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A characterization of the two-weight inequality for Riesz potentials on cones of radially decreasing functions

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Abstract
We establish necessary and sufficient conditions on a weight pair (v,w) governing the boundedness of the Riesz potential operator I_α defined on a homogeneous group G from $L^p_{\mathrm{dec},r}(w,G)$ to $L^q(v,G)$, where $L^p_{\mathrm{dec},r}(w,G)$ is the Lebesgue space defined for non-negative radially decreasing functions on G. The same problem is also studied for the potential operator with product kernels I_{α_1,α_2} defined on a product of two homogeneous groups $G_1 \times G_2$. In the latter case weights, in general, are not of product type. The derived results are new even for Euclidean spaces. To get the main results we use Sawyer-type duality theorems (which are also discussed in this paper) and two-weight Hardy-type inequalities on G and $G_1 \times G_2$, respectively.
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1 Introduction

A homogeneous group is a simply connected nilpotent Lie group G on a Lie algebra g with the one-parameter group of transformations $\delta_t = \exp(A \log t)$, $t > 0$, where A is a diagonalized linear operator in G with positive eigenvalues. In the homogeneous group G the mappings $\exp o \delta_t o \exp^{-1}$, $t > 0$, are automorphisms in G, which will be again denoted by δ_t. The number $Q = \operatorname{tr} A$ is the homogeneous dimension of G. The symbol e will stand for the neutral element in G.

It is possible to equip G with a homogeneous norm $r : G \to [0,\infty)$ which is continuous on G, smooth on $G\backslash\{e\}$, and satisfies the conditions:

(i) $r(x) = r(x^{-1})$ for every $x \in G$;
(ii) $r(\delta_t x) = t\,r(x)$ for every $x \in G$ and $t > 0$;
(iii) $r(x) = 0$ if and only if $x = e$;
(iv) there exists $c_0 > 0$ such that

$$r(xy) \le c_0\big(r(x) + r(y)\big), \quad x, y \in G.$$

In the sequel we denote by $B(a,t)$ an open ball with the center a and radius $t > 0$, *i.e.*

$$B(a,t) := \big\{y \in G; r\big(ay^{-1}\big) < t\big\}.$$

It can be observed that $\delta_t B(e,1) = B(e,t)$.

Let us fix a Haar measure $|\cdot|$ in G such that $|B(e,1)| = 1$. Then $|\delta_t E| = t^Q|E|$. In particular, $|B(x,t)| = t^Q$ for $x \in G$, $t > 0$.

Examples of homogeneous groups are: the Euclidean n-dimensional space $\mathbb{R}^n$, the Heisenberg group, upper triangular groups, *etc.* For the definition and basic properties of the homogeneous group we refer to [1, p.12].

An everywhere positive function ρ on G will be called a weight. Denote by $L^p(\rho, G)$ $(1 < p < \infty)$ the weighted Lebesgue space, which is the space of all measurable functions $f : G \to \mathbb{C}$ defined by the norm

$$\|f\|_{L^p(\rho,G)} = \left(\int_G |f(x)|^p \rho(x)\,dx\right)^{\frac{1}{p}} < \infty.$$

If $\rho \equiv 1$, then we use the notation $L^p(G)$.

Denote by $\mathcal{DR}(G)$ the class of all radially decreasing functions on G with values in $\mathbb{R}_+$, *i.e.* the fact that $\phi \in \mathcal{DR}(G)$ means that there is a decreasing $\bar{\phi} : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that $\varphi(x) = \bar{\phi}(r(x))$. In the sequel we will use the symbol ϕ itself for $\bar{\phi}$; $\phi \in \mathcal{DR}(G)$ will be written also by the symbol $\varphi \downarrow r$. Let G_1 and G_2 be homogeneous groups. We say that a function $\psi : G_1 \times G_2 \mapsto \mathbb{R}_+$ is radially decreasing if it is such in each variable separately uniformly to another one. The fact that ψ is radially decreasing on $G_1 \times G_2$ will be denoted as $\psi \in \mathcal{DR}(G_1 \times G_2)$.

Let

$$(I_\alpha f)(x) = \int_G f(y)\big(r\big(xy^{-1}\big)\big)^{\alpha-Q}\,dy, \quad 0 < \alpha < Q,$$

be the Riesz potential defined on G, where r is the homogeneous norm and dy is the normalized Haar measure on G. The operator I_α plays a fundamental role in harmonic analysis, *e.g.*, in the theory of Sobolev embeddings, in the theory of sublaplacians on nilpotent groups *etc.* Weighted estimates for multiple Riesz potentials can be applied, for example, to establish Sobolev and Poincaré inequalities on product spaces (see, *e.g.*, [2]).

Let G_1 and G_2 be homogeneous groups with homogeneous norms r_1 and r_2 and homogeneous dimensions Q_1 and Q_2, respectively. We define the potential operator on $G_1 \times G_2$ as follows:

$$(I_{\alpha,\beta} f)(x,y) = \iint_{G_1\times G_2} f(t,\tau)\big(r_1\big(xt^{-1}\big)\big)^{\alpha-Q_1}\big(r_2\big(y\tau^{-1}\big)\big)^{\beta-Q_2}\,dt\,d\tau,$$
$$(x,y) \in G_1 \times G_2, 0 < \alpha < Q_1, 0 < \beta < Q_2.$$

Our aim is to derive two-weight criteria for I_α on the cone of radially decreasing functions on G. The same problem is also studied for the potential operator with product kernels $I_{\alpha,\beta}$ defined on a product of two homogeneous groups, where only the right-hand side weight is of product type. As far as we know the derived results for $I_{\alpha,\beta}$ are new, even in the case of Euclidean spaces. The proofs of the main results are based on Sawyer (see [3]) type duality theorem which is also true for homogeneous groups (see Propositions C and E below) and Hardy-type two-weight inequalities in homogeneous groups. Analogous results for multiple potential operators defined on $\mathbb{R}^n_+$ with respect to the cone of non-negative decreasing functions on $\mathbb{R}^n_+$ were studied in [4, 5]. It should be emphasized that the two-weight problem for a multiple Hardy operator for the cone of decreasing functions on $\mathbb{R}^n_+$ was investigated by Barza, Heinig and Persson [6] under the restriction that both weights are of product type.

Historically, the one-weight inequality for the classical Hardy operator on decreasing functions was characterize by Arino and Muckenhoupt [7] under the so called B_p condition. The same problem for multiple Hardy transform was studied by Arcozzi, Barza, Garcia-Domingo and Soria [8]. This problem in the two-weight setting was solved by Sawyer [3]. Some sufficient conditions guaranteeing the two-weight inequality for the Riesz potential I_α on $\mathbb{R}^n$ were given by Rakotondratsimba [9]. In particular, the author showed that I_α is bounded from $L^p_{\mathrm{dec},r}(w,\mathbb{R}^n)$ to $L^q(v,\mathbb{R}^n)$ if the weighted Hardy operators $(\mathcal{H}f)(x) = \frac{1}{|x|^{n-\alpha}}\int_{|y|<|x|} f(y)\,dy$ and $(\mathcal{H}'f)(x) = \int_{|y|>|x|}\frac{f(y)}{|y|^{n-\alpha}}\,dy$ are bounded from $L^p(w,\mathbb{R}^n)$ to $L^q(v,\mathbb{R}^n)$. In fact, the author studied the problem on the cone of monotone decreasing functions.

Now we give some comments regarding the notation: in the sequel under the symbol $A \approx B$ we mean that there are positive constants c_1 and c_2 (depending on appropriate parameters) such that $c_1 A \le B \le c_2 A$; $A \ll B$ means that there is a positive constant c such that $A \le cB$; integral over a product set $E_1 \times E_2$ from g will be denoted by $\iint_{E_1\times E_2} g(x,y)\,dx\,dy$ or $\int_{E_1}\int_{E_2} g(x,y)\,dx\,dy$; for a weight functions w and w_i on G, by the symbols $W(t)$ and $W_i(t)$ will be denoted the integrals $\int_{B(e,t)} w(x)\,dx$ and $\int_{B(e_i,t)} w_i(x)\,dx$ respectively; for a weight w on $G_1 \times G_2$, we denote $W(t,\tau) := \int_{B(e_1,t)\times B(e_2,\tau)} w(x,y)\,dx\,dy$, where e_1 and e_1 are neutral elements in G_1 and G_2, respectively. Finally, we mention that constants (often different constants in one and the same line of inequalities) will be denoted by c or C. The symbol p' stands for the conjugate number of p: $p' = p/(p-1)$, where $1 < p < \infty$.

2 Preliminaries

We begin this section with the statements regarding polar coordinates in G (see *e.g.*, [1, p.14]).

