a$ provided $r_a < \frac{1}{2}D(q_a)$, where

$$D(q_a) = \min_{\lambda \in \Lambda_a, \lambda \neq 0} |\lambda|,$$

is the minimal lattice distance. Introduce a complex sewing parameter ϵ, where $|\epsilon| \le r_1 r_2 < \frac{1}{4}D(q_1)D(q_2)$ and excise the disk $\{z_a, |z_a| \le |\epsilon|/r_{\bar{a}}\}$ centered at $z_a = 0$ to form a punctured torus

$$\hat{\mathcal{S}}_a = \mathcal{S}_a \backslash \{z_a, |z_a| \le |\epsilon|/r_{\bar{a}}\}, \tag{19}$$

where we use the convention

$$\overline{1} = 2, \quad \overline{2} = 1. \tag{20}$$

Defining the annulus

$$\mathcal{A}_a = \{z_a, |\epsilon|/r_{\bar{a}} \le |z_a| \le r_a\} \subset \hat{\mathcal{S}}_a, \tag{21}$$

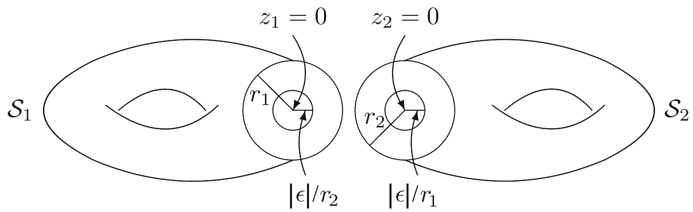

Fig. 1. Sewing Two Tori

we identify $\mathcal{A}_1$ with $\mathcal{A}_2$ via the sewing relation

$$z_1 z_2 = \epsilon. \tag{22}$$

The genus two Riemann surface is parameterized by the domain

$$\mathcal{D}^\epsilon = \{(\tau_1, \tau_2, \epsilon) \in \mathbb{H}_1 \times \mathbb{H}_1 \times \mathbb{C} \mid |\epsilon| < \frac{1}{4} D(q_1) D(q_2)\}. \tag{23}$$

We next introduce the infinite dimensional matrix $A_a(\tau_a, \epsilon) = (A_a(k, l, \tau_a, \epsilon))$ for $k, l \geq 1$, where

$$A_a(k, l, \tau_a, \epsilon) = \frac{\epsilon^{(k+l)/2}}{\sqrt{kl}} C(k, l, \tau_a). \tag{24}$$

The matrices A_1, A_2 play a dominant role both here and in our later discussion of the free bosonic VOA and its modules on a genus two Riemann surface. In particular, the matrix $I - A_1 A_2$ and $\det(I - A_1 A_2)$ (where I denotes the infinite identity matrix) play an important role where $\det(I - A_1 A_2)$ is defined by

$$\begin{aligned} \log \det(I - A_1 A_2) &= \operatorname{Tr} \log(I - A_1 A_2) \\ &= - \sum_{n \geq 1} \frac{1}{n} \operatorname{Tr}((A_1 A_2)^n). \end{aligned} \tag{25}$$

One finds

Theorem 1. (a) (*op. cit., Proposition* 1) *The infinite matrix*

$$(I - A_1 A_2)^{-1} = \sum_{n \geq 0} (A_1 A_2)^n, \tag{26}$$

is convergent for $(\tau_1, \tau_2, \epsilon) \in \mathcal{D}^\epsilon$.

(b) (*op. cit., Theorem* 2 *&* *Proposition* 3) $\det(I - A_1 A_2)$ *is non-vanishing and holomorphic for* $(\tau_1, \tau_2, \epsilon) \in \mathcal{D}^\epsilon$. □

The bilinear two form $\omega^{(2)}(x, y)$, the holomorphic one forms $\nu_i(x)$ and the period matrix Ω_{ij} are given in terms of the matrices A_a and holomorphic one forms on the punctured torus $\hat{\mathcal{S}}_a$ given by

$$a_a(k, x) = \sqrt{k} \epsilon^{k/2} P_{k+1}(\tau_a, x) dx. \tag{27}$$

Letting $a_a(x)$, $a_a^T(x)$ denote the infinite row, respectively column vector with elements (27) we have:

Theorem 2. (*op. cit., Lemma* 2*, Proposition* 1*, Theorem* 4)

$$\omega^{(2)}(x,y)=\begin{cases}P_2(\tau_a,x-y)dxdy+a_a(x)A_{\bar{a}}(I-A_aA_{\bar{a}})^{-1}a_a^T(y), & x,y\in\hat{\mathcal{S}}_a,\\ -a_a(x)(I-A_{\bar{a}}A_a)^{-1}a_{\bar{a}}^T(y), & x\in\hat{\mathcal{S}}_a,\ y\in\hat{\mathcal{S}}_{\bar{a}}.\end{cases} \tag{28}$$

□

Applying (17) we then find (op. cit., Theorem 4)

$$\nu_a(x)=\begin{cases}dx+\epsilon^{1/2}(a_a(x)A_{\bar{a}}(I-A_aA_{\bar{a}})^{-1})(1), & x\in\hat{\mathcal{S}}_a,\\ -\epsilon^{1/2}(a_{\bar{a}}(x)(I-A_aA_{\bar{a}})^{-1})(1), & x\in\hat{\mathcal{S}}_{\bar{a}},\end{cases} \tag{29}$$

where (1) refers to the (1)-entry of a vector. Furthermore applying (18) we have

Theorem 3. (*op. cit., Theorem* 4) *The* ϵ*-formalism determines a holomorphic map*

$$\begin{aligned}F^\epsilon:\mathcal{D}^\epsilon&\to\mathbb{H}_2,\\ (\tau_1,\tau_2,\epsilon)&\mapsto\Omega(\tau_1,\tau_2,\epsilon),\end{aligned} \tag{30}$$

where $\Omega=\Omega(\tau_1,\tau_2,\epsilon)$ *is given by*

$$2\pi i\Omega_{11}=2\pi i\tau_1+\epsilon(A_2(I-A_1A_2)^{-1})(1,1), \tag{31}$$

$$2\pi i\Omega_{22}=2\pi i\tau_2+\epsilon(A_1(I-A_2A_1)^{-1})(1,1), \tag{32}$$

$$2\pi i\Omega_{12}=-\epsilon(I-A_1A_2)^{-1}(1,1). \tag{33}$$

Here (1, 1) *refers to the* (1, 1)*-entry of a matrix.* □

$\mathcal{D}^\epsilon$ is preserved under the action of $G\simeq(SL(2,\mathbb{Z})\times SL(2,\mathbb{Z}))\rtimes\mathbb{Z}_2$, the direct product of two copies of $SL(2,\mathbb{Z})$ (the left and right torus modular groups) which are interchanged upon conjugation by an involution β as follows:

$$\begin{aligned}\gamma_1.(\tau_1,\tau_2,\epsilon)&=\left(\frac{a_1\tau_1+b_1}{c_1\tau_1+d_1},\tau_2,\frac{\epsilon}{c_1\tau_1+d_1}\right),\\ \gamma_2.(\tau_1,\tau_2,\epsilon)&=\left(\tau_1,\frac{a_2\tau_2+b_2}{c_2\tau_2+d_2},\frac{\epsilon}{c_2\tau_2+d_2}\right),\\ \beta.(\tau_1,\tau_2,\epsilon)&=(\tau_2,\tau_1,\epsilon),\end{aligned} \tag{34}$$

for $(\gamma_1,\gamma_2)\in SL(2,\mathbb{Z})\times SL(2,\mathbb{Z})$ with $\gamma_i=\begin{pmatrix}a_i & b_i\\ c_i & d_i\end{pmatrix}$. There is a natural injection $G\to Sp(4,\mathbb{Z})$ in which the two $SL(2,\mathbb{Z})$ subgroups are mapped to

$$\Gamma_1=\left\{\begin{bmatrix}a_1&0&b_1&0\\0&1&0&0\\c_1&0&d_1&0\\0&0&0&1\end{bmatrix}\right\},\quad \Gamma_2=\left\{\begin{bmatrix}1&0&0&0\\0&a_2&0&b_2\\0&0&1&0\\0&c_2&0&d_2\end{bmatrix}\right\}, \tag{35}$$

and the involution is mapped to

$$\beta=\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}. \tag{36}$$

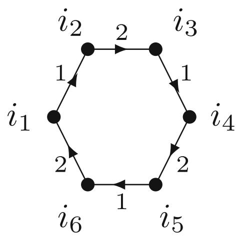

Fig. 2. Chequered Cycle

Thus as a subgroup of $Sp(4, \mathbb{Z})$, G also has a natural action on the Siegel upper half plane $\mathbb{H}_2$, where for $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(4, \mathbb{Z})$,

$$\gamma.\Omega = (A\Omega + B)(C\Omega + D)^{-1}. \tag{37}$$

One then finds

Theorem 4. (*op. cit., Theorem* 5) *F^{ϵ} is equivariant with respect to the action of G, i.e. there is a commutative diagram for $\gamma \in G$,*

$$\begin{array}{ccc} \mathcal{D}^{\epsilon} & \stackrel{F^{\epsilon}}{\to} & \mathbb{H}_2 \\ \gamma \downarrow & & \downarrow \gamma \\ \mathcal{D}^{\epsilon} & \stackrel{F^{\epsilon}}{\to} & \mathbb{H}_2 \end{array}$$

□

3. Graphical Expansions

3.1. Rotationless and Chequered Cycles. We set up some notation and discuss certain types of labeled graphs. These arise directly from consideration of the terms that appear in the expressions for $\omega^{(2)}(x, y)$, $\nu_i(x)$ and Ω_{ij} reviewed in the last section, and will later play an important rôle in the analysis of genus two partition functions for vertex operator algebras.

We introduce the notion of a *chequered cycle* as a (clockwise) oriented, labeled polygon L with $2n$ nodes for some integer $n \geq 0$, and nodes labeled by arbitrary positive integers. Moreover, edges carry a label 1 or 2 which alternate as one moves around the polygon (Fig. 2). A chequered cycle is said to be *rotationless* when its graph admits no non-trivial rotations where a rotation is an orientation-preserving automorphism of the graph which preserves the node labels. (See the Appendix for more details.)

We call a node with label 1 *distinguished* if its abutting edges are of type $\stackrel{2}{\longrightarrow} \stackrel{1}{\bullet} \stackrel{1}{\longrightarrow}$. Set

$$\begin{aligned} \mathcal{R} &= \{\text{isomorphism classes of rotationless chequered cycles}\}, \\ \mathcal{R}_{21} &= \{\text{isomorphism classes of rotationless chequered cycles} \\ &\qquad \text{with a distinguished node}\}, \\ \mathcal{L}_{21} &= \{\text{isomorphism classes of chequered cycles with a} \\ &\qquad \textit{unique} \text{ distinguished node}\}. \end{aligned} \tag{38}$$

$$\overset{1}{\bullet} \overset{a}{\longrightarrow} \overset{i}{\bullet} \quad \ldots \quad \overset{j}{\bullet} \overset{b}{\longrightarrow} \overset{1}{\bullet}$$

Fig. 3. Chequered necklace

Let S be a commutative ring and $S[t]$ the polynomial ring with coefficients in S. Let M_1 and M_2 be infinite matrices with (k,l)-entries

$$M_a(k,l) = t^{k+l} s_a(k,l) \tag{39}$$

for $a = 1, 2$ and $k, l \geq 1$, where $s_a(k,l) \in S$. Given this data, we define a map, or *weight function*,

$$\zeta : \{\text{chequered cycles}\} \longrightarrow S[t]$$

as follows: if L is a chequered cycle then L has edges E labeled as $\overset{k}{\bullet} \overset{a}{\longrightarrow} \overset{l}{\bullet}$. Then set $\zeta(E) = M_a(k,l)$ and

$$\zeta(L) = \prod \zeta(E), \tag{40}$$

where the product is taken over all edges of L.

It is useful to also introduce a variation on the theme of chequered cycles namely *oriented chequered necklaces*. These are connected graphs with $n \geq 3$ nodes, $(n-2)$ of which have valency 2 and two of which have valency 1 (these latter are the *end nodes*) together with an orientation, say from left to right. There is also a degenerate necklace N_0 with a single node and no edges. As before, nodes are labeled with arbitrary positive integers and edges are labeled with an index 1 or 2 which alternate along the necklace. For such a necklace N, we define the weight function $\zeta(N)$ as a product of edge weights as in (40), with $\zeta(N_0) = 1$.

Among all chequered necklaces there is a distinguished set for which both end nodes are labeled by 1. There are four types of such chequered necklaces, which may be further distinguished by the labels of the two edges at the extreme left and right. Using the convention (20) we say that the chequered necklace of Fig. 3 is of *type ab* for $a, b \in \{1, 2\}$, and set

$$\mathcal{N}_{ab} = \{\text{isomorphism classes of oriented chequered necklaces of type } ab\}, \tag{41}$$

$$\zeta_{ab} = \sum_{N \in \mathcal{N}_{ab}} \zeta(N). \tag{42}$$

3.2. Necklace graphical expansions for $\omega^{(2)}$, ν_i and Ω_{ij}. We now apply the formalism of the previous Subsection to the expressions for $\omega^{(2)}(x,y)$, $\nu_i(x)$ and Ω_{ij} in the ϵ -formalism reviewed in Sect. 2. We begin with the period matrix Ω_{ij}. Here the ring S is taken to be the product $S_1 \times S_2$, where for $a = 1, 2$, S_a is the ring of quasi-modular forms $\mathbb{C}[E_2(\tau_a), E_4(\tau_a), E_6(\tau_a)]$, and $t = \epsilon^{1/2}$. The matrices M_a are taken to be the A_a defined in (24). Thus

$$s_a(k,l) = \frac{C(k,l,\tau_a)}{\sqrt{kl}}, \tag{43}$$

and for the edge E labeled as $\overset{k}{\bullet} \overset{a}{\longrightarrow} \overset{l}{\bullet}$ we have

$$\zeta(E) = A_a(k,l). \tag{44}$$

Recalling the notation (42), we find

Proposition 1. ([MT2], *Prop.* 4) *For* $a = 1, 2$,

$$\Omega_{aa} = \tau_a + \frac{\epsilon}{2\pi i}\zeta_{\bar{a}\bar{a}},$$
$$\Omega_{a\bar{a}} = -\frac{\epsilon}{2\pi i}\zeta_{\bar{a}a}.$$

□

Furthermore, in the notation of Sect. 3.1 we have

Proposition 2.

$$\zeta_{12} = \zeta_{21} = \prod_{L\in\mathcal{R}_{21}} (1-\zeta(L))^{-1}. \tag{45}$$

Beyond the intrinsic interest of this product formula, our main use of it will be to provide an alternate proof of Theorem 8 below. We therefore relegate the proof of Proposition 13 to the Appendix.

We can similarly obtain necklace graphical expansions for the bilinear form $\omega^{(2)}(x, y)$ and the holomorphic one forms $\nu_i(x)$. We introduce further distinguished valence one nodes labeled by $1, x$ for $x \in \hat{\mathcal{S}}_a$, the punctured torus (19). The set of edges $\{E\}$ is augmented by edges with weights defined by:

$$\begin{aligned} \zeta(\overset{1,x}{\bullet} \overset{a}{\longrightarrow} \overset{1,y}{\bullet}) &= P_2(\tau_a, x-y), \quad x, y \in \hat{\mathcal{S}}_a, \\ \zeta(\overset{1,x}{\bullet} \overset{a}{\longrightarrow} \overset{k}{\bullet}) = \zeta(\overset{k}{\bullet} \overset{a}{\longrightarrow} \overset{1,x}{\bullet}) &= \sqrt{k}\epsilon^{k/2}P_{k+1}(\tau_a, x), \quad x \in \hat{\mathcal{S}}_a, \end{aligned} \tag{46}$$

for elliptic functions (10).

