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*Research Article*

# A New System of Nonlinear Fuzzy Variational Inclusions Involving $(A, \eta)$-Accretive Mappings in Uniformly Smooth Banach Spaces

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A new system of nonlinear fuzzy variational inclusions involving $(A, \eta)$-accretive mappings in uniformly smooth Banach spaces is introduced and studied many fuzzy variational and variational inequality (inclusion) problems as special cases of this system. By using the resolvent operator technique associated with $(A, \eta)$-accretive operator due to Lan et al. and Nadler's fixed points theorem for set-valued mappings, an existence theorem of solutions for this system of fuzzy variational inclusions is proved. We also construct some new iterative algorithms for the solutions of this system of nonlinear fuzzy variational inclusions in uniformly smooth Banach spaces and discuss the convergence of the sequences generated by the algorithms in uniformly smooth Banach spaces. Our results extend, improve, and unify many known results in the recent literatures.



## 1. Introduction

Variational inequality was initially studied by Stampacchia [1] in 1964. In order to study many kinds of problems arising in industrial, physical, regional, economical, social, pure, and applied sciences, the classical variational inequality problems have been extended and generalized in many directions. Among these generalizations, variational inclusion introduced and studied by Hassouni and Moudafi [2] is of interest and importance. It provides us with a unified, natural, novel innovative, and general technique to study a wide class of the problems arising in different branches of mathematical and engineering sciences (see, e.g., [3–7]).

Next, the development of variational inequality is to design efficient iterative algorithms to compute approximate solutions for variational inequalities and their generalizations. Up to now, many authors have presented implementable and significant numerical methods such as projection method, and its variant forms, linear approximation, descent method, Newton's method and the method based on the auxiliary principle technique. In particular, the method based on the resolvent operator technique is a generalization of the projection method and has been widely used to solve variational inclusions.

Some new and interesting problems, which are called the *systems of variational inequality problems*, were introduced and studied. Pang [8], Cohen and Chaplais [9], Bianchi [10], and Ansari and Yao [11] considered some systems of scalar variational inequalities and Pang showed that the traffic equilibrium problem, the spatial equilibrium problem, the Nash equilibrium, and the general equilibrium programming problems can be modelled as variational inequalities. He decomposed the original variational inequality into a system of variational inequalities which are easy to solve and studied the convergence of such methods. Ansari et al. [12] introduced and studied a system of vector variational inequalities by a fixed point theorem. Allevi et al. [13] considered a system of generalized vector variational inequalities and established some existence results under relative pseudomonotonicity. Kassay and Kolumbán [14] introduced a system of variational inequalities and proved an existence theorem by the Ky Fan lemma. Kassay et al. [15] studied Minty and Stampacchia variational inequality systems with the help of the Kakutani-Fan-Glicksberg fixed point theorem. Peng [16, 17] Peng and Yang [18] introduced a system of quasivariational inequality problems and proved its existence theorem by maximal element theorems. Verma [19–23] introduced and studied some systems of variational inequalities and developed some iterative algorithms for approximating the solution for this system of generalized nonlinear quasivariational inequalities in Hilbert spaces. J. K. Kim and D. S. Kim [24] introduced a new system of generalized nonlinear quasivariational inequalities and obtained some existence and uniqueness results of solution for this system of generalized nonlinear quasivariational inequalities in Hilbert spaces. Cho et al. [25] introduced a new system of nonlinear variational inequalities and proved some existence and uniqueness theorems of solutions for the system of nonlinear variational inequalities in Hilbert spaces. As generalizations of system of variational inequalities, Agarwal et al. [26] introduced a system of generalized nonlinear mixed quasivariational inclusions and investigated the sensitivity of solutions for this system of generalized nonlinear mixed quasivariational inclusions in Hilbert spaces. Kazmi and Bhat [27] introduced a system of nonlinear variational-like inclusions and gave an iterative algorithm for finding its approximate solution. It is known that accretivity of the underlying operator plays indispensable roles in the theory of variational inequality and its generalizations.

In 2001, Huang and Fang [28] were the first to introduce generalized $m$-accretive mapping and give the definition of the resolvent operator for generalized $m$-accretive mappings in Banach spaces. They also proved some properties of the resolvent operator for generalized $m$-accretive mappings in Banach spaces. Subsequently, Fang and Huang [29], Yan et al. [30], Fang et al. [31], Lan et al. [32, 33], Fang and Huang [34], and Peng et al. [35] introduced and investigated many new systems of variational inclusions involving $H$-monotone operators and $(H,\eta)$-monotone operators in Hilbert spaces, generalized $m$-accretive mappings, $H$-accretive mappings and $(H,\eta)$-accretive mappings in Banach spaces, respectively.

In 2004, Verma in [36, 37] introduced new notions of $A$-monotone and $(A,\eta)$-monotone operators and studied some properties of $A$-monotone and $(A,\eta)$-monotone operators in Hilbert spaces. In [38], Lan et al. first introduced a new concept of $(A,\eta)$-accretive mappings, which generalizes the existing monotone or accretive operators and studied some properties of $(A,\eta)$-accretive mappings and defined resolvent operators associated with $(A,\eta)$-accretive mappings. They also investigated a class of variational inclusions using the resolvent operator associated with $(A,\eta)$-accretive mappings. Subsequently, Lan [39], by using the concept of $(A,\eta)$-accretive mappings and the new resolvent operator technique associated with $(A,\eta)$-accretive mappings, introduced and studied a system of general mixed quasivariational inclusions involving $(A,\eta)$-accretive mappings in Banach spaces and constructed a perturbed iterative algorithm with mixed errors for this system of nonlinear $(A,\eta)$-accretive variational inclusions in $q$-uniformly smooth Banach spaces.

On the other hand, the fuzzy set theory introduced by Zadeh [40] has emerged as an interesting and fascinating branch of pure and applied sciences. The application of the fuzzy set theory can be found in many branches of regional, physical, mathematical, and engineering sciences (see [41–45] and the references therein).

In 1989, Chang and Zhu [46] first introduced the classes of variational inequalities for fuzzy mappings. In subsequent years, several classes of variational inequalities, variational inclusions, and complementarity problems for fuzzy mappings were investigated by many authors, in particular, by Chang and Haung [47, 48], Lan et al. [49], Noor [50–52], Noor and Al-said [53], and many others.

Recently, Lan and Verma [54], by using the concept of $(A,\eta)$-accretive mappings, the resolvent operator technique associated with $(A,\eta)$-accretive mappings, introduced and studied a new class of nonlinear fuzzy variational inclusion systems with $(A,\eta)$-accretive mappings in Banach spaces and construct some new iterative algorithms to approximate the solutions of the nonlinear fuzzy variational inclusion systems.

Inspired and motivated by recent research works in these fields, in this paper, we introduce and study a new system of nonlinear fuzzy variational inclusions with $(A,\eta)$-accretive mappings in Banach spaces. By using the resolvent operator associated with $(A,\eta)$-mappings due to Lan et al. and Nadler's fixed points theorem, we construct some new iterative algorithms for approximating the solutions of this system of nonlinear fuzzy variational inclusions in Banach spaces and prove the existence of solutions and the convergence of the sequences generated by the algorithms in $q$-uniformly smooth Banach spaces. The results presented in this paper improve and extend the corresponding results of [29–35, 38, 39, 55–60] and many other recent works.

## 2. Preliminaries

Let $X$ be a real Banach space with dual space $X^*$, $\langle\cdot,\cdot\rangle$ be the dual pair between $X$ and $X^*$, $2^X$ denote the family of all nonempty subsets of $X$, and let $CB(X)$ denote the family of all nonempty closed bounded subsets of $X$. The *generalized duality mapping* $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \left\{ f^* \in X^* : \langle x, f^* \rangle = \|x\|^q, \left\| f^* \right\| = \|x\|^{q-1} \right\}, \quad \forall x \in X, \tag{2.1}$$

where $q > 1$ is a constant. In particular, $J_2$ is the usual normalized duality mapping.

It is known that, in general, $J_q(x) = \|x\|^{q-2}J_2(x)$ for all $x \neq 0$ and $J_q$ is single valued if $X^*$ is strictly convex. In the sequel, we always assume that $X$ is a real Banach space such that $J_q$ is single-valued. If $X$ is a Hilbert space, then $J_2$ becomes the identity mapping on $X$.

The *modulus of smoothness* of $X$ is the function $\rho_X : [0,\infty) \to [0,\infty)$ defined by

$$\rho_X(t) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| \leq 1,\ \|y\| \leq t\right\}. \tag{2.2}$$

A Banach space $X$ is said to be *uniformly smooth* if

$$\lim_{t\to 0}\frac{\rho_X(t)}{t} = 0. \tag{2.3}$$

$X$ is called *$q$-uniformly smooth* if there exists a constant $c > 0$ such that

$$\rho_X(t) \leq ct^q, \quad \forall q > 1. \tag{2.4}$$

Note that $J_q$ is single-valued if $X$ is uniformly smooth. Concerned with the characteristic inequalities in $q$-uniformly smooth Banach spaces, Xu [61] proved the following result.

**Lemma 2.1.** *A real Banach space $X$ is $q$-uniformly smooth if and only if there exists a constant $c_q > 0$ such that, for all $x, y \in X$,*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q. \tag{2.5}$$

*Definition 2.2.* A set-valued mapping $T : X \to 2^X$ is said to be *$\xi$-$\widehat{H}$-Lipschitz continuous* if there exists a constant $\xi > 0$ such that

$$\widehat{H}(T(x), T(y)) \leq \xi\|x-y\|, \quad \forall x, y \in X, \tag{2.6}$$

where $\widehat{H} : 2^X \times 2^X \to \mathbb{R} \cup \{+\infty\}$ is the Hausdorff pseudo-metric, that is,

$$\widehat{H}(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\}, \quad \forall A, B \in 2^X, \tag{2.7}$$

where $d(u,K) = \inf_{v\in K}\|u-v\|$.

It should be pointed that if domain of $\widehat{H}$ is restricted to closed bounded subsets $CB(X)$, then $\widehat{H}$ is the Hausdorff metric.

**Lemma 2.3** (see [62]). *Let $(X,d)$ be a complete metric space and let $T : X \to CB(X)$ be a set-valued mapping satisfying*

$$\widehat{H}(T(x),T(y)) \le kd(x,y), \quad \forall x,y \in X, \tag{2.8}$$

*where $k \in (0,1)$ is a constant. Then the mapping $T$ has a fixed point in $X$.*

**Lemma 2.4** (see [62]). *Let $(X,d)$ be a complete metric space and let $T : X \to CB(X)$ ba a set-valued mapping. Then for any $\varepsilon > 0$ and any $x,y \in X$, $u \in T(x)$, there exists $v \in T(y)$ such that*

$$d(u,v) \le (1+\varepsilon)\widehat{H}(T(x),T(y)). \tag{2.9}$$

*Definition 2.5.* Let X be a $q$-uniformly smooth Banach space, $T, A : X \to X$ and let $\eta : X\times X \to X$ be single-valued mappings.

(i) $T$ is said to be *accretive* if

$$\langle T(x) - T(y), J_q(x-y)\rangle \ge 0, \quad \forall x,y \in X; \tag{2.10}$$

(ii) $T$ is said to be *strictly accretive* if $T$ is accretive and

$$\langle T(x) - T(y), J_q(x-y)\rangle = 0 \tag{2.11}$$

if and only if $x = y$;

(iii) $T$ is said to be *$r$-strongly accretive* if there exists a constant $r > 0$ such that

$$\langle T(x) - T(y), J_q(x-y)\rangle \ge r\|x-y\|^q, \quad \forall x,y \in X; \tag{2.12}$$

(iv) $T$ is said to be *$m$-relaxed accretive* if there exists a constant $m > 0$ such that

$$\langle T(x) - T(y), J_q(x-y)\rangle \ge -m\|x-y\|^q, \quad \forall x,y \in X; \tag{2.13}$$

(v) $T$ is said to be *$(\zeta,\varsigma)$-relaxed cocoercive* if there exist constants $\zeta,\varsigma > 0$ such that

$$\langle T(x) - T(y), J_q(x-y)\rangle \ge -\zeta\|T(x)-T(y)\|^q + \varsigma\|x-y\|^q, \quad \forall x,y \in X; \tag{2.14}$$

(vi) $T$ is said to be *$\gamma$-Lipschitz continuous* if there exists a constant $\gamma > 0$ such that

$$\|T(x) - T(y)\| \le \gamma\|x-y\|, \quad \forall x,y \in X; \tag{2.15}$$

(vii) $\eta$ is said to be *$\tau$-Lipschitz continuous* if there exists a constant $\tau$ such that

$$\|\eta(x,y)\| \le \tau\|x-y\|, \quad \forall x,y \in X; \tag{2.16}$$

(viii) $\eta(\cdot,\cdot)$ is said to be *$\epsilon$-Lipschitz continuous in the first variable* if there exists a constant $\epsilon > 0$ such that

$$\|\eta(x,u) - \eta(y,u)\| \le \epsilon\|x - y\|, \quad \forall x,y,u \in X; \tag{2.17}$$

(ix) $\eta(\cdot,u)$ is said to be *$(\rho,\xi)$-relaxed cocoercive with respect to $A$* if there exist constants $\rho,\xi > 0$ such that

$$\langle \eta(x,u) - \eta(y,u), J_q(A(x) - A(y))\rangle \ge -\rho\|\eta(x,u) - \eta(y,u)\|^q + \xi\|x - y\|^q, \quad \forall x,y,u \in X. \tag{2.18}$$

In a similar way to (viii) and (ix), we can define the Lipschitz continuity of the mapping $\eta(\cdot,\cdot)$ in the second variable and relaxed cocoercivity of $\eta(u,\cdot)$ with respect to $A$.

*Definition 2.6.* Let $X$ be a $q$-uniformly smooth Banach space, $\eta : X \times X \to X$ and let $H, A : X \to X$ be three single-valued mappings. Set-valued mapping $M : X \to 2^X$ is said to be

(i) *accretive* if

$$\langle u - v, J_q(x - y)\rangle \ge 0, \quad \forall x,y \in X,\ u \in Mx,\ v \in My; \tag{2.19}$$

(ii) *$\eta$-accretive* if

$$\langle u - v, J_q(\eta(x,y))\rangle \ge 0, \quad \forall x,y \in X,\ u \in Mx,\ v \in My; \tag{2.20}$$

(iii) *strictly $\eta$-accretive* if $M$ is $\eta$-accretive and the equality holds if and only if $x = y$;

(iv) *$r$-strongly $\eta$-accretive* if there exists a constant $r > 0$ such that

$$\langle u - v, J_q(\eta(x,y))\rangle \ge r\|x - y\|^q, \quad \forall x,y \in X,\ u \in Mx,\ v \in My; \tag{2.21}$$

(v) *$\alpha$-relaxed $\eta$-accretive* if there exists a constant $\alpha > 0$ such that

$$\langle u - v, J_q(\eta(x,y))\rangle \ge -\alpha\|x - y\|^q, \quad \forall x,y \in X,\ u \in Mx,\ v \in My; \tag{2.22}$$

(vi) *$m$-accretive* if $M$ is accretive and $(I + \lambda M)(X) = X$ for all $\lambda > 0$, where $I$ denotes the identity operator on $X$;

(vii) *generalized $m$-accretive* if $M$ is $\eta$-accretive and $(I + \lambda \mathrm{M})(X) = X$ for all $\lambda > 0$;

(viii) *$H$-accretive* if $M$ is accretive and $(H + \lambda M)(X) = X$ for all $\lambda > 0$;

(ix) *$(H,\eta)$-accretive* if $M$ is $\eta$-accretive and $(H + \lambda M)(X) = X$ for all $\lambda > 0$.