Proposition A *Let G be a homogeneous group and let $S = \{x \in G : r(x) = 1\}$. There is a* (*unique*) *Radon measure σ on S such that for all $u \in L^1(G)$*,

$$\int_G u(x)\,dx = \int_0^\infty \int_S u(\delta_t \bar{y}) t^{Q-1}\,d\sigma(\bar{y})\,dt.$$

Let a be a positive number. The two-weight inequality for the Hardy-type transforms

$$\big(H^a f\big)(x) = \int_{B(e,ar(x))} f(y)\,dy, \quad x \in G, \qquad (\widetilde{H}^a f(x) = \int_{G\setminus B(e,ar(x))} f(y)\,dy, \quad x \in G,$$

reads as follows (see [10], Chapter 1 for more general case, in particular, for quasi-metric measure spaces):

Theorem A *Let $1 < p \le q < \infty$ and let a be a positive number. Then*

(i) *The operator H^a is bounded from $L^p(u_1,G)$ to $L^q(u_2,G)$ if and only if*

$$\sup_{t>0}\bigg(\int_{G\setminus B(e,t)} u_2(x)\,dx\bigg)^{1/q}\bigg(\int_{B(e,at)} u_1^{1-p'}(x)\,dx\bigg)^{1/p'} < \infty.$$

(ii) *The operator* $\widetilde{H}^a$ *is bounded from* $L^p(u_1, G)$ *to* $L^q(u_2, G)$ *if and only if*

$$\sup_{t>0}\left(\int_{B(e,t)} u_2(x)\,dx\right)^{1/q}\left(\int_{G\setminus B(e,at)} u_1^{1-p'}(x)\,dx\right)^{1/p'} < \infty.$$

We refer also to [11] for the Hardy inequality written for balls with center at the origin. In the sequel we denote H^1 by H.

The following statement for Euclidean spaces was derived by Barza, Johansson and Persson [12].

Proposition B *Let* w *be a weight function on* G *and let* $1 < p < \infty$. *If* $f \in \mathcal{DR}(G)$ *and* $g \geq 0$, *then*

$$\sup_{f\downarrow r} \frac{\int_G f(x)g(x)\,dx}{(\int_G f(x)^p w(x)\,dx)^{1/p}} \approx \|w\|_{L^1(G)}^{-1/p}\|g\|_{L^1(G)} + \left(\int_G H^{p'}\bigl(r(x)\bigr)W^{-p'}\bigl(r(x)\bigr)w(x)\,dx\right)^{1/p'},$$

where $H(t) = \int_{B(e,t)} g(x)\,dx$, $W(t) = \int_{B(e,t)} w(x)\,dx$.

The proof of Proposition B repeats the arguments (for $\mathbb{R}^n$) used in the proof of Theorem 3.1 of [12] taking Proposition A and the following lemma into account.

Lemma A *Let* $1 < p < \infty$. *For a weight function* w, *the inequality*

$$\int_G w(x)\left(\int_{G\setminus B(e,r(x))} f(y)\,dy\right)^p dx \leq p\int_G f^p(x)W^p\bigl(r(x)\bigr)w^{1-p}(x)\,dx, \quad f \geq 0,$$

holds.

Proof The proof of this lemma is based on Theorem A (part (ii)) taking $a = 1$, $p = q$, $u_2(x) = v(x)$, $u_1 = w^{1-p}(x)W^p(r(x))$ there. Details are omitted. □

Corollary A *Let the conditions of Proposition* B *be satisfied and let* $\int_G w(x)\,dx = \infty$. *Then the following relation holds*:

$$\sup_{f\downarrow r} \frac{\int_G f(x)g(x)\,dx}{(\int_G f^p(x)w(x)\,dx)^{1/p}} \approx \left(\int_G H^{p'}\bigl(r(x)\bigr)W\bigl(r(x)\bigr)w(x)\,dx\right)^{1/p}.$$

Corollary A implies the following duality result, which follows in the standard way (see [3, 12] for details).

Proposition C *Let* $1 < p, q < \infty$ *and let* v, w *be weight functions on* G *with* $\int_G w(x)\,dx = \infty$. *Then the integral operator* T *defined on functions on* G *is bounded from* $L^p_{\mathrm{dec},r}(w, G)$ *to* $L^q(v, G)$ *if and only if*

$$\left(\int_G\left(\int_{B(e,r(x))}\bigl(T^*g\bigr)(y)\,dy\right)^{p'}W^{-p'}\bigl(r(x)\bigr)w(x)\,dx\right)^{1/p'} \leq C\left(\int_G g^{q'}(x)v^{1-q'}(x)\,dx\right)^{1/q'} \tag{2.1}$$

holds for every positive measurable g *on* G.

The next statement yields the criteria for the two-weight boundedness of the operator H on the cone $\mathcal{DR}(G)$. In particular the following statement is true.

Theorem B *Let $1 < p \le q < \infty$ and let v and w be weights on G such that $\|w\|_{L^1(G)} = \infty$. Then H is bounded from $L^p_{\mathrm{dec},r}(w,G)$ to $L^q_v(v,G)$ if and only if*

(i)

$$\sup_{t>0}\left(\int_{B(e,t)} w(x)\,dx\right)^{-1/p}\left(\int_{B(e,t)} v(x)r^{Qq}(x)\,dx\right)^{1/q} < \infty;$$

(ii)

$$\sup_{t>0}\left(\int_{B(e,t)} r^{Qp'}(x)W^{-p'}\big(r(x)\big)w(x)\,dx\right)^{1/p'}\left(\int_{G\setminus B(e,t)} v(x)\,dx\right)^{1/q} < \infty.$$

Proof The proof of this statement follows by the standard way applying Proposition C (see *e.g.* [3, 12]). □

Definition 2.1 Let ρ be a locally integrable a.e. positive function on G. We say that ρ satisfies the doubling condition at e ($\rho \in DC(G)$) if there is a positive constant $b > 1$ such that for all $t > 0$ the following inequality holds:

$$\int_{B(e,2t)} \rho(x)\,dx \le b\int_{B(e,t)} \rho(x)\,dx.$$

Further, we say that $w \in DC^{\gamma,p}(G)$, where $1 < p < \infty$, $0 < \gamma < Q/p$, if there is a positive constant b such that for all $t > 0$

$$\int_{G\setminus B(e,t)} r^{\gamma p'}(x)W^{-p'}\big(r(x)\big)w(x)\,dx \le b\int_{G\setminus B(e,2t)} r^{\gamma p'}(x)W^{-p'}\big(r(x)\big)w(x)\,dx.$$

Remark 2.1 It can be checked that if a weight w satisfies the doubling condition et e in the strong sense, *i.e.*, $w \in DC(G)$ and $\int_{B(e,2t)} w(x)\,dx \le c\int_{B(e,2t)\setminus B(e,t)} w(x)\,dx$ with a constant c independent of t, then $w \in DC^{\gamma,p}(G)$.

Definition 2.2 We say that a locally integrable a.e. positive function ρ on $G_1 \times G_2$ satisfies the doubling condition with respect to the second variable ($\rho \in DC(y)$) uniformly to the first one if there is a positive constant c such that for all $t > 0$ and almost every $x \in G_1$ the following inequality holds:

$$\int_{B(e_2,2t)} \rho(x,y)\,dy \le c\int_{B(e_2,t)} \rho(x,y)\,dy.$$

Analogously is defined the class of weights $DC(x)$.

3 Riesz potentials on G

The main result of this section reads as follows.

Theorem 3.1 *Let $1 < p \le q < \infty$ and let v and w be weights such that either $w \in DC^{\alpha,p}(G)$ or $v \in DC(G)$; let $\|w\|_{L^1(G)} = \infty$. Then the operator I_α is bounded from $L^p_{\mathrm{dec},r}(w,G)$ to $L^q(v,G)$*

if and only if

(i)

$$\sup_{t>0}\left(\int_{B(e,t)} w(x)\,dx\right)^{-1/p}\left(\int_{B(e,t)} r^{\alpha q}(x)v(x)\,dx\right)^{1/q} < \infty; \tag{3.1}$$

(ii)

$$\sup_{t>0}\left(\int_{B(e,t)} r^{p'Q}(x)W^{-p'}\big(r(x)\big)w(x)\,dx\right)^{1/p'}\left(\int_{G\setminus B(e,t)} r^{(\alpha-Q)q}(x)v(x)\,dx\right)^{1/q} < \infty; \tag{3.2}$$

(iii)

$$\sup_{t>0}\left(\int_{B(e,t)} v(x)\,dx\right)^{1/q}\left(\int_{G\setminus B(e,t)} r^{\alpha p'}(x)W^{-p'}\big(r(x)\big)w(x)\,dx\right)^{1/p'} < \infty. \tag{3.3}$$

To prove this result we need to prove some auxiliary statements.