Similarly to (41) we consider chequered necklaces where one or both end points are $1, x$-type labeled nodes. We thus define for $x \in \hat{\mathcal{S}}_a$ and $y \in \hat{\mathcal{S}}_b$ three isomorphism classes of oriented chequered necklaces denoted $\mathcal{N}_{ab}^{x,1}, \mathcal{N}_{ab}^{1,y}$ and $\mathcal{N}_{ab}^{x,y}$ with the following respective configurations

$$\{\overset{1,x}{\bullet} \overset{a}{\longrightarrow} \overset{i}{\bullet}\ldots\overset{j}{\bullet} \overset{b}{\longrightarrow} \overset{1}{\bullet}\}, \tag{47}$$

$$\{\overset{1}{\bullet} \overset{a}{\longrightarrow} \overset{i}{\bullet}\ldots\overset{j}{\bullet} \overset{b}{\longrightarrow} \overset{1,y}{\bullet}\}, \tag{48}$$

$$\{\overset{1,x}{\bullet} \overset{a}{\longrightarrow} \overset{i}{\bullet}\ldots\overset{j}{\bullet} \overset{b}{\longrightarrow} \overset{1,y}{\bullet}\}. \tag{49}$$

Let $\zeta_{ab}^{x,1}, \zeta_{ab}^{1,y}$ and $\zeta_{ab}^{x,y}$ denote the respective sum of the weights for each class. Comparing to (28) and (29) and applying (17) we find the following graphical expansions for the bilinear form $\omega^{(2)}(x, y)$ and the holomorphic one forms $\nu_i(x)$:

Proposition 3. *For* $a = 1, 2$,

$$\omega^{(2)}(x,y) = \begin{cases} \zeta_{aa}^{x,y}dxdy, & x, y \in \hat{\mathcal{S}}_a, \\ -\zeta_{a\bar{a}}^{x,y}dxdy, & x \in \hat{\mathcal{S}}_a,\ y \in \hat{\mathcal{S}}_{\bar{a}}, \end{cases} \tag{50}$$

$$\nu_a(x) = \begin{cases} (1+\epsilon^{1/2}\zeta_{aa}^{x,1})dx, & x \in \hat{\mathcal{S}}_a, \\ -\epsilon^{1/2}\zeta_{\bar{a}a}^{x,1}dx, & x \in \hat{\mathcal{S}}_{\bar{a}}. \end{cases} \tag{51}$$

4. Vertex Operator Algebras and the Li-Zamolodchikov Metric

4.1. Vertex operator algebras. We review some relevant aspects of vertex operator algebras ([FHL,FLM,Ka,LL,MN,MT6]). A vertex operator algebra (VOA) is a quadruple $(V, Y, \mathbf{1}, \omega)$ consisting of a $\mathbb{Z}$-graded complex vector space $V = \bigoplus_{n\in\mathbb{Z}} V_n$, a linear map $Y : V \to (\mathrm{End}\, V)[[z, z^{-1}]]$, for formal parameter z, and a pair of distinguished vectors (states): the vacuum $\mathbf{1} \in V_0$, and the conformal vector $\omega \in V_2$. For each state $v \in V$ the image under the Y map is the vertex operator

$$Y(v, z) = \sum_{n\in\mathbb{Z}} v(n)z^{-n-1}, \tag{52}$$

with modes $v(n) \in \mathrm{End}V$, where $\mathrm{Res}_{z=0}z^{-1}Y(v, z)\mathbf{1} = v(-1)\mathbf{1} = v$. Vertex operators satisfy the Jacobi identity or equivalently, operator locality or Borcherds's identity for the modes (loc. cit.).

The vertex operator for the conformal vector ω is defined as

$$Y(w, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}.$$

The modes $L(n)$ satisfy the Virasoro algebra of central charge c:

$$[L(m), L(n)] = (m - n)L(m + n) + (m^3 - m)\frac{c}{12}\delta_{m,-n}.$$

We define the homogeneous space of weight k to be $V_k = \{v \in V | L(0)v = kv\}$, where we write $\mathrm{wt}(v) = k$ for v in V_k. Then as an operator on V we have

$$v(n) : V_m \to V_{m+k-n-1}.$$

In particular, the *zero mode* $o(v) = v(\mathrm{wt}(v) - 1)$ is a linear operator on V_m. A state v is said to be *quasi-primary* if $L(1)v = 0$ and *primary* if additionally $L(2)v = 0$.

The subalgebra $\{L(-1), L(0), L(1)\}$ generates a natural action on vertex operators associated with $SL(2,\mathbb{C})$ Möbius transformations on z ([B1,DGM,FHL,Ka]). In particular, we note the inversion $z \mapsto 1/z$ for which

$$Y(v, z) \mapsto Y^\dagger(v, z) = Y(e^{zL(1)}(-\frac{1}{z^2})^{L(0)}v, \frac{1}{z}). \tag{53}$$

$Y^\dagger(v, z)$ is the *adjoint* vertex operator [FHL]. Under the dilatation $z \mapsto az$ we have

$$Y(v, z) \mapsto a^{L(0)}Y(v, z)a^{-L(0)} = Y(a^{L(0)}v, az). \tag{54}$$

We also note ([BPZ,Z2]) that under a general origin-preserving conformal map $z \mapsto w = \phi(z)$,

$$Y(v, z) \mapsto Y((\phi'(z))^{L(0)}v, w), \tag{55}$$

for any primary vector v.

We consider some particular VOAs, namely Heisenberg free boson and lattice VOAs. Consider an l-dimensional complex vector space (i.e., abelian Lie algebra) $\mathfrak{H}$ equipped with a non-degenerate, symmetric, bilinear form (,) and a distinguished orthonormal basis $a_1, a_2, \ldots a_l$. The corresponding affine Lie algebra is the Heisenberg Lie algebra $\hat{\mathfrak{H}} = \mathfrak{H} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}k$ with brackets $[k, \hat{\mathfrak{H}}] = 0$ and

$$[a_i \otimes t^m, a_j \otimes t^n] = m\delta_{i,j}\delta_{m,-n}k. \tag{56}$$

Corresponding to an element λ in the dual space $\mathfrak{H}^*$ we consider the Fock space defined by the induced (Verma) module

$$M^{(\lambda)} = U(\hat{\mathfrak{H}}) \otimes_{U(\mathfrak{H}\otimes\mathbb{C}[t]\oplus\mathbb{C}k)} \mathbb{C},$$

where $\mathbb{C}$ is the 1-dimensional space annihilated by $\mathfrak{H} \otimes t\mathbb{C}[t]$ and on which k acts as the identity and $\mathfrak{H} \otimes t^0$ via the character λ; U denotes the universal enveloping algebra. There is a canonical identification of linear spaces

$$M^{(\lambda)} = S(\mathfrak{H} \otimes t^{-1}\mathbb{C}[t^{-1}]),$$

where S denotes the (graded) symmetric algebra. The Heisenberg free boson VOA M^l corresponds to the case $\lambda = 0$. The Fock states

$$v = a_1(-1)^{e_1}.a_1(-2)^{e_2} \ldots a_1(-n)^{e_n} \ldots a_l(-1)^{f_1}.a_l(-2)^{f_2} \ldots a_l(-p)^{f_p}.\mathbf{1}, \tag{57}$$

for non-negative integers $e_i, \ldots, f_j$ form a basis of M^l with $a_i(n) \equiv a_i \otimes t^n$. The vacuum $\mathbf{1}$ is canonically identified with the identity of $M_0 = \mathbb{C}$, while the weight 1 subspace M_1 may be naturally identified with $\mathfrak{H}$. M^l is a simple VOA of central charge l.

Next we consider the case of a lattice vertex operator algebra V_L associated to a positive-definite even lattice L (cf. [B1,FLM]). Thus L is a free abelian group of rank l equipped with a positive definite, integral bilinear form $(\ ,\ ) : L \otimes L \to \mathbb{Z}$ such that (α, α) is even for $\alpha \in L$. Let $\mathfrak{H}$ be the space $\mathbb{C} \otimes_{\mathbb{Z}} L$ equipped with the $\mathbb{C}$-linear extension of $(\ ,\ )$ to $\mathfrak{H} \otimes \mathfrak{H}$ and let M^l be the corresponding Heisenberg VOA. The Fock space of the lattice theory may be described by the linear space

$$V_L = M^l \otimes \mathbb{C}[L] = \sum_{\alpha\in L} M^l \otimes e^\alpha, \tag{58}$$

where $\mathbb{C}[L]$ denotes the group algebra of L with canonical basis e^α, $\alpha \in L$. M^l may be identified with the subspace $M^l \otimes e^0$ of V_L, in which case M^l is a subVOA of V_L and the rightmost equation of (58) then displays the decomposition of V_L into irreducible M^l-modules. V_L is a simple VOA of central charge l. Each $\mathbf{1} \otimes e^\alpha \in V_L$ is a primary state of weight $\frac{1}{2}(\alpha, \alpha)$ with vertex operator (loc. cit.)

$$\begin{aligned} Y(\mathbf{1} \otimes e^\alpha, z) &= Y_-(\mathbf{1} \otimes e^\alpha, z)Y_+(\mathbf{1} \otimes e^\alpha, z)e^\alpha z^\alpha, \\ Y_\pm(\mathbf{1} \otimes e^\alpha, z) &= \exp(\mp \sum_{n>0} \frac{\alpha(\pm n)}{n} z^{\mp n}). \end{aligned} \tag{59}$$

The operators $e^\alpha \in \mathbb{C}[L]$ obey

$$e^\alpha e^\beta = \epsilon(\alpha, \beta)e^{\alpha+\beta} \tag{60}$$

for 2-cocycle $\epsilon(\alpha, \beta)$ satisfying $\epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = (-1)^{(\alpha,\beta)}$.

4.2. The Li-Zamolodchikov metric. A bilinear form $\langle\ ,\ \rangle : V \times V \longrightarrow \mathbb{C}$ is called *invariant* in case the following identity holds for all $a, b, c \in V$ ([FHL]):

$$\langle Y(a, z)b, c\rangle = \langle b, Y^{\dagger}(a, z)c\rangle, \tag{61}$$

with $Y^{\dagger}(a, z)$ the adjoint operator (53).

Remark 1. Note that

$$\begin{aligned} \langle a, b\rangle &= \mathrm{Res}_{w=0} w^{-1} \mathrm{Res}_{z=0} z^{-1} \langle Y(a, w)\mathbf{1}, Y(b, z)\mathbf{1}\rangle \\ &= \mathrm{Res}_{w=0} w^{-1} \mathrm{Res}_{z=0} z^{-1} \langle \mathbf{1}, Y^{\dagger}(a, w)Y(b, z)\mathbf{1}\rangle \\ &= \text{“}\langle \mathbf{1}, Y(a, z = \infty)Y(b, z = 0)\mathbf{1}\rangle\text{”}, \end{aligned} \tag{62}$$

with $w = 1/z$, following (53). Thus the invariant bilinear form is equivalent to what is known as the (chiral) Zamolodchikov metric in Conformal Field Theory ([BPZ,P]).

First note that any invariant bilinear form on V is necessarily symmetric by a theorem of [FHL]. Generally a VOA may have *no* non-zero invariant bilinear form, even if it is well-behaved in other ways. Examples where V is rational can be found in [DM]. Results of Li [Li] guarantee that if V_0 is spanned by the vacuum vector $\mathbf{1}$ then the following hold: (a) V has *at most one* nonzero invariant bilinear form up to scalars; (b) if V *has* a nonzero invariant bilinear form $\langle\ ,\ \rangle$ then the radical $\mathrm{Rad}\langle\ ,\ \rangle$ is the unique maximal ideal of V, and in particular V is simple if, and only if, $\langle\ ,\ \rangle$ is non-degenerate. In this case, V is *self-dual* in the sense that V is isomorphic to the contragredient module V' as a V-module. Conversely, if V is a self-dual VOA then it has a nondegenerate invariant bilinear form. All of the VOAs that occur in this paper satisfy these conditions, i.e., they are simple and self-dual with $V_0 = \mathbb{C}\mathbf{1}$. Then if we normalize so that $\langle \mathbf{1}, \mathbf{1}\rangle = 1$ then $\langle\ ,\ \rangle$ is unique and nondegenerate. We refer to this particular bilinear form as the *Li-Zamolodchikov metric* on V, or LiZ-metric for short.

Remark 2. Uniqueness entails that the LiZ-metric on the tensor product $V_1 \otimes V_2$ of a pair of simple VOAs satisfying the appropriate conditions is just the tensor product of the LiZ metrics on V_1 and V_2.

If a is a homogeneous, quasi-primary state, the component form of (61) reads

$$\langle a(n)b, c\rangle = (-1)^{\mathrm{wt}(a)} \langle b, a(2\mathrm{wt}(a) - n - 2)c\rangle. \tag{63}$$

In particular, since the conformal vector ω is quasi-primary of weight 2 we may take ω in place of a in (63) and obtain

$$\langle L(n)b, c\rangle = \langle b, L(-n)c\rangle. \tag{64}$$

The case $n = 0$ of (64) shows that the homogeneous spaces V_n, V_m are orthogonal if $n \neq m$. Taking $u = \mathbf{1}$ and using $a = a(-1)\mathbf{1}$ in (63) yields

$$\langle a, b\rangle = (-1)^{\mathrm{wt}(a)} \langle \mathbf{1}, a(2\mathrm{wt}(a) - 1)b\rangle, \tag{65}$$

for a quasi-primary, and this affords a practical way to compute the LiZ-metric.

Consider the rank one Heisenberg (free boson) VOA $M = M^1$ generated by a weight one state a with $(a, a) = 1$. Then $\langle a, a\rangle = -\langle \mathbf{1}, a(1)a(-1)\mathbf{1}\rangle = -1$. Using (56), it is straightforward to verify that in general the Fock basis consisting of vectors of the form

$$v = a(-1)^{e_1} \ldots a(-p)^{e_p}.\mathbf{1}, \tag{66}$$

for non-negative integers $\{e_i\}$ is orthogonal with respect to the LiZ-metric, and that

$$\langle v, v\rangle = \prod_{1\leq i\leq p} (-i)^{e_i} e_i!. \tag{67}$$

This result generalizes in an obvious way for a rank l free boson VOA M^l with Fock basis (57) following Remark 2.

5. Partition and *n*-Point Functions for Vertex Operator Algebras on a Riemann Surface

In this section we consider the partition and n-point functions for a VOA on a Riemann surface of genus zero, one or two. Our definitions are based on sewing schemes for the given Riemann surface in terms of one or more surfaces of lower genus and are motivated by ideas in conformal field theory especially [FS,So1,P]. We assume throughout that V has a non-degenerate LiZ metric $\langle\ ,\ \rangle$. Then for any V basis $\{u^{(a)}\}$, we may define the *dual basis* $\{\bar{u}^{(a)}\}$ with respect to the LiZ metric where

$$\langle u^{(a)}, \bar{u}^{(b)}\rangle = \delta_{ab}. \tag{68}$$

5.1. Genus zero case. We begin with the definition of the genus zero n-point function given by:

$$Z_V^{(0)}(v_1, z_1; \ldots v_n, z_n) = \langle \mathbf{1}, Y(v_1, z_1) \ldots Y(v_n, z_n)\mathbf{1}\rangle, \tag{69}$$

for $v_1, \ldots v_n \in V$. In particular, the genus zero partition (or 0-point) function is $Z_V^{(0)} = \langle \mathbf{1}, \mathbf{1}\rangle = 1$. The genus zero n-point function is a rational function of $z_1, \ldots z_n$, which we refer to as the insertion points, with possible poles at $z_i = z_j, i \neq j$ determined from the locality of the vertex operators. Thus we may consider $z_1, \ldots z_n \in \mathbb{C} \cup \{\infty\}$, the Riemann sphere, with $Z_V^{(0)}(v_1, z_1; \ldots; v_n, z_n)$ evaluated for $|z_1| > |z_2| > \cdots > |z_n|$ (e.g. [FHL,Z2,GG]). The n-point function has a canonical geometric interpretation for primary vectors v_i of $L(0)$ weight $\mathrm{wt}(v_i)$. Then $Z_V^{(0)}(v_1, z_1; \ldots; v_n, z_n)$ parameterizes a global meromorphic differential form on the Riemann sphere,

$$\mathcal{F}_V^{(0)}(v_1, \ldots v_n) = Z_V^{(0)}(v_1, z_1; \ldots; v_n, z_n) \prod_{1\leq i\leq n} (dz_i)^{\mathrm{wt}(v_i)}. \tag{70}$$

It follows from (55) that $\mathcal{F}_V^{(0)}$ is conformally invariant. This is the starting point of various algebraic-geometric approaches to n-point functions at higher genera e.g. [TUY,Z2]. However, it is important to note that the n-point function is intrinsically defined by its meromorphic pole structure in these approaches. Thus the partition or 0-point function is an undetermined overall normalization factor which is conventionally chosen to be unity (op. cit.).