*Remark 2.7.* The following should be noticed.

(1) The class of generalized $m$-accretive operators was first introduced by Huang and Fang [28] and includes that of $m$-accretive operators as a special case. The class of

$H$-accretive operators was first introduced and studied by Fang and Huang [63] and also includes that of $m$-accretive operators as a special case.

(2) When $X = \mathscr{H}$ is a Hilbert space, (i)–(ix) of Definition 2.6 reduce to the definitions of monotone operators, $\eta$-monotone operators, strictly $\eta$-monotone operators, strongly $\eta$-monotone operators, relaxed $\eta$-monotone operators, maximal monotone operators, maximal $\eta$-monotone operators, $H$-monotone operators, and $(H,\eta)$-monotone operators, respectively.

*Definition 2.8.* Let $A : X \to X$, $\eta : X \times X \to X$ be two single-valued mappings and let $M : X \to 2^X$ be a set-valued mapping. Then $M$ is said to be *$(A,\eta)$-accretive* with constant $m$ if $M$ is $m$-relaxed $\eta$-accretive and $(A + \lambda M)(X) = X$ for all $\lambda > 0$.

*Remark 2.9.* For appropriate and suitable choices of $m$, $A$, $\eta$, and the space $X$, it is easy to see that Definition 2.8 includes a number of definitions of monotone operators and accretive operators (see [38]).

In [38], Lan et al. showed that $(A + \rho M)^{-1}$ is a single-valued operator if $M : X \to 2^X$ is an $(A,\eta)$-accretive mapping and $A : X \to X$ an $r$-strongly $\eta$-accretive mapping. Based on this fact, we can define the resolvent operator $R^{\eta,A}_{M,\rho}$ associated with an $(A,\eta)$-accretive mapping $M$ as follows.

*Definition 2.10.* Let $A : X \to X$ be a strictly $\eta$-accretive mapping and let $M : X \to 2^X$ be an $(A,\eta)$-accretive mapping. The *resolvent operator* $R^{\eta,A}_{M,\rho} : X \to X$ associated with $A$ and $M$ is defined by

$$R^{\eta,A}_{M,\rho}(x) = \left(A + \rho M\right)^{-1}(x), \quad \forall x \in X. \tag{2.23}$$

**Proposition 2.11** (see [38]). *Let $X$ be a $q$-uniformly smooth Banach space, let $\eta : X \times X \to X$ be $\tau$-Lipschitz continuous, let $A : X \to X$ be a $r$-strongly $\eta$-accretive mapping and let $M : X \to 2^X$ be an $(A,\eta)$-accretive mapping with constant $m$. Then the resolvent operator $R^{\eta,A}_{M,\rho} : X \to X$ is $(\tau^{q-1}/(r - \rho m))$-Lipschitz continuous, that is,*

$$\left\| R^{\eta,A}_{M,\rho}(x) - R^{\eta,A}_{M,\rho}(y) \right\| \leq \frac{\tau^{q-1}}{r - \rho m} \|x - y\|, \quad \forall x, y \in X, \tag{2.24}$$

*where $\rho \in (0, r/m)$ is a constant.*

In what follows, we denote the collection of all fuzzy sets on $X$ by $\mathfrak{F}(X) = \{A \mid A : X \to [0,1]\}$. A mapping $\mathcal{S}$ from $X$ to $\mathfrak{F}(X)$ is called a *fuzzy mapping*. If $\mathcal{S} : X \to \mathfrak{F}(X)$ is a fuzzy mapping, then the set $\mathcal{S}(x)$ for any $x \in X$ is a fuzzy set on $\mathfrak{F}(X)$ (in the sequel we denote $\mathcal{S}(x)$ by $\mathcal{S}_x$) and $\mathcal{S}_x(y)$ for any $y \in X$ is the degree of membership of $y$ in $\mathcal{S}_x$. For any $A \in \mathfrak{F}(X)$ and $\alpha \in [0,1]$, the set

$$(A)_\alpha = \{x \in X : A(x) \geq \alpha\} \tag{2.25}$$

is called a *$\alpha$-cut* set of $A$.

A fuzzy mapping $\mathcal{S} : X \to \mathfrak{F}(X)$ is said to satisfy the condition $(*)$ if there exists a function $a : X \to [0,1]$ such that for each $x \in X$ the set

$$(\mathcal{S}_x)_{a(x)} := \{y \in X : \mathcal{S}_x(y) \geq a(x)\} \tag{2.26}$$

is a nonempty closed and bounded subset of $X$, that is, $(\mathcal{S}_x)_{a(x)} \in CB(X)$.

By using the fuzzy mapping $\mathcal{S}$ satisfying the condition $(*)$ with corresponding function $a : X \to [0,1]$, we can define a set-valued mapping $S$ as follows:

$$S : X \longrightarrow CB(X), \quad x \longmapsto (\mathcal{S}_x)_{a(x)}. \tag{2.27}$$

In the sequel, $S$, $T$, $L$, $D$, $G$, $W$, and $K$ are called *the set-valued mappings induced by the fuzzy mappings* $\mathcal{S}$, $\mathfrak{T}$, $\mathcal{L}$, $\mathfrak{D}$, $\mathcal{G}$, $\mathcal{W}$, and $\mathcal{K}$, respectively.

## 3. A New System of Fuzzy Variational Inclusions

In this section, we introduce some systems of fuzzy variational inclusions $q$-uniformly smooth Banach spaces $X$ and their relations.

Let $X_1$ be a $q_1$-uniformly smooth Banach space with $q_1 > 1$, let $X_2$ be a $q_2$-uniformly smooth Banach space with $q_2 > 1$, let $E, P : X_1 \times X_2 \to X_1$, $F, Q : X_1 \times X_2 \to X_2$, $A_1 : X_1 \to X_1$, $A_2 : X_2 \to X_2$, $f, p, l : X_1 \to X_1$, $g, h, k : X_2 \to X_2$, $\eta_1 : X_1 \times X_1 \to X_1$, $\eta_2 : X_2 \times X_2 \to X_2$ be single-valued mappings, and let $\mathcal{S}, \mathfrak{T}, \mathcal{L}, \mathfrak{D} : X_1 \to \mathfrak{F}(X_1)$ and $\mathcal{G}, \mathcal{W}, \mathcal{K} : X_2 \to \mathfrak{F}(X_2)$ be fuzzy mappings. Further, suppose that $M : X_1 \times X_1 \to 2^{X_1}$ and $N : X_2 \times X_2 \to 2^{X_2}$ are any nonlinear operators such that for all $z \in X_1$, $M(\cdot, z) : X_1 \to 2^{X_1}$ is an $(A_1, \eta_1)$-accretive with $f(x) - y \in \operatorname{dom}(M(\cdot, z))$ for all $x, y \in X_1$ and for all $t \in X_2$, $N(\cdot, t) : X_2 \to 2^{X_2}$ is an $(A_2, \eta_2)$-accretive with $g(u) \in \operatorname{dom}(N(\cdot, t))$ for all $u \in X_2$. Now, for given mappings $\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d} : X_1 \to [0,1]$ and $\widetilde{e}, \widetilde{f}, \widetilde{g} : X_2 \to [0,1]$, we consider the following system.

*System 3.1.* For any given $a \in X_1$, $b \in X_2$, $\lambda_1 > 0$, $\lambda_2 > 0$, our problem is as follows:

Find $x, z, u, v, m \in X_1$ and $y, w, t, s \in X_2$ such that $\mathcal{S}_x(u) \geq \widetilde{a}(x)$, $\mathfrak{T}_x(v) \geq \widetilde{b}(x)$, $\mathcal{L}_x(z) \geq \widetilde{c}(x)$, $\mathfrak{D}_x(m) \geq \widetilde{d}(x)$, $\mathcal{G}_y(w) \geq \widetilde{e}(y)$, $\mathcal{W}_y(t) \geq \widetilde{f}(y)$, $\mathcal{K}_y(s) \geq \widetilde{g}(y)$, and

$$\begin{cases} a \in E(p(x), w) + P(l(z), t) + \lambda_1 M(f(x) - v, x), \\ b \in F(u, h(y)) + Q(m, k(s)) + \lambda_2 N(g(y), y). \end{cases} \tag{3.1}$$

This system is called a *system of nonlinear fuzzy variational inclusions involving* $(A, \eta)$*-accretive mappings* in uniformly smooth Banach spaces.

*Remark 3.2.* For appropriate and suitable choices of $X_1$, $X_2$, $q_1$, $q_2$, $E$, $P$, $F$, $Q$, $A_1$, $A_2$, $f$, $g$, $h$, $k$, $l$, $p$, $\eta_1$, $\eta_2$, $\mathcal{S}$, $\mathfrak{T}$, $\mathcal{L}$, $\mathfrak{D}$, $\mathcal{G}$, $\mathcal{W}$, $\mathcal{K}$, $M$, $N$, $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$, $\widetilde{d}$, $\widetilde{e}$, $\widetilde{f}$, and $\widetilde{g}$ one can obtain many known and new classes of (fuzzy) variational inequalities and (fuzzy) variational inclusions as special cases of System 3.1.

Now, we consider some special cases of System 3.1.

*System 3.3.* Let $S, T, L, D : X_1 \to CB(X_1)$ and $G, W, K : X_2 \to CB(X_2)$ be classical set-valued mappings and let $M$, $N$, $f$, $g$, $E$, $P$, $F$, $Q$, $p$, $l$, $h$, $k$ be the mappings as in System 3.1. Now, by using $S$, $T$, $L$, $D$, $G$, $W$, and $K$, we define fuzzy mappings $\mathcal{S}, \mathcal{T}, \mathcal{L}, \mathfrak{D} : X_1 \to 2^{X_1}$ and $\mathcal{G}, \mathcal{W}, \mathcal{K} : X_2 \to 2^{X_2}$ as follows:

$$\begin{aligned} &\mathcal{S}_x = \chi_{S(x)}, \quad \mathcal{T}_x = \chi_{T(x)}, \quad \mathcal{L}_x = \chi_{L(x)} \\ &\mathfrak{D}_x = \chi_{D(x)}, \mathcal{G}_x = \chi_{G(x)}, \quad \mathcal{W}_x = \chi_{W(x)}, \quad \mathcal{K}_x = \chi_{K(x)}, \end{aligned} \tag{3.2}$$

where $\chi_{S(x)}$, $\chi_{T(x)}$, $\chi_{L(x)}$, $\chi_{D(x)}$, $\chi_{G(x)}$, $\chi_{W(x)}$, and $\chi_{K(x)}$ are the characteristic functions of the sets $S(x)$, $T(x)$, $L(x)$, $D(x)$, $G(x)$, $W(x)$, and $K(x)$, respectively.

It is easy to see that $\mathcal{S}$, $\mathcal{T}$, $\mathcal{L}$, and $\mathfrak{D}$ are fuzzy mappings satisfying the condition $(*)$ with constant functions $\tilde{a}(x) = 1$, $\tilde{b}(x) = 1$, $\tilde{c}(x) = 1$, $\tilde{d}(x) = 1$ for all $x \in X_1$, respectively, and $\mathcal{G}$, $\mathcal{W}$, and $\mathcal{K}$ are fuzzy mappings satisfying the condition $(*)$ with constant functions $\tilde{e}(y) = 1$, $\tilde{f}(y) = 1$, $\tilde{g}(y) = 1$ for all $y \in X_2$, respectively. Also

$$\begin{aligned} (\mathcal{S})_{\tilde{a}(x)} &= \left(\chi_{S(x)}\right)_1 = \left\{r \in X_1 : \chi_{S(x)}(r) = 1\right\} = S(x), \\ (\mathcal{T})_{\tilde{b}(x)} &= \left(\chi_{T(x)}\right)_1 = \left\{r \in X_1 : \chi_{T(x)}(r) = 1\right\} = T(x), \\ (\mathcal{L})_{\tilde{c}(x)} &= \left(\chi_{L(x)}\right)_1 = \left\{r \in X_1 : \chi_{L(x)}(r) = 1\right\} = L(x), \\ (\mathfrak{D})_{\tilde{d}(x)} &= \left(\chi_{D(x)}\right)_1 = \left\{r \in X_1 : \chi_{D(x)}(r) = 1\right\} = D(x), \\ (\mathcal{G})_{\tilde{e}(y)} &= \left(\chi_{G(y)}\right)_1 = \left\{t \in X_2 : \chi_{G(y)}(t) = 1\right\} = G(y), \\ (\mathcal{W})_{\tilde{f}(y)} &= \left(\chi_{W(y)}\right)_1 = \left\{t \in X_2 : \chi_{W(y)}(t) = 1\right\} = W(y), \\ (\mathcal{K})_{\tilde{g}(y)} &= \left(\chi_{K(y)}\right)_1 = \left\{t \in X_2 : \chi_{K(y)}(t) = 1\right\} = K(y). \end{aligned} \tag{3.3}$$

Then System 3.1 is equivalent to the following:

Find $x, z, u, v, m \in X_1$, $y, w, t, s \in X_2$ such that $u \in S(x)$, $v \in T(x)$, $z \in L(x)$, $m \in D(x)$, $w \in G(y)$, $t \in W(y)$, $s \in K(y)$, and

$$\begin{cases} a \in E(p(x), w) + P(l(z), t) + \lambda_1 M(f(x) - v, x), \\ b \in F(u, h(y)) + Q(m, k(s)) + \lambda_2 N(g(y), y). \end{cases} \tag{3.4}$$

System 3.3 is called a *system of nonlinear set-valued variational inclusions with* $(A, \eta)$*-accretive mappings*.