Lemma 3.1 *Let $0 < \alpha < Q$ and let c_0 be the constant from the triangle inequality of r. Then there is a positive constant c depending only on Q, α, and c_0 such that for all $s \in B(e, r(x)/2)$,*

$$I(x,y) := \int_{B(e,r(x))\setminus B(e,2c_0r(y))} r\big(ty^{-1}\big)^{\alpha-Q}\,dt \le cr\big(xy^{-1}\big)^{\alpha}. \tag{3.4}$$

Proof We have

$$\begin{aligned} I(x,y) &= \int_0^\infty \big|\big\{t \in G : r\big(ty^{-1}\big)^{\alpha-Q} > \lambda\big\} \cap B\big(e,r(x)\big) \setminus B\big(e,2c_0r(y)\big)\big|\,d\lambda \\ &= \int_0^{r(xy^{-1})^{\alpha-Q}} (\cdots) + \int_{r(xy^{-1})^{\alpha-Q}}^\infty (\cdots) =: I^{(1)}(x,y) + I^{(2)}(x,y). \end{aligned}$$

Observe that, by the triangle inequality for r, we have $r^Q(x) \le c_0^Q 2^{Q-1}(r^Q(xy^{-1}) + r^Q(y))$. This implies that $r^Q(x) - (2c_0)^Q r^Q(y) \le c_0^Q 2^{Q-1} r^Q(xy^{-1})$. Hence,

$$\begin{aligned} I^{(1)}(x,y) &\le r\big(xy^{-1}\big)^{\alpha-Q}\big|B\big(e,r(x)\big) \setminus B\big(e,2c_0r(y)\big)\big| \\ &= r\big(xy^{-1}\big)^{\alpha-Q}\big(r^Q(x) - (2c_0)^Q r^Q(y)\big) \le cr\big(xy^{-1}\big)^{\alpha}. \end{aligned}$$

Further, it is easy to see that

$$I^{(2)}(x,y) \le cr\big(xy^{-1}\big)^{\alpha}.$$

Finally we have (3.4). □

Let us introduce the following potential operators:

$$(J_\alpha f)(x) = \int_{B(e,2c_0r(x))} f(y)r^{\alpha-Q}\big(xy^{-1}\big)\,dy, \qquad (S_\alpha f)(x) = \int_{G\setminus B(e,2c_0r(x))} f(y)r^{\alpha-Q}\big(xy^{-1}\big)\,dy,$$

$$x \in G,\ 0 < \alpha < Q.$$

It is easy to see that

$$I_\alpha f = J_\alpha f + S_\alpha f. \tag{3.5}$$

We need also to introduce the following weighted Hardy operator:

$$(H_\alpha f)(x) = r(x)^{\alpha-Q}(Hf)(x).$$

Proposition 3.1 *The following relation holds for all $f \in \mathcal{DR}(G)$:*

$$J_\alpha f \approx H_\alpha f. \tag{3.6}$$

Proof We have

$$\begin{aligned}(J_\alpha f)(x) &= \int_{B(e,r(x)/2c_0)} f(y) r^{\alpha-Q}\bigl(xy^{-1}\bigr)\,dy + \int_{B(e,2c_0 r(x))\setminus B(e,r(x)/(2c_0))} f(y) r^{\alpha-Q}\bigl(xy^{-1}\bigr)\,dy \\ &=: \bigl(J_\alpha^{(1)} f\bigr)(x) + \bigl(J_\alpha^{(2)} f\bigr)(x).\end{aligned}$$

If $y \in B(e, r(x)/2c_0)$, then $r(x) \le c_0(r(xy^{-1}) + r(y)) \le c_0 r(xy^{-1}) + r(x)/2$. Hence $r(x) \le 2c_0 r(xy^{-1})$. Consequently,

$$\bigl(J_\alpha^{(1)} f\bigr)(x) \le c(H_\alpha f)(x).$$

Applying now the fact that $f \in DR(G)$ we see that

$$\begin{aligned}\bigl(J_\alpha^{(2)} f\bigr)(x) &\le f\bigl(r(x)/2c_0\bigr) \int_{B(e,r(x)/2c_0)\setminus B(e,2c_0 r(x))} r^{\alpha-Q}\bigl(xy^{-1}\bigr)\,dy \\ &\le cf\bigl(r(x)/2c_0\bigr) r(x)^\alpha \le c(H_\alpha f)(x).\end{aligned}$$

□

Lemma 3.2 *Let $1 < p \le q < \infty$ and let v and w be weights on G such that $\|w\|_{L^1(G)} = \infty$. Then the operator S_α is bounded from $L^p_{\mathrm{dec},r}(w, G)$ to $L^q(v, G)$ if*

$$\sup_{t>0} \biggl(\int_{G\setminus B(e,t)} r^{\alpha p'}(x) W^{-p'}\bigl(r(x)\bigr) w(x)\,dx\biggr)^{1/p'} \biggl(\int_{B(e,t/(2c_0))} v(x)\,dx\biggr)^{1/q} < \infty.$$

Conversely, if S_α is bounded from $L^p_{\mathrm{dec},r}(w, G)$ to $L^q(v, G)$, then the condition

$$\sup_{t>0} \biggl(\int_{G\setminus B(e,t)} r^{\alpha p'}(x) W^{-p'}(x) w(x)\,dx\biggr)^{1/q} \biggl(\int_{B(e,t/(4c_0))} v(x)\,dx\biggr)^{1/p'} < \infty$$

is satisfied. Furthermore, if either $w \in DC^{\alpha,p}(G)$ or $v \in DC(G)$, then the operator S_α is bounded from $L^p_{\mathrm{dec},r}(w, G)$ to $L^q(v, G)$ if and only if

$$\sup_{t>0} \biggl(\int_{G\setminus B(e,t)} r^{\alpha p'}(x) W^{-p'}\bigl(r(x)\bigr) w(x)\,dx\biggr)^{1/q} \biggl(\int_{B(e,t)} v(x)\,dx\biggr)^{1/p'} < \infty.$$

Proof Applying Proposition C, S_α is bounded from $L^p_{\mathrm{dec},r}(w,G)$ to $L^q(v,G)$ if and only if

$$\left(\int_G\left(\int_{B(e,r(x))}(S^*_\alpha f)(y)\,dy\right)^{p'}W^{-p'}\big(r(x)\big)w(x)\,dx\right)^{1/p'}\le c\left(\int_G g^{q'}(x)v^{1-q'}(x)\,dx\right)^{1/q'},$$

where

$$\big(S^*_\alpha f\big)(x)=\int_{B(e,r(x)/(2c_0))}f(y)r^{\alpha-Q}\big(xy^{-1}\big)\,dy.$$

Now we show that

$$\begin{aligned}c_1r^\alpha(x)\int_{B(e,r(x)/(4c_0))}g(s)\,ds&\le\int_{B(e,r(x))}\big(S^*_\alpha g\big)(y)\,dy\\&\le c_2r^\alpha(x)\int_{B(e,r(x)/(2c_0))}g(s)\,ds,\quad g\ge0.\end{aligned}\tag{3.7}$$

To prove the right-hand side estimate in (3.7) observe that by Tonelli's theorem and Lemma 3.1 we have

$$\begin{aligned}\int_{B(e,r(x))}\big(S^*_\alpha g\big)(y)\,dy&=\int_{B(e,r(x)/(2c_0))}f(s)\left(\int_{B(e,r(x))\setminus B(e,2c_0r(s))}r^{\alpha-Q}\big(sy^{-1}\big)\,dy\right)ds\\&\le c_2r(x)^\alpha\int_{B(e,r(x)/(2c_0))}f(s)\,ds.\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{B(e,r(x))}\big(S^*_\alpha g\big)(y)\,dy&\ge cr^{\alpha-Q}(x)\left(\int_{B(e,r(x))\setminus B(e,r(x)/2)}\left(\int_{B(e,r(y)/(2c_0))}f(s)\,ds\right)dy\right)\\&\ge c_1r^\alpha(x)\left(\int_{B(e,r(x)/(4c_0))}f(s)\,ds\right).\end{aligned}$$

Thus, Theorem A completes the proof. □

Proof of Theorem 3.1 By (3.5) it is enough to estimate the terms with $J_\alpha f$ and $S_\alpha f$. By applying Proposition 3.1 and Theorem B we find that J_α is bounded from $L^p_{\mathrm{dec},r}(w,G)$ to $L^q(v,G)$ if and only if the conditions (ii) and (iii) are satisfied. Now by Lemma 3.2 and the equality (which is a consequence of Proposition A)

$$\left(\int_{G\setminus B(e,t)}W\big(r(x)\big)w(x)\,dx\right)^{1/p'}=\left(\int_{B(e,t)}w(x)\,dx\right)^{-1/p}$$

we see that S_α is bounded from $L^p_{\mathrm{dec},r}(w,G)$ to $L^q(v,G)$ if and only if (i) is satisfied. □

4 Multiple potentials on $G_1\times G_2$

Let us now investigate the two-weight problem for the operator I_{α_1,α_2} on the cone $\mathcal{DR}(G_1\times G_2)$. In the sequel without loss of generality we denote the triangle inequality constants for G_1 and G_2 by one and the same symbol c_0.