It is instructive to consider $\mathcal{F}_V^{(0)}$ in the context of a trivial sewing of two Riemann spheres parameterized by z_1 and z_2 to form another Riemann sphere as follows. For $r_a > 0$, $a = 1, 2$, and a complex parameter ϵ satisfying $|\epsilon| \leq r_1 r_2$, excise the open disks

$|z_a| < |\epsilon| r_{\bar{a}}^{-1}$ (recall convention (20)) and identify the annular regions $r_a \geq |z_a| \geq |\epsilon| r_{\bar{a}}^{-1}$ via the sewing relation

$$z_1 z_2 = \epsilon. \tag{71}$$

Consider $Z_V^{(0)}(v_1, x_1; \ldots v_n, x_n)$ for quasi-primary v_i with $r_1 \geq |x_i| \geq |\epsilon| r_2^{-1}$ and let $y_i = \epsilon/x_i$. Then for $0 \leq k \leq n-1$ we find from (68) that

$$\begin{aligned} &Y(v_{k+1}, x_{k+1}) \ldots Y(v_n, x_n)\mathbf{1} \\ &\quad = \sum_{r \geq 0} \sum_{u \in V_r} \langle \bar{u}, Y(v_{k+1}, x_{k+1}) \ldots Y(v_n, x_n)\mathbf{1} \rangle u, \end{aligned} \tag{72}$$

where the inner sum is taken over any basis for V_r. Thus

$$\begin{aligned} &Z_V^{(0)}(v_1, x_1; \ldots v_n, x_n) \\ &\quad = \sum_{r \geq 0} \sum_{u \in V_r} \langle \mathbf{1}, Y(v_1, x_1) \ldots Y(v_k, x_k) u \rangle \langle \bar{u}, Y(v_{k+1}, x_{k+1}) \ldots Y(v_n, x_n)\mathbf{1} \rangle. \end{aligned}$$

But

$$\langle \mathbf{1}, Y(v_1, x_1) \ldots Y(v_k, x_k) u \rangle = \mathrm{Res}_{z_1=0} z_1^{-1} Z_V^{(0)}(v_1, x_1; \ldots v_k, x_k; u, z_1),$$

and

$$\begin{aligned} &\langle \bar{u}, Y(v_{k+1}, x_{k+1}) \ldots Y(v_n, x_n)\mathbf{1} \rangle \\ &\quad = \langle \mathbf{1}, Y^{\dagger}(v_n, x_n) \ldots Y^{\dagger}(v_{k+1}, x_{k+1}) \bar{u} \rangle \\ &\quad = \langle \mathbf{1}, \epsilon^{L(0)} Y^{\dagger}(v_n, x_n) \epsilon^{-L(0)} \ldots \epsilon^{L(0)} Y^{\dagger}(v_{k+1}, x_{k+1}) \epsilon^{-L(0)} \epsilon^{L(0)} \bar{u} \rangle \\ &\quad = \epsilon^r \mathrm{Res}_{z_2=0} z_2^{-1} Z_V^{(0)}(v_n, y_n; \ldots v_{k+1}, y_{k+1}; \bar{u}, z_2) \prod_{k+1 \leq i \leq n} (-\frac{\epsilon}{x_i^2})^{\mathrm{wt}(v_i)}. \end{aligned}$$

The last equation holds since for quasiprimary states v_i, the Mö bius transformation $x \mapsto y = \epsilon/x$ induces

$$Y(v_i, x_i) \mapsto \epsilon^{L(0)} Y^{\dagger}(v_i, x_i) \epsilon^{-L(0)} = (-\frac{\epsilon}{x_i^2})^{\mathrm{wt}(v_i)} Y(v_i, y_i). \tag{73}$$

Thus we find:

Proposition 4. *For homogeneous quasiprimary states v_i with the sewing scheme (71), we have*

$$\begin{aligned} &\mathcal{F}_V^{(0)}(v_1, \ldots, v_n) \\ &\quad = \sum_{r \geq 0} \epsilon^r \sum_{u \in V_r} \mathrm{Res}_{z_1=0} z_1^{-1} Z_V^{(0)}(v_1, x_1; \ldots v_k, x_k; u, z_1), \\ &\mathrm{Res}_{z_2=0} z_2^{-1} Z_V^{(0)}(v_n, y_n; \ldots v_{k+1}, y_{k+1}; \bar{u}, z_2) \prod_{1 \leq i \leq k} (dx_i)^{wt(v_i)} \prod_{k+1 \leq i \leq n} (dy_i)^{wt(v_i)}, \end{aligned}$$

for any k, $0 \leq k \leq n-1$, i.e. the RHS is independent of the choice of Riemann sphere on which the insertion point of each state v_i lies. □

5.2. Genus one case. We now consider genus one n-point functions defined in terms of a self-sewing of a Riemann sphere where punctures are located at the origin and the point at infinity [MT2]. Choose local coordinates $z_1 = z$ in the neighborhood of the origin and $z_2 = 1/z'$ for z' in the neighborhood of the point at infinity. For $a = 1, 2$ and $r_a > 0$, identify the annular regions $|q|r_{\bar{a}}^{-1} \leq |z_a| \leq r_a$ for complex q satisfying $|q| \leq r_1 r_2$ via the sewing relation $z_1 z_2 = q$, i.e. $z = qz'$. Then it is straightforward to show that the annuli do not intersect for $|q| < 1$, and that $q = \exp(2\pi i \tau)$, where τ is the torus modular parameter (e.g. [MT2], Prop. 8).

We define the genus one partition function by

$$\begin{aligned} Z_V^{(1)}(q) &= Z_V^{(1)}(\tau) \\ &= q^{-c/24} \sum_{n \geq 0} q^n \sum_{u \in V_n} \mathrm{Res}_{z_2=0} z_2^{-1} \mathrm{Res}_{z_1=0} z_1^{-1} \langle \mathbf{1}, Y^{\dagger}(u, z_2) Y(\bar{u}, z_1) \mathbf{1} \rangle, \end{aligned} \tag{74}$$

where the inner sum is taken over any basis for V_n. The external factor of $q^{-c/24}$ is introduced in the usual way to enhance the modular properties of $Z_V^{(1)}(q)$ [Z1]. From (62) and (68) it follows that

$$Z_V^{(1)}(\tau) = \sum_{n \geq 0} \dim V_n q^{n-c/24} = \mathrm{Tr}_V(q^{L(0)-c/24}), \tag{75}$$

the standard graded trace definition.

The genus one n-point function might similarly be defined by

$$\begin{aligned} &\sum_{r \geq 0} q^{r-c/24} \sum_{u \in V_r} \mathrm{Res}_{z_2=0} z_2^{-1} \mathrm{Res}_{z_1=0} z_1^{-1} \langle \mathbf{1}, Y^{\dagger}(u, z_2) Y(v_1, x_1) \ldots Y(v_n, x_n) Y(\bar{u}, z_1) \mathbf{1} \rangle \\ &= \mathrm{Tr}_V(Y(v_1, x_1) \ldots Y(v_n, x_n) q^{L(0)-c/24}). \end{aligned} \tag{76}$$

However, it is natural to consider the conformal map $x = q_z \equiv \exp(z)$ in order to describe the elliptic properties of the n-point function [Z1]. Since from (55), for a primary state v, $Y(v, w) \to Y(q_z^{L(0)} v, q_z)$ under this conformal map, we are led to the following definition of the genus one n-point function (op. cit.):

$$\begin{aligned} &Z_V^{(1)}(v_1, z_1; \ldots v_n, z_n; \tau) \\ &= \mathrm{Tr}_V(Y(q_{z_1}^{L(0)} v_1, q_{z_1}) \ldots Y(q_{z_n}^{L(0)} v_n, q_{z_n}) q^{L(0)-c/24}), \end{aligned} \tag{77}$$

which agrees with (76) for homogeneous primary states v_i. Furthermore, for primary v_i of weight $\mathrm{wt}(v_i)$, $Z_V^{(1)}$ parameterizes a global meromorphic differential form on the torus

$$\mathcal{F}_V^{(1)}(v_1, \ldots v_n; \tau) = Z_V^{(1)}(v_1, z_1; \ldots v_n, z_n; \tau) \prod_{1 \leq i \leq n} (dz_i)^{\mathrm{wt}(v_i)}. \tag{78}$$

Zhu introduced ([Z1]) a second VOA $(V, Y[\,,\,], \mathbf{1}, \tilde{\omega})$ which is isomorphic to $(V, Y(\,,\,), \mathbf{1}, \omega)$. It has vertex operators

$$Y[v, z] = \sum_{n \in \mathbb{Z}} v[n] z^{-n-1} = Y(q_z^{L(0)} v, q_z - 1), \tag{79}$$

and conformal vector $\tilde{\omega} = \omega - \frac{c}{24}\mathbf{1}$. Let

$$Y[\tilde{\omega}, z] = \sum_{n\in\mathbb{Z}} L[n] z^{-n-2}, \tag{80}$$

and write $\mathrm{wt}[v] = k$ if $L[0]v = kv$, $V_{[k]} = \{v \in V | \mathrm{wt}[v] = k\}$. Only primary vectors are homogeneous with respect to both $L(0)$ and $L[0]$, in which case $\mathrm{wt}(v) = \mathrm{wt}[v]$. Similarly, we define the square bracket LiZ metric $\langle\ ,\ \rangle_{\mathrm{sq}}$ which is invariant with respect to the square bracket adjoint.

We denote 1-point functions by

$$Z_V^{(1)}(v, \tau) = Z_V^{(1)}(v, z; \tau) = \mathrm{Tr}_V(o(v) q^{L(0)-c/24}). \tag{81}$$

($Z_V^{(1)}(v, \tau)$ is necessarily z independent.) Any n-point function can be expressed in terms of 1-point functions ([MT1], Lemma 3.1) as follows:

$$\begin{aligned} &Z_V^{(1)}(v_1, z_1; \ldots v_n, z_n; \tau) \\ &= Z_V^{(1)}(Y[v_1, z_1] \ldots Y[v_{n-1}, z_{n-1}] Y[v_n, z_n]\mathbf{1}, \tau) \qquad (82) \\ &= Z_V^{(1)}(Y[v_1, z_{1n}] \ldots Y[v_{n-1}, z_{n-1n}] v_n, \tau), \qquad (83) \end{aligned}$$

where $z_{in} = z_i - z_n$.

We may consider a trivial sewing of a torus with local coordinate z_1 to a Riemann sphere with local coordinate z_2 by identifying the annuli $r_a \geq |z_a| \geq |\epsilon| r_{\bar{a}}^{-1}$ via the sewing relation $z_1 z_2 = \epsilon$. Consider $Z_V^{(1)}(v_1, x_1; \ldots v_n, x_n)$ for quasi-primary v_i of $L[0]$ weight $\mathrm{wt}[v_i]$, with $r_1 \geq |x_i| \geq |\epsilon| r_2^{-1}$, and let $y_i = \epsilon / x_i$. Using (82), and employing the square bracket version of (72) with square bracket LiZ metric $\langle\ ,\ \rangle_{\mathrm{sq}}$, we have

$$\begin{aligned} &Z_V^{(1)}(v_1, x_1; \ldots v_n, x_n; \tau) \\ &= \sum_{r\geq 0} \sum_{u\in V_{[r]}} Z_V^{(1)}(Y[v_1, x_1] \ldots Y[v_k, x_k] u; \tau) \langle \bar{u}, Y[v_{k+1}, x_{k+1}] \ldots Y[v_n, x_n]\mathbf{1}\rangle_{\mathrm{sq}}, \end{aligned}$$

where the inner sum is taken over any basis $\{u\}$ of $V_{[r]}$, and $\{\bar{u}\}$ is the dual basis with respect to $\langle\ ,\ \rangle_{\mathrm{sq}}$. Now

$$Z_V^{(1)}(Y[v_1, x_1] \ldots Y[v_k, x_k] u; \tau) = \mathrm{Res}_{z_1=0} z_1^{-1} Z_V^{(1)}(v_1, x_1; \ldots v_k, x_k; u, z_1; \tau).$$

Using the isomorphism between the round and square bracket formalisms, we find as before that

$$\begin{aligned} &\langle \bar{u}, Y[v_{k+1}, x_{k+1}] \ldots Y[v_n, x_n]\mathbf{1}\rangle_{\mathrm{sq}} \\ &= \epsilon^r \mathrm{Res}_{z_2=0} z_2^{-1} Z_V^{(0)}(v_n, y_n; \ldots v_{k+1}, y_{k+1}; \bar{u}, z_2) \prod_{k+1\leq i\leq n} (-\frac{\epsilon}{x_i^2})^{\mathrm{wt}[v_i]}. \end{aligned}$$

We thus obtain a natural analogue of Proposition 4:

Proposition 5. *For square bracket homogeneous quasiprimary states v_i with the above sewing scheme, then we have*

$$\begin{aligned}&\mathcal{F}_V^{(1)}(v_1,\ldots,v_n;\tau)\\&\quad=\sum_{r\geq 0}\epsilon^r\sum_{u\in V_{[r]}}\mathrm{Res}_{z_1=0}z_1^{-1}Z_V^{(1)}(v_1,x_1;\ldots v_k,x_k;u,z_1;\tau),\\&\mathrm{Res}_{z_2=0}z_2^{-1}Z_V^{(0)}(v_n,y_n;\ldots v_{k+1},y_{k+1};\bar{u},z_2)\prod_{1\leq i\leq k}(dx_i)^{wt[v_i]}\prod_{k+1\leq i\leq n}(dy_i)^{wt[v_i]},\end{aligned}$$

and is independent of $k=0,1,\ldots n-1$, where the inner sum is taken over any basis $\{u\}$ for $V_{[r]}$, $\{\bar{u}\}$ is the dual basis with respect to $\langle\;,\;\rangle_{\mathrm{sq}}$. □

We note that all the above definitions can be naturally extended for any V-module N with vertex operators $Y_N(v,x)$, where the trace in (83) is taken over N and $o(v)$ is replaced by $o_N(v)$ the Virasoro level preserving part of $Y_N(v,x)$.

5.3. Genus two case. Motivated by Proposition 5, we now discuss the formal definition of the genus two n-point function associated with the genus two ϵ-sewing scheme reviewed in Sect. 2.2. Recall that we sew together a pair of punctured tori $\hat{\mathcal{S}}_a$ of (19) with modular parameters τ_a for $a=1,2$ via the sewing relation (22). We define the genus two n-point function for $v_1,\ldots v_k$ inserted at $x_1,\ldots,x_k\in\hat{\mathcal{S}}_1$ and $v_{k+1},\ldots v_n$ inserted at $y_{k+1},\ldots,y_n\in\hat{\mathcal{S}}_2$ for $k=0,1,\ldots n-1$ by

$$\begin{aligned}&Z_V^{(2)}(v_1,x_1;\ldots v_k,x_k|v_{k+1},y_{k+1};\ldots v_n,y_n;\tau_1,\tau_2,\epsilon)\\&\quad=\sum_{r\geq 0}\epsilon^r\sum_{u\in V_{[r]}}\mathrm{Res}_{z_1=0}z_1^{-1}Z_V^{(1)}(v_1,x_1;\ldots v_k,x_k;u,z_1;\tau_1)\\&\qquad\cdot\mathrm{Res}_{z_2=0}z_2^{-1}Z_V^{(1)}(v_n,y_n;\ldots v_{k+1},y_{k+1};\bar{u},z_2;\tau_2),\\&\quad=\sum_{r\geq 0}\epsilon^r\sum_{u\in V_{[r]}}Z_V^{(1)}(Y[v_1,x_1]\ldots Y[v_k,x_k]u,z_1;\tau_1)\\&\qquad\cdot Z_V^{(1)}(Y[v_n,y_n]\ldots Y[v_{k+1},y_{k+1}]\bar{u},z_2;\tau_2),\end{aligned}\tag{84}$$

where the inner sum is taken over any basis $V_{[r]}$ and $\bar{u}$ is the dual of u with respect to $\langle,\rangle_{\mathrm{sq}}$. The last expression in (84) follows from (83).

Remark 3. Following Remark 2 it is clear that the genus two n-point function on the tensor product $V_1\otimes V_2$ of a pair of simple VOAs is just the product of n-point functions on V_1 and V_2.

In this paper we mainly concentrate on the genus two partition function (i.e. the 0-point function) given by

$$Z_V^{(2)}(\tau_1,\tau_2,\epsilon)=\sum_{n\geq 0}\epsilon^n\sum_{u\in V_{[n]}}Z_V^{(1)}(u,\tau_1)Z_V^{(1)}(\bar{u},\tau_2).\tag{85}$$

Some examples of n-point functions will also be computed. A general discussion of all genus two n-point functions for the Heisenberg VOA and its modules will appear elsewhere [MT5].