*System 3.4.* If $T : X_1 \to X_1$ is a single-valued mapping, then System 3.3 collapses to the following system of nonlinear variational inclusions:

Find $x, z, u, m \in X_1$, $y, w, t, s \in X_2$ such that $u \in S(x)$, $z \in L(x)$, $m \in D(x)$, $w \in G(y)$, $t \in W(y)$, $s \in K(y)$, and

$$\begin{cases} a \in E(p(x), w) + P(l(z), t) + \lambda_1 M(f(x) - T(x), x), \\ b \in F(u, h(y)) + Q(m, k(s)) + \lambda_2 N(g(y), y). \end{cases} \tag{3.5}$$

*System 3.5.* If $X_i = \mathscr{H}_i$ $(i = 1, 2)$ are two Hilbert spaces, $S : \mathscr{H}_1 \to \mathscr{H}_1$ and $G : \mathscr{H}_2 \to \mathscr{H}_2$ are two single-valued mappings, $p = h = l = k \equiv I$(: the identity mapping), $\lambda_1 = \lambda_2 = 1$, $T \equiv 0$(: the zero mapping), $a = b = 0$, $M(x, y) = M(x)$ for all $(x, y) \in \mathscr{H}_1 \times \mathscr{H}_1$, $N(x, y) = N(x)$ for all $(x, y) \in \mathscr{H}_2 \times \mathscr{H}_2$, then System 3.4 reduces to the following system:

Find $(x, y, z, m, t, s)$ such that $(x, y) \in \mathscr{H}_1 \times \mathscr{H}_2$, $z \in L(x)$, $m \in D(x)$, $t \in W(y)$, $s \in K(y)$, and

$$\begin{cases} 0 \in E(x, y) + P(z, t) + M(f(x)), \\ 0 \in F(x, y) + Q(m, s) + N(g(y)). \end{cases} \tag{3.6}$$

System 3.5 was introduced and studied by Peng and Zhu in [59].

*System 3.6.* If $a = b = 0$, $\lambda_1 = \lambda_2 = 1$, and $P = Q \equiv 0$, then System 3.3 can be replaced by the following:

Find $u \in \mathrm{S}(x)$, $v \in \mathrm{T}(x)$ and $w \in \mathrm{G}(y)$ such that

$$\begin{cases} 0 \in E(p(x), w) + M(f(x) - v, x), \\ 0 \in F(u, h(y)) + N(g(y), y), \end{cases} \tag{3.7}$$

which is studied by Lan and Verma [54].

*System 3.7.* If $T : X_1 \to X_1$ is a single-valued mapping, then System 3.6 collapses to the following system of nonlinear variational inclusions:

Find $x, u \in X_1$, $y, w \in X_2$ such that $u \in S(x)$, $w \in G(y)$ and

$$\begin{cases} 0 \in E(p(x), w) + M(f(x) - T(x), x), \\ 0 \in F(u, h(y)) + N(g(y), y), \end{cases} \tag{3.8}$$

which is studied by Lan and Verma [54].

*System 3.8.* If $p = h \equiv I$, $T \equiv 0$, $M(x, y) = M(x)$ for all $(x, y) \in X_1 \times X_1$, $N(x, y) = N(x)$ for all $(x, y) \in X_2 \times X_2$, $S : X_1 \to X_1$, and $G : X_2 \to X_2$ are identity mappings, then System 3.7 reduces to the following system:

Find $(x, y) \in X_1 \times X_2$ such that

$$\begin{cases} 0 \in E(x, G(y)) + \mathrm{M}(f(x)), \\ 0 \in F(S(x), y) + N(g(y)). \end{cases} \tag{3.9}$$

System 3.8 is investigated by Jin [55] when $S$ and $G$ are the identity mappings.

*System 3.9.* If $f - T = g \equiv I$, $P = Q \equiv 0$, $\lambda_1 = \lambda_2 = 1$, $S : X_1 \to X_1$, and $G : X_2 \to X_2$ are two single-valued mappings, then System 3.4 is equivalent to the following:

Find $(x, y) \in X_1 \times X_2$ such that

$$\begin{cases} a \in E(p(x), G(y)) + M(x, x), \\ b \in F(S(x), h(y)) + N(y, y), \end{cases} \tag{3.10}$$

which is introduced and studied by Lan [39] when $S$ and $G$ are identity mappings.

*System 3.10.* When $p = h = S = G \equiv I$, $a = b = 0$, System 3.9 can be replaced to the following:
Find $(x, y) \in X_1 \times X_2$ such that

$$\begin{cases} 0 \in E(x, y) + M(x, x), \\ 0 \in F(x, y) + N(y, y), \end{cases} \tag{3.11}$$

which is studied by Jin [56].

*System 3.11.* If $X_i = \mathcal{H}_i$ $(i = 1, 2)$ is two Hilbert spaces, $M(x, y) = M(x)$ for all $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_1$ and $N(x, y) = N(x)$ for all $(x, y) \in \mathcal{H}_2 \times \mathcal{H}_2$, then System 3.7 reduces to the following generalized system of set-valued variational inclusions:
Find $x, u \in \mathcal{H}_1$, $y, w \in \mathcal{H}_2$ such that $u \in S(x)$, $w \in G(y)$, and

$$\begin{cases} 0 \in E(p(x), w) + M(f(x) - T(x)), \\ 0 \in F(u, h(y)) + N(g(y)), \end{cases} \tag{3.12}$$

which is studied by Lan et al. [57] when $M$, $N$ are $A$-monotone mappings and there exists single-valued mapping $\psi : \mathcal{H}_1 \to \mathcal{H}_1$ such that $\psi(x) = f(x) - T(x)$ for all $x \in \mathcal{H}_1$.

*System 3.12.* If $g = p = h = f - T \equiv I$, then System 3.11 collapses to the following system of nonlinear variational inclusions:
Find $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$, $u \in S(x)$, $w \in G(y)$ such that

$$\begin{cases} 0 \in E(x, w) + M(x), \\ 0 \in F(u, y) + N(y), \end{cases} \tag{3.13}$$

which is considered by Huang and Fang [28].

*System 3.13.* If $S : \mathcal{H}_1 \to \mathcal{H}_1$ and $G : \mathcal{H}_2 \to \mathcal{H}_2$ are two single-valued mappings, then System 3.12 is equivalent to the following:
Find $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$ such that

$$\begin{cases} 0 \in E(x, G(y)) + M(x), \\ 0 \in F(S(x), y) + N(y), \end{cases} \tag{3.14}$$

which is investigated by Fang et al. [31] and Peng et al. [35], Fang and Huang [34], and Verma [36] with $S = G \equiv I$.

*System 3.14.* If $M(x) = \partial\varphi(x)$ and $N(y) = \partial\phi(y)$ for all $x \in \mathscr{H}_1$ and $y \in \mathscr{H}_2$, where $\varphi : \mathscr{H}_1 \to \mathbb{R}\cup\{+\infty\}$ and $\phi : \mathscr{H}_2 \to \mathbb{R}\cup\{+\infty\}$ are two proper, convex, and lower semicontinuous functionals, and $\partial\varphi$ and $\partial\phi$ denote subdifferential operators of $\varphi$ and $\phi$, respectively, then System 3.13 reduces to the following system:

Find $(x,y) \in \mathscr{H}_1 \times \mathscr{H}_2$ such that

$$\begin{cases} \langle E(x,G(y)), s-x\rangle + \varphi(s) - \varphi(x) \geq 0, & \forall s \in \mathscr{H}_1, \\ \langle F(S(x),y), t-y\rangle + \phi(t) - \phi(y) \geq 0, & \forall t \in \mathscr{H}_2, \end{cases} \tag{3.15}$$

which is called a *system of nonlinear mixed variational inequalities*. Some special cases of System 3.7 can be found in [22]. Further, if $S = G \equiv I$, then System 3.7 reduces to the system of nonlinear variational inequalities considered by Cho et al. [25].

*System 3.15.* If $M(x) = \partial\delta_{K_1}(x)$ and $N(y) = \partial\delta_{K_2}(y)$ for all $x \in K_1$ and $y \in K_2$, where $K_1$ and $K_2$ are nonempty closed convex subsets of $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively, and $\delta_{K_1}$ and $\delta_{K_2}$ denote indicator functions of $K_1$ and $K_2$, respectively, then System 3.14 becomes the following problem:

Find $(x,y) \in K_1 \times K_2$ such that

$$\begin{cases} \langle E(x,G(y)), s-x\rangle \geq 0, & \forall s \in K_1, \\ \langle F(S(x),y), t-y\rangle \geq 0, & \forall t \in K_2, \end{cases} \tag{3.16}$$

which is the just system in [24] when $S$ and $G$ are singlevalued and $S = G \equiv I$.

*System 3.16.* If $\mathscr{H}_1 = \mathscr{H}_2 = \mathscr{H}$, $K_1 = K_2 = K$, $E(x,G(y)) = \rho_1 G(y) + x - y$, and $F(S(x),y) = \rho_2 S(x) + y - x$ for all $x, y \in \mathscr{H}$, where $\rho_1 > 0$ and $\rho_2 > 0$ are two constants, then System 3.15 is equivalent to the following:

Find an element $(x,y) \in K \times K$ such that

$$\begin{cases} \langle \rho_1 G(y) + x - y, s-x\rangle \geq 0, & \forall s \in K, \\ \langle \rho_2 S(x) + y - x, t-y\rangle \geq 0, & \forall t \in K, \end{cases} \tag{3.17}$$

which is the system of nonlinear variational inequalities considered by Verma [22] with $S = G$.

*Remark 3.17.* If $x = y$, $S = G$ and $\rho_1 = \rho_2$, then System 3.16 reduces to the following classical nonlinear variational inequality problem:

Find an element $x \in K$ such that

$$\langle S(x), z-x\rangle \geq 0, \quad \forall z \in K. \tag{3.18}$$

## 4. Existence Theorems

In this section, we prove the existence theorem for solutions of Systems 3.1. For our main results, we have the following lemma which offers a good approach to solve System 3.1.

**Lemma 4.1.** *Let $X_i$, $A_i$, $\eta_i$, $\lambda_i$ $(i = 1,2)$, $E$, $F$, $P$, $Q$, $\mathcal{S}$, $\mathcal{T}$, $\mathcal{L}$, $\mathfrak{D}$, $\mathcal{G}$, $\mathcal{W}$, $\mathcal{K}$, $M$, $N$, $f$, $g$, $h$, $p$, $l$, $k$, $a$, and $b$ be the same as in System 3.1. Then, for any given $x,z,u,v,m \in X_1$ and $y,w,t,s \in X_2$, $(x,y,z,t,m,s,u,v,w)$ is a solution of System 3.1 if and only if*

$$\begin{aligned} f(x) &= v + R^{\eta_1,A_1}_{M(\cdot,x),\rho_1}\left[A_1(f(x)-v) - \frac{\rho_1}{\lambda_1}(E(p(x),w) + P(l(z),t) - a)\right], \\ g(y) &= R^{\eta_2,A_2}_{N(\cdot,y),\rho_2}\left[A_2(g(y)) - \frac{\rho_2}{\lambda_2}(F(u,h(y)) + Q(m,k(s)) - b)\right], \end{aligned} \tag{4.1}$$

*where $\rho_1 > 0$ and $\rho_2 > 0$ are two constants.*

*Proof.* The conclusion follows directly from Definition 2.10 and some simple arguments. □

From Lemma 4.1, we have the following.

**Theorem 4.2.** *Let $X_1$ and $X_2$ be the same as in Lemma 4.1, let $\mathcal{S},\mathcal{T},\mathcal{L},\mathfrak{D} : X_1 \to \mathfrak{F}(X_1)$ and $\mathcal{G},\mathcal{W},\mathcal{K} : X_2 \to \mathfrak{F}(X_2)$ be fuzzy mappings satisfying the condition $(*)$ with the corresponding functions $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$, $\tilde{e}$, $\tilde{f}$ and $\tilde{g}$, respectively, $S,T,L,D : X_1 \to CB(X_1)$, and let $G,W,K : X_2 \to CB(X_2)$ be $\xi$-$\widehat{H}_1$-Lipschitz continuous, $\zeta$-$\widehat{H}_1$-Lipschitz continuous, $\gamma$-$\widehat{H}_1$-Lipschitz continuous, $\varpi$-$\widehat{H}_1$-Lipschitz continuous, $\xi'$-$\widehat{H}_2$-Lipschitz continuous, $\zeta'$-$\widehat{H}_2$-Lipschitz continuous, and $\gamma'$-$\widehat{H}_2$-Lipschitz continuous, respectively, where $\widehat{H}_i$ is the Hausdorff pseudometric on $2^{X_i}$ for $i = 1,2$. Assume that $\eta_i : X_i \times X_i \to X_i$ is $\tau_i$-Lipschitz continuous, $A_i : X_i \to X_i$ is $r_i$-strongly $\eta_i$-accretive and $\beta_i$-Lipschitz continuous for $i = 1,2$, $p,l : X_1 \to X_1$ are $\delta_1$-Lipschitz continuous, and $\delta_2$-Lipschitz continuous, respectively, $h,k : X_2 \to X_2$ are $\pi_1$-Lipschitz continuous and $\pi_2$-Lipschitz continuous, respectively, $f : X_1 \to X_1$ is $(\kappa, e_1)$-relaxed cocoercive, $\mu$-Lipschitz continuous and $g : X_2 \to X_2$ is $(\sigma, e_2)$-relaxed cocoercive, $\epsilon$-Lipschitz continuous. Suppose that $M(\cdot,z) : X_1 \to 2^{X_1}$ is an $(A_1,\eta_1)$-accretive operator with constant $m_1$ for all $z \in X_1$ and $N(\cdot,t) : X_2 \to 2^{X_2}$ is an $(A_2,\eta_2)$-accretive operator with constant $m_2$ for all $t \in X_2$, and $E,P : X_1 \times X_2 \to X_1$ are two single-valued mappings such that $E(\cdot,y)$ and $P(\cdot,y)$ are $\nu_1$-Lipschitz continuous and $\nu_2$-Lipschitz continuous in the first variable, respectively, $E(x,\cdot)$, $P(x,\cdot)$ are $\iota_1$-Lipschitz continuous $\iota_2$-Lipschitz continuous in the second variable, respectively, for all $(x,y) \in X_1 \times X_2$, and $E(p_1(\cdot),y)$ is $(\theta_1,s_1)$-relaxed cocoercive with respect to $f'$, where $f' : X_1 \to X_1$ is defined by $f'(x) = A_1 \circ (f(x) - v) = A_1(f(x) - v)$ for all $x \in X_1$, $\tilde{b} : X_1 \to [0,1]$ and $\mathcal{T}_x(v) \geq \tilde{b}(x)$. Further, suppose that $F,Q : X_1 \times X_2 \to X_2$ are two nonlinear mappings such that $F(\cdot,y)$, $Q(\cdot,y)$ are $\rho_1$-Lipschitz continuous and $\rho_2$-Lipschitz continuous in the first variable, respectively, $F(x,\cdot)$ and $Q(x,\cdot)$ are $\upsilon_1$-Lipschitz continuous and $\upsilon_2$-Lipschitz continuous in the second variable, respectively, and $F(x,h(\cdot))$ is $(\theta_2,s_2)$-relaxed cocoercive with respect to $g'$, where $g' : X_2 \to X_2$ is defined by $g'(x) = A_2 \circ g(x) = A_2(g(x))$ for all $x \in X_2$.*