The following statement can be derived just in the same way as Theorem 3.1 was obtained in [6]. The proof is omitted to avoid repeating those arguments.

Proposition D *Let $1 < p < \infty$ and let $w(x,y) = w_1(x)w_2(y)$ be a product weight on $G_1 \times G_2$. Then the relation*

$$\sup_{0 \le f \downarrow r} \frac{\iint_{G_1 \times G_2} f(x,y)g(x,y)\,dx\,dy}{(\iint_{G_1 \times G_2} f^p(x,y)w(x,y))^{1/p}} \approx \sum_{i=1}^{4} I_k$$

holds for a non-negative measurable function g, where

$$I_1 := \|w\|_{L^1(G_1 \times G_2)}^{-1/p} \|g\|_{L^1(G_1 \times G_2)},$$

$$I_2 := \|w_2\|_{L^1(G_1)}^{-1/p} \left(\int_{G_1} \left(\int_{B(e_1, r_1(x))} \|g(t,\cdot)\|_{L^1(G_2)}\,dt \right)^{p'} W_1^{-p'}\big(r_1(x)\big) w_1(x)\,dx \right)^{1/p'},$$

$$I_3 := \|w_1\|_{L^1(G_1)}^{-1/p} \left(\int_{G_2} \left(\int_{B(e_2, r_2(y))} \|g(\cdot,\tau)\|_{L^1(G_1)}\,d\tau \right)^{p'} W_2^{-p'}\big(r_2(y)\big) w_2(y)\,dy \right)^{1/p'},$$

$$I_4 := \left(\int_{G_1 \times G_2} \left(\int_{G_1 \times G_2} g(t,\tau)\,dt\,d\tau \right)^{p'} W^{-p'}\big(r_1(x), r_2(y)\big) w(x,y)\,dx\,dy \right)^{1/p'}.$$

Applying Proposition D together with the duality arguments we can get the following statement (*cf.* [6]).

Proposition E *Let $1 < p < \infty$ and let v and w be weights on $G_1 \times G_2$ such that $w(x,y) = w_1(x)w_2(y)$, $\|w\|_{L^1(G_1 \times G_2)} = \infty$. Then an integral operator T defined for functions from $\mathcal{DR}(G_1 \times G_2)$ is bounded from $L^p_{\mathrm{dec},r}(w, G_1 \times G_2)$ to $L^p(v, G_1 \times G_2)$ if and only if for all non-negative measurable g on $G_1 \times G_2$,*

$$\left(\iint_{G_1 \times G_2} \left(\iint_{B(e_1, r_1(x)) \times B(e_2, r_2(y))} \big(T^* g\big)(t,\tau)\,dt\,d\tau \right)^{p'} W^{-p'}(x,y) w(x,y)\,dx\,dy \right)^{1/p'}$$
$$\le C \left(\iint_{G_1 \times G_2} g^{q'}(x,y) v^{1-q'}(x,y)\,dx\,dy \right)^{1/q'}.$$

The next statements deal with the double Hardy-type operators defined on $G_1 \times G_2$:

$$\big(H^{a,b} f\big)(x,y) = \int_{B(e_1, ar_1(x))} \int_{B(e_2, br_2(x))} f(t,\tau)\,dt\,d\tau, \quad (x,y) \in G_1 \times G_2,$$

$$\big(\tilde{H}^{a,b} f\big)(x,y) = \int_{G_1 \setminus B(e_1, ar_1(x))} \int_{G_2 \setminus B(e_2, br_2(x))} f(t,\tau)\,dt\,d\tau, \quad (x,y) \in G_1 \times G_2,$$

$$\big(H_1^{a,b} f\big)(x,y) = \int_{B(e_1, ar_1(x))} \int_{G_2 \setminus B(e_2, br_2(y))} f(t,\tau)\,dt\,d\tau, \quad (x,y) \in G_1 \times G_2,$$

$$\big(H_2^{a,b} f\big)(x,y) = \int_{G_1 \setminus B(e_1, ar_1(x))} \int_{B(e_2, br_2(y))} f(t,\tau)\,dt\,d\tau, \quad (x,y) \in G_1 \times G_2.$$

Proposition 4.1 *Let $1 < p \le q < \infty$. Suppose that v and w be weights on $G_1 \times G_2$ such that either $w(x,y) = w_1(x)w_2(y)$ or $v(x,y) = v_1(x)v_2(y)$. Then:*

(i) *The operator* $H^{a,b}$ *is bounded from* $L^p(w, G_1 \times G_2)$ *to* $L^q(v, G_1 \times G_2)$ *if and only if*

$$A := \sup_{t>0,\tau>0} \left(\int_{G_1 \setminus B(e_1,t)} \int_{G_2 \setminus B(e_2,\tau)} v(x,y)\,dx\,dy \right)^{1/q} \times \left(\int_{B(e_1,at)} \int_{B(e_2,b\tau)} w^{1-p'}(x,y)\,dx\,dy \right)^{1/p'} < \infty.$$

(ii) *The operator* $\tilde{H}^{a,b}$ *is bounded from* $L^p(w, G_1 \times G_2)$ *to* $L^q(v, G_1 \times G_2)$ *if and only if*

$$\sup_{t>0,\tau>0} \left(\int_{B(e_1,t)} \int_{B(e_2,\tau)} v(x,y)\,dx\,dy \right)^{1/q} \left(\int_{G_1 \setminus B(e_1,at)} \int_{G_2 \setminus B(e_2,b\tau)} w^{1-p'}(x,y)\,dx\,dy \right)^{1/p'} < \infty.$$

(iii) *The operator* $H_1^{a,b}$ *is bounded from* $L^p(w, G_1 \times G_2)$ *to* $L^q(v, G_1 \times G_2)$ *if and only if*

$$\sup_{t>0,\tau>0} \left(\int_{G_1 \setminus B(e_1,t)} \int_{B(e_2,\tau)} v(x,y)\,dx\,dy \right)^{1/q} \left(\int_{B(e_1,at)} \int_{G_2 \setminus B(e_2,b\tau)} w^{1-p'}(x,y)\,dx\,dy \right)^{1/p'} < \infty.$$

(iv) *The operator* $H_2^{a,b}$ *is bounded from* $L^p(w, G_1 \times G_2)$ *to* $L^q(v, G_1 \times G_2)$ *if and only if*

$$\sup_{t>0,\tau>0} \left(\int_{B(e_1,t)} \int_{G_2 \setminus B(e_2,\tau)} v(x,y)\,dx\,dy \right)^{1/q} \left(\int_{G_1 \setminus B(e_1,at)} \int_{B(e_2,b\tau)} w^{1-p'}(x,y)\,dx\,dy \right)^{1/p'} < \infty.$$

Proof Let $w(x,y) = w_1(x)w_2(y)$. Then the proposition follows in the same way as the appropriate statements regarding the Hardy operators defined on $\mathbb{R}_+^2$ in [13, 14] (see also Theorem 1.1.6 of [15]). If v is a product weight, *i.e.* $v(x,y) = v_1(x)v_2(y)$, then the result follows from the duality arguments. We give the proof, for example, for $H^{a,b}$ in the case when $w(x,y) = w_1(x)w_2(y)$.

First suppose that $S := \int_{G_2} w_2^{1-p'}(y)\,dy = \infty$. Let $\{x_k\}_{k=-\infty}^{+\infty}$ be a sequence of positive numbers for which the equality

$$2^k = \int_{B(e_2,bx_k)} w_2^{1-p'}(y)\,dy \tag{4.1}$$

holds for all $k \in \mathbb{Z}$. This equality follows because of the continuity in t of the integral over the ball $B(e_2, bt)$. It is clear that $\{x_k\}$ is increasing and $\mathbb{R}_+ = \bigcup_{k\in\mathbb{Z}} [x_k, x_{k+1})$. Moreover, it is easy to verify that

$$2^k = \int_{B(e_2,bx_{k+1}) \setminus B(e_2,bx_k)} w_2^{1-p'}(y)\,dy.$$

Let $f \geq 0$. We have

$$\begin{aligned} \|H^{a,b}f\|^q_{L^q_v(G_1\times G_2)} &= \iint_{G_1\times G_2} v(x,y)\big(H^{a,b}f\big)^q(x,y)\,dx\,dy \leq \sum_{k\in\mathbb{Z}} \int_{G_1} \int_{B(e_2,x_{k+1})\setminus B(e_2,x_k)} v(x,y) \\ &\quad \times \left(\iint_{B(e_1,ar_1(x))\times B(e_2,br_2(x))} f(t,\tau)\,dt\,d\tau \right)^q dx\,dy \\ &\leq \sum_{k\in\mathbb{Z}} \int_{G_1} \left(\int_{B(e_2,x_{k+1})\setminus B(e_2,x_k)} v(x,y)\,dy \right) \end{aligned}$$

$$\times \left(\int_{B(e_1,ar_1(x))} \left(\int_{B(e_2,bx_{k+1})} f(t,\tau)\,d\tau \right) dt \right)^q dx$$
$$= \sum_{k\in\mathbb{Z}} \int_{G_1} V_k(x) \left(\int_{B(e_1,ar_1(x))} F_k(t)\,dt \right)^q dx,$$

where

$$V_k(x) := \int_{B(e_2,x_{k+1})\setminus B(e_2,x_k)} v(x,y)\,dy; \qquad F_k(t) := \int_{B(e_2,bx_{k+1})} f(t,\tau)\,d\tau.$$