Clearly the definition of the n-point function (84) depends on the choice of punctured torus on which the insertion points lie. However, by defining an associated formal differential form, we find the following genus two analogue of Propositions 4 and 5:

Proposition 6. *For $x_i \in \hat{\mathcal{S}}_1$ and $y_i \in \hat{\mathcal{S}}_2$ with $x_i y_i = \epsilon$ and square bracket homogeneous quasiprimary states v_i, the formal differential form*

$$\begin{aligned}&\mathcal{F}_V^{(2)}(v_1, \ldots, v_n; \tau_1, \tau_2, \epsilon)\\ &\quad\equiv Z_V^{(2)}(v_1, x_1; \ldots v_k, x_k | v_{k+1}, y_{k+1}; \ldots v_n, y_n; \tau_1, \tau_2, \epsilon)\\ &\quad\quad\cdot \prod_{1\le i\le k} (dx_i)^{wt[v_i]} \prod_{k+1\le i\le n} (dy_i)^{wt[v_i]},\end{aligned} \tag{86}$$

is independent of $k = 0, 1, \ldots n-1$.

Proof. Consider the left torus contribution in the summand of (84) and expand $Y[v_k, x_k]u$ in a square bracket homogeneous basis:

$$\begin{aligned}&Z_V^{(1)}(Y[v_1, x_1] \ldots Y[v_k, x_k]u; \tau_1)\\ &\quad= \sum_{s\ge 0} \sum_{w\in V_{[s]}} Z_V^{(1)}(Y[v_1, x_1] \ldots Y[v_{k-1}, x_{k-1}]w; \tau_1)\langle \bar{w}, Y[v_k, x_k]u\rangle_{\mathrm{sq}}.\end{aligned}$$

But for quasi-primary v_k and using (73) we find

$$\epsilon^r \langle \bar{w}, Y[v_k, x_k]u\rangle_{\mathrm{sq}} = \langle \epsilon^{L[0]} Y^{\dagger}[v_k, x_k]\bar{w}, u\rangle_{\mathrm{sq}} = \epsilon^s (-\frac{\epsilon}{x_k^2})^{\mathrm{wt}[v_k]} \langle Y[v_k, y_k]\bar{w}, u\rangle_{\mathrm{sq}},$$

where $x_k y_k = \epsilon$. Noting that

$$\begin{aligned}&\sum_{r\ge 0} \sum_{u\in V_{[r]}} Z_V^{(1)}(Y[v_n, y_n] \ldots Y[v_{k+1}, y_{k+1}]\bar{u}; \tau_2)\langle u, Y[v_k, y_k]\bar{w}\rangle_{\mathrm{sq}}\\ &\quad= Z_V^{(1)}(Y[v_n, y_n] \ldots Y[v_{k+1}, y_{k+1}]Y[v_k, y_k]\bar{w}; \tau_2),\end{aligned}$$

we therefore find that

$$\begin{aligned}&Z_V^{(2)}(v_1, x_1; \ldots v_k, x_k | v_{k+1}, y_{k+1}; \ldots v_n, y_n; \tau_1, \tau_2, \epsilon)\\ &\quad= (-\frac{\epsilon}{x_k^2})^{\mathrm{wt}[v_k]} Z_V^{(2)}(v_1, x_1; \ldots v_{k-1}, x_{k-1} | v_k, y_k; \ldots v_n, y_n; \tau_1, \tau_2, \epsilon).\end{aligned}$$

Hence

$$\begin{aligned}&Z_V^{(2)}(v_1, x_1; \ldots v_k, x_k | v_{k+1}, y_{k+1}; \ldots v_n, y_n; \tau_1, \tau_2, \epsilon)\\ &\quad\cdot \prod_{1\le i\le k} (dx_i)^{\mathrm{wt}[v_i]} \prod_{k+1\le i\le n} (dy_i)^{\mathrm{wt}[v_i]}\\ &= Z_V^{(2)}(v_1, x_1; \ldots v_{k-1}, x_{k-1} | v_k, y_k; \ldots v_n, y_n; \tau_1, \tau_2, \epsilon)\\ &\quad\cdot \prod_{1\le i\le k-1} (dx_i)^{\mathrm{wt}[v_i]} \prod_{k\le i\le n} (dy_i)^{\mathrm{wt}[v_i]}.\end{aligned}$$

The result follows by repeated application of this identity. □

Remark 4. If $Z_V^{(2)}(\tau_1, \tau_2, \epsilon)$ is convergent on $\mathcal{D}^\epsilon$, we conjecture that for primary states $v_1, \ldots v_n$ then $\mathcal{F}_V^{(2)}(v_1, \ldots, v_n; \tau_1, \tau_2, \epsilon)$ is a genus two global meromorphic form with possible poles only at coincident insertion points.

Finally, note that all the above definitions can be naturally extended for any pair of V-modules N_1, N_2, where the left (right) 1-point function in (84) is considered for N_1 (respectively N_2).

6. The Heisenberg VOA

In this section we compute closed formulas for the genus two partition function for the rank one Heisenberg VOA M and compute the n-point function for n Heisenberg vectors and the Virasoro vector 1-point function. We also discuss the modular properties of the partition function in some detail.

6.1. The genus two partition function $Z_M^{(2)}(\tau_1, \tau_2, \epsilon)$.

We wish to establish a closed formula for the genus two partition function $Z_M^{(2)}(\tau_1, \tau_2, \epsilon)$ of (85) in terms of the infinite matrices A_1, A_2 introduced in (24) of Sect. 2. Recalling the definition (25) we have:

Theorem 5. *Let M be the vertex operator algebra of one free boson. Then*

$$Z_M^{(2)}(\tau_1, \tau_2, \epsilon) = Z_M^{(1)}(\tau_1) Z_M^{(1)}(\tau_2)(\det(I - A_1 A_2))^{-1/2}, \tag{87}$$

where $Z_M^{(1)}(\tau) = 1/\eta(\tau)$.

Remark 5. From Remark 3 it follows that the genus two partition function for l free bosons M^l is just the l^{th} power of (87).

Proof of Theorem. The genus two partition function $Z_M^{(2)}(\tau_1, \tau_2, \epsilon)$ of (85) is V basis independent. We choose the standard Fock vectors (in the square bracket formulation)

$$v = a[-1]^{e_1} \ldots a[-p]^{e_p} \mathbf{1}. \tag{88}$$

Of course, these Fock vectors correspond in a natural 1-1 manner with unrestricted partitions, the state v (88) corresponding to a partition $\lambda = \{1^{e_1} \ldots p^{e_p}\}$ with $|\lambda| = \sum_i e_i$ elements of $n = \sum_{1 \leq i \leq p} i e_i$. We sometimes write $v = v(\lambda)$ to indicate this correspondence. Furthermore, following (67),

$$v(\lambda) = (-1)^{|\lambda|} \left(\prod_{1 \leq i \leq p} i^{e_i} e_i! \right) \bar{v}(\lambda).$$

Thus with this diagonal basis we have

$$Z_M^{(2)}(\tau_1, \tau_2, \epsilon) = \sum_{\lambda = \{i^{e_i}\}} \frac{(-1)^{|\lambda|}}{\prod_i i^{e_i} e_i!} \epsilon^{\sum i e_i} Z_M^{(1)}(v(\lambda), \tau_1) Z_M^{(1)}(v(\lambda), \tau_2). \tag{89}$$

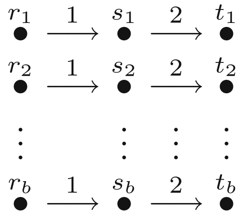

Fig. 4. Two complete matchings

As discussed at length in [MT1], the partition λ may be thought of as a labeled set $\Phi = \Phi_\lambda$ with e_i elements labeled i. One of the main results of [MT1] (loc.cit. Corollary 1 and Eq. (53)) is that for even $|\lambda|$,

$$Z_M^{(1)}(v(\lambda), \tau) = Z_M^{(1)}(\tau) \sum_{\phi \in F(\Phi_\lambda)} \Gamma(\phi), \tag{90}$$

with

$$\Gamma(\phi, \tau) = \Gamma(\phi) = \prod_{(r,s)} C(r, s, \tau), \tag{91}$$

for C of (11), where ϕ ranges over the elements of $F(\Phi_\lambda)$ (the fixed-point-free involutions in $\Sigma(\Phi_\lambda)$) and (r, s) ranges over the orbits of ϕ on Φ_λ. If $|\lambda|$ is odd then $Z_M^{(1)}(v(\lambda), \tau) = 0$. With this notation, (89) reads

$$Z_M^{(2)}(\tau_1, \tau_2, \epsilon) = Z_M^{(1)}(\tau_1) Z_M^{(1)}(\tau_2) \sum_{\lambda = \{i^{e_i}\}} \frac{E(\lambda)}{\prod_i i^{e_i} e_i!} \epsilon^{\sum i e_i}, \tag{92}$$

where λ ranges over all even $|\lambda|$ unrestricted partitions and where we have set

$$E(\lambda) = \sum_{\phi, \psi \in F(\Phi_\lambda)} \Gamma_1(\phi) \Gamma_2(\psi), \tag{93}$$

$$\Gamma_i(\phi) = \Gamma(\phi, \tau_i). \tag{94}$$

We now analyze the nature of the expression $E(\lambda)$ more closely. This will lead us to the connection between $Z^{(2)}(\tau_1, \tau_2, \epsilon)$ and the chequered cycles discussed in Sect. 3.1. The idea is to use the technique employed in the proof of Proposition 4 of [MT1]. If we fix for a moment a partition λ then a pair of fixed-point-free involutions ϕ, ψ correspond (loc.cit.) to a pair of complete matchings μ_ϕ, μ_ψ on the labeled set Φ_λ which we may represent pictorially as Fig. 4.

Here, μ_ϕ is the matching with edges labeled 1, μ_ψ the matching with edges labeled 2, and where we denote the (labeled) elements of Φ_λ by $\{r_1, s_1, \ldots, r_b, s_b\} = \{s_1, t_1, \ldots, s_b, t_b\}$. From this data we may create a chequered cycle in a natural way: starting with some node of Φ_λ, apply the involutions ϕ, ψ successively and repeatedly until the initial node is reached, using the complete matchings to generate a chequered cycle. The resulting chequered cycle corresponds to an orbit of $\langle \psi\phi \rangle$ considered as a cyclic subgroup of $\Sigma(\Phi_\lambda)$. Repeat this process for each such orbit to obtain a *chequered diagram D* consisting of the union of the chequered cycles corresponding to all of the orbits of $\langle \psi\phi \rangle$ on

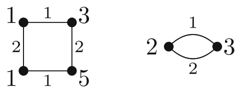

Fig. 5. Chequered diagram

Φ_λ. To illustrate, for the partition $\lambda = \{1^2.2.3^2.5\}$ with matchings $\mu_\phi = (13)(15)(23)$ and $\mu_\psi = (11)(35)(23)$, the corresponding chequered diagram is shown in Fig. 5

Two chequered diagrams are isomorphic if there is a bijection on the nodes which preserves edges and labels of nodes and edges. If $\lambda = \{1^{e_1} \ldots p^{e_p}\}$, then $\Sigma(\Phi_\lambda)$ acts on the chequered diagrams which have Φ_λ as an underlying set of labeled nodes. The *Automorphism subgroup* $\mathrm{Aut}(D)$, consisting of the elements of $\Sigma(\Phi_\lambda)$ which preserves node labels, is isomorphic to $\Sigma_{e_1} \times \cdots \times \Sigma_{e_p}$. It induces all isomorphisms among these chequered diagrams. Of course $|\mathrm{Aut}(D)| = \prod_{1 \le i \le p} e_i!$. We have almost established the first step in the proof of Theorem 5, namely

Proposition 7. *We have*

$$Z_M^{(2)}(\tau_1, \tau_2, \epsilon) = Z_M^{(1)}(\tau_1) Z_M^{(1)}(\tau_2) \sum_D \frac{\gamma(D)}{|\mathrm{Aut}(D)|}, \tag{95}$$

where D ranges over isomorphism classes of chequered configurations and

$$\gamma(D) = \frac{E(\lambda)}{\prod_i i^{e_i}} \epsilon^{\sum i e_i}. \tag{96}$$

Proposition 7 follows from what we have said together with (92). It is only necessary to point out that because the label subgroup induces all isomorphisms of chequered diagrams, when we sum over isomorphism classes of such diagrams in (92) the term $\prod_i e_i!$ must be replaced by $|\mathrm{Aut}(D)|$. □

Recalling the weights (40), we define

$$\zeta(D) = \Pi_E \zeta(E),$$

where the product is taken over the edges E of D and $\zeta(E)$ is as in (44).

Lemma 1. *For all D we have*

$$\zeta(D) = \gamma(D). \tag{97}$$

Proof. Let D be determined by a partition $\lambda = \{1^{e_1} \ldots p^{e_p}\}$ and a pair of involutions $\phi, \psi \in F(\Phi_\lambda)$, and let (a, b), (r, s) range over the orbits of ϕ resp. ψ on Φ_λ. Then we find

$$\begin{aligned} \frac{E(\lambda)}{\prod_i i^{e_i}} \epsilon^{\sum i e_i} &= \frac{\prod_{(a,b)} C(a, b, \tau_1) \prod_{(r,s)} C(r, s, \tau_2)}{\prod_i i^{e_i}} \epsilon^{\sum i e_i} \\ &= \prod_{(ab)} \frac{\epsilon^{(a+b)/2}}{\sqrt{ab}} C(a, b, \tau_1) \prod_{(rs)} \frac{\epsilon^{(r+s)/2}}{\sqrt{rs}} C(r, s, \tau_2) \\ &= \prod_{(ab)} A_1(a, b) \prod_{(rs)} A_2(r, s) = \zeta(D). \end{aligned}$$

□

We may represent a chequered diagram formally as a product

$$D = \prod_i L_i^{m_i} \tag{98}$$

in case D is the disjoint union of unoriented chequered cycles L_i with multiplicity m_i. Then $\mathrm{Aut}(D)$ is isomorphic to the direct product of the groups $\mathrm{Aut}(L_i^{m_i})$ of order $\left|\mathrm{Aut}(L_i^{m_i})\right| = |\mathrm{Aut}(L_i)|^{m_i}\, m_i!$ so that

$$|\mathrm{Aut}(D)| = \prod_i |\mathrm{Aut}(L_i^{m_i})| m_i!.$$

Noting that the expression $\zeta(D)$ is multiplicative over disjoint unions of diagrams, we calculate

$$\begin{aligned}\sum_D \frac{\zeta(D)}{|\mathrm{Aut}(D)|} &= \prod_L \sum_{k\geq 0} \frac{\zeta(L)^k}{|\mathrm{Aut}(L)|^k k!}\\ &= \prod_L \exp\left(\frac{\zeta(L)}{|\mathrm{Aut}(L)|}\right)\\ &= \exp\left(\sum_L \frac{\zeta(L)}{|\mathrm{Aut}(L)|}\right),\end{aligned}$$

where L ranges over isomorphism classes of unoriented chequered cycles. Now $\mathrm{Aut}(L)$ is either a dihedral group of order $2r$ or a cyclic group of order r for some $r \geq 1$, depending on whether L admits a reflection symmetry or not. If we now *orient* our cycles, say in a clockwise direction, then we can replace the previous sum over L by a sum over the set of (isomorphism classes of) *oriented* chequered cycles $\mathcal{O}$ to obtain

$$\sum_D \frac{\zeta(D)}{|\mathrm{Aut}(D)|} = \exp\left(\frac{1}{2}\sum_{M\in\mathcal{O}} \frac{\zeta(M)}{|\mathrm{Aut}(M)|}\right). \tag{99}$$

Let $\mathcal{O}_{2n} \subset \mathcal{O}$ be denoted the set of oriented chequered cycles with $2n$ nodes. Then we have

Lemma 2.

$$\mathrm{Tr}((A_1A_2)^n) = \sum_{M\in\mathcal{O}_{2n}} \frac{n}{|\mathrm{Aut}(M)|}\zeta(M). \tag{100}$$