*In addition, if there exist constants $\rho_1 \in (0, r_1/m_1)$ and $\rho_2 \in (0, r_2/m_2)$ such that*

$$\left\| R^{\eta_1,A_1}_{M(\cdot,x),\rho_1}(z) - R^{\eta_1,A_1}_{M(\cdot,y),\rho_1}(z) \right\| \leq \varsigma \|x-y\|, \quad \forall x,y,z \in X_1, \tag{4.2}$$

$$\left\| R^{\eta_2,A_2}_{N(\cdot,x),\rho_2}(z) - R^{\eta_2,A_2}_{N(\cdot,y),\rho_2}(z) \right\| \leq \vartheta \|x-y\|, \quad \forall x,y,z \in X_2, \tag{4.3}$$

$$
\begin{gathered}
\varsigma+\zeta+\sqrt[q_1]{1-q_1e_1+(c_{q_1}+q_1\kappa)\mu^{q_1}}<1,\\
\vartheta+\sqrt[q_2]{1-q_2e_2+(c_{q_2}+q_2\sigma)\epsilon^{q_2}}<1,\\
\sqrt[q_1]{\beta_1^{q_1}(\mu+\zeta)^{q_1}-q_1\frac{\rho_1}{\lambda_1}(-\theta_1\nu_1{}^{q_1}\delta_1{}^{q_1}+s_1)+\frac{c_{q_1}\rho_1{}^{q_1}\nu_1{}^{q_1}\delta_1{}^{q_1}}{\lambda_1{}^{q_1}}}<\frac{\tau_1^{1-q_1}\lambda_1}{\rho_1}(r_1-\rho_1m_1)\chi_1-\nu_2\delta_2\gamma,\\
\sqrt[q_2]{\beta_2^{q_2}\epsilon^{q_2}-q_2\frac{\rho_2}{\lambda_2}(-\theta_2\upsilon_1{}^{q_2}\pi_1{}^{q_2}+s_2)+\frac{c_{q_2}\rho_2{}^{q_2}\upsilon_1{}^{q_2}\pi_1{}^{q_2}}{\lambda_2{}^{q_2}}}<\frac{\tau_2^{1-q_2}\lambda_2}{\rho_2}(r_2-\rho_2m_2)\chi_2-\upsilon_2\pi_2\gamma',
\end{gathered}
\tag{4.4}
$$

*where*

$$
\begin{aligned}
\chi_1&=1-\left(\varsigma+\zeta+\sqrt[q_1]{1-q_1e_1+(c_{q_1}+q_1\kappa)\mu^{q_1}}\right)-\frac{\rho_2\tau_2^{q_2-1}(\rho_1\xi+\rho_2\varpi)}{\lambda_2(r_2-\rho_2m_2)},\\
\chi_2&=1-\left(\vartheta+\sqrt[q_2]{1-q_2e_2+(c_{q_2}+q_2\sigma)\epsilon^{q_2}}\right)-\frac{\rho_1\tau_1^{q_1-1}(\iota_1\xi'+\iota_2\zeta')}{\lambda_1(r_1-\rho_1m_1)},
\end{aligned}
\tag{4.5}
$$

*$\lambda_1$, $\lambda_2$ are the same as in System 3.1, and $c_{q_1}$, $c_{q_2}$ are two constants guaranteed by Lemma 2.1, then System 3.1 admits a solution.*

*Proof.* For any given $\rho_1>0$ and $\rho_2>0$, define mappings $\Phi_{\rho_1}: X_1\times X_1\times X_1\times X_2\times X_2\to X_1$ and $\Psi_{\rho_2}: X_1\times X_1\times X_2\times X_2\to X_2$ as follows:

$$
\begin{aligned}
&\Phi_{\rho_1}(x,z,v,t,w)\\
&\quad=x-f(x)+v+R^{\eta_1,A_1}_{M(\cdot,x),\rho_1}\left[A_1(f(x)-v)-\frac{\rho_1}{\lambda_1}(E(p(x),w)+P(l(z),t)-a)\right],\\
&\Psi_{\rho_2}(u,m,s,y)\\
&\quad=y-g(y)+R^{\eta_2,A_2}_{N(\cdot,y),\rho_2}\left[A_2(g(y))-\frac{\rho_2}{\lambda_2}(F(u,h(y))+Q(m,k(s))-b)\right]
\end{aligned}
\tag{4.6}
$$

for all $(x,y,z,t,m,s,u,v,w)\in X_1\times X_2\times X_1\times X_2\times X_1\times X_2\times X_1\times X_1\times X_2$, where $a\in X_1$ and $b\in X_2$ are the same as in System 3.1, and let $\tilde{a},\tilde{b},\tilde{c},\tilde{d}: X_1\to[0,1]$ and $\tilde{e},\tilde{f},\tilde{g}: X_2\to[0,1]$ be mappings such that $\mathcal{S}_x(u)\geq\tilde{a}(x)$, $\mathfrak{T}_x(v)\geq\tilde{b}(x)$, $\mathcal{L}_x(z)\geq\tilde{c}(x)$, $\mathfrak{D}_x(m)\geq\tilde{d}(x)$, $\mathcal{G}_y(w)\geq\tilde{e}(y)$, $\mathcal{W}_y(t)\geq\tilde{f}(y)$, and $\mathcal{K}_y(s)\geq\tilde{g}(y)$.

Now, define a norm $\|\cdot\|_*$ on $X_1\times X_2$ by

$$
\|(u,v)\|_*=\|u\|+\|v\|,\quad\forall(u,v)\in X_1\times X_2.
\tag{4.7}
$$

It is easy to see that $(X_1\times X_2,\|\cdot\|_*)$ is a Banach space (see [34]). For any given $\rho_1>0$ and $\rho_2>0$, define a mapping $Q_{\rho_1,\rho_2}: X_1\times X_2\times X_1\times X_1\times X_1\times X_1\times X_2\times X_2\times X_2\to X_1\times X_2$ by

$$
Q_{\rho_1,\rho_2}(x,y,z,u,v,m,s,t,w)=(\Phi_{\rho_1}(x,z,v,t,w),\Psi_{\rho_2}(u,m,s,y))
\tag{4.8}
$$

for all $(x, y, z, u, v, m, s, t, w) \in X_1 \times X_2 \times X_1 \times X_1 \times X_1 \times X_1 \times X_2 \times X_2 \times X_2$ and let

$$\begin{aligned}\mathfrak{R}_{\rho_1,\rho_2}(x,y) = \Big\{ & Q_{\rho_1,\rho_2}(x,y,z,u,v,m,s,t,w) : \\ & \mathcal{S}_x(u) \geq \tilde{a}(x), \mathcal{T}_x(v) \geq \tilde{b}(x), \mathcal{L}_x(z) \geq \tilde{c}(x), \mathfrak{D}_x(m) \geq \tilde{d}(x), \mathcal{G}_y(w) \geq \tilde{e}(y), \\ & \mathcal{W}_y(t) \geq \tilde{f}(y), \mathcal{K}_y(s) \geq \tilde{g}(y), \text{ where } \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} : X_1 \to [0,1], \\ & \tilde{e}, \tilde{f}, \tilde{g} : X_2 \to [0,1] \Big\}\end{aligned} \tag{4.9}$$

for all $(x, y) \in X_1 \times X_2$. Then, for any given $(x, y), (x', y') \in X_1 \times X_2$, $\varepsilon > 0$ and $Q_{\rho_1,\rho_2}(x, y, z, u, v, m, s, t, w) \in \mathfrak{R}_{\rho_1,\rho_2}(x, y)$, there exists $(z, u, v, m, s, t, w) \in X_1 \times X_1 \times X_1 \times X_1 \times X_2 \times X_2 \times X_2$ such that

$$\begin{gathered}\mathcal{S}_x(\mathrm{u}) \geq \tilde{a}(x), \qquad \mathcal{T}_x(v) \geq \tilde{b}(x), \qquad \mathcal{L}_x(z) \geq \tilde{c}(x), \qquad \mathfrak{D}_x(m) \geq \tilde{d}(x), \\ \mathcal{G}_y(w) \geq \tilde{e}(y), \qquad \mathcal{W}_y(t) \geq \tilde{f}(y), \qquad \mathcal{K}_y(s) \geq \tilde{g}(y),\end{gathered} \tag{4.10}$$

where $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} : X_1 \to [0,1], \tilde{e}, \tilde{f}, \tilde{g} : X_2 \to [0,1]$ and (4.6) holds. Since $\mathcal{S}_x(u) \geq \tilde{a}(x), \mathcal{T}_x(v) \geq \tilde{b}(x), \mathcal{L}_x(z) \geq \tilde{c}(x), \mathfrak{D}_x(m) \geq \tilde{d}(\mathrm{x}), \mathcal{G}_y(w) \geq \tilde{e}(y), \mathcal{W}_y(t) \geq \tilde{f}(y), \mathcal{K}_y(s) \geq \tilde{g}(y)$, that is, $u \in S(x) \in CB(X_1)$, $v \in T(x) \in CB(X_1)$, $z \in L(x) \in CB(X_1)$, $m \in D(x) \in CB(X_1)$, $w \in G(y) \in CB(X_2)$, $t \in W(y) \in CB(X_2)$, $s \in K(y) \in CB(X_2)$, it follows from Lemma 2.4 that there exist $u' \in S(x')$, $v' \in T(x')$, $z' \in L(x')$, $m' \in D(x')$, $w' \in G(y')$, $t' \in W(y')$, $s' \in K(y')$, that is, $\mathcal{S}_{x'}(u') \geq \tilde{a}(x'), \mathcal{T}_{x'}(v') \geq \tilde{b}(x'), \mathcal{L}_{x'}(z') \geq \tilde{c}(x'), \mathfrak{D}_{x'}(m') \geq \tilde{d}(x'), \mathcal{G}_{y'}(w') \geq \tilde{e}(y'), \mathcal{W}_{y'}(t') \geq \tilde{f}(y'), \mathcal{K}_{y'}(s') \geq \tilde{g}(y')$ such that

$$\begin{gathered}\|u - u'\| \leq (1+\varepsilon)\widehat{H}_1(S(x), S(x')), \qquad \|v - v'\| \leq (1+\varepsilon)\widehat{H}_1(T(x), T(x')), \\ \|z - z'\| \leq (1+\varepsilon)\widehat{H}_1(L(x), L(x')), \qquad \|m - m'\| \leq (1+\varepsilon)\widehat{H}_1(D(x), D(x')), \\ \|w - w'\| \leq (1+\varepsilon)\widehat{H}_2(G(y), G(y')), \qquad \|t - t'\| \leq (1+\varepsilon)\widehat{H}_2(W(y), W(y')), \\ \|s - s'\| \leq (1+\varepsilon)\widehat{H}_2(K(y), K(y')).\end{gathered} \tag{4.11}$$

Letting

$$\begin{aligned}&\Phi_{\rho_1}(x', z', v', t', w') \\ &\quad = x' - f(x') + v' + R^{\eta_1,A_1}_{M(\cdot,x'),\rho_1}\left[A_1(f(x') - v') - \frac{\rho_1}{\lambda_1}(E(p(x'), w') + P(l(z'), t') - a)\right], \\ &\Psi_{\rho_2}(u', m', s', y') \\ &\quad = y' - g(y') + R^{\eta_2,A_2}_{N(\cdot,y'),\rho_2}\left[A_2(g(y')) - \frac{\rho_2}{\lambda_2}(F(u', h(y')) + Q(m', k(s')) - b)\right],\end{aligned} \tag{4.12}$$

we have

$$\left(\Phi_{\rho_1}(x',z',v',t',w'),\Psi_{\rho_2}(u',m',s',y')\right)=Q_{\rho_1,\rho_2}(x',y',z',u',v',m',s',t',w'). \tag{4.13}$$

Now, it follows from (4.2) and Proposition 2.11 that

$$\begin{aligned}
&\left\|\Phi_{\rho_1}(x,z,v,t,w)-\Phi_{\rho_1}(x',z',v',t',w')\right\| \\
&\quad\le \left\|x-x'-\left(f(x)-f(x')\right)\right\|+\left\|v-v'\right\| \\
&\qquad+\left\|R^{\eta_1,A_1}_{M(\cdot,x),\rho_1}\left[A_1\left(f(x)-v\right)-\frac{\rho_1}{\lambda_1}\left(E\left(p(x),w\right)+P\left(l(z),t\right)-a\right)\right]\right. \\
&\qquad\qquad\left.-R^{\eta_1,A_1}_{M(\cdot,x'),\rho_1}\left[A_1\left(f(x')-v'\right)-\frac{\rho_1}{\lambda_1}\left(E\left(p(x'),w'\right)+P\left(l(z'),t'\right)-a\right)\right]\right\| \\
&\quad\le \left\|x-x'-\left(f(x)-f(x')\right)\right\|+\left\|v-v'\right\| \\
&\qquad+\left\|R^{\eta_1,A_1}_{M(\cdot,x),\rho_1}\left[A_1\left(f(x)-v\right)-\frac{\rho_1}{\lambda_1}\left(E\left(p(x),w\right)+P\left(l(z),t\right)-a\right)\right]\right. \\
&\qquad\qquad\left.-R^{\eta_1,A_1}_{M(\cdot,x),\rho_1}\left[A_1\left(f(x')-v'\right)-\frac{\rho_1}{\lambda_1}\left(E\left(p(x'),w'\right)+P\left(l(z'),t'\right)-a\right)\right]\right\| \\
&\qquad+\left\|R^{\eta_1,A_1}_{M(\cdot,x),\rho_1}\left[A_1\left(f(x')-v'\right)-\frac{\rho_1}{\lambda_1}\left(E\left(p(x'),w'\right)+P\left(l(z'),t'\right)-a\right)\right]\right. \\
&\qquad\qquad\left.-R^{\eta_1,A_1}_{M(\cdot,x'),\rho_1}\left[A_1\left(f(x')-v'\right)-\frac{\rho_1}{\lambda_1}\left(E\left(p(x'),w'\right)+P\left(l(z'),t'\right)-a\right)\right]\right\| \\
&\quad\le \left\|x-x'-\left(f(x)-f(x')\right)\right\|+\left\|v-v'\right\|+\varsigma\left\|x-x'\right\| \\
&\qquad+\frac{\tau_1^{q_1-1}}{r_1-\rho_1 m_1}\left\|A_1\left(f(x)-v\right)-\frac{\rho_1}{\lambda_1}\left(E\left(p(x),w\right)+P\left(l(z),t\right)-a\right)\right. \\
&\qquad\qquad\left.-\left(A_1\left(f(x')-v'\right)-\frac{\rho_1}{\lambda_1}\left(E\left(p(x'),w'\right)+P\left(l(z'),t'\right)-a\right)\right)\right\| \\
&\quad\le \left\|x-x'-\left(f(x)-f(x')\right)\right\|+\left\|v-v'\right\|+\varsigma\left\|x-x'\right\| \\
&\qquad+\frac{\tau_1^{q_1-1}}{r_1-\rho_1 m_1}\left\{\frac{\rho_1}{\lambda_1}\left[\left\|E\left(p(x),w\right)-E\left(p(x),w'\right)\right\|+\left\|P\left(l(z),t\right)-P\left(l(z),t'\right)\right\|\right.\right. \\
&\qquad\qquad\left.+\left\|P\left(l(z),t'\right)-P\left(l(z'),t'\right)\right\|\right] \\
&\qquad\qquad\left.+\left\|A_1\left(f(x)-v\right)-A_1\left(f(x')-v'\right)-\frac{\rho_1}{\lambda_1}\left(E\left(p(x),w'\right)-E\left(p(x'),w'\right)\right)\right\|\right\}.
\end{aligned} \tag{4.14}$$