It is obvious that

$$A^q \ge \sup_{\substack{a>0\\ j\in\mathbb{Z}}} \left(\int_{G_1\setminus B(e_1,t)} v_j(y)\,dy \right) \left(\iint_{B(e_1,at)\times B(e_2,bx_j)} w^{1-p'}(x,y)\,dx\,dy \right)^{q/p'}.$$

Hence, by Theorem A

$$\|H^{a,b} f\|^q_{L^q_v(G_1\times G_2)} \le cA^q \sum_{j\in\mathbb{Z}} \left[\int_{G_1} w_1(x) \left(\int_{B(e_2,bx_j)} w_2^{1-p'}(y)\,dy \right)^{1-p} \big(F_k(x)\big)^p dx \right]^{q/p}$$
$$\le cA^q \left[\int_{G_1} w_1(x) \sum_{j\in Z} \left(\int_{B(e_2,bx_j)} w_2^{1-p'}(y)\,dy \right)^{1-p} \right.$$
$$\left. \times \left(\sum_{k=-\infty}^{j} \int_{B(e_2,bx_{k+1})\setminus B(e_2,bx_k)} f(x,\tau)\,d\tau \right)^p dx \right]^{q/p}.$$

On the other hand, (4.1) yields

$$\sum_{k=n}^{+\infty} \left(\int_{B(e_2,bx_k)} w_2^{1-p'}(y)\,dy \right)^{1-p} \left(\sum_{k=-\infty}^{n} \int_{B(e_2,bx_{k+1})\setminus B(e_2,bx_k)} w_2^{1-p'}(y)\,dy \right)^{p-1}$$
$$= \sum_{k=n}^{+\infty} \left(\int_{B(e_2,bx_k)} w_2^{1-p'}(y)\,dy \right)^{1-p} \left(\int_{B(e_2,bx_{n+1})} w_2^{1-p'}(y)\,dy \right)^{p-1}$$
$$= \left(\sum_{k=n}^{+\infty} 2^{k(1-p)} \right) 2^{(n+1)(p-1)} \le c$$

for all $n \in \mathbb{Z}$. Hence by the discrete Hardy inequality (see *e.g.* [16]) and Hölder's inequality we have

$$\|H^{a,b} f\|^q_{L^q_v(G_1\times G_2)} \le cA^q \left[\int_{G_1} w_1(x) \sum_{j\in\mathbb{Z}} \left(\int_{B(e_2,bx_{j+1})\setminus B(e_2,bx_j)} w_2^{1-p'}(y)\,dy \right)^{1-p} \right.$$
$$\left. \times \left(\int_{B(e_2,bx_{j+1})\setminus B(e_2,bx_j)} f(x,\tau)\,d\tau \right)^p dx \right]^{q/p}$$
$$\le cA^q \left[\int_{G_1} w_1(x) \sum_{j\in\mathbb{Z}} \left(\int_{B(e_2,bx_{j+1})\setminus B(e_2,bx_j)} w_2(\tau) f^p(x,\tau)\,d\tau \right) dx \right]^{q/p}$$
$$= cA^q \|f\|^q_{L^p_w(G_1\times G_2)}.$$

If $S < \infty$, then without loss of generality we can assume that $S = 1$. In this case we choose the sequence $\{x_k\}_{k=-\infty}^{0}$ for which (4.1) holds for all $k \in \mathbb{Z}_-$. Arguing as in the case $S = \infty$ and using slight modification of the discrete Hardy inequality (see also [15], Chapter 1 for similar arguments), we finally obtain the desired result.

Finally we notice that the part (i) can also be proved if we first establish the boundedness of the operator $(\mathcal{H}^{a,b}\varphi)(t,\tau) = \int_0^{at}\int_0^{b\tau}\varphi(s,r)\,ds\,dr$ in the spirit of Theorem 1.1.6 in [15] and then pass to the case of $G_1 \times G_2$ by Proposition A. □

The next statement will be useful for us.

Proposition 4.2 *Let $1 < p \le q < \infty$. Assume that v and w are weights on $G_1 \times G_2$. Suppose that $w(x,y) = w_1(x)w_2(y)$ and that $W_i(\infty) = \infty$, $i = 1,2$. Then the operator $H^{1,1}$ is bounded from $L^p_{\mathrm{dec},r}(w, G_1 \times G_2)$ to $L^q(v, G_1 \times G_2)$ if and only if the following four conditions are satisfied*:

(i)

$$\sup_{a_1,a_2>0}\left(\int_{B(e_1,a_1)}\int_{B(e_2,a_2)} w(x,y)\,dx\,dy\right)^{-1/p} \times\left(\int_{B(e_1,a_1)}\int_{B(e_2,a_2)} r_1^{Q_1q}(x)r_2(y)^{Q_2q}v(x,y)\,dx\,dy\right)^{1/q} < \infty;$$

(ii)

$$\sup_{a_1,a_2>0}\left(\int_{B(e_1,a_1)}\int_{B(e_2,a_2)} r_1^{Q_1p'}(x)r_2(y)^{Q_2p'}W^{-p'}\big(r_1(x),r_2(y)\big)w(x,y)\,dx\,dy\right)^{1/p'} \times\left(\int_{G_1\setminus B(e_1,a_1)}\int_{G_2\setminus B(e_2,a_2)} v(x,y)\,dx\,dy\right)^{1/q} < \infty;$$

(iii)

$$\sup_{a_1,a_2>0}\left(\int_{B(e_1,a_1)} w_1\big(r_1(x)\big)\,dx\right)^{-1/p}\left(\int_{B(e_2,a_2)} r_2(y)^{Q_2p'}W_2^{-p'}\big(r_2(y)\big)w_2(y)\,dy\right)^{1/p'} \times\left(\int_{B(e_1,a_1)}\int_{G_2\setminus B(e_2,a_2)} r_1(x)^{Q_1q}v(x,y)\,dx\,dy\right)^{1/q} < \infty;$$

(iv)

$$\sup_{a_1,a_2>0}\left(\int_{B(e_1,a_1)} r_1(x)^{Q_1p'}W_1^{-p'}\big(r_1(x)\big)w_1(x)\,dt_1\right)^{1/p'}\left(\int_{B(e_2,a_2)} w_2(y)\,dy\right)^{-1/p} \times\left(\int_{G_1\setminus B(e_1,a_1)}\int_{B(e_2,a_2)} r_2(y)^{Q_2q}v(x,y)\,dx\,dy\right)^{1/q} < \infty.$$

Proof We follow the proof of Theorem 5.3 in [6]. First of all observe that by Proposition E, if w is a product weight, *i.e.*, $w(x_1,x_2) = w_1(x_1)w_2(x_2)$, such that $W_i(\infty) = \infty$, $i = 1,2$, and v is any weight on $G_1 \times G_2$, then $H^{1,1}$ is bounded from $L^p_{\mathrm{dec},r}(w, G_1)$ to $L^q(v, G_2)$ if and only if

$$\begin{aligned}&\biggl(\iint_{G_1\times G_2}\biggl(\int_{B(e_1,r_1(x))}\int_{B(e_2,r_2(x))}\biggl[\int_{G_1\backslash B(e_1,r_1(t))}\int_{G_1\backslash B(e_2,r_2(\tau))}g(s,\varepsilon)\,ds\,d\varepsilon\biggr]\,dt\,d\tau\biggr)^{p'}\\&\quad\times W^{-p'}\bigl(r_1(x),r_2(y)\bigr)w(x,y)\,dx\,dy\biggr)^{1/p'}\\&\quad\le c\biggl(\iint_{G_1\times G_2}g^{q'}(x,y)v^{1-q'}(x,y)\,dx\,dy\biggr)^{1/q'},\quad g\ge 0.\end{aligned}\tag{4.2}$$