Proof. The contribution $A_1(i_1, i_2)A_2(i_2, i_3)\ldots A_2(i_{2n}, i_1)$ to the left-hand-side of (100) is equal to the weight $\zeta(M)$ for some $M \in \mathcal{O}_{2n}$ with vertices $i_1, i_2, \ldots i_{2n}$. Let $\sigma = \begin{pmatrix} i_1 & \ldots & i_k & \ldots & i_{2n} \\ i_3 & \ldots & i_{k+2} & \ldots & i_2\end{pmatrix}$ denote the order n permutation of the indices which generates rotations of M. Then $\mathrm{Aut}(M) = \langle \sigma^m \rangle$ for some $m = n/|\mathrm{Aut}(M)|$. Now sum over all i_k to compute $\mathrm{Tr}((A_1A_2)^n)$, noting that for inequivalent M the weight $\zeta(M)$ occurs with multiplicity m. The lemma follows. □

We may now complete the proof of Theorem 5. From (99) and (100) we obtain

$$\begin{aligned}\sum_D \frac{\zeta(D)}{|\mathrm{Aut}(D)|} &= \exp\left(\frac{1}{2}\mathrm{Tr}(\sum_n \frac{1}{n}(A_1A_2)^n)\right)\\ &= \exp(-\frac{1}{2}\mathrm{Tr}(\log(1-A_1A_2)))\\ &= \det(\exp(-\frac{1}{2}(\log(1-A_1A_2))))\\ &= (\det(1-A_1A_2))^{-1/2}.\end{aligned}$$

□

We may also obtain a product formula for $Z_M^{(2)}(\tau_1,\tau_2,\epsilon)$ as follows. Recalling the notation (38), for each oriented chequered cycle M, $\mathrm{Aut}(M)$ is a cyclic group of order r for some $r \geq 1$. Furthermore it is evident that there is a rotationless chequered cycle N with $\zeta(M)=\zeta(N)^r$. Indeed, N may be obtained by taking a suitable consecutive sequence of n/r nodes of M, where n is the total number of nodes of M. We thus see that

$$\begin{aligned}\sum_{M\in\mathcal{O}} \frac{\zeta(M)}{|\mathrm{Aut}(M)|} &= \sum_{N\in\mathcal{R}}\sum_{r\geq 1}\frac{\zeta(N)^r}{r}\\ &= -\sum_{N\in\mathcal{R}} \log(1-\zeta(N)).\end{aligned}$$

Then (99) implies

$$\det(1-A_1A_2) = \prod_{N\in\mathcal{R}}(1-\zeta(N)), \tag{101}$$

and thus we obtain

Theorem 6. *Let M be the vertex operator algebra of one free boson. Then*

$$Z_M^{(2)}(\tau_1,\tau_2,\epsilon) = \frac{Z_M^{(1)}(\tau_1)Z_M^{(1)}(\tau_2)}{\prod_{N\in\mathcal{R}}(1-\zeta(N))^{1/2}}. \tag{102}$$

6.2. Holomorphic and modular invariance properties. In Sect. 2.2 we reviewed the genus two ϵ-sewing formalism and introduced the domain $\mathcal{D}^\epsilon$ parameterizing the genus two surface. An immediate consequence of Theorem 5 and Theorem 1(b) is the following:

Theorem 7. $Z_M^{(2)}(\tau_1,\tau_2,\epsilon)$ *is holomorphic on the domain $\mathcal{D}^\epsilon$.* □

We next consider the automorphic properties of the genus two partition function with respect to the group G reviewed in Sect. 2.2. For two free bosons the genus one partition function is

$$Z_{M^2}^{(1)}(\tau) = \frac{1}{\eta(\tau)^2}. \tag{103}$$

Let χ be the character of $SL(2,\mathbb{Z})$ defined by its action on $\eta(\tau)^{-2}$, i.e.

$$\eta(\gamma\tau)^{-2} = \chi(\gamma)\eta(\tau)^{-2}(c\tau+d)^{-1}, \tag{104}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$. Recall (e.g. [Se]) that $\chi(\gamma)$ is a twelfth root of unity. For a function $f(\tau)$ on $\mathbb{H}_1$, $k \in \mathbb{Z}$ and $\gamma \in SL(2,\mathbb{Z})$, we define

$$f(\tau)|_k\gamma = f(\gamma\tau)\,(c\tau+d)^{-k}, \tag{105}$$

so that

$$Z^{(1)}_{M^2}(\tau)|_{-1}\gamma = \chi(\gamma)Z^{(1)}_{M^2}(\tau). \tag{106}$$

The genus two partition function for two free bosons is

$$Z^{(2)}_{M^2}(\tau_1,\tau_2,\epsilon) = \frac{1}{\eta(\tau_1)^2\eta(\tau_2)^2\det(I-A_1A_2)}. \tag{107}$$

Analogously to (105), we define

$$f(\tau_1,\tau_2,\epsilon)|_k\gamma = f(\gamma(\tau_1,\tau_2,\epsilon))\det(C\Omega+D)^{-k}. \tag{108}$$

Here, the action of γ on the right-hand-side is as in (35). We have abused notation by adopting the following conventions in (108), which we continue to use below:

$$\Omega = F^{\epsilon}(\tau_1,\tau_2,\epsilon),\ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(4,\mathbb{Z}), \tag{109}$$

where F^{ϵ} is as in Theorem 3, and γ is identified with an element of $Sp(4,\mathbb{Z})$ via (35)-(36). Note that (108) defines a right action of G on functions $f(\tau_1,\tau_2,\epsilon)$. We will establish the natural extension of (106) to the genus 2 case. To describe this, introduce the character $\chi^{(2)}$ of G defined by

$$\chi^{(2)}(\gamma_1\gamma_2\beta^m) = (-1)^m\chi(\gamma_1\gamma_2), \quad \gamma_i \in \Gamma_i,\ i=1,2,$$

(notation as in (35), (36)). Thus $\chi^{(2)}$ takes values which are twelfth roots of unity, and we have

Theorem 8. *If $\gamma \in G$ then*

$$Z^{(2)}_{M^2}(\tau_1,\tau_2,\epsilon)|_{-1}\gamma = \chi^{(2)}(\gamma)Z^{(2)}_{M^2}(\tau_1,\tau_2,\epsilon).$$

Corollary 1. *For the rank 24 Heisenberg VOA M^{24} we have*

$$Z^{(2)}_{M^{24}}(\tau_1,\tau_2,\epsilon)|_{-12}\gamma = Z^{(2)}_{M^{24}}(\tau_1,\tau_2,\epsilon),$$

for $\gamma \in G$.

Proof. We will give two different proofs of this result. Using the convention (109), we have to show that

$$Z^{(2)}_{M^2}(\gamma(\tau_1,\tau_2,\epsilon))\det(C\Omega+D)=\chi^{(2)}(\gamma)Z^{(2)}_{M^2}(\tau_1,\tau_2,\epsilon) \tag{110}$$

for $\gamma\in G$, and it is enough to do this for a generating set of G. If $\gamma=\beta$ then the result is clear since $\det(C\Omega+D)=\chi^{(2)}(\beta)=-1$ and β exchanges τ_1 and τ_2. So we may assume that $\gamma=(\gamma_1,\gamma_2)\in\Gamma_1\times\Gamma_2$.

Our first proof utilizes the determinant formula (87) as follows. For $\gamma_1\in\Gamma_1$, define $A'_a(k,l,\tau_a,\epsilon)=A_a(k,l,\gamma_1\tau_a,\frac{\epsilon}{c_1\tau_1+d_1})$ following (35). We find from Sect. 4.4 of [MT2] that

$$\begin{aligned}I-A'_1A'_2&=I-A_1A_2-\kappa\Delta A_2\\&=(I-\kappa S).(I-A_1A_2),\end{aligned}$$

where $\Delta(k,l)=\delta_{k1}\delta_{l1}$, $\kappa=-\frac{\epsilon}{2\pi i}\frac{c_1}{c_1\tau_1+d_1}$ and $S(k,l)=\delta_{k1}(A_2(I-A_1A_2)^{-1})(1,l)$. Since $\det(I-A_1A_2)$ and $\det(I-A'_1A'_2)$ are convergent on $\mathcal{D}^\epsilon$ we find

$$\det(I-A'_1A'_2)=\det(I-\kappa S)\det(I-A_1A_2).$$

But $\det(I-\kappa S)=1-\kappa S(1,1)=\frac{c_1\Omega_{11}+d_1}{c_1\tau_1+d_1}$ which implies (110) for $\gamma_1\in\Gamma_1$. A similar proof applies for $\gamma_2\in\Gamma_2$.

The second proof uses Proposition 2 together with (45), which tell us that

$$Z^{(2)}_{M^2}(\tau_1,\tau_2,\epsilon)=\frac{-2\pi i\Omega_{12}}{\epsilon\eta(\tau_1)^2\eta(\tau_2)^2}\prod_{\mathcal{R}'}(1-\zeta(L))^{-1}, \tag{111}$$

where $\mathcal{R}'=\mathcal{R}\backslash\mathcal{R}_{21}$. Now in general a term $\zeta(L)$ will not be invariant under the action of γ. This is because of the presence of quasi-modular terms $A_a(1,1)$ arising from $E_2(\tau_a)$. But it is clear from (35) and the definition (11) of $C(k,l,\tau)$ together with its modular-invariance properties that if $L\in\mathcal{R}'$ then such terms are absent and $\zeta(L)$ *is* invariant. So the product term in (111) is invariant under the action of γ.

Next, we see from (35) that the expression $\epsilon\eta(\tau_1)^2\eta(\tau_2)^2$ is invariant under the action of γ up to a scalar $\chi(\gamma_1)\chi(\gamma_2)=\chi^{(2)}(\gamma)$. This reduces the proof of (110) to showing that

$$(\gamma_1,\gamma_2):\Omega_{12}\mapsto\Omega_{12}\ \det(C\Omega+D)^{-1},$$

and this is implicit in (37) upon applying Theorem 4. This completes the second proof of Theorem 8. □

Remark 6. An unusual feature of the formulas in Theorem 8 and Corollary 1 is that the definition of the automorphy factor $\det(C\Omega+D)$ requires the map $F^\epsilon:\mathcal{D}^\epsilon\to\mathbb{H}_2$. Thus although the automorphy factor resembles that of a Siegel modular form on $\mathbb{H}_2$, the partition function is *not* a function on $\mathbb{H}_2$ but rather on $\mathcal{D}^\epsilon$.

6.3. Some genus two n-point functions. In this section we calculate some examples of genus two n-point functions for the rank one Heisenberg VOA M. A general analysis of all such functions will appear elsewhere [MT5]. We consider here the examples of the n-point function for the Heisenberg vector a and the 1-point function for the Virasoro vector $\tilde{\omega}$. We find that the formal differential form (86) associated with the Heisenberg n-point function is described in terms of the global symmetric two form $\omega^{(2)}$ [TUY], whereas the Virasoro 1-point function is described by the genus two projective connection [Gu]. These results illustrate the general conjecture made in Remark 4.

We first consider the example of the Heisenberg vector 1-point function where a is inserted at x on the left torus (say). Since $Z_M^{(1)}(Y[a,x]v;\tau) = 0$ for a Fock vector $v = v(\lambda)$ for even $|\lambda|$ and $Z_M^{(1)}(v;\tau) = 0$ for odd $|\lambda|$ [MT1] we find from (84) that $Z_M^{(2)}(a, x|\tau_1, \tau_2, \epsilon) = 0$.

Consider next the 2-point function for two Heisenberg vectors inserted on the left torus at $x_1, x_2 \in \hat{\mathcal{S}}_1$ with

$$Z_M^{(2)}(a, x_1; a, x_2|\tau_1, \tau_2, \epsilon) = \sum_{r\geq 0} \epsilon^r \sum_{v\in M_{[r]}} Z_M^{(1)}(Y[a, x_1]Y[a, x_2]v; \tau_1) Z_M^{(1)}(\bar{v}; \tau_2). \tag{112}$$

Following (86) of Proposition 6, we consider the associated formal differential form $\mathcal{F}^{(2)}(a, a; \tau_1, \tau_2, \epsilon)$ for (112) and find that it is determined by the bilinear form $\omega^{(2)}$ of (15):

Theorem 9. *The genus two Heisenberg vector 2-point function is*

$$\mathcal{F}_M^{(2)}(a, a; \tau_1, \tau_2, \epsilon) = \omega^{(2)} Z_M^{(2)}(\tau_1, \tau_2, \epsilon). \tag{113}$$

Proof. The proof proceeds along the same lines as Theorem 5. As before, we let $v(\lambda)$ denote a Heisenberg Fock vector (88) determined by an unrestricted partition $\lambda = \{1^{e_1} \ldots p^{e_p}\}$ with label set Φ_λ. Define a label set for the three vectors $a, a, v(\lambda)$ given by $\Phi = \Phi_1 \cup \Phi_2 \cup \Phi_3$ for $\Phi_1, \Phi_2 = \{1\}$ and $\Phi_3 = \Phi_\lambda$ and let $F(\Phi)$ denote the set of fixed point free involutions on Φ. For $\phi = \ldots (rs) \ldots \in F(\Phi)$, let $\Gamma_1(x_1, x_2, \phi) = \prod_{(r,s)} \gamma(r, s)$, where for $r \in \Phi_i$ and $s \in \Phi_j$,

$$\gamma(r, s) = \begin{cases} D(1, 1, x_1 - x_2, \tau_1) = P_2(\tau_1, x_1 - x_2), & i = 1;\ j = 2 \\ D(1, s, x_i, \tau_1) = s P_{s+1}(\tau_1, x_i), & i = 1, 2;\ j = 3 \\ C(r, s, \tau_1), & i, j = 3, \end{cases} \tag{114}$$

for C, D of (11) and (12). Then following Corollary 1 of [MT1] we find for even $|\lambda|$ that

$$Z_M^{(1)}(Y[a, x_1]Y[a, x_2]v(\lambda), \tau_1) = Z_M^{(1)}(\tau_1) \sum_{\phi\in F(\Phi)} \Gamma_1(x_1, x_2, \phi).$$

Recalling that $\mathcal{F}^{(2)}(a, a; \tau_1, \tau_2, \epsilon) = Z_M^{(2)}(a, x_1; a, x_2|\tau_1, \tau_2, \epsilon) dx_1 dx_2$ we then obtain the following analogue of (92):

$$\mathcal{F}^{(2)}(a, a; \tau_1, \tau_2, \epsilon) = Z_M^{(1)}(\tau_1) Z_M^{(1)}(\tau_2) \sum_{\lambda=\{i^{e_i}\}} \frac{E(x_1, x_2, \lambda)}{\prod_i i^{e_i} e_i!} \epsilon^{\sum i e_i} dx_1 dx_2, \tag{115}$$

where

$$E(x_1, x_2, \lambda) = \sum_{\phi \in F(\Phi),\ \psi \in F(\Phi_\lambda)} \Gamma_1(x_1, x_2, \phi)\Gamma_2(\psi),$$

with $\Gamma_2(\psi)$ as before.