Thus, by Lemma 2.1, we have

$$\begin{aligned}&\|x-x'-(f(x)-f(x'))\|^{q_1}\\&\quad\le\|x-x'\|^{q_1}-q_1\langle f(x)-f(x'),J_{q_1}(x-x')\rangle+c_{q_1}\|f(x)-f(x')\|^{q_1}.\end{aligned}\tag{4.15}$$

Since $f$ is $(\kappa,e_1)$-relaxed cocoercive and $\mu$-Lipschitz continuous, we conclude that

$$\begin{aligned}\|x-x'-(f(\mathrm{x})-f(x'))\|^{q_1}&\le\|x-x'\|^{q_1}-q_1e_1\|x-x'\|^{q_1}+(c_{q_1}+q_1\kappa)\mu^{q_1}\|x-x'\|^{q_1}\\&=(1-q_1e_1+(c_{q_1}+q_1\kappa)\mu^{q_1})\|x-x'\|^{q_1}.\end{aligned}\tag{4.16}$$

By (4.11) and $\zeta$-$\widehat{H}_1$-Lipschitz continuity of $T$,

$$\|v-v'\|\le(1+\varepsilon)\widehat{H}_1(T(x),T(x'))\le\zeta(1+\varepsilon)\|x-x'\|.\tag{4.17}$$

Since $E(x,\cdot)$ is $\iota_1$-Lipschitz continuous in the second variable and $G$ is $\xi'$-$\widehat{H}_2$-Lipschitz continuous, by (4.11), we have

$$\begin{aligned}\|E(p(x),w)-E(p(x),w')\|&\le\iota_1\|w-w'\|\le\iota_1(1+\varepsilon)\widehat{H}_2(G(y),G(y'))\\&\le\iota_1\xi'(1+\varepsilon)\|y-y'\|.\end{aligned}\tag{4.18}$$

Since $P(x,\cdot)$ is $\iota_2$-Lipschitz continuous in the second variable, $W$ is $\zeta'$-$\widehat{H}_2$-Lipschitz continuous, $p_2$ is $\delta_2$-Lipschitz continuous, $P(\cdot,y)$ is $\nu_2$-Lipschitz continuous in the first variable, and $L$ is $\gamma$-$\widehat{H}_1$-Lipschitz continuous, using (4.11), we deduce that

$$\begin{aligned}\|P(l(z),t)-P(l(z),t')\|&\le\iota_2\|t-t'\|\le\iota_2(1+\varepsilon)\widehat{H}_2(W(y),W(y'))\\&\le\iota_2\zeta'(1+\varepsilon)\|y-y'\|,\end{aligned}\tag{4.19}$$

and also

$$\begin{aligned}\|P(l(z),t')-P(l(z'),t')\|&\le\nu_2\|l(z)-l(z')\|\le\nu_2\delta_2\|z-z'\|\\&\le\nu_2\delta_2(1+\varepsilon)\widehat{H}_1(L(x),L(x'))\\&\le\nu_2\delta_2\gamma(1+\varepsilon)\|x-x'\|.\end{aligned}\tag{4.20}$$

Again, by Lemma 2.1, it follows that

$$\begin{aligned}
&\left\| A_1(f(x)-v) - A_1(f(x')-v') - \frac{\rho_1}{\lambda_1}(E(p(x),w') - E(p(x'),w')) \right\|^{q_1} \\
&\quad \le \left\| A_1(f(x)-v) - A_1(f(x')-v') \right\|^{q_1} - q_1 \frac{\rho_1}{\lambda_1} \\
&\quad\quad \times \left\langle E(p(x),w') - E(p(x'),w'), J_{q_1}(A_1(f(x)-v) - A_1(f(x')-v')) \right\rangle \\
&\quad\quad + c_{q_1} \frac{\rho_1{}^{q_1}}{\lambda_1{}^{q_1}} \left\| E(p(x),w') - E(p(x'),w') \right\|^{q_1}.
\end{aligned} \tag{4.21}$$

Since $A_1$ is $\beta_1$-Lipschitz continuous, $f$ is $\mu$-Lipschitz continuous, and $T$ is $\zeta$-Lipschitz continuous, by (4.11), we get

$$\begin{aligned}
\left\| A_1(f(x)-v) - A_1(f(x')-v') \right\| &\le \beta_1 \left\| f(x) - f(x') - (v - v') \right\| \\
&\le \beta_1 \left( \left\| f(x) - f(x') \right\| + \left\| v - v' \right\| \right) \\
&\le \beta_1 (\mu + \zeta(1+\varepsilon)) \left\| x - x' \right\|.
\end{aligned} \tag{4.22}$$

Since $E(p(\cdot),y)$ is $(\theta_1, s_1)$-relaxed cocoercive with respect to $f'$, where $f'(x) = A_1 \circ (f(x) - v) = A_1(f(x)-v)$, $E(\cdot,y)$ is $\nu_1$-Lipschitz continuous in the first variable and $p$ is $\delta_1$-Lipschitz continuous, we have

$$\begin{aligned}
&\left\langle E(p(x),w') - E(p(x'),w'), J_{q_1}(A_1(f(x)-v) - A_1(f(x')-v')) \right\rangle \\
&\quad \le -\theta_1 \left\| E(p(x),w') - E(p(x'),w') \right\|^{q_1} + s_1 \left\| x - x' \right\|^{q_1} \\
&\quad \le -\theta_1 \nu_1{}^{q_1} \left\| p(x) - p(x') \right\|^{q_1} + s_1 \left\| x - x' \right\|^{q_1} \\
&\quad \le \left( -\theta_1 \nu_1{}^{q_1} \delta_1{}^{q_1} + s_1 \right) \left\| x - x' \right\|^{q_1},
\end{aligned} \tag{4.23}$$

and also

$$\left\| E(p(x),w') - E(p(x'),w') \right\| \le \nu_1 \left\| p(x) - p(x') \right\| \le \nu_1 \delta_1 \left\| x - x' \right\|. \tag{4.24}$$

Hence, using (4.21)–(4.24), we have

$$\begin{aligned}
&\left\| A_1(f(x)-v) - A_1(f(x')-v') - \frac{\rho_1}{\lambda_1}(E(p(x),w') - E(p(x'),w')) \right\|^{q_1} \\
&\quad \le \left( \beta_1{}^{q_1} (\mu + \zeta(1+\varepsilon))^{q_1} - q_1 \frac{\rho_1}{\lambda_1} (-\theta_1 \nu_1{}^{q_1} \delta_1{}^{q_1} + s_1) + \frac{c_{q_1} \rho_1{}^{q_1} \nu_1{}^{q_1} \delta_1{}^{q_1}}{\lambda_1{}^{q_1}} \right) \left\| x - x' \right\|^{q_1},
\end{aligned} \tag{4.25}$$

where $c_{q_1}$ is the constant as in Lemma 2.1. Using (4.14)–(4.20), and (4.25), it follows that

$$\left\| \Phi_{\rho_1}(x,z,v,t,w) - \Phi_{\rho_1}(x',z',v',t',w') \right\| \le \varphi_1(\varepsilon) \left\| x - x' \right\| + \phi_1(\varepsilon) \left\| y - y' \right\|, \tag{4.26}$$

where

$$\begin{aligned}
\varphi_1(\varepsilon) &= \varsigma + \zeta(1+\varepsilon) + \sqrt[q_1]{1 - q_1 e_1 + (c_{q_1} + q_1\kappa)\mu^{q_1}} + \frac{\rho_1 \tau_1^{q_1-1}(\nu_2\delta_2\gamma(1+\varepsilon) + \psi_1(\varepsilon))}{\lambda_1(r_1 - \rho_1 m_1)}, \\
\psi_1(\varepsilon) &= \sqrt[q_1]{\beta_1{}^{q_1}(\mu + \zeta(1+\varepsilon))^{q_1} - q_1\frac{\rho_1}{\lambda_1}\left(-\theta_1\nu_1{}^{q_1}\delta_1{}^{q_1} + s_1\right) + \frac{c_{q_1}\rho_1{}^{q_1}\nu_1{}^{q_1}\delta_1{}^{q_1}}{\lambda_1{}^{q_1}}}, \qquad (4.27)\\
\phi_1(\varepsilon) &= \frac{\rho_1\tau_1^{q_1-1}(\iota_1\xi' + \iota_2\zeta')(1+\varepsilon)}{\lambda_1(r_1 - \rho_1 m_1)}.
\end{aligned}$$

Similarly, for any $(u, m, s, y), (u', m', s', y') \in X_1 \times X_1 \times X_2 \times X_2$, it follows from (4.3) and Proposition 2.11 that

$$\begin{aligned}
&\left\|\Psi_{\rho_2}(u, m, s, y) - \Psi_{\rho_2}(u', m', s', y')\right\| \\
&\quad \le \left\|y - y' - (g(y) - g(y'))\right\| \\
&\qquad + \Bigg\| R^{\eta_2, A_2}_{N(\cdot, y), \rho_2}\left[A_2(g(y)) - \frac{\rho_2}{\lambda_2}(F(u, h(y)) + Q(m, k(s)) - b)\right] \\
&\qquad\qquad - R^{\eta_2, A_2}_{N(\cdot, y'), \rho_2}\left[A_2(g(y')) - \frac{\rho_2}{\lambda_2}(F(u', h(y')) + Q(m', k(s')) - b)\right]\Bigg\| \\
&\quad \le \left\|y - y' - (g(y) - g(y'))\right\| \\
&\qquad + \Bigg\| R^{\eta_2, A_2}_{N(\cdot, y), \rho_2}\left[A_2(g(y)) - \frac{\rho_2}{\lambda_2}(F(u, h(y)) + Q(m, k(s)) - b)\right] \\
&\qquad\qquad - R^{\eta_2, A_2}_{N(\cdot, y), \rho_2}\left[A_2(g(y')) - \frac{\rho_2}{\lambda_2}(F(u', h(y')) + Q(m', k(s')) - b)\right]\Bigg\| \\
&\qquad + \Bigg\| R^{\eta_2, A_2}_{N(\cdot, y), \rho_2}\left[A_2(g(y')) - \frac{\rho_2}{\lambda_2}(F(u', h(y')) + Q(m', k(s')) - b)\right] \\
&\qquad\qquad - R^{\eta_2, A_2}_{N(\cdot, y'), \rho_2}\left[A_2(g(y')) - \frac{\rho_2}{\lambda_2}(F(u', h(y')) + Q(m', k(s')) - b)\right]\Bigg\| \\
&\quad \le \left\|y - y' - (g(y) - g(y'))\right\| + \vartheta\left\|y - y'\right\| \\
&\qquad + \frac{\tau_2^{q_2-1}}{r_2 - \rho_2 m_2}\Bigg\| A_2(g(y)) - \frac{\rho_2}{\lambda_2}(F(u, h(y)) + Q(m, k(s)) - b) \\
&\qquad\qquad - \left(A_2(g(y')) - \frac{\rho_2}{\lambda_2}(F(u', h(y')) + Q(m', k(s')) - b)\right)\Bigg\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \|y-y'-(g(y)-g(y'))\| + \vartheta\|y-y'\| + \frac{\tau_2^{q_2-1}}{r_2-\rho_2 m_2} \\
&\quad\times\Big\{\frac{\rho_2}{\lambda_2}\big(\|F(u,h(y))-F(u',h(y))\| + \|Q(m,k(s))-Q(m',k(s))\| \\
&\qquad\qquad + \|Q(m',k(s))-Q(m',k(s'))\|\big) \\
&\qquad + \left\|A_2(g(y))-A_2(g(y'))-\frac{\rho_2}{\lambda_2}\big(F(u',h(y))-F(u',h(y'))\big)\right\|\Big\}.
\end{aligned}
\tag{4.28}
$$

Thus, by Lemma 2.1, we have

$$
\begin{aligned}
&\|y-y'-(g(y)-g(y'))\|^{q_2} \\
&\quad \leq \|y-y'\|^{q_2} - q_2\langle g(y)-g(y'), J_{q_2}(y-y')\rangle + c_{q_2}\|g(y)-g(y')\|^{q_2}.
\end{aligned}
\tag{4.29}
$$

Since $g$ is $(\sigma, e_2)$-relaxed cocoercive and $\epsilon$-Lipschitz continuous, we have

$$
\begin{aligned}
\|y-y'-(g(y)-g(y'))\|^{q_2} &\leq \|y-y'\|^{q_2} - q_2 e_2\|y-y'\|^{q_2} + (c_{q_2}+q_2\sigma)\epsilon^{q_2}\|y-y'\|^{q_2} \\
&= \big(1-q_2e_2+(c_{q_2}+q_2\sigma)\epsilon^{q_2}\big)\|y-y'\|^{q_2}.
\end{aligned}
\tag{4.30}
$$

Since $F(\cdot, y)$ is $\rho_1$-Lipschitz continuous in the first variable and $S$ is $\xi$-$\widehat{H}_1$-Lipschitz continuous, by (4.11), we obtain

$$
\begin{aligned}
\|F(u,h(y))-F(u',h(y))\| &\leq \rho_1\|u-u'\| \leq \rho_1(1+\varepsilon)\widehat{H}_1(S(x),S(x')) \\
&\leq \rho_1\xi(1+\varepsilon)\|x-x'\|.
\end{aligned}
\tag{4.31}
$$

Since $Q(x,\cdot)$ is $\upsilon_2$-Lipschitz continuous in the second variable, $k$ is $\pi_2$-Lipschitz continuous, $Q(\cdot, y)$ is $\rho_2$-Lipschitz continuous in the first variable, $D$ is $\varpi$-$\widehat{H}_1$-Lipschitz continuous, and $K$ is $\gamma'$-$\widehat{H}_2$-Lipschitz continuous, using (4.11), we conclude that

$$
\begin{aligned}
\|Q(m,k(s))-Q(m',k(s))\| &\leq \rho_2\|m-m'\| \\
&\leq \rho_2(1+\varepsilon)\widehat{H}_1(D(x),D(x')) \\
&\leq \rho_2\varpi(1+\varepsilon)\|x-x'\|,
\end{aligned}
\tag{4.32}
$$

$$
\begin{aligned}
\|Q(m',k(s))-Q(m',k(s'))\| &\leq \upsilon_2\|k(s)-k(s')\| \\
&\leq \upsilon_2\pi_2\|s-s'\| \\
&\leq \upsilon_2\pi_2(1+\varepsilon)\widehat{H}_2(K(y),K(y')) \\
&\leq \upsilon_2\pi_2\gamma'(1+\varepsilon)\|y-y'\|.
\end{aligned}
\tag{4.33}
$$