Further, we have

$$\begin{aligned}&\iint_{B(e_1,r_1(x))\times B(e_2,r_2(x))}\biggl(\int_{G_1\backslash B(e_1,r_1(t))}\int_{G_2\backslash B(e_2,r_2(t))}g(s,\varepsilon)\,ds\,d\varepsilon\biggr)\,dt\,d\tau\\&\quad=\int_{B(e_1,r_1(x))}\int_{B(e_2,r_2(x))}r_1^{Q_1}(t)r_2^{Q_2}(\tau)g(t,\tau)\,dt\,d\tau\\&\qquad+r_1^{Q_1}(x)\int_{G_1\backslash B(e_1,r_1(x))}\int_{B(e_2,r_2(y))}r_2^{Q_2}(\tau)g(t,\tau)\,dt\,d\tau\\&\qquad+r_2^{Q_2}(y)\int_{B(e_1,r_1(x))}\int_{G_2\backslash B(e_2,r_2(y))}r_1^{Q_1}(t)g(t,\tau)\,dt\,d\tau\\&\qquad+r_1^{Q_1}(x)r_2^{Q_2}(y)\int_{G_1\backslash B(e_1,r_1(x))}\int_{G_2\backslash B(e_2,r_2(y))}g(t,\tau)\,dt\,d\tau\\&\quad=:I^{(1)}(x,y)+I^{(2)}(x,y)+I^{(3)}(x,y)+I^{(4)}(x,y).\end{aligned}$$

It is obvious that (4.2) holds if and only if

$$\begin{aligned}&\biggl(\iint_{G_1\times G_2}\bigl(I^{(j)}\bigr)^{p'}(x,y)W^{-p'}\bigl(r_1(x),r_2(y)\bigr)w(x,y)\,dx\,dy\biggr)^{1/p'}\\&\quad\le c\biggl(\iint_{G_1\times G_2}g^{q'}(x,y)v^{1-q'}(x,y)\,dx\,dy\biggr)^{1/q'}\end{aligned}\tag{4.3}$$

for $j=1,2,3,4$. By using Proposition 4.1 (part (i)) we find that

$$\begin{aligned}&\biggl(\iint_{G_1\times G_2}\bigl(I^{(1)}\bigr)^{p'}(x,y)W^{-p'}\bigl(r_1(x),r_2(y)\bigr)w(x,y)\,dx\,dy\biggr)^{1/p'}\\&\quad\le c\biggl(\iint_{G_1\times G_2}g^{q'}(x,y)v^{1-q'}(x,y)\,dx\,dy\biggr)^{1/q'}\end{aligned}$$

if and only if

$$\begin{aligned}&\biggl(\int_{G_1\backslash B(e_1,t)}\int_{G_2\backslash B(e_2,\tau)}W^{-p'}\bigl(r_1(x),r_2(y)\bigr)w(x,y)\,dx\,dy\biggr)^{1/p'}\\&\quad\times\biggl(\iint_{B(e_1,t)\times B(e_2,\tau)}\biggl(\frac{v^{1-q'}(x,y)}{r_1^{Q_1q'}(x)r_2^{Q_2q'}(y)}\biggr)^{1-q}\,dx\,dy\biggr)^{1/q}\\&\quad=c_p\biggl(\iint_{B(e_1,t)\times B(e_2,\tau)}w(x,y)\,dx\,dy\biggr)^{-1/p}\end{aligned}$$

$$\times \left(\iint_{B(e_1,t)\times B(e_2,\tau)} v(x,y) r_1^{Q_1 q}(x) r_2^{Q_2 q}(y)\, dx\, dy \right)^{1/q}$$
$$\le C.$$

In the latter equality we used the equality

$$\left(\int_{G_i \setminus B(e_i,t)} W_i^{-p'}\big(r_i(x)\big) w_i(x)\, dx \right)^{1/p'} = \left(\int_{B(e_i,t)} w_i(x)\, dx \right)^{-1/p}, \quad i = 1, 2,$$

which is a direct consequence of integration by parts and Proposition A. Taking now Proposition 4.1 (part (ii)) into account we find that (4.3) holds for $j = 4$ if and only if condition (ii) is satisfied, while Proposition 4.1 (parts (iii) and (iv)) and the following observation:

$$\sup_{a_1,a_2>0} \left(\int_{G_1\setminus B(e_1,a_1)} w_1(x) W_1^{-p'}\big(r_1(x)\big)\, dx \right)^{1/p'} \left(\int_{B(e_2,a_2)} r_2^{p' Q_2}(y) W_2^{-p'}\big(r_2(y)\big) w_2(y)\, dy \right)^{1/p'}$$
$$\times \left(\int_{B(e_1,a_1)} \int_{G_2\setminus B(e_2,a_2)} r_1^{Q_1 q}(x) v(x,y)\, dx\, dy \right)^{1/q}$$
$$= c_p \sup_{a_1,a_2>0} \left(\int_{B(e_1,a_1)} w_1(x)\, dx \right)^{-1/p} \left(\int_{B(e_2,a_2)} r_2^{Q_2 p'}(y) W_2^{-p'}\big(r_2(y)\big) w_2(y)\, dy \right)^{1/p'}$$
$$\times \left(\int_{B(e_1,a_1)} \int_{G_2\setminus B(e_2,a_2)} r_1^{Qq}(x) v(x,y)\, dx\, dy \right)^{1/q} < \infty;$$
$$\sup_{a_1,a_2>0} \left(\int_{B(e_1,a_1)} r_1^{Q_1 p'}(x) W_1^{-p'}\big(r_1(x)\big) w_1(x)\, dx \right)^{1/p'} \left(\int_{G_2\setminus B(e_2,a_2)} w_2(y) W_2^{-p'}\big(r_2(y)\big)\, dy \right)^{1/p'}$$
$$\times \left(\int_{G_1\setminus B(e_1,a_1)} \int_{B(e_2,a_2)} r_2^{Q_2 q}(y) v(x,y)\, dx\, dy \right)^{1/q}$$
$$= c_p \sup_{a_1,a_2>0} \left(\int_{B(e_1,a_1)} r_1^{Q_1 p'}(x) W_1^{-p'}\big(r_1(x)\big) w_1(x)\, dx \right)^{1/p'} \left(\int_{B(e_2,a_2)} w_2(t_2)\, dt_2 \right)^{-1/p}$$
$$\times \left(\int_{G_1\setminus B(e_1,a_1)} \int_{B(e_2,a_2)} r_2^{Q_2 q}(y) v(x,y)\, dx\, dy \right)^{1/q} < \infty$$

yield (4.3) for $j = 2, 3$. □

Let

$$(J_{\alpha_1,\alpha_2} f)(x,y) = \int_{B(e_1,2c_0 r_1(x))} \int_{B(e_2,2c_0 r_2(y))} f(t,\tau) r_1\big(xt^{-1}\big)^{\alpha_1 - Q_1} r_2\big(y\tau^{-1}\big)^{\alpha_2 - Q_2}\, dt\, d\tau,$$
$$(J_{\alpha_1} S_{\alpha_2} f)(x,y) = \int_{B(e_1,2c_0 r_1(x))} \int_{G_2\setminus B(e_2,2c_0 r_2(y))} f(t,\tau) r_1\big(xt^{-1}\big)^{\alpha_1 - Q_1} r_2\big(y\tau^{-1}\big)^{\alpha_2 - Q_2}\, dt\, d\tau,$$
$$(S_{\alpha_1} J_{\alpha_2} f)(x,y) = \int_{G_1\setminus B(e_1,2c_0 r_1(x))} \int_{B(e_2,2c_0 r_2(y))} f(t,\tau) r_1\big(xt^{-1}\big)^{\alpha_1 - Q_1} r_2\big(y\tau^{-1}\big)^{\alpha_2 - Q_2}\, dt\, d\tau,$$
$$(S_{\alpha_1,\alpha_2} f)(x,y) = \int_{G_1\setminus B(e_1,2c_0 r_1(x))} \int_{G_2\setminus B(e_2,2c_0 r_2(y))} f(t,\tau) r_1\big(xt^{-1}\big)^{\alpha_1 - Q_1} r_2\big(y\tau^{-1}\big)^{\alpha_2 - Q_2}\, dt\, d\tau,$$

where c_0 is the constant from the triangle inequality for the homogeneous norms r_1 and r_2.

It is obvious that

$$I_{\alpha_1,\alpha_2}f = J_{\alpha_1,\alpha_2}f + J_{\alpha_1}S_{\alpha_2}f + S_{\alpha_1}J_{\alpha_2}f + S_{\alpha_1,\alpha_2}f. \tag{4.4}$$

Now we formulate the main result of this section.