The expression (115) can be interpreted as a sum of weights $\zeta(D)$ associated with isomorphism classes of chequered configurations D where, in this case, **each** configuration includes two distinguished valence one nodes of type 1, x_i (see Sect. 3.2) corresponding to the label sets $\Phi_1, \Phi_2 = \{1\}$. As before, $\zeta(D) = \prod_E \zeta(E)$ for standard chequered edges E (44) augmented by the contributions for edges connected to the two valence one nodes with weights as in (46) (for $a = 1$). Then we find, as in Proposition 7, that

$$\mathcal{F}^{(2)}(a, a; \tau_1, \tau_2, \epsilon) = Z_M^{(1)}(\tau_1)Z_M^{(1)}(\tau_2) \sum_D \frac{\zeta(D)}{\prod_i e_i!} dx_1 dx_2.$$

Each D can be decomposed into *exactly* one necklace configuration N of type $\mathcal{N}_{11}^{x_1,x_2}$ of (49) connecting the two distinguished nodes and a standard configuration $\hat{D}$ of the type appearing in Subsect. 6.1 so that $\zeta(D) = \zeta(N)\zeta(\hat{D})$. Furthermore, if $\lambda' = \{1^{e'_1} \ldots p^{e'_p}\}$ is the subset of λ that labels $\hat{D}$ then the necklace contribution $\zeta(N)$ occurs with multiplicity $\prod_i \frac{e_i!}{e'_i!} = \frac{|\mathrm{Aut}(D)|}{|\mathrm{Aut}(\hat{D})|}$. It follows that

$$\begin{aligned}
\mathcal{F}^{(2)}(a, a; \tau_1, \tau_2, \epsilon) &= Z_M^{(1)}(\tau_1)Z_M^{(1)}(\tau_2) \sum_{\hat{D}} \frac{\zeta(\hat{D})}{|\mathrm{Aut}(\hat{D})|} \sum_{N \in \mathcal{N}_{11}^{x_1,x_2}} \zeta(N) dx_1 dx_2 \\
&= Z_M^{(2)}(\tau_1, \tau_2, \epsilon)\zeta_{11}^{x_1,x_2} dx_1 dx_2 \\
&= Z_M^{(2)}(\tau_1, \tau_2, \epsilon)\omega^{(2)}(x_1, x_2),
\end{aligned}$$

using (50) of Proposition 3. Applying Proposition 6, the same two form arises for the other possible insertions of two Heisenberg vectors. Alternatively, a similar explicit calculation can be carried out in each case leading to the expressions for $\omega^{(2)}$ described by (50). □

In a similar fashion one can generally show that the n-point function for n Heisenberg vectors vanishes for n odd and for n even is determined by the global symmetric meromorphic n form given by the symmetric (tensor) product

$$\mathrm{Sym}_n \omega^{(2)} = \sum_{\psi} \prod_{(r,s)} \omega^{(2)}(x_r, x_s), \tag{116}$$

where the sum is taken over the set of fixed point free involutions $\psi = \ldots (rs) \ldots$ of the labels $\{1, \ldots, n\}$. Then one finds

Theorem 10. *The genus two Heisenberg vector n-point function is given by the global symmetric meromorphic n-form*

$$\mathcal{F}_M^{(2)}(a, \ldots, a; \tau_1, \tau_2, \epsilon) = \mathrm{Sym}_n \omega^{(2)} Z_M^{(2)}(\tau_1, \tau_2, \epsilon). \tag{117}$$

Theorem 10 is in agreement with earlier results in [TUY] based on an assumed analytic structure for the ratio $\mathcal{F}_M^{(2)}(a,\ldots,a;\tau_1,\tau_2,\epsilon)/Z_M^{(2)}(\tau_1,\tau_2,\epsilon)$.

Using the associativity property of a VOA, the genus two Heisenberg n -point function (117) is a generator of *all* genus two n -point functions for M in an analogous way to that described for genus one in [MT1]. This will be further developed elsewhere [MT5]. We illustrate this by computing the 1-point function for the Virasoro vector $\tilde{\omega} = \frac{1}{2}a[-1]a$. This is determined by the genus two projective connection defined by e.g. [Gu]

$$s^{(2)}(x) = 6 \lim_{x\to y} \left(\omega^{(2)}(x,y) - \frac{dxdy}{(x-y)^2} \right). \tag{118}$$

We then find

Proposition 8. *The genus two* 1*-point function for the Virasoro vector* $\tilde{\omega}$ *is*

$$\mathcal{F}_M^{(2)}(\tilde{\omega};\tau_1,\tau_2,\epsilon) = \frac{1}{12} s^{(2)} Z_M^{(2)}(\tau_1,\tau_2,\epsilon). \tag{119}$$

Proof. Using the associativity property of a VOA we have [MT1]

$$\begin{aligned} Z_M^{(1)}(Y[a,x_1]Y[a,x_2]v;\tau_1) &= Z_M^{(1)}(Y[Y[a,x_1-x_2]a,x_2]v;\tau_1) \\ &= \frac{Z_M^{(1)}(v;\tau_1)}{(x_1-x_2)^2} + 2Z_M^{(1)}(Y[\tilde{\omega},x_2]v;\tau_1) + \cdots . \end{aligned}$$

Hence using the Heisenberg 2-point function (112) we find

$$\begin{aligned} \mathcal{F}^{(2)}(\omega;\tau_1,\tau_2,\epsilon) &= \lim_{x_1\to x_2} \frac{1}{2} \left(Z_M^{(2)}(a,x_1;a,x_2|\tau_1,\tau_2,\epsilon) - \frac{Z_M^{(2)}(\tau_1,\tau_2,\epsilon)}{(x_1-x_2)^2} \right) dx_1 dx_2 \\ &= \frac{1}{12} s^{(2)}(x_1) Z_M^{(2)}(\tau_1,\tau_2,\epsilon). \end{aligned}$$

□

Notice that $\mathcal{F}^{(2)}(\omega;\tau_1,\tau_2,\epsilon)$ is not a global differential 2-form since $s^{(2)}(x)$ transforms under a general conformal transformation $\phi(x)$ ([Gu]) as

$$s^{(2)}(\phi(x)) = s^{(2)}(x) - \{\phi;x\}dx^2, \tag{120}$$

where $\{\phi;x\} = \frac{\phi'''}{\phi'} - \frac{3}{2}\left(\frac{\phi''}{\phi'}\right)^2$ is the usual Schwarzian derivative. This property of the Virasoro 1-point function has previously been discussed many times in the physics and mathematics literature based on a variety of stronger assumptions e.g. [EO,TUY,FS,U,Z2].

7. Heisenberg Modules, Lattice VOAs and Theta Series

In this section we generalize the methods of Sect. 6 to compute the genus two partition function for a pair of Heisenberg modules. We consider the genus two n-point function for the Heisenberg vector and the Virasoro 1-point function. We apply these results to obtain closed formulas for the genus two partition function for a lattice VOA V_L (in terms of the genus two Siegel theta function for L) and the ‘twisted’ genus two partition function for the $\mathbb{Z}$ -lattice VOA (in terms of the genus two Riemann theta function with characters). We finally derive a genus two Ward identity for the Virasoro 1-point function for these theories.

7.1. Heisenberg modules. In this section we discuss the genus two partition function for a pair of simple Heisenberg modules $M \otimes e^{\alpha_1}$ and $M \otimes e^{\alpha_2}$ for $\alpha_1, \alpha_2 \in \mathbb{C}$. The partition function is then

$$Z^{(2)}_{\alpha_1,\alpha_2}(\tau_1, \tau_2, \epsilon) = \sum_{n \geq 0} \epsilon^n \sum_{u \in M_{[n]}} Z^{(1)}_{M \otimes e^{\alpha_1}}(u, \tau_1) Z^{(1)}_{M \otimes e^{\alpha_2}}(\bar{u}, \tau_2), \tag{121}$$

where u ranges over any basis for $M_{[n]}$. An explicit formula for $Z^{(1)}_{M \otimes e^{\alpha}}(u, \tau)$ was given in [MT1] (Corollary 3 and Theorem 1). We are going to use these results, together with graphical techniques similar to those employed for free bosons in Sect. 6 to establish a closed formula for (121). Letting $\alpha.\Omega.\alpha = \sum_{i,j=1,2} \alpha_i \Omega_{ij} \alpha_j$, where Ω_{ij} is the genus two period matrix we find

Theorem 11. *We have*

$$Z^{(2)}_{\alpha_1,\alpha_2}(\tau_1, \tau_2, \epsilon) = e^{i\pi\alpha.\Omega.\alpha} Z^{(2)}_M(\tau_1, \tau_2, \epsilon). \tag{122}$$

$Z^{(2)}_{\alpha_1,\alpha_2}(\tau_1, \tau_2, \epsilon)$ *is holomorphic on the domain* $\mathcal{D}^{\epsilon}$.

Remark 7. This is a natural generalization of the genus one partition function relation $Z^{(1)}_{M \otimes e^{\alpha}}(\tau) = q^{\alpha^2/2} Z^{(1)}_M(\tau)$.

Proof. Consider the Fock basis vectors $v = v(\lambda)$ (cf. (88)) identified with partitions $\lambda = \{i^{e_i}\}$ as in Sect. 6. Recall that λ defines a labeled set Φ_λ with e_i nodes labeled i. It is useful to re-state Corollary 3 of [MT1] in the following form:

$$Z^{(1)}_{M \otimes e^{\alpha}}(v, \tau) = Z^{(1)}_M(\tau) q^{\alpha^2/2} \sum_{\phi} \Gamma_{\lambda,\alpha}(\phi). \tag{123}$$

Here, ϕ ranges over the set of involutions

$$\mathrm{Inv}_1(\Phi_\lambda) = \{\phi \in \mathrm{Inv}(\Phi_\lambda) |\ p \in \mathrm{Fix}(\phi) \Rightarrow p \text{ has label } 1\}. \tag{124}$$

In words, ϕ is an involution in the symmetric group $\Sigma(\Phi_\lambda)$ such that all fixed-points of ϕ carry the label 1. Note that this includes the fixed-point-free involutions, which were the only involutions which played a role in the case of free bosons. The main difference between the free bosonic VOA and its modules is the need to include additional involutions in the latter case. In particular, we note that permutations with $|\lambda|$ odd can contribute in this case for $\lambda = \{i^{e_i}\}$ with e_1 odd. Finally,

$$\Gamma_{\lambda,\alpha}(\phi, \tau) = \Gamma_{\lambda,\alpha}(\phi) = \prod_{\Xi} \Gamma(\Xi), \tag{125}$$

where Ξ ranges over the orbits (of length ≤ 2) of ϕ acting on Φ_λ and

$$\Gamma(\Xi) = \begin{cases} C(r, s, \tau), & \text{if } \Xi = \{r, s\}, \\ \alpha, & \text{if } \Xi = \{1\}. \end{cases} \tag{126}$$

From (143)-(125) we get

$$Z^{(2)}_{\alpha_1,\alpha_2}(\tau_1, \tau_2, \epsilon) \\ = Z^{(1)}_M(\tau_1) Z^{(1)}_M(\tau_2) \sum_{\lambda = \{i^{e_i}\}} \frac{(-1)^{|\lambda|} E_{\alpha_1,\alpha_2}(\lambda)}{\prod_i i^{e_i} e_i!} q_1^{\alpha_1^2/2} q_2^{\alpha_2^2/2} \epsilon^{\sum i e_i}, \tag{127}$$

where

$$E_{\alpha_1,\alpha_2}(\lambda) = \sum_{\phi,\psi \in \mathrm{Inv}_1(\Phi_\lambda)} \Gamma_{\lambda,\alpha_1}(\phi, \tau_1)\Gamma_{\lambda,\alpha_2}(\psi, \tau_2). \tag{128}$$

(Compare with Eqs. (92)–(93).)

Now we follow the proof of Proposition 7 to obtain an expression analogous to (95), namely

$$Z^{(2)}_{\alpha_1,\alpha_2}(\tau_1, \tau_2, \epsilon) = Z^{(1)}_M(\tau_1)Z^{(1)}_M(\tau_2) \sum_D \frac{\gamma^0_{\alpha_1,\alpha_2}(D)}{|\mathrm{Aut}(D)|} q_1^{\alpha_1^2/2} q_2^{\alpha_2^2/2}, \tag{129}$$

the meaning of which we now enlarge upon. Compared to (95), the chequered diagrams D which occur in (129) are more general than before, in that they reflect the fact that the relevant involutions may now have fixed-points. Thus D is the union of its connected (as yet unoriented) components which are either chequered cycles as before or else chequered necklaces (see Sect. 3.2). Necklaces arise from orbits of the group $\langle\psi\phi\rangle$ on Φ_λ in which one of the nodes in the orbit is a fixed-point of ϕ or ψ. In that case the orbit will generally contain two such nodes which comprise the end nodes of the necklace. Note that these end nodes necessarily carry the label 1 (cf. (124)). There is degeneracy when *both* ϕ and ψ fix the node, in which case the degenerate necklace is obtained.

Similarly to (96), the term $\gamma^0_{\alpha_1,\alpha_2}(D)$ in (129) is given by

$$\gamma^0_{\alpha_1,\alpha_2}(D) = (-1)^{|\lambda|} \frac{\prod_{\Xi_1} \Gamma(\Xi_1) \prod_{\Xi_2} \Gamma(\Xi_2)}{\prod_i i^{e_i}} \epsilon^{\sum i e_i}, \tag{130}$$

where Ξ_1, Ξ_2 range over the orbits of ϕ, ψ respectively on Φ_λ. As usual the summands in (129) are multiplicative over connected components of the chequered diagram. This applies, in particular, to the chequered cycles which occur, and these are independent of the lattice elements. As a result, (129) factors as a product of two expressions, the first a sum over diagrams consisting only of chequered cycles and the second a sum over diagrams consisting only of chequered necklaces. However, the first expression corresponds precisely to the genus two partition function for the free boson (Proposition 7). We thus obtain

$$\frac{Z^{(2)}_{\alpha_1,\alpha_2}(\tau_1, \tau_2, \epsilon)}{Z^{(2)}_M(\tau_1, \tau_2, \epsilon)} = \sum_{D^N} \frac{\gamma^0_{\alpha_1,\alpha_2}(D^N)}{|\mathrm{Aut}(D^N)|} q_1^{\alpha_1^2/2} q_2^{\alpha_2^2/2}, \tag{131}$$

where here D^N ranges over all chequered diagrams all of whose connected components are chequered necklaces. So Theorem 11 is reduced to establishing

Proposition 9. *We have*

$$e^{i\pi\alpha.\Omega.\alpha} = \sum_{D^N} \frac{\gamma^0_{\alpha_1,\alpha_2}(D^N)}{|\mathrm{Aut}(D^N)|} q_1^{\alpha_1^2/2} q_2^{\alpha_2^2/2}. \tag{132}$$

We may apply the argument of (98) et. seq. to the inner sum in (132) to write it as an exponential expression

$$\exp\{i\pi(\alpha_1^2\tau_1 + \alpha_2^2\tau_2) + \sum_N \frac{\gamma^0_{\alpha_1,\alpha_2}(N)}{|\mathrm{Aut}(N)|}\}, \tag{133}$$

where N ranges over all unoriented chequered necklaces.

Recall the isomorphism class $\mathcal{N}_{ab}$ of oriented chequered necklaces of type ab as displayed in Fig. 3 of Sect. 3.2. Then (133) can be written as

$$\exp\{i\pi(\alpha_1^2\tau_1+\alpha_2^2\tau_2)+\frac{1}{2}\sum_{a,b\in\{1,2\}}\sum_{N_{ab}}\gamma^0_{\alpha_1,\alpha_2}(N_{ab})\}, \tag{134}$$

where here N ranges over *oriented* chequered necklaces of type ab.

From (126) and (130) we see that the contribution of the end nodes to $\gamma^0_{\alpha_1,\alpha_2}(N)$ is equal to $\epsilon\alpha_{\bar{a}}\alpha_{\bar{b}}$ for a type ab necklace. The remaining edge factors of $\gamma^0_{\alpha_1,\alpha_2}(N)$ have product $\gamma(N)=\zeta(N)$ by Lemma 1. Finally, necklaces of type 11 and 22 arise from Fock vectors with an even number $|\lambda|$ of permutation symbols whereas necklaces of type 12 and 21 arise from Fock vectors for odd $|\lambda|$ leading to a further -1 contribution in (130) in these cases. Overall we find that

$$\sum_{N_{ab}}\gamma^0_{\alpha_1,\alpha_2}(N_{ab})=(-1)^{a+b}\epsilon\alpha_{\bar{a}}\alpha_{\bar{b}}\zeta_{ab},$$

recalling $\zeta_{ab}=\sum_{N\in\mathcal{N}_{ab}}\zeta(N)$. Hence (134) may be re-expressed as

$$\exp\{\frac{\alpha_1^2}{2}(2\pi i\tau_1+\epsilon\zeta_{22})+\frac{\alpha_2^2}{2}(2\pi i\tau_2+\epsilon\zeta_{11})-\alpha_1\alpha_2\epsilon\zeta_{21}\}, \tag{135}$$

where $\zeta_{12}=\zeta_{21}$. Expression (135) reproduces (145) on applying Proposition 1.

Finally we note from Theorems 3 and 7 that $Z^{(2)}_{\alpha_1,\alpha_2}(\tau_1,\tau_2,\epsilon)$ is holomorphic on the domain $\mathcal{D}^\epsilon$. This completes the proof of Theorem 11. □

7.2. Some genus two n-point functions. In this section we consider the genus two n-point functions for the Heisenberg vector a and the 1-point function for the Virasoro vector $\tilde{\omega}$ for a pair of Heisenberg modules $M\otimes e^{\alpha_i}$. We again express each n-point function in terms of the associated formal differential form following (86) of Proposition 6. The results generalize those of Sect. 6.3. They are established by making use of similar methods, so that detailed proofs will not be given.