Again, by Lemma 2.1, it follows that

$$\begin{aligned}
&\left\| A_2(g(y)) - A_2(g(y')) - \frac{\rho_2}{\lambda_2}(F(u',h(y)) - F(u',h(y'))) \right\|^{q_2} \\
&\quad \le \| A_2(g(y)) - A_2(g(y')) \|^{q_2} \\
&\qquad - q_2 \frac{\rho_2}{\lambda_2} \langle F(u',h(y)) - F(u',h(y')), J_{q_2}(A_2(g(y)) - A_2(g(y'))) \rangle \\
&\qquad + c_{q_2} \frac{\rho_2{}^{q_2}}{\lambda_2{}^{q_2}} \| F(u',h(y)) - F(u',h(y')) \|^{q_2}.
\end{aligned} \tag{4.34}$$

Since $A_2$ is $\beta_2$-Lipschitz continuous and $g$ is $\epsilon$-Lipschitz continuous, we have

$$\| A_2(g(y)) - A_2(g(y')) \| \le \beta_2 \| g(y) - g(y') \| \le \beta_2 \epsilon \| y - y' \|. \tag{4.35}$$

Since $F(u, h(\cdot))$ is $(\theta_2, s_2)$-relaxed cocoercive with respect to $g' = A_2 \circ g$, $F(x, \cdot)$ is $\upsilon_1$-Lipschitz continuous in the second variable, and $h$ is $\pi_1$-Lipschitz continuous, we get

$$\begin{aligned}
&\langle F(u',h(y)) - F(u',h(y')), J_{q_2}(A_2(g(y)) - A_2(g(y'))) \rangle \\
&\quad \le -\theta_2 \| F(u',h(y)) - F(u',h(y')) \|^{q_2} + s_2 \| y - y' \|^{q_2} \\
&\quad \le -\theta_2 \upsilon_1{}^{q_2} \| h(y) - h(y') \|^{q_2} + s_2 \| y - y' \|^{q_2} \\
&\quad \le (-\theta_2 \upsilon_1{}^{q_2} \pi_1{}^{q_2} + s_2) \| y - y' \|^{q_2},
\end{aligned} \tag{4.36}$$

$$\begin{aligned}
&\| F(u',h(y)) - F(u',h(y')) \| \le \upsilon_1 \| h(y) - h(y') \| \\
&\quad \le \upsilon_1 \pi_1 \| y - y' \|.
\end{aligned} \tag{4.37}$$

Therefore, it follows from (4.34)–(4.37) that

$$\begin{aligned}
&\left\| A_2(g(y)) - A_2(g(y')) - \frac{\rho_2}{\lambda_2}(F(u',h(y)) - F(u',h(y'))) \right\|^{q_2} \\
&\quad \le \left( \beta_2{}^{q_2} \epsilon^{q_2} - q_2 \frac{\rho_2}{\lambda_2}(-\theta_2 \upsilon_1{}^{q_2} \pi_1{}^{q_2} + s_2) + \frac{c_{q_2} \rho_2{}^{q_2} \upsilon_1{}^{q_2} \pi_1{}^{q_2}}{\lambda_2{}^{q_2}} \right) \| y - y' \|^{q_2},
\end{aligned} \tag{4.38}$$

where $c_{q_2}$ is the constant as in Lemma 2.1. From (4.28)–(4.33), and (4.38), it follows that

$$\| \Psi_{\rho_2}(u, m, s, y) - \Psi_{\rho_2}(u', m', s', y') \| \le \varphi_2(\varepsilon) \| x - x' \| + \phi_2(\varepsilon) \| y - y' \|, \tag{4.39}$$

where

$$\begin{aligned}
\varphi_2(\varepsilon) &= \frac{\rho_2 \tau_2^{q_2-1}(\rho_1 \xi + \rho_2 \varpi)(1+\varepsilon)}{\lambda_2(r_2 - \rho_2 m_2)}, \\
\phi_2(\varepsilon) &= \vartheta + \sqrt[q_2]{1 - q_2 e_2 + (c_{q_2} + q_2 \sigma)\epsilon^{q_2}} + \frac{\rho_2 \tau_2^{q_2-1}(v_2 \pi_2 \gamma'(1+\varepsilon) + \psi_2)}{\lambda_2(r_2 - \rho_2 m_2)}, \\
\psi_2 &= \sqrt[q_2]{\beta_2{}^{q_2} \epsilon^{q_2} - q_2 \frac{\rho_2}{\lambda_2}(-\theta_2 v_1{}^{q_2} \pi_1{}^{q_2} + s_2) + \frac{c_{q_2} \rho_2{}^{q_2} v_1{}^{q_2} \pi_1{}^{q_2}}{\lambda_2{}^{q_2}}}.
\end{aligned} \tag{4.40}$$

It follows from (4.26) and (4.39) that

$$\begin{aligned}
&\|\Phi_{\rho_1}(x,z,v,t,w) - \Phi_{\rho_1}(x',z',v',t',w')\| + \|\Psi_{\rho_2}(u,m,s,y) - \Psi_{\rho_2}(u',m',s',y')\| \\
&\quad \le \omega(\varepsilon)(\|x - x'\| + \|y - y'\|),
\end{aligned} \tag{4.41}$$

where $\omega(\varepsilon) = \max\{\varphi_1(\varepsilon) + \varphi_2(\varepsilon), \phi_1(\varepsilon) + \phi_2(\varepsilon)\}$. Using (4.8) and (4.41), we deduce that

$$\begin{aligned}
&\|Q_{\rho_1,\rho_2}(x,y,z,u,v,m,s,t,w) - Q_{\rho_1,\rho_2}(x',y',z',u',v',m',s',t',w')\|_* \\
&\quad \le \omega(\varepsilon)\|(x,y) - (x',y')\|_*,
\end{aligned} \tag{4.42}$$

that is,

$$\begin{aligned}
&\sup_{Q_{\rho_1,\rho_2}(x,y,z,u,v,m,s,t,w) \in \mathfrak{R}_{\rho_1,\rho_2}(x,y)} d(Q_{\rho_1,\rho_2}(x,y,z,u,v,m,s,t,w), \mathfrak{R}_{\rho_1,\rho_2}(x',y')) \\
&\quad \le \omega(\varepsilon)\|(x,y) - (x',y')\|_*.
\end{aligned} \tag{4.43}$$

Similarly, we have

$$\begin{aligned}
&\sup_{Q_{\rho_1,\rho_2}(x',y',z',u',v',m',s',t',w') \in \mathfrak{R}_{\rho_1,\rho_2}(x',y')} d(Q_{\rho_1,\rho_2}(x',y',z',u',v',m',s',t',w'), \mathfrak{R}_{\rho_1,\rho_2}(x,y)) \\
&\quad \le \omega(\varepsilon)\|(x,y) - (x',y')\|_*.
\end{aligned} \tag{4.44}$$

By (4.43), (4.44), and the definition of Hausdorff pseudo-metric, we have

$$\widehat{H}(\mathfrak{R}_{\rho_1,\rho_2}(x,y), \mathfrak{R}_{\rho_1,\rho_2}(x',y')) \le \omega(\varepsilon)\|(x,y) - (x',y')\|_*, \quad \forall (x,y), (x',y') \in X_1 \times X_2. \tag{4.45}$$

Letting $\varepsilon \to 0$, one has

$$\widehat{H}(\mathfrak{R}_{\rho_1,\rho_2}(x,y), \mathfrak{R}_{\rho_1,\rho_2}(x',y')) \le \omega\|(x,y) - (x',y')\|_*, \quad \forall (x,y), (x',y') \in X_1 \times X_2, \tag{4.46}$$

where

$$\omega = \max\{\varphi_1 + \varphi_2, \phi_1 + \phi_2\}, \tag{4.47}$$

$$\varphi_1 = \varsigma + \zeta + \sqrt[q_1]{1 - q_1 e_1 + (c_{q_1} + q_1\kappa)\mu^{q_1}} + \frac{\rho_1 \tau_1^{q_1-1}(\nu_2\delta_2\gamma + \psi_1)}{\lambda_1(r_1 - \rho_1 m_1)},$$

$$\psi_1 = \sqrt[q_1]{\beta_1{}^{q_1}(\mu+\zeta)^{q_1} - q_1\frac{\rho_1}{\lambda_1}(-\theta_1\nu_1{}^{q_1}\delta_1{}^{q_1} + s_1) + \frac{c_{q_1}\rho_1{}^{q_1}\nu_1{}^{q_1}\delta_1{}^{q_1}}{\lambda_1{}^{q_1}}},$$

$$\phi_2 = \vartheta + \sqrt[q_2]{1 - q_2 e_2 + (c_{q_2} + q_2\sigma)\epsilon^{q_2}} + \frac{\rho_2\tau_2^{q_2-1}(\upsilon_2\pi_2\gamma' + \psi_2)}{\lambda_2(r_2 - \rho_2 m_2)}, \tag{4.48}$$

$$\varphi_2 = \frac{\rho_2\tau_2^{q_2-1}(\rho_1\xi + \rho_2\varpi)}{\lambda_2(r_2 - \rho_2 m_2)},$$

$$\phi_1 = \frac{\rho_1\tau_1^{q_1-1}(\iota_1\xi' + \iota_2\zeta')}{\lambda_1(r_1 - \rho_1 m_1)}$$

and $\psi_2$ is the constant as in (4.40). From (4.4), we know that $0 \le \omega < 1$ and so it follows from (4.46) that $\Re_{\rho_1,\rho_2} : X_1 \times X_2 \to X_1 \times X_2$ is a contractive mapping. Hence Lemma 2.3 implies that $\Re_{\rho_1,\rho_2}$ has a fixed point in $X_1 \times X_2$; that is, there exists a point $(x^*, y^*) \in X_1 \times X_2$ such that $(x^*, y^*) \in \Re_{\rho_1,\rho_2}(x^*, y^*)$. Now, it follows from (4.6), (4.8), and Lemma 4.1 that $(x^*, y^*, z^*, u^*, v^*, m^*, n^*, t^*, w^*)$ is a solution of System 3.1 and this is the desired result. This completes the proof. □

By using Theorem 4.2, we can derive the following.

**Theorem 4.3.** *Let $X_i$, $A_i$, $\eta_i$ $(i = 1, 2)$, $\mathcal{S}$, $\mathcal{T}$, $\mathcal{L}$, $\mathfrak{D}$, $\mathcal{G}$, $\mathcal{W}$, $\mathcal{K}$, $S$, $T$, $L$, $D$, $G$, $W$, $K$, $M$, $N$, $E$, $P$, $F$, $Q$, $p$, $l$, $h$, $k$, $f'$, and $g'$ be the same as in Theorem 4.2. Assume that $f : X_1 \to X_1$ is $\kappa$-strongly accretive $\mu$-Lipschitz continuous and $g : X_2 \to X_2$ is $\sigma$-strongly accretive $\epsilon$-Lipschitz continuous.*

*Further, if there exist constants $\rho_1 \in (0, r_1/m_1)$ and $\rho_2 \in (0, r_2/m_2)$ such that* (4.2) *and* (4.3) *hold and*

$$\varsigma + \zeta + \sqrt[q_1]{1 - q_1\kappa + c_{q_1}\mu^{q_1}} < 1,$$

$$\vartheta + \sqrt[q_2]{1 - q_2\sigma + c_{q_2}\epsilon^{q_2}} < 1,$$

$$\sqrt[q_1]{\beta_1{}^{q_1}(\mu+\zeta)^{q_1} - q_1\frac{\rho_1}{\lambda_1}(-\theta_1\nu_1{}^{q_1}\delta_1{}^{q_1} + s_1) + \frac{c_{q_1}\rho_1{}^{q_1}\nu_1{}^{q_1}\delta_1{}^{q_1}}{\lambda_1{}^{q_1}}} < \frac{\tau_1^{1-q_1}\lambda_1}{\rho_1}(r_1 - \rho_1 m_1)\chi_1 - \nu_2\delta_2\gamma,$$

$$\sqrt[q_2]{\beta_2{}^{q_2}\epsilon^{q_2} - q_2\frac{\rho_2}{\lambda_2}(-\theta_2\upsilon_1{}^{q_2}\pi_1{}^{q_2} + s_2) + \frac{c_{q_2}\rho_2{}^{q_2}\upsilon_1{}^{q_2}\pi_1{}^{q_2}}{\lambda_2{}^{q_2}}} < \frac{\tau_2^{1-q_2}\lambda_2}{\rho_2}(r_2 - \rho_2 m_2)\chi_2 - \upsilon_2\pi_2\gamma', \tag{4.49}$$

*where*

$$\chi_1 = 1 - \left(\varsigma + \zeta + \sqrt[q_1]{1 - q_1\kappa + c_{q_1}\mu^{q_1}}\right) - \frac{\rho_2\tau_2^{q_2-1}(\rho_1\xi + \rho_2\varpi)}{\lambda_2(r_2 - \rho_2 m_2)},$$
$$\chi_2 = 1 - \left(\vartheta + \sqrt[q_2]{1 - q_2\sigma + c_{q_2}\epsilon^{q_2}}\right) - \frac{\rho_1\tau_1^{q_1-1}(\iota_1\xi' + \iota_2\zeta')}{\lambda_1(r_1 - \rho_1 m_1)}, \tag{4.50}$$

*$\lambda_1$, $\lambda_2$ are the same as in System 3.1, and $c_{q_1}$, $c_{q_2}$ are two constants guaranteed by Lemma 2.1, then System 3.1 admits a solution.*

**Theorem 4.4.** *Let $X_i$, $A_i$, $\eta_i$ $(i = 1,2)$, $p$, $l$, $h$, $k$, $f$, $g$, $F$, $Q$, $M$, $N$, and $P$ be the same as in Theorem 4.2. Assume that $T : X_1 \to X_1$ is $\zeta$-Lipschitz continuous, and $S, L, D : X_1 \to CB(X_1)$ and $G, W, K : X_2 \to CB(X_2)$ are $\xi$-$\widehat{H}_1$-Lipschitz continuous, $\gamma$-$\widehat{H}_1$-Lipschitz continuous, $\varpi$-$\widehat{H}_1$-Lipschitz continuous, $\xi'$-$\widehat{H}_2$-Lipschitz continuous, $\zeta'$-$\widehat{H}_2$-Lipschitz continuous, and $\gamma'$-$\widehat{H}_2$-Lipschitz continuous, respectively. Suppose that $E : X_1 \times X_2 \to X_1$ is a single-valued mapping such that $E(\cdot, y)$ is $\nu_1$-Lipschitz continuous in the first variable and $E(x, \cdot)$ is $\iota_1$-Lipschitz continuous in the second variable for all $(x, y) \in X_1 \times X_2$, and $E(p(\cdot), y)$ is a $(\theta_1, s_1)$-relaxed cocercive mapping with respect to $f' = A_1 \circ (f - T)$ defined by $f'(x) = A_1 \circ (f(x) - T(x)) = A_1(f(x) - T(x))$ for all $x \in X_1$. If there exist constants $\rho_1 \in (0, r_1/m_1)$ and $\rho_2 \in (0, r_2/m_2)$ such that conditions* (4.2)–(4.4) *hold, then System 3.4 has a solution $(x^*, y^*, z^*, u^*, m^*, s^*, t^*, w^*)$.*

**Theorem 4.5.** *Let $X_i$, $A_i$, $\eta_i$ $(i = 1,2)$, $p$, $l$, $h$, $k$, $E$, $F$, $P$, $Q$, $M$, $N$, $S$, $T$, $L$, $D$, $G$, $W$, and $K$ be the same as in Theorem 4.4. Suppose that $f : X_1 \to X_1$ is $\kappa$-strongly accretive and $\mu$-Lipschitz continuous and $g : X_2 \to X_2$ is $\sigma$-strongly accretive and $\epsilon$-Lipschitz continuous. If there exist constants $\rho_1 \in (0, r_1/m_1)$ and $\rho_2 \in (0, r_2/m_2)$ such that conditions* (4.2), (4.3), *and* (4.49) *hold, then System 3.3 has a solution $(x^*, y^*, z^*, u^*, m^*, s^*, t^*, w^*)$.*

## 5. Iterative Algorithm and Convergence

In this section, motivated by Theorems 4.2 and 4.4, Lemmas 4.1 and 2.4, we construct the following iterative algorithms for approximating solutions of Systems 3.1 and 3.3 and discuss the convergence analysis of the algorithms.