Theorem 4.1 *Let $1 < p \le q < \infty$. Assume that v and w are weights on $G_1 \times G_2$ such that $w(x,y) = w_1(x)w_2(y)$. Suppose that either $w_i \in DC^{\alpha_i,p}(G)$, $i = 1,2$, or $v \in DC(x) \cap DC(y)$. Then the operator I_{α_1,α_2} is bounded from $L^p_{\mathrm{dec},r}(w, G_1 \times G_2)$ to $L^q(v, G_1 \times G_2)$ if and only if the following conditions are satisfied*:

(i)

$$\begin{aligned} A_1 := \sup_{a_1,a_2>0} &\left(\int_{B(e_1,a_1)}\int_{B(e_2,a_2)} w(x,y)\,dx\,dy\right)^{-1/p} \\ &\times \left(\int_{B(e_1,a_1)}\int_{B(e_2,a_2)} \left(r_1^{\alpha_1}(x)r_2^{\alpha_2}(y)\right)^q v(x,y)\,dx\,dy\right)^{1/q} < \infty; \end{aligned}$$

(ii)

$$\begin{aligned} A_2 := \sup_{a_1,a_2>0} &\left(\int_{B(e_1,a_1)}\int_{B(e_2,a_2)} r_1^{Q_1p'}(x)r_2^{Q_2p'}(y)\,W^{-p'}\big(r_1(x),r_2(y)\big)w(x,y)\,dx\,dy\right)^{1/p'} \\ &\times \left(\int_{G_1\setminus B(e_1,a_1)}\int_{G_2\setminus B(e_2,a_2)} \left(r_1^{\alpha_1-Q_1}(x)r_2^{\alpha_2-Q_2}(y)\right)^q v(x,y)\,dx\,dy\right)^{1/q} < \infty; \end{aligned}$$

(iii)

$$\begin{aligned} A_3 := \sup_{a_1,a_2>0} &\left(\int_{B(e_1,a_1)} w_1(x)\,dx\right)^{-1/p}\left(\int_{B(e_2,a_2)} r_2^{Q_2p'}(y)\,W_2^{-p'}\big(r_2(y)\big)w_2(y)\,dy\right)^{1/p'} \\ &\times \left(\int_{B(e_1,a_1)}\int_{G_2\setminus B(e_2,a_2)} r_1^{\alpha_1 q}(x)r_2^{q(\alpha_2-Q_2)}(y)v(x,y)\,dx\,dy\right)^{1/q} < \infty; \end{aligned}$$

(iv)

$$\begin{aligned} A_4 := \sup_{a_1,a_2>0} &\left(\int_{B(e_1,a_1)} r_1^{Q_1p'}(x)\,W_1^{-p'}\big(r_1(x)\big)w_1(x)\,dx\right)^{1/p'}\left(\int_{B(e_2,a_2)} w_2(y)\,dy\right)^{-1/p} \\ &\times \left(\int_{G_1\setminus B(e_1,a_1)}\int_{B(e_2,a_2)} r_1^{q(\alpha_1-Q_1)}(x)r_2^{q\alpha_2}(y)v(x,y)\,dx\,dy\right)^{1/q} < \infty; \end{aligned}$$

(v)

$$\begin{aligned} A_5 := \sup_{a_1,a_2>0} &\left(\int_{G_1\setminus B(e_1,a_1)}\int_{G_2\setminus B(e_2,a_2)} r_1^{\alpha_1p'}(x)r_2^{\alpha_2p'}(y)\,W^{-p'}\big(r_1(x),r_2(y)\big)w(x,y)\,dx\,dy\right)^{1/p'} \\ &\times \left(\int_{B(e_1,a_1)}\int_{B(e_2,a_2)} v(x,y)\,dx\,dy\right)^{1/q} < \infty; \end{aligned}$$

(vi)

$$A_6 := \sup_{a_1,a_2>0} \left(\int_{B(e_1,a_1)} w_1(x)\,dx \right)^{-1/p} \left(\int_{G_2 \setminus B(e_2,a_2)} r_2^{\alpha_2 p'}(y) W_2^{-p'}\bigl(r_2(y)\bigr) w_2(y)\,dy \right)^{1/p'} \times \left(\int_{B(e_1,a_1)} \int_{B(e_2,a_2)} r_1^{\alpha_1 q}(x) v(x,y)\,dx\,dy \right)^{1/q} < \infty;$$

(vii)

$$A_7 := \sup_{a_1,a_2>0} \left(\int_{B(e_1,a_1)} r_1^{Q_1 p'}(x) W_1^{-p'}\bigl(r_1(x)\bigr) w_1(x)\,dx \right)^{1/p'} \times \left(\int_{G_2 \setminus B(e_2,a_2)} r_2^{\alpha_2 p'}(y) W_2^{-p'}\bigl(r_2(y)\bigr) w_2(y)\,dy \right)^{1/p'} \times \left(\int_{G_1 \setminus B(e_1,a_1)} \int_{B(e_2,a_2)} r_1^{(\alpha_1 - Q_1) q}(x) v(x,y)\,dx\,dy \right)^{1/q} < \infty;$$

(viii)

$$A_8 := \sup_{a_1,a_2>0} \left(\int_{G_2 \setminus B(e_1,a_1)} r_1^{\alpha_1 p'}(x) W_1^{-p'}\bigl(r_1(x)\bigr) w_1(x)\,dx \right)^{-1/p} \left(\int_{B(e_2,a_2)} w_2(y)\,dy \right)^{1/p'} \times \left(\int_{B(e_1,a_1)} \int_{B(e_2,a_2)} r_2^{\alpha_2 q}(x) v(x,y)\,dx\,dy \right)^{1/q} < \infty;$$

(ix)

$$A_9 := \sup_{a_1,a_2>0} \left(\int_{B(e_1,a_1)} r_2^{Q_2 p'}(y) W_2^{-p'}\bigl(r_2(y)\bigr) w_2(y)\,dy \right)^{1/p'} \times \left(\int_{G_1 \setminus B(e_1,a_1)} r_1^{\alpha_1 p'}(x) W_1^{-p'}\bigl(r_1(x)\bigr) w_1(x)\,dx \right)^{1/p'} \times \left(\int_{B(e_1,a_1)} \int_{G_2 \setminus B(e_2,a_2)} r_2^{(\alpha_2 - Q_2) q}(y) v(x,y)\,dx\,dy \right)^{1/q} < \infty.$$

Proof Let us assume that $v \in DC(x) \cap DC(y)$. The case when $w_i \in DC^{\alpha_i,p}(G_i)$, $i = 1, 2$, follows analogously. By using representation (4.4) we have to investigate the boundedness of the operators $J_{\alpha_1,\alpha_2} f$, $J_{\alpha_1} S_{\alpha_2} f$, $S_{\alpha_1} J_{\alpha_2} f$, $S_{\alpha_1,\alpha_2} f$ separately.

Since $f \in \mathcal{DR}(G_1 \times G_2)$ by using the arguments of the proof of Proposition 3.1 it can be checked that

$$(J_{\alpha_1,\alpha_2} f)(x,y) \approx r_1^{\alpha_1 - Q_1}(x) r_2^{\alpha_2 - Q_2}(y) \iint_{B(e_1,r_1(x)) \times B(e_2,r_2(y))} f(t,\tau)\,dt\,d\tau$$

(see also [4] for a similar estimate in the case of the multiple one-sided potentials on $\mathbb{R}^2_+$). Hence, by Proposition 4.2 we find that J_{α_1,α_2} is bounded from $L^p_{\mathrm{dec},r}(w, G_1 \times G_2)$ to $L^q(v, G_1 \times G_2)$ if and only if conditions (i)- (iv) hold.

Observe that the dual to S_{α_1,α_2} is given by

$$\big(S^*_{\alpha_1,\alpha_2}g\big)(x,y) = \iint_{B(e_1,r_1(x)/(2c_0))\times B(e_2,r_2(y)/(2c_0))} g(t,\tau) r_1^{\alpha_1-Q_1}\big(xt^{-1}\big) r_2^{\alpha_2-Q_2}\big(y\tau^{-1}\big)\,dt\,d\tau.$$

Further, Tonelli's theorem together with Lemma 3.1 for both variables implies that there are positive constants c_1 and c_2 such that for all $(x,y)\in G_1\times G_2$ for the dual (see also the proof of Lemma 3.2),

$$\begin{aligned} &r_1^{\alpha_1}(x)r_2^{\alpha_2}(y)\iint_{B(e_1,r_1(x)/(4c_0))\times B(e_2,r_2(y)/(4c_0))} g(t,\tau)\,dt\,d\tau \\ &\quad\le c_1\iint_{B(e_1,r_1(x))\times B(e_2,r_2(y))} \big(S^*_{\alpha_1,\alpha_2}g\big)(t,\tau)\,dt\,d\tau \\ &\quad\le c_2 r_1^{\alpha_1}(x)r_2^{\alpha_2}(y)\iint_{B(e_1,r_1(x)/(2c_0))\times B(e_2,r_2(y)/(2c_0))} g(t,\tau)\,dt\,d\tau. \end{aligned}$$

Applying Propositions 4.1 and 4.2 with the condition that $v\in DC(G_1\times G_2)$ we find that the operator S_{α_1,α_2} is bounded from $L^p_{\mathrm{dec},r}(w,G_1\times G_2)$ to $L^q(v,G_1\times G_2)$ if and only if condition (v) is satisfied.