We first consider the example of the Heisenberg vector a inserted on the left torus (say). Then $\mathcal{F}^{(2)}_{\alpha_1,\alpha_2}(a;\tau_1,\tau_2,\epsilon)=Z^{(2)}_{\alpha_1,\alpha_2}(a,x_1|\tau_1,\tau_2,\epsilon)dx_1$ is the corresponding differential form. Defining $\nu_\alpha=\alpha_1\nu_1+\alpha_2\nu_2$, for holomorphic 1-forms ν_i, we find

Theorem 12. *The Heisenberg vector* 1*-point function for a pair of modules* $M\otimes e^{\alpha_1}$, $M\otimes e^{\alpha_2}$ *is*

$$\mathcal{F}^{(2)}_{\alpha_1,\alpha_2}(a;\tau_1,\tau_2,\epsilon)=\nu_\alpha Z^{(2)}_{\alpha_1,\alpha_2}(\tau_1,\tau_2,\epsilon). \tag{136}$$

Proof. The proof proceeds along the same lines as Theorems 9 and 11. We find that

$$\mathcal{F}^{(2)}_{\alpha_1,\alpha_2}(a;\tau_1,\tau_2,\epsilon)=Z^{(1)}_M(\tau_1)Z^{(1)}_M(\tau_2)\sum_D\frac{\zeta(D)}{\prod_i e_i!}dx_1,$$

where the sum is taken over isomorphism classes of chequered configurations D where, in this case, each configuration includes one distinguished valence one node of type $1,x_1$. Each D can be decomposed into exactly one necklace configuration of type $\mathcal{N}^{x_1,1}_{11}$ of (47), standard configurations of the type appearing in Theorem 5 and necklace contributions of type $\mathcal{N}_{ab}$ of (41) as in Theorem 11. The result then follows on applying the graphical expansion for $\nu_i(x_1)$ of (51). □

In a similar fashion one can generalize Theorem 10 concerning the n-point function for n Heisenberg vectors. This is determined by the global symmetric meromorphic n form given by a symmetric (tensor) product of ν_α and $\omega^{(2)}$ defined by

$$\mathrm{Sym}_n\left(\omega^{(2)}, \nu_\alpha\right) = \sum_{\psi} \prod_{(r,s)} \omega^{(2)}(x_r, x_s) \prod_{(t)} \nu_\alpha(x_t), \tag{137}$$

where the sum is taken over the set of involutions $\psi = \ldots(rs)\ldots(t)\ldots$ of the labels $\{1, \ldots, n\}$. Then one finds

Theorem 13. *The genus two Heisenberg vector n-point function for a pair of modules $M \otimes e^{\alpha_1}$, $M \otimes e^{\alpha_2}$ is given by the global symmetric meromorphic n-form*

$$\mathcal{F}^{(2)}_{\alpha_1,\alpha_2}(a, \ldots, a; \tau_1, \tau_2, \epsilon) = \mathrm{Sym}_n\left(\omega^{(2)}, \nu_\alpha\right) Z^{(2)}_{\alpha_1,\alpha_2}. \tag{138}$$

Theorem 13 is a natural generalization of Corollary 4 of [MT1] concerning genus one n-point functions for a Heisenberg module.

Similarly to Proposition 8 it follows that

Proposition 10. *The genus two 1-point function for a pair of modules $M \otimes e^{\alpha_1}$, $M \otimes e^{\alpha_2}$ for the Virasoro vector $\tilde{\omega}$ is*

$$\mathcal{F}^{(2)}_{\alpha_1,\alpha_2}(\tilde{\omega}; \tau_1, \tau_2, \epsilon) = \left(\frac{1}{2}\nu_\alpha^2 + \frac{1}{12}s^{(2)}\right) Z^{(2)}_{\alpha_1,\alpha_2}(\tau_1, \tau_2, \epsilon). \tag{139}$$

Finally, let us introduce the differential operator [Fa, U]

$$\mathcal{D} = \frac{1}{2\pi i} \sum_{1 \le i \le j \le 2} \nu_i \nu_j \frac{\partial}{\partial \Omega_{ij}}. \tag{140}$$

$\mathcal{D}$ maps differentiable functions on $\mathbb{H}_2$ to the space of holomorphic 2-forms (spanned by $\nu_1^2, \nu_2^2, \nu_1\nu_2$) and is $Sp(4, \mathbb{Z})$ invariant. It follows from Theorem 11 that (139) can be rewritten as a Ward identity

$$\mathcal{F}^{(2)}_{\alpha_1,\alpha_2}(\tilde{\omega}; \tau_1, \tau_2, \epsilon) = Z^{(2)}_M(\tau_1, \tau_2, \epsilon)\left(\mathcal{D} + \frac{1}{12}s^{(2)}\right) e^{i\pi\alpha.\Omega.\alpha}. \tag{141}$$

Remark 8. In theoretical physics, a conformal Ward identity is an identity between different correlation functions following from conformal invariance e.g. [EO, DFMS]. Thus in (141) the Virasoro 1-point function is related to the normalized partition function $Z^{(2)}_{\alpha_1,\alpha_2}/Z^{(2)}_M = e^{i\pi\alpha.\Omega.\alpha}$.

7.3. Lattice VOAs. Let L be an even lattice of dimension l with V_L the corresponding lattice VOA. The underlying Fock space is

$$V_L = M^l \otimes C[L] = \oplus_{\alpha \in L} M^l \otimes e^\alpha, \tag{142}$$

where M^l is the corresponding rank l Heisenberg free boson theory. We follow Sect. 4.1 and [MT1] concerning further notation for lattice theories.

The general shape of $Z^{(2)}_{V_L}(\tau_1, \tau_2, \epsilon)$ is as in (85). Note that the modes of a state $u \otimes e^{\alpha}$ map $M^l \otimes e^{\beta}$ to $M^l \otimes e^{\alpha+\beta}$. Thus if $\alpha \neq 0$ then $Z^{(1)}_{V_L}(u \otimes e^{\alpha}, \tau)$ vanishes, and as a result we see that

$$\begin{aligned} Z^{(2)}_{V_L}(\tau_1, \tau_2, \epsilon) &= \sum_{n \geq 0} \epsilon^n \sum_{u \in M^l_{[n]}} Z^{(1)}_{V_L}(u, \tau_1) Z^{(1)}_{V_L}(\bar{u}, \tau_2) \\ &= \sum_{\alpha, \beta \in L} \sum_{n \geq 0} \epsilon^n \sum_{u \in M^l_{[n]}} Z^{(1)}_{M^l \otimes e^{\alpha}}(u, \tau_1) Z^{(1)}_{M^l \otimes e^{\beta}}(\bar{u}, \tau_2). \end{aligned} \tag{143}$$

Here, u ranges over any basis for $M^l_{[n]}$. Viewing $M^l \otimes e^{\alpha}$ as a simple module for M^l we may employ Theorem 11 for each component to obtain

Theorem 14. *We have*

$$Z^{(2)}_{V_L}(\tau_1, \tau_2, \epsilon) = Z^{(2)}_{M^l}(\tau_1, \tau_2, \epsilon) \theta^{(2)}_L(\Omega), \tag{144}$$

where $\theta^{(2)}_L(\Omega)$ is the (genus two) Siegel theta function associated to L (e.g. [Fr])

$$\theta^{(2)}_L(\Omega) = \sum_{\alpha, \beta \in L} \exp(\pi i ((\alpha, \alpha)\Omega_{11} + 2(\alpha, \beta)\Omega_{12} + (\beta, \beta)\Omega_{22})). \tag{145}$$

We can similarly compute n-point functions for n Heisenberg vectors $a_1, \ldots a_l$ using Theorem 13. We can also employ Proposition 10 and the Ward identity (141) to obtain the 1-point function for the Virasoro vector $\tilde{\omega} = \frac{1}{2} \sum_i a_i[-1]a_i$ as follows:

Proposition 11. *The Virasoro 1-point function for a lattice VOA satisfies a genus two Ward identity*

$$\mathcal{F}^{(2)}_{V_L}(\tilde{\omega}; \tau_1, \tau_2, \epsilon) = Z^{(2)}_{M^l}(\tau_1, \tau_2, \epsilon) \left(\mathcal{D} + \frac{l}{12} s^{(2)} \right) \theta^{(2)}_L(\Omega). \tag{146}$$

The Ward identity (146) is reminiscent of some earlier results in physics and mathematics, e.g. [EO, KNTY].

We briefly discuss the holomorphic and automorphic properties of $Z^{(2)}_{V_L}(\tau_1, \tau_2, \epsilon)$ and $\mathcal{F}^{(2)}_L(\tilde{\omega}; \tau_1, \tau_2, \epsilon)$. There is more that one can say here, but a fuller discussion must wait for another time [MT5]. The function $\theta^{(2)}_L(\Omega)$ is a Siegel modular form of weight $l/2$ ([Fr]) for some subgroup of $Sp(4, \mathbb{Z})$, in particular it is holomorphic on the Siegel upper half-space $\mathbb{H}_2$. From Theorems 3, 7 and 14, we deduce

Theorem 15. *$Z^{(2)}_{V_L}(\tau_1, \tau_2, \epsilon)$ is holomorphic on the domain $\mathcal{D}^{\epsilon}$.* □

We can obtain the automorphic properties of $Z^{(2)}_{V_L}(\tau_1, \tau_2, \epsilon)$ in the same way using that for $\theta^{(2)}_L(\Omega)$ together with Theorem 8. Rather than do this explicitly, let us introduce a variation of the partition function, namely the *normalized partition function*

$$\hat{Z}^{(2)}_{V_L}(\tau_1, \tau_2, \epsilon) = \frac{Z^{(2)}_{V_L}(\tau_1, \tau_2, \epsilon)}{Z^{(2)}_{M^l}(\tau_1, \tau_2, \epsilon)}. \tag{147}$$

Bearing in mind the convention (109), what (144) says is that there is a commuting diagram of holomorphic maps

$$\begin{array}{ccc} \mathcal{D}^{\epsilon} & \xrightarrow{F^{\epsilon}} & \mathbb{H}_2 \\ \hat{Z}^{(2)}_{V_L} \searrow & & \swarrow \theta^{(2)}_L \\ & \mathbb{C} & \end{array} . \tag{148}$$

Furthermore, the G-actions on the two functions in question are compatible. More precisely, if $\gamma \in G$ then we have

$$\begin{aligned} \hat{Z}^{(2)}_{V_L}(\tau_1, \tau_2, \epsilon)|_{l/2}\, \gamma &= \hat{Z}^{(2)}_{V_L}(\gamma(\tau_1, \tau_2, \epsilon)) \det(C\Omega + D)^{-l/2} \\ &= \theta^{(2)}_L(F^{\epsilon}(\gamma(\tau_1, \tau_2, \epsilon))) \det(C\Omega + D)^{-l/2} \qquad \text{(from (148))} \\ &= \theta^{(2)}_L(\gamma(F^{\epsilon}(\tau_1, \tau_2, \epsilon))) \det(C\Omega + D)^{-l/2} \qquad \text{(from Theorem 4)} \\ &= \theta^{(2)}_L(\gamma\Omega) \det(C\Omega + D)^{-l/2} \qquad \text{(from (109))} \\ &= \theta^{(2)}_L(\Omega)|_{l/2}\, \gamma. \end{aligned} \tag{149}$$

For example, if the lattice L is *unimodular* as well as even then $\theta^{(2)}_L$ is a Siegel modular form of weight $l/2$ on the full group $Sp(4, \mathbb{Z})$. Then (149) informs us that

$$\hat{Z}^{(2)}_{V_L}(\tau_1, \tau_2, \epsilon)|_{l/2}\gamma = \hat{Z}^{(2)}_{V_L}(\tau_1, \tau_2, \epsilon), \;\; \gamma \in G,$$

i.e. $\hat{Z}^{(2)}_{V_L}(\tau_1, \tau_2, \epsilon)$ is automorphic of weight $l/2$ with respect to the group G.

Similar remarks may be made about the *normalized Virasoro 1-point function* defined by

$$\hat{\mathcal{F}}^{(2)}_{V_L}(\tilde{\omega}; \tau_1, \tau_2, \epsilon) = \frac{\mathcal{F}^{(2)}_{V_L}(\tilde{\omega}; \tau_1, \tau_2, \epsilon)}{Z^{(2)}_{M^l}(\tau_1, \tau_2, \epsilon)}, \tag{150}$$

which obeys the Ward identity

$$\hat{\mathcal{F}}^{(2)}_{V_L}(\tilde{\omega}; \tau_1, \tau_2, \epsilon) = \left(\mathcal{D} + \frac{l}{12} s^{(2)}\right) \hat{Z}^{(2)}_{V_L}(\tau_1, \tau_2, \epsilon). \tag{151}$$

Using the modular transformation properties of the projective connection (e.g. [Fa,U]) one finds that (151) enjoys the same modular properties as $\hat{Z}^{(2)}_{V_L}(\tau_1, \tau_2, \epsilon)$ i.e.

Proposition 12. *The normalized Virasoro* 1*-point function for a lattice VOA obeys*

$$\hat{\mathcal{F}}^{(2)}_{V_L}(\tilde{\omega}; \tau_1, \tau_2, \epsilon)|_{l/2}\gamma = \left(\mathcal{D} + \frac{l}{12} s^{(2)}\right) \left(\hat{Z}^{(2)}_{V_L}(\tau_1, \tau_2, \epsilon)|_{l/2}\gamma\right), \tag{152}$$

for $\gamma \in G$.

7.4. Rank two fermion vertex super algebra and the genus two riemann theta series. As a last application of Theorem 11, we briefly consider the rank two fermion Vertex Operator Super Algebra (VOSA) $V = V(H, +\frac{1}{2})^2$. V can be decomposed in terms of a Heisenberg subVOA generated by a Heisenberg state a and irreducible modules $M \otimes e^m$ for $m \in \mathbb{Z}$ e.g. [Ka]. One can construct orbifold n-point functions for a pair g, h of commuting V automorphisms generated by $a(0)$ [MTZ]. In particular, consider the 1-point function (which is non-vanishing only for $u \in M$) for a g-twisted sector for $g = e^{-2\pi i\lambda a(0)}$ together with an automorphism $h = e^{2\pi i\mu a(0)}$ (for real λ, μ) which can be expressed as (op. cit.)

$$\begin{aligned} Z_V^{(1)}((g,h);u,\tau) &= \mathrm{Tr}_V(ho(u)q^{L(0)+\lambda^2/2+\lambda a(0)-1/24}) \\ &= \sum_{m\in\mathbb{Z}} e^{2\pi i m\mu}\mathrm{Tr}_{M\otimes e^{m+\lambda}}(o(u)q^{L(0)-1/24}), \end{aligned} \tag{153}$$

utilizing the Heisenberg decomposition. In particular, the orbifold partition function is expressed in terms of the Jacobi theta series

$$\begin{aligned} Z_V^{(1)}((g,h);\tau) &= \frac{e^{-2\pi i\lambda\mu}}{\eta(\tau)}\vartheta\begin{bmatrix}\lambda\\ \mu\end{bmatrix}(\tau), \\ \vartheta\begin{bmatrix}\lambda\\ \mu\end{bmatrix}(\tau) &= \sum_{m\in\mathbb{Z}} e^{i\pi(m+\lambda)^2\tau+2\pi i(m+\lambda)\mu}. \end{aligned} \tag{154}$$

Similarly to (121), it is natural to define the genus two orbifold partition function for a pair of g_i-twisted sectors together with commuting automorphisms h_i parameterized by λ_i, μ_i for $i = 1, 2$ with

$$Z_V^{(2)}((g_i,h_i);\tau_1,\tau_2,\epsilon) = \sum_{n\geq 0}\epsilon^n \sum_{u\in M_{[n]}} Z_V^{(1)}((g_1,h_1);u,\tau_1)Z_V^{(1)}((g_2,h_2);\bar{u},\tau_2), \tag{155}$$

where u ranges over any basis for $M_{[n]}$. A more detailed description of this and an alternative fermionic VOSA approach to this will be described elsewhere [TZ]. Here we decompose the genus one 1-point functions of (155) in terms of Heisenberg modules $M \otimes e^{m_i+\lambda_i}$ to find, in the notation of (121), that

$$Z_V^{(2)}((g_i,h_i);\tau_1,\tau_2,\epsilon) = \sum_{m\in\mathbb{Z}^2} e^{2\pi i m.\mu} Z^{(2)}_{m_1+\lambda_1,m_2+\lambda_2}(\tau_1,\tau_2,\epsilon), \tag{156}$$

where here $\lambda = (\lambda_1, \lambda_2)$, $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2$ and $m = (m_1, m_2) \in \mathbb{Z}^2$. Theorem 11 implies

Theorem 16. *We have*

$$Z_V^{(2)}((g_i,h_i);\tau_1,\tau_2,\epsilon) = e^{-2\pi i\lambda.\mu} Z_M^{(2)}(\tau_1,\tau_2,\epsilon)\theta^{(2)}\begin{bmatrix}\lambda\\ \mu\end{bmatrix}(\Omega), \tag{157}$$

for genus two Riemann theta function (*e.g.* [Mu])

$$\theta^{(2)}\begin{bmatrix}\lambda\\ \mu\end{bmatrix}(\Omega) = \sum_{m\in\mathbb{Z}^2} e^{i\pi(m+\lambda).\Omega.(m+\lambda)+2\pi i(m+\lambda).\mu}. \tag{158}$$

As already described for lattice VOAs, one can similarly obtain a Ward identity for the Virasoro 1-point function analogous to (146) and (151) and analyze the modular properties of (157) and the Virasoro 1-point function under the action of G.