*Algorithm 5.1.* Let $X_i$, $A_i$, $\eta_i$, $\lambda_i$ $(i = 1,2)$, $E$, $P$, $F$, $Q$, $p$, $l$, $h$, $k$, $f$, $g$, $M$, $N$, $\mathcal{S}$, $\mathfrak{T}$, $\mathcal{L}$, $\mathfrak{D}$, $\mathcal{G}$, $\mathcal{W}$, $\mathcal{K}$, $S$, $T$, $L$, $D$, $G$, $W$, $K$, $a$ and $b$ be the same as in System 3.1. For any given $(x_0, y_0) \in X_1 \times X_2$, $\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d} : X_1 \to [0,1]$ and $\widetilde{e}, \widetilde{f}, \widetilde{g} : X_2 \to [0,1]$ for all $n \geq 0$ and an element $(x, y, z, u, v, m, s, t, w) \in X_1 \times X_2 \times X_1 \times X_1 \times X_1 \times X_1 \times X_2 \times X_2 \times X_2$, define the iterative sequence $\{(x_n, y_n, z_n, u_n, v_n, m_n, s_n, t_n, w_n)\}_{n=0}^{\infty}$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n\left(x_n - f(x_n) + v_n + R^{\eta_1, A_1}_{M(\cdot, x_n), \rho_1}(\Theta_n)\right) + \alpha_n e_n + r_n,$$
$$y_{n+1} = (1 - \alpha_n)y_n + \alpha_n\left(y_n - g(y_n) + R^{\eta_2, A_2}_{N(\cdot, y_n), \rho_2}(\Omega_n)\right) + \alpha_n f_n + k_n,$$
$$\mathcal{S}_{x_n}(u_n) \geq \widetilde{a}(x_n), \qquad \|u_n - u\| \leq \left(1 + \frac{1}{1+n}\right)\widehat{H}_1(S(x_n), S(x)),$$

$$\begin{aligned}
&\mathcal{T}_{x_n}(v_n) \geq \widetilde{b}(x_n), && \|v_n - v\| \leq \left(1 + \frac{1}{1+n}\right)\widehat{H_1}(T(x_n), T(x)),\\
&\mathcal{L}_{x_n}(z_n) \geq \widetilde{c}(x_n), && \|z_n - z\| \leq \left(1 + \frac{1}{1+n}\right)\widehat{H_1}(L(x_n), L(x)),\\
&\mathfrak{D}_{x_n}(m_n) \geq \widetilde{d}(x_n), && \|m_n - m\| \leq \left(1 + \frac{1}{1+n}\right)\widehat{H_1}(D(x_n), D(x)),\\
&\mathcal{G}_{y_n}(w_n) \geq \widetilde{e}(y_n), && \|w_n - w\| \leq \left(1 + \frac{1}{1+n}\right)\widehat{H_2}(G(y_n), G(y)),\\
&\mathcal{W}_{y_n}(t_n) \geq \widetilde{f}(y_n), && \|t_n - t\| \leq \left(1 + \frac{1}{1+n}\right)\widehat{H_2}(W(y_n), W(y)),\\
&\mathcal{K}_{y_n}(s_n) \geq \widetilde{g}(y_n), && \|s_n - s\| \leq \left(1 + \frac{1}{1+n}\right)\widehat{H_2}(K(y_n), K(y)),
\end{aligned} \tag{5.1}$$

where

$$\begin{aligned}
\Theta_n &= A_1(f(x_n) - v_n) - \frac{\rho_1}{\lambda_1}(E(p(x_n), w_n) + P(l(z_n), t_n) - a),\\
\Omega_n &= A_2(g(y_n)) - \frac{\rho_2}{\lambda_2}(F(u_n, h(y_n)) + Q(m_n, k(s_n)) - b),
\end{aligned} \tag{5.2}$$

$\rho_1$ and $\rho_2$ are constants, $\{\alpha_n\}$ is a sequence in $[0,1]$ with $\sum_{n=0}^{\infty}\alpha_n = \infty$, and $\{(e_n, f_n)\}_{n=0}^{\infty}$ and $\{(r_n, k_n)\}_{n=0}^{\infty}$ are two sequences in $X_1 \times X_2$ to take into account a possible inexact computation of the resolvent operator point satisfying the following conditions:

$$\begin{gathered}
e_n = e'_n + e''_n, \qquad f_n = f'_n + f''_n,\\
\lim_{n\to\infty}\left\|(e'_n, f'_n)\right\|_* = 0,\\
\sum_{n=0}^{\infty}\left\|(e''_n, f''_n)\right\|_* < \infty, \qquad \sum_{n=0}^{\infty}\|(r_n, k_n)\|_* < \infty.
\end{gathered} \tag{5.3}$$

*Algorithm 5.2.* Assume that $X_i$, $A_i$, $\eta_i$, $\lambda_i$ $(i = 1,2)$, $E$, $P$, $F$, $Q$, $p$, $l$, $h$, $k$, $f$, $g$, $M$, $N$, $S$, $T$, $L$, $D$, $G$, $W$, $K$, $a$ and $b$ are the same as in System 3.4. For any given $(x_0, y_0) \in X_1 \times X_2$, $n \geq 0$ and an element $(x, y, z, u, m, s, t, w) \in X_1 \times X_2 \times X_1 \times X_1 \times X_1 \times X_2 \times X_2 \times X_2$, define the iterative sequence $\{(x_n, y_n, z_n, u_n, m_n, s_n, t_n, w_n)\}_{n=0}^{\infty}$ by

$$\begin{aligned}
x_{n+1} &= (1-\alpha_n)x_n + \alpha_n\left(x_n - f(x_n) + T(x_n) + R^{\eta_1, A_1}_{M(\cdot, x_n), \rho_1}(\Theta'_n)\right) + \alpha_n e_n + r_n,\\
y_{n+1} &= (1-\alpha_n)y_n + \alpha_n\left(y_n - g(y_n) + R^{\eta_2, A_2}_{N(\cdot, y_n), \rho_2}(\Omega_n)\right) + \alpha_n f_n + k_n,\\
u_n &\in S(x_n), \quad \|u_n - u\| \leq \left(1 + \frac{1}{1+n}\right)\widehat{H_1}(S(x_n), S(x)),
\end{aligned}$$

$$
\begin{aligned}
&z_n \in L(x_n), \quad \|z_n - z\| \le \left(1 + \frac{1}{1+n}\right)\widehat{H}_1(L(x_n), L(x)),\\
&m_n \in D(x_n), \quad \|m_n - m\| \le \left(1 + \frac{1}{1+n}\right)\widehat{H}_1(D(x_n), D(x)),\\
&w_n \in G(y_n), \quad \|w_n - w\| \le \left(1 + \frac{1}{1+n}\right)\widehat{H}_2(G(y_n), G(y)),\\
&t_n \in W(y_n), \quad \|t_n - t\| \le \left(1 + \frac{1}{1+n}\right)\widehat{H}_2(W(y_n), W(y)),\\
&s_n \in K(y_n), \quad \|s_n - s\| \le \left(1 + \frac{1}{1+n}\right)\widehat{H}_2(K(y_n), K(y)),
\end{aligned}
\tag{5.4}
$$

where $\Theta'_n = A_1(f(x_n) - T(x_n)) - (\rho_1/\lambda_1)(E(p(x_n), w_n) + P(l(z_n), t_n) - a)$, $\Omega_n$, $\rho_1$, $\rho_2$, $\{\alpha_n\}$, $\{(e_n, f_n)\}_{n=0}^{\infty}$ and $\{(r_n, k_n)\}_{n=0}^{\infty}$ are the same as in Algorithm 5.1.

*Remark 5.3.* If $e_n = f_n = 0$ for all $n \ge 0$, $L = D = W = K \equiv 0$, $P = Q \equiv 0$, $a = b = 0$, and $\lambda_1 = \lambda_2 = 1$, then Algorithms 5.1 and 5.2 reduce to Algorithms 4.1 and 4.2 of [38]. In particular, when we choose suitable $\alpha_n$, $e_n$, $f_n$, $r_n$, $k_n$, $A_i$, $\eta_i$ $(i = 1, 2)$, $E$, $P$, $F$, $Q$, $p$, $l$, $h$, $k$, $f$, $g$, $M$, $N$, $\mathcal{S}$, $\mathcal{T}$, $\mathcal{L}$, $\mathfrak{D}$, $\mathcal{G}$, $\mathcal{W}$, $\mathcal{K}$, $S$, $T$, $L$, $D$, $G$, $W$, $K$, and the spaces $X_1$, $X_2$, then Algorithms 5.1 and 5.2 can be degenerated to a number of algorithms involving many known algorithms due to classes of variational inequalities and variational inclusions (see, e.g., [38, 55, 56, 58–60] and the references therein).

**Lemma 5.4.** *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three nonnegative real sequences satisfying the following condition: there exists a natural number $n_0$ such that*

$$
a_{n+1} \le (1 - t_n)a_n + b_n t_n + c_n, \quad \forall n \ge n_0,
\tag{5.5}
$$

*where $t_n \in [0, 1]$, $\sum_{n=0}^{\infty} t_n = \infty$, $\lim_{n\to\infty} b_n = 0$, $\sum_{n=0}^{\infty} c_n < \infty$. Then $\lim_{n\to 0} a_n = 0$.*

*Proof.* The proof directly follows from Liu [64, Lemma 2]. □

**Theorem 5.5.** *Let $X_i$, $A_i$, $\eta_i$ $(i = 1, 2)$, $E$, $P$, $F$, $Q$, $p$, $l$, $h$, $k$, $f$, $g$, $M$, $N$, $\mathcal{S}$, $\mathcal{T}$, $\mathcal{L}$, $\mathfrak{D}$, $\mathcal{G}$, $\mathcal{W}$, $\mathcal{K}$, $S$, $T$, $L$, $D$, $G$, $W$, and $K$ be the same as in Theorem 4.2. Suppose that all the conditions of Theorem 4.2 hold. Then the iterative sequence $\{(x_n, y_n, z_n, u_n, v_n, m_n, s_n, t_n, w_n)\}_{n=0}^{\infty}$ generated by Algorithm 5.1 converges strongly to a solution $(x^*, y^*, z^*, u^*, v^*, m^*, s^*, t^*, w^*)$ of System 3.1.*

*Proof.* It follows from Theorem 4.2 that System 3.1 has a solution $(x^*, y^*, z^*, u^*, v^*, m^*, s^*, t^*, w^*)$. Hence, by Lemma 4.1, we have

$$
\begin{aligned}
f(x^*) &= v^* + R^{\eta_1, A_1}_{M(\cdot, x^*), \rho_1}\left[A_1(f(x^*) - v^*) - \frac{\rho_1}{\lambda_1}\left(E(p(x^*), w^*) + P(l(z^*), t^*) - a\right)\right],\\
g(y^*) &= R^{\eta_2, A_2}_{M(\cdot, y^*), \rho_2}\left[A_2(g(y^*)) - \frac{\rho_2}{\lambda_2}\left(F(u^*, h(y^*)) + Q(m^*, k(s^*)) - b\right)\right].
\end{aligned}
\tag{5.6}
$$

Using (5.1), (5.6), and our assumptions, it follows that

$$
\begin{aligned}
\|x_{n+1}-x^*\| \le{}& (1-\alpha_n)\|x_n-x^*\| + \alpha_n \\
&\times \Bigg( \big\|x_n - x^* - \big(f(x_n)-f(x^*)\big)\big\| + \|v_n - v^*\| \\
&\quad + \Bigg\| R^{\eta_1,A_1}_{M(\cdot,x_n),\rho_1}\left[A_1\big(f(x_n)-v_n\big) - \frac{\rho_1}{\lambda_1}\big(E\big(p(x_n),w_n\big) + P(l(z_n),t_n) - a\big)\right] \\
&\qquad - R^{\eta_1,A_1}_{M(\cdot,x^*),\rho_1}\left[A_1\big(f(x^*)-v^*\big) - \frac{\rho_1}{\lambda_1}\big(E\big(p(x^*),w^*\big) + P(l(z^*),t^*) - a\big)\right]\Bigg\|\Bigg) \\
&+ \alpha_n\|e_n\| + \|r_n\| \\
\le{}& (1-\alpha_n)\|x_n-x^*\| + \alpha_n \\
&\times \Bigg( \big\|x_n - x^* - \big(f(x_n)-f(x^*)\big)\big\| + \|v_n - v^*\| \\
&\quad + \Bigg\| R^{\eta_1,A_1}_{M(\cdot,x_n),\rho_1}\left[A_1\big(f(x_n)-v_n\big) - \frac{\rho_1}{\lambda_1}\big(E\big(p(x_n),w_n\big) + P(l(z_n),t_n) - a\big)\right] \\
&\qquad - R^{\eta_1,A_1}_{M(\cdot,x_n),\rho_1}\left[A_1\big(f(x^*)-v^*\big) - \frac{\rho_1}{\lambda_1}\big(E\big(p(x^*),w^*\big) + P(l(z^*),t^*) - a\big)\right]\Bigg\| \\
&\quad + \Bigg\| R^{\eta_1,A_1}_{M(\cdot,x_n),\rho_1}\left[A_1\big(f(x^*)-v^*\big) - \frac{\rho_1}{\lambda_1}\big(E\big(p(x^*),w^*\big) + P(l(z^*),t^*) - a\big)\right] \\
&\qquad - R^{\eta_1,A_1}_{M(\cdot,x^*),\rho_1}\left[A_1\big(f(x^*)-v^*\big) - \frac{\rho_1}{\lambda_1}\big(E\big(p(x^*),w^*\big) + P(l(z^*),t^*) - a\big)\right]\Bigg\|\Bigg) \\
&+ \alpha_n\big(\|e'_n\| + \|e''_n\|\big) + \|r_n\| \\
\le{}& (1-\alpha_n)\|x_n-x^*\| + \alpha_n \\
&\times \Bigg\{ \big\|x_n - x^* - \big(f(x_n)-f(x^*)\big)\big\| + \|v_n - v^*\| + \varsigma\|x_n - x^*\| + \frac{\tau_1^{q_1-1}}{r_1-\rho_1 m_1} \\
&\quad \times \Bigg( \frac{\rho_1}{\lambda_1}\Big(\big\|E\big(p(\mathrm{x}_n),w_n\big) - E\big(p(x_n),w^*\big)\big\| + \|P(l(z_n),t_n) - P(l(z_n),t^*)\| \\
&\qquad\qquad + \|P(l(z_n),t^*) - P(l(z^*),t^*)\|\Big) \\
&\qquad + \Bigg\| A_1\big(f(x_n)-v_n\big) - A_1\big(f(x^*)-v^*\big) - \frac{\rho_1}{\lambda_1}\big(E\big(p(x_n),w^*\big) - E\big(p(x^*),w^*\big)\big)\Bigg\|\Bigg)\Bigg\} \\
&+ \alpha_n\|e'_n\| + \|e''_n\| + \|r_n\| \\
\le{}& (1-\alpha_n)\|x_n-x^*\| + \alpha_n\big(\varphi_1(n)\|x_n-x^*\| + \phi_1(n)\|y_n-y^*\|\big) + \alpha_n\|e'_n\| + \|e''_n\| + \|r_n\|,
\end{aligned} \tag{5.7}
$$

where

$$\varphi_1(n) = \varsigma + \zeta\left(1 + \frac{1}{1+n}\right) + \sqrt[q_1]{1 - q_1 e_1 + (c_{q_1} + q_1\kappa)\mu^{q_1}} + \frac{\rho_1 \tau_1^{q_1-1}(\nu_2\delta_2\gamma(1+1/(1+n)) + \psi_1(n))}{\lambda_1(r_1 - \rho_1 m_1)},$$

$$\psi_1(n) = \sqrt[q_1]{\beta_1^{q_1}\left(\mu + \zeta\left(1 + \frac{1}{1+n}\right)\right)^{q_1} - q_1\frac{\rho_1}{\lambda_1}\left(-\theta_1\nu_1^{q_1}\delta_1^{q_1} + s_1\right) + \frac{c_{q_1}\rho_1^{q_1}\nu_1^{q_1}\delta_1^{q_1}}{\lambda_1^{q_1}}}, \tag{5.8}$$

$$\phi_1(n) = \frac{\rho_1\tau_1^{q_1-1}(\iota_1\xi' + \iota_2\zeta')(1+1/(1+n))}{\lambda_1(r_1 - \rho_1 m_1)}.$$

Similarly, we have

$$\begin{aligned}
\|y_{n+1} - y^*\| &\leq (1-\alpha_n)\|y_n - y^*\| + \alpha_n \\
&\quad\times \Bigg( \|y_n - y^* - (g(y_n) - g(y^*))\| \\
&\qquad + \Bigg\| R^{\eta_2, A_2}_{N(\cdot, y_n), \rho_2}\left[A_2(g(y_n)) - \frac{\rho_2}{\lambda_2}(F(u_n, h(y_n)) + Q(m_n, k(s_n)) - b)\right] \\
&\qquad\quad - R^{\eta_2, A_2}_{N(\cdot, y^*), \rho_2}\left[A_2(g(y^*)) - \frac{\rho_2}{\lambda_2}(F(u^*, h(y^*)) + Q(m^*, k(s^*)) - b)\right]\Bigg\|\Bigg) \\
&\quad + \alpha_n\|f_n\| + \|k_n\| \\
&\leq (1-\alpha_n)\|y_n - y^*\| + \alpha_n \\
&\quad\times \Bigg( \|y_n - y^* - (g(y_n) - g(y^*))\| \\
&\qquad + \Bigg\| R^{\eta_2, A_2}_{N(\cdot, y_n), \rho_2}\left[A_2(g(y_n)) - \frac{\rho_2}{\lambda_2}(F(u_n, h(y_n)) + Q(m_n, k(s_n)) - b)\right] \\
&\qquad\quad - R^{\eta_2, A_2}_{N(\cdot, y_n), \rho_2}\left[A_2(g(y^*)) - \frac{\rho_2}{\lambda_2}(F(u^*, h(y^*)) + Q(m^*, k(s^*)) - b)\right]\Bigg\| \\
&\qquad + \Bigg\| R^{\eta_2, A_2}_{N(\cdot, y^*), \rho_2}\left[A_2(g(y^*)) - \frac{\rho_2}{\lambda_2}(F(u^*, h(y^*)) + Q(m^*, k(s^*)) - b)\right] \\
&\qquad\quad - R^{\eta_2, A_2}_{N(\cdot, y^*), \rho_2}\left[A_2(g(y^*)) - \frac{\rho_2}{\lambda_2}(F(u^*, h(y^*)) + Q(m^*, k(s^*)) - b)\right]\Bigg\|\Bigg) \\
&\quad + \alpha_n(\|f_n'\| + \|f_n''\|) + \|k_n\|
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\alpha_n)\left\|y_n - y^*\right\| + \alpha_n \\
&\quad \times \left\{ \left\|y_n - y^* - \left(g(y_n) - g(y^*)\right)\right\| + \vartheta\left\|y_n - y^*\right\| + \frac{\tau_2^{q_2-1}}{r_2 - \rho_2 m_2} \right. \\
&\qquad \times \left( \frac{\rho_2}{\lambda_2}\left(\left\|F(u_n, h(y_n)) - F(u^*, h(y_n))\right\| + \|Q(m_n, k(s_n)) - Q(m^*, k(s_n))\| \right.\right. \\
&\qquad\qquad \left. +\|Q(m^*, k(s_n)) - Q(m^*, k(s^*))\|\right) \\
&\qquad\qquad \left.\left. +\left\| A_2(g(y_n)) - A_2(g(y^*)) - \frac{\rho_2}{\lambda_2}\left(F(u^*, h(y_n)) - F(u^*, h(y^*))\right)\right\| \right) \right\} \\
&\quad + \alpha_n\left\|f_n'\right\| + \left\|f_n''\right\| + \|k_n\| \\
&\leq (1-\alpha_n)\left\|y_n - y^*\right\| + \alpha_n\left(\varphi_2(n)\|x_n - x^*\| + \phi_2(n)\left\|y_n - y^*\right\|\right) \\
&\quad + \alpha_n\left\|f_n'\right\| + \left\|f_n''\right\| + \|k_n\|,
\end{aligned}
\tag{5.9}
$$

where

$$
\begin{aligned}
\phi_2(n) &= \vartheta + \sqrt[q_2]{1 - q_2 e_2 + (c_{q_2} + q_2\sigma)\epsilon^{q_2}} + \frac{\rho_2\tau_2^{q_2-1}\left(\upsilon_2\pi_2\gamma'(1 + 1/(1+n)) + \psi_2\right)}{\lambda_2(r_2 - \rho_2 m_2)}, \\
\varphi_2(n) &= \frac{\rho_2\tau_2^{q_2-1}(\rho_1\xi + \rho_2\varpi)(1 + 1/(1+n))}{\lambda_2(r_2 - \rho_2 m_2)},
\end{aligned}
\tag{5.10}
$$

and $\psi_2$ is the same as (4.40). By (5.7) and (5.9), we obtain

$$
\begin{aligned}
\left\|(x_{n+1}, y_{n+1}) - (x^*, y^*)\right\|_* = \|x_{n+1} - x^*\| + \left\|y_{n+1} - y^*\right\| &\leq (1-\alpha_n)\left\|(x_n, y_{\mathrm{n}}) - (x^*, y^*)\right\|_* \\
&\quad + \alpha_n \max\left\{\varphi_1(n) + \varphi_2(n), \phi_1(n) + \phi_2(n)\right\}\left\|(x_n, y_n) - (x^*, y^*)\right\|_* \\
&\quad + \alpha_n\left\|(e_n', f_n')\right\|_* + \left\|(e_n'', f_n'')\right\|_* + \|(r_n, k_n)\|_* \\
&= (1 - (1 - \omega(n))\alpha_n)\left\|(x_n, y_n) - (x^*, y^*)\right\|_* \\
&\quad + \alpha_n\left\|(e_n', f_n')\right\|_* + \left\|(e_n'', f_n'')\right\|_* + \|(r_n, k_n)\|_*,
\end{aligned}
\tag{5.11}
$$

where

$$
\omega(n) = \max\left\{\varphi_1(n) + \varphi_2(n), \phi_1(n) + \phi_2(n)\right\}. \tag{5.12}
$$

Now, $\omega(n) \to \omega = \max\{\varphi_1 + \varphi_2, \phi_1 + \phi_2\}$ as $n \to \infty$, where $\varphi_1$, $\varphi_2$, $\phi_1$, and $\phi_2$ are the constants as in (4.48).

Since $\widehat{\omega} = (1/2)(\omega + 1) \in (\omega, 1)$, deduce that there exists $n_0 \geq 1$ such that $\omega(n) < \widehat{\omega}$, for all $n \geq n_0$. Accordingly, it follows from (5.11) that for all $n \geq n_0$,

$$\begin{aligned} \left\|(x_{n+1}, y_{n+1}) - (x^*, y^*)\right\|_* \leq{}& (1 - (1-\widehat{\omega})\alpha_n)\left\|(x_n, y_n) - (x^*, y^*)\right\|_* + \alpha_n\left\|(e'_n, f'_n)\right\|_* \\ &+ \left\|(e''_n, f''_n)\right\|_* + \|(r_n, k_n)\|_*. \end{aligned} \tag{5.13}$$

Letting

$$\begin{aligned} a_n &= \left\|(x_{n+1}, y_{n+1}) - (x^*, y^*)\right\|_*, \qquad t_n = (1-\widehat{\omega})\alpha_n, \\ b_n &= \frac{\left\|(e'_n, f'_n)\right\|_*}{1-\widehat{\omega}}, \qquad c_n = \left\|(e''_n, f''_n)\right\|_* + \|(r_n, k_n)\|_*, \end{aligned} \tag{5.14}$$

then (5.13) can be written as

$$a_{n+1} \leq (1 - t_n)a_n + b_n t_n + c_n, \quad \forall n \geq 0. \tag{5.15}$$

Therefore, it follows from Lemma 5.4 that $\lim_{n\to\infty} a_n = 0$ and so the sequence $\{(x_n, y_n, z_n, u_n, v_n, m_n, s_n, t_n, w_n)\}_{n=0}^{\infty}$ defined by Algorithm 5.1 converges strongly to a solution $(x^*, y^*, z^*, u^*, v^*, m^*, s^*, t^*, w^*)$ of System 3.1. □

**Theorem 5.6.** *Suppose that* $X_i$, $A_i$, $\eta_i$ $(i = 1, 2)$, $E$, $P$, $F$, $Q$, $p$, $l$, $h$, $k$, $f$, $g$, $M$, $N$, $\mathcal{S}$, $\mathfrak{T}$, $\mathcal{L}$, $\mathfrak{D}$, $\mathcal{G}$, $\mathcal{W}$, $\mathcal{K}$, $S$, $T$, $L$, $D$, $G$, $W$, *and* $K$ *are the same as in Theorem 4.3. Assume that all the conditions of Theorem 4.3 hold. Then the iterative sequence* $\{(x_n, y_n, z_n, u_n, v_n, m_n, s_n, t_n, w_n)\}_{n=0}^{\infty}$ *generated by Algorithm 5.1 converges strongly to the solution* $(x^*, y^*, z^*, u^*, v^*, m^*, s^*, t^*, w^*)$ *of System 3.1.*

**Theorem 5.7.** *Assume that* $X_i$, $A_i$, $\eta_i$ $(i = 1, 2)$, $E, P, F, Q, p, l, h, k, f, g, M, N, S, T, L, D, G, W$, *and* $K$ *are the same as in Theorem 4.4. Suppose that all the conditions of Theorem 4.4 hold. Then the iterative sequence* $\{(x_n, y_n, z_n, u_n, m_n, s_n, t_n, w_n)\}_{n=0}^{\infty}$ *generated by Algorithm 5.2 converges strongly to a solution* $(x^*, y^*, z^*, u^*, m^*, s^*, t^*, w^*)$ *of System 3.3.*

**Theorem 5.8.** *Let* $X_i$, $A_i$, $\eta_i$ $(i = 1, 2)$, $E$, $P$, $F$, $Q$, $p$, $l$, $h$, $k$, $f$, $g$, $M$, $N$, $S$, $T$, $L$, $D$, $G$, $W$, *and* $K$ *be the same as in Theorem 4.5. Suppose that all the conditions of Theorem 4.5 hold. Then the iterative sequence* $\{(x_n, y_n, z_n, u_n, m_n, s_n, t_n, w_n)\}_{n=0}^{\infty}$ *generated by Algorithm 5.2 converges strongly to a solution* $(x^*, y^*, z^*, u^*, m^*, s^*, t^*, w^*)$ *of System 3.3.*

*Remark 5.9.* The following should be noticed.

(1) Theorem 3.1 in [54] is a special case of the Theorems 4.2 and 4.3. Moreover, Theorems 4.4 and 4.5 improve and extend Theorem 3.2 [54].

(2) In view of Remark 5.3, Theorems 5.5 and 5.6 improve and generalize Theorem 4.1 in [54]. Also, Theorems 5.7 and 5.8 are extensions of Theorem 4.2 in [54].

*Remark 5.10.* When $M$ and $N$ are $(A, \eta)$-monotone operators, $A$-accretive mappings, $A$-monotone operators, $(H, \eta)$-accretive mappings, $(H, \eta)$-monotone operators, or $H$-monotone operators, respectively, from Theorems 4.2–4.5 and 5.5–5.8, we can obtain the existence and convergence results of solutions for Systems 3.1 and 3.4. In brief, for a suitable and appropriate choice of the mappings $A_i$, $\eta_i$ $(i = 1, 2)$, $E$, $P$, $F$, $Q$, $p$, $l$, $h$, $k$, $f$, $g$, $M$, $N$, $\mathcal{S}$, $\mathcal{T}$, $\mathcal{L}$, $\mathfrak{D}$, $\mathcal{G}$, $\mathcal{W}$, $\mathcal{K}$, $S$, $T$, $L$, $D$, $G$, $W$, $K$, and the spaces $X_1$, $X_2$, Theorems 4.2–4.5 and 5.5–5.8 include many known results of the generalized variational inclusions as special cases (see [29–35, 38, 39, 55–60] and the references therein).

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