8. Appendix - A Product Formula

Here we continue the discussion initiated in Subsect. 3.1, with a view to proving Proposition 2. Consider a set of independent (non-commuting) variables x_i indexed by the elements of a finite set $I = \{1, \ldots, N\}$. The set of all distinct monomials $x_{i_1} \ldots x_{i_n} (n \geq 0)$ may be considered as a basis for the tensor algebra associated with an N dimensional vector space. Call n the degree of the monomial $x_{i_1} \ldots x_{i_n}$.

Let $\rho = \rho_n$ be the standard cyclic permutation which acts on monomials of degree n via $\rho : x_{i_1} \ldots x_{i_n} \mapsto x_{i_n} x_{i_1} \ldots x_{i_{n-1}}$. The *rotation group* of a given monomial $x = x_{i_1} \ldots x_{i_n}$ is the subgroup of $\langle \rho_n \rangle$ that leaves x invariant. Call x *rotationless* in case its rotation group is trivial. Let us say that two monomials x, y of degree n are *equivalent* in case $y = \rho_n^r(x)$ for some $r \in Z$, and denote the corresponding equivalence class by (x). We call these *cycles*. Note that equivalent monomials have the same rotation group, so we may meaningfully refer to the rotation group of a cycle. In particular, a *rotationless cycle* is a cycle whose representative monomials are themselves rotationless. Let C_n be the set of inequivalent cycles of degree n.

It is convenient to identify a cycle $(x_{i_1} \ldots x_{i_n})$ with a *cyclic labeled graph*, that is, a graph with n vertices labeled $x_{i_1}, \ldots, x_{i_n}$ and with edges $x_{i_1} x_{i_2}, \ldots, x_{i_n} x_{i_1}$. We will sometimes afflict the graph with one of the two possible orientations.

Let $\mathcal{M}(I)$ be the (multiplicative semigroup generated by) the rotationless cycles in the symbols x_i, $i \in I$. There is an injection

$$\iota : \bigcup_{n \geq 0} C_n \longrightarrow \mathcal{M}(I) \tag{159}$$

defined as follows. If $(x) \in C_n$ has rotation group of order r then $r|n$ and there is a rotationless monomial y such that $x = y^r$. We then map $(x) \mapsto (y)^r$. It is readily verified that this is well-defined. In this way, each cycle is mapped to a power of a rotationless cycle in $\mathcal{M}(I)$. A typical element of $\mathcal{M}(I)$ is uniquely expressible in the form

$$p_1^{f_1} p_2^{f_2} \ldots p_k^{f_k}, \tag{160}$$

where $p_1, \ldots, p_k$ are distinct rotationless cycles and $f_1, \ldots, f_k$ are non-negative integers. We call (160) the *reduced form* of an element in $\mathcal{M}(I)$. A general element of $\mathcal{M}(I)$ is then essentially a labeled graph, each of whose connected components are rotationless labeled cycles as discussed in Subsect. 3.1.

Now consider a second finite set T together with a map

$$F : T \longrightarrow I. \tag{161}$$

Thus elements of I label elements of T via the map F. F induces a natural map

$$\overline{F} : \Sigma(T) \longrightarrow \mathcal{M}(I)$$

from the symmetric group $\Sigma(T)$ as follows. For an element $\tau \in \Sigma(T)$, write τ as a product of disjoint cycles $\tau = \sigma_1.\sigma_2 \ldots$. We set $\overline{F}(\tau) = \overline{F}(\sigma_1)\overline{F}(\sigma_2) \ldots$, so it suffices to define $\overline{F}(\sigma)$ for a cycle $\sigma = (s_1 s_2 \ldots)$ with $s_1, s_2, \ldots \in T$. In this case we set

$$\overline{F}(\sigma) = \iota((x_{F(s_1)} x_{F(s_2)} \ldots)),$$

where ι is as in (159). When written in the form (160), we call $\overline{F}(\tau)$ the *reduced F-form* of τ.

For $i \in I$, let $s_i = |F^{-1}(i)|$ be the number of elements in T with label i. So the number of elements in T is equal to $\sum_{i\in I} s_i$. We say that two elements $\tau_1, \tau_2 \in \Sigma(T)$ are F-equivalent if they have the same reduced F-form, i.e. $\overline{F}(\tau_1) = \overline{F}(\tau_2)$. We will show that each equivalence class contains the same number of elements. Precisely,

Lemma 3. *Each F-equivalence class contains precisely $\prod_{i\in I} s_i$ elements. In particular, the number of F-equivalence classes is $|T|!/\prod_{i\in I} s_i$.*

Proof. An element $\tau \in \Sigma(T)$ may be represented uniquely as

$$\begin{pmatrix} 0 & 1 & \cdots & M \\ \tau(0) & \tau(1) & \cdots & \tau(M) \end{pmatrix}$$

so that

$$\overline{F}(\tau) = \begin{pmatrix} F(0) & F(1) & \cdots & F(M) \\ F(\tau(0)) & F(\tau(1)) & \cdots & F(\tau(M)) \end{pmatrix}$$

with an obvious notation. Exactly s_i of the $\tau(j)$ satisfy

$$\overline{F}(\tau(j)) = x_i$$

so that there are $\prod_{i\in I} s_i$ choices of τ which have a given image under $\overline{F}$. The lemma follows. □

The next results employs notation introduced in Subsects. 3.1 and 3.2.

Lemma 4. *We have*

$$(I - M_1 M_2)^{-1}(1, 1) = (1 - \sum_{L\in\mathcal{L}_{21}} \zeta(L))^{-1}. \tag{162}$$

As before, the left-hand-side of (162) means $\sum_{n\geq 0}(M_1 M_2)^n(1, 1)$. It is a certain power series with entries being quasi-modular forms.

Proof of Lemma. We have

$$(M_1 M_2)^n(1, 1) = \sum M_1(1, k_1)M_2(k_1, k_2) \ldots M_2(k_{2n-1}, 1), \tag{163}$$

where the sum ranges over all choices of positive integers $k_1, \ldots, k_{2n-1}$. Such a choice corresponds to a (isomorphism class of) chequered cycle L with $2n$ nodes and with at least one distinguished node, so that the left-hand-side of (162) is equal to

$$\sum_L \zeta(L)$$

summed over all such L. We can formally write L as a product $L = L_1 L_2 \ldots L_p$, where each $L_i \in \mathcal{L}_{21}$. This indicates that L has p distinguished nodes and that the L_i are the edges of L between consecutive distinguished nodes, which can be naturally thought of as chequered cycles in $\mathcal{L}_{21}$. Note that in the representation of L as such a product, the L_i do not commute unless they are equal, moreover ζ is multiplicative. Then

$$(I - M_1 M_2)^{-1}(1, 1) = \sum_{L_i\in\mathcal{L}_{21}} \zeta(L_1 \ldots L_p) = (1 - \sum_{L\in\mathcal{L}_{21}} \zeta(L))^{-1},$$

as required. □

Proposition 13. *We have*

$$(I - M_1M_2)^{-1}(1,1) = \prod_{L\in\mathcal{R}_{21}} (1-\zeta(L))^{-1}. \tag{164}$$

Proof. By Lemma 4 we have

$$(I - M_1M_2)^{-1}(1,1) = \sum m(e_1,\ldots,e_k)\zeta(L_1)^{e_1}\ldots\zeta(L_k)^{e_k}, \tag{165}$$

where the sum ranges over distinct elements $L_1, \ldots L_k$ of $\mathcal{L}_{21}$ and all k-tuples of non-negative integers $e_1, \ldots, e_k$, and where the multiplicity is

$$m(e_1,\ldots,e_k) = \frac{(\sum_i e_i)!}{\prod_i (e_i!)}.$$

Let S be the set consisting of e_i copies of L_i, $1 \le i \le k$, let I be the integers between 1 and k, and let $F : S \longrightarrow I$ be the obvious labelling map. A reduced F-form is then an element of $\mathcal{M}(I)$, where the variables x_i are now the L_i. The free generators of $\mathcal{M}(I)$, i.e. rotationless cycles in the x_i, are naturally identified *precisely* with the elements of $\mathcal{R}_{21}$, and Lemma 3 implies that each element of $\mathcal{M}(I)$ corresponds to just one term under the summation in (165). Equation (164) follows immediately from this and the multiplicativity of ζ, and the proposition is proved. □

References

[B1] Borcherds, R.E.: Vertex algebras, kac-moody algebras and the monster. Proc. Nat. Acad. Sc. **83**, 3068–3071 (1986)

[B2] Borcherds, R.E.: Monstrous moonshine and monstrous lie superalgebras. Inv. Math. **109**, 405–444 (1992)

[BK] Belavin, A., Knizhnik, V.: Algebraic geometry and the geometry of strings. Phys. Lett. **168B**, 201–206 (1986)

[BPZ] Belavin, A., Polyakov, A., Zamolodchikov, A.: Infinite conformal symmetry in two-dimensional quantum field theory. Nucl. Phys. **B241**, 333–380 (1984)

[CN] Conway, J.H., Norton, S.P.: Monstrous moonshine. Bull. Lond. Math. Soc. **12**, 308–339 (1979)

[DFMS] Di Francesco, P., Mathieu, P., Senechal, D.: *Conformal Field Theory*. Springer-Verlag, New York (1997)

[DGM] Dolan, L., Goddard, P., Montague, P.: Conformal field theories, representations and lattice constructions. Commun. Math. Phys. **179**, 61–120 (1996)

[DM] Dong, C., Mason, G.: Shifted vertex operator algebras. Proc. Camb. Phil. Math. Soc. **141**, 67–80 (2006)

[DP] D'Hoker, E., Phong, D.H.: The geometry of string perturbation theory. Rev. Mod. Phys. **60**, 917–1065 (1988)

[EO] Eguchi, T., Ooguri, H.: Conformal and current algebras on a general riemann surface. Nucl. Phys. B **282**, 308–328 (1987)

[Fa] Fay, J.: *Theta functions on Riemann surfaces*, Lecture Notes in Mathematics **352**, Berlin-New York: Springer-Verlag, 1973

[FHL] Frenkel, I., Huang, Y., Lepowsky, J.: On axiomatic approaches to vertex operator algebras and modules. Mem. Amer. Math. Soc. **104** (1993)

[FK] Farkas, H.M., Kra, I.: *Riemann surfaces*, New York: Springer-Verlag, 1980

[FLM] Frenkel, I., Lepowsky, J., Meurman,A.: *Vertex operator algebras and the Monster*, New York: Academic Press, 1988

[Fr] Freitag, E.: *Siegelische modulfunktionen*, Berlin and New York: Springer-Verlag, 1983

[FS] Freidan, D., Shenker, S.: The analytic geometry of two dimensional conformal field theory. Nucl. Phys. **B281**, 509–545 (1987)

[GG] Gaberdiel, M., Goddard, P.: Axiomatic conformal field theory. Commun. Math. Phys. **209**, 549–594 (2000)

[GSW] Green, M., Schwartz, J., Witten, E.: *Superstring theory Vol. 1*, Cambridge: Cambridge University Press, 1987

[Gu] Gunning, R.C.: *Lectures on Riemann surfaces* . Princeton, NJ: Princeton Univ. Press, 1966

[Ka] Kac, V.: *Vertex operator algebras for beginners*. University Lecture Series, Vol. **10**, Providence, RI: Amer. Math. Soc., 1998

[Kn] Knizhnik, V.G.: Multiloop amplitudes in the theory of quantum strings and complex geometry. Sov. Phys. Usp. **32**, 945–971 (1989)

[KNTY] Kawamoto, N., Namikawa, Y., Tsuchiya, A., Yamada, Y.: Geometric realization of conformal field theory on riemann surfaces. Commun. Math. Phys. **116**, 247–308 (1988)

[KZ] Kaneko, M., Zagier, D.: A generalized Jacobi theta function and quasimodular forms. In: *The Moduli Space of Curves* (Texel Island, 1994), Progr. in Math. **129**, Boston: Birkhauser, 1995

[Li] Li, H.: Symmetric invariant bilinear forms on vertex operator algebras. J. Pure. Appl. Alg. **96**, 279–297 (1994)

[LL] Lepowsky, J., Li, H.: *Introduction to vertex operator algebras and their representations*. Boston: Birkhäuser, 2004

[MN] Matsuo, A., Nagatomo, K,: Axioms for a vertex algebra and the locality of quantum fields. Math. Soc. Jap. Mem. **4** (1999)

[MT1] Mason, G., Tuite, M.P.: Torus chiral n-point functions for free boson and lattice vertex operator algebras. Commun. Math. Phys. **235**, 47–68 (2003)

[MT2] Mason, G., Tuite, M.P.: On genus two riemann surfaces formed from sewn tori. Commun. Math. Phys. **270**, 587–634 (2007)

[MT3] Mason, G., Tuite, M.P.: Partition functions and chiral algebras. In: *Lie algebras, vertex operator algebras and their applications* (in honor of Jim Lepowsky and Robert L. Wilson), Contemporary Mathematics **442**, Providence, RI: Amer. Math. Soc., 2007, pp. 401–410

[MT4] Mason, G., Tuite, M.P.: Free bosonic vertex operator algebras on genus two Riemann surfaces II. to appear

[MT5] Mason, G., Tuite, M.P.: In preparation

[MT6] Mason, G., Tuite, M.P.: Vertex operators and modular forms. In: Kirsten, K., Williams, F. (eds.) *A Window into Zeta and Modular Physics*, vol. 57, pp. 183–278. MSRI Publications, Cambridge University Press, Cambridge (2010)

[MTZ] Mason, G., Tuite, M.P., Zuevsky, A.: Torus *n*-point functions for $\mathbb{R}$-graded vertex operator superalgebras and continuous fermion orbifolds. Commun. Math. Phys. **283**, 305–342 (2008)

[Mu] Mumford, D.: *Tata lectures on Theta I and II.* Birkhäuser, Boston (1983)

[P] Polchinski, J.: *String theory*. Volume **I**, Cambridge: Cambridge University Press, 1998

[Se] Serre, J-P.: *A course in arithmetic*. Springer-Verlag, Berlin (1978)

[So1] Sonoda, H.: Sewing conformal field theories i. Nucl. Phys. **B311**, 401–416 (1988)

[So2] Sonoda, H.: Sewing conformal field theories ii. Nucl. Phys. **B311**, 417–432 (1988)

[T] Tuite, M.P.: *Genus two meromorphic conformal field theory.* CRM Proceedings and Lecture Notes **30**, Providence, RI: Amer. Math. Soc., 2001, pp. 231–251

[TUY] Tsuchiya, A., Ueno, K., Yamada, Y.: Conformal field theory on universal family of stable curves with gauge symmetries. Adv. Stud. Pure. Math. **19**, 459–566 (1989)

[TZ] Tuite, M.P., Zuevsky, A.: Genus two partition and correlation functions for fermionic vertex operator superalgebras I, arXiv:1007.5203

[U] Ueno, K.: Introduction to conformal field theory with gauge symmetries. In: *Geometry and Physics - Proceedings of the conference at Aarhus Univeristy, Aaarhus, Denmark*, New York: Marcel Dekker, 1997

[Y] Yamada, A.: Precise variational formulas for abelian differentials. Kodai Math. J. **3**, 114–143 (1980)

[Z1] Zhu, Y.: Modular invariance of characters of vertex operator algebras. J. Amer. Math. Soc. **9**, 237–302 (1996)

[Z2] Zhu, Y.: Global vertex operators on riemann surfaces. Commun. Math. Phys. **165**, 485–531 (1994)

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