Further, observe that due to the fact that f is radially decreasing with respect to the first variable we have

$$(J_{\alpha_1}S_{\alpha_2}f)(x,y)\approx(\mathcal{H}_{\alpha_1}S_{\alpha_2}f)(x,y),$$

where

$$(\mathcal{H}_{\alpha_1}S_{\alpha_2}f)(x,y) = r_1^{\alpha_1-Q_1}(x)\int_{B(e_1,2c_0r_1(x))}\int_{G_2\setminus B(e_2,2c_0r_2(y))} f(t,\tau)r_2\big(y\tau^{-1}\big)^{\alpha_2-Q_2}\,dt\,d\tau.$$

Dual of $\mathcal{H}_{\alpha_1}S_{\alpha_2}$ is given by

$$\big(\mathcal{H}^*_{\alpha_1}S^*_{\alpha_2}g\big)(t,\tau) = \int_{G_1\setminus B(e_1,r(t))}\int_{B(e_2,r(\tau)/2c_0)} r_1^{\alpha_1-Q_1}(s)r_2^{\alpha_2-Q_2}\big(\varepsilon\tau^{-1}\big)f(s,\varepsilon)\,ds\,d\varepsilon.$$

Further, we have

$$\begin{aligned} T(x,y) &:= \iint_{B(e_1,r_1(x))\times B(e_2,r_2(y))}\big(\mathcal{H}^*_{\alpha_1}S^*_{\alpha_2}g\big)(t,\tau)\,dt\,d\tau \\ &= \iint_{B(e_1,r_1(x))\times B(e_2,r_2(y))}\bigg(\int_{B(e_1,r_1(x))\setminus B(e_1,r(t))}\int_{B(e_2,r(\tau)/2c_0)} r_1^{\alpha_1-Q_1}(s) \\ &\qquad\times r_2^{\alpha_2-Q_2}\big(\tau\varepsilon^{-1}\big)f(s,\varepsilon)\,ds\,d\varepsilon\bigg)\,dt\,d\tau \\ &\qquad+\iint_{B(e_1,r_1(x))\times B(e_2,r_2(y))}\bigg(\int_{G_1\setminus B(e_1,r_1(x))}\int_{B(e_2,r(\tau)/(2c_0))} r_1^{\alpha_1-Q_1}(s) \\ &\qquad\times r_2^{\alpha_2-Q_2}\big(\tau\varepsilon^{-1}\big)f(s,\varepsilon)\,ds\,d\varepsilon\bigg)\,dt\,d\tau \\ &=: T_1(x,y)+T_2(x,y). \end{aligned}$$

Tonelli's theorem for G_1, the inequality $r_2^{\alpha_2-Q_2}(\tau\varepsilon^{-1}) \geq cr_2^{\alpha_2-Q_2}(y)$ for $\tau \in B(e_2, r(y))$, $\varepsilon \in B(e_2, r(\tau)/(2c_0))$, and the fact that the integral $\int_{B(e_1,\tau)} f(s,\varepsilon)\,ds$ is decreasing in τ uniformly to ε yield

$$\begin{aligned}
T_1(x,y) &\geq cr_2^{\alpha_2-Q_2}(y) \\
&\quad \times \int_{B(e_1,r_1(x))} \int_{B(e_2,r_2(y))\setminus B(e_2,r_2(y)/2)} \left(\int_{B(e_1,r_1(x))\setminus B(e_1,r(t))} \int_{B(e_2,r_2(y)/(4c_0))} r_1^{\alpha_1-Q_1}(s) \right. \\
&\quad \left. \times f(s,\varepsilon)\,ds\,d\varepsilon \right) dt\,d\tau \\
&= cr_2^{\alpha_2}(y) \int_{B(e_1,r_1(x))} \left(\int_{B(e_1,r_1(x))\setminus B(e_1,r(t))} r_1^{\alpha_1-Q_1}(s) \right. \\
&\quad \left. \times \left(\int_{B(e_2,r_2(y)/(4c_0))} f(s,\varepsilon)\,d\varepsilon \right) ds \right) dt \\
&= cr_2^{\alpha_2}(y) \int_{B(e_1,r_1(x))} \left(\int_{B(e_1,r_1(x))\setminus B(e_1,r(t))} F(s,y)\,ds \right) dt \\
&= cr_2^{\alpha_2}(y) \int_{B(e_1,r_1(x))} F(s,y) \left(\int_{B(e_1,r(s))} dt \right) ds \\
&= cr_2^{\alpha_2}(y) \int_{B(e_1,r_1(x))} \int_{B(e_2,r_2(y)/(4c_0))} r_1^{\alpha_1}(s) f(t,\tau)\,d\varepsilon\,ds.
\end{aligned}$$

Here we used the notation

$$F(s,y) := \int_{B(e_2,r_2(y)/(4c_0))} f(s,\varepsilon)\,d\varepsilon.$$

Taking into account that the function $\int_{B(e_2,2c_0\lambda)} f(s,\varepsilon)\,d\varepsilon$ is decreasing in λ uniformly to s, the inequality $r_2(\tau\varepsilon^{-1}) \leq cr_2(y)$ for $\tau \in B(e_2, r(y))$, $\varepsilon \in B(e_2, r(\tau)/(2c_0))$, and Tonelli's theorem for G_1 we find that

$$T_2(x,y) \geq cr_1^{Q_1}(x) r_2^{\alpha_2}(y) \int_{G_1\setminus B(e_1,r_1(x))} \int_{B(e_2,r_2(y)/(4c_0))} r_1^{\alpha_1-Q_1}(s) f(t,\tau)\,d\varepsilon\,ds.$$

To get the upper estimate, observe that Tonelli's theorem for $G_1 \times G_2$ and Lemma 3.1 for r_2 yield

$$\begin{aligned}
T_1(x,y) &\leq \int_{B(e_1,r_1(x))} \int_{B(e_2,r_2(y)/(2c_0))} r_1^{\alpha_1-Q_1}(s) f(s,\varepsilon) \\
&\quad \times \left(\int_{B(e_1,r_1(s))} \int_{B(e_2,r_2(y))\setminus B(e_2,2c_0 r_2(\varepsilon))} r_2^{\alpha_2-Q_2}\left(\tau\varepsilon^{-1}\right) dt\,d\tau \right) ds\,d\varepsilon \\
&\leq cr_2^{\alpha_2}(y) \iint_{B(e_1,r_1(x))\times B(e_2,r_2(y)/(2c_0))} r_1^{\alpha_1}(s) f(s,\varepsilon)\,ds\,d\varepsilon.
\end{aligned}$$

Similarly,

$$T_2(x,y) \leq cr_1^{Q_1}(x) r_2^{\alpha_2}(y) \iint_{G_1\setminus B(e_1,r_1(x))\times B(e_2,r_2(y)/(2c_0))} r_1^{\alpha_1-Q_1}(s) f(s,\varepsilon)\,ds\,d\varepsilon.$$

Summarizing these estimates we see that there are positive constants c_1 and c_2 depending only on α_1, α_2, Q_1, and Q_2 such that

$$\begin{aligned} & r_2^{\alpha_2}(y) \iint_{B(e_1,r_1(x))\times B(e_2,r_2(y)/(4c_0))} r_1^{\alpha_1}(s) f(s,\varepsilon)\,ds\,d\varepsilon \\ &\quad + r_1^{Q_1}(x) r_2^{\alpha_2}(y) \iint_{G_1\setminus B(e_1,r_1(x))\times B(e_2,r_1(y)/(4c_0))} r_1^{\alpha_1-Q_1}(s) f(s,\varepsilon)\,ds\,d\varepsilon. \\ &\leq c_1 T(x,y) \leq r_2^{\alpha_2}(y) \iint_{B(e_1,r_1(x))\times B(e_2,r2(y)/(2c_0))} r_1^{\alpha_1}(s) f(s,\varepsilon)\,ds\,d\varepsilon \\ &\quad + r_1^{Q_1}(x) r_2^{\alpha_2}(y) \iint_{G_1\setminus B(e_1,r_1(x))\times B(e_2,r_1(y)/(2c_0))} r_1^{\alpha_1-Q_1}(s) f(s,\varepsilon)\,ds\,d\varepsilon. \end{aligned}$$

Taking Propositions 4.1 and E into account together with the doubling condition for v with respect to the second variable we see that the operator $J_{\alpha_1}S_{\alpha_2}$ is bounded from $L^p_{\mathrm{dec},r}(w,G_1)$ to $L^q(v,G_2)$ if and only if the conditions (vi) and (vii) are satisfied.

In a similar manner (changing the roles of the first and second variables) we can see that $S_{\alpha_1}J_{\alpha_2}$ is bounded from $L^p_{\mathrm{dec},r}(w,G_1)$ to $L^q(v,G_2)$ if and only if the conditions (viii) and (ix) are satisfied. □

Theorem 4.1 and Remark 2.1 imply criteria for the trace inequality for I_{α_1,α_2}. Namely the following statement holds.

Theorem 4.2 *Let* $1 < p \leq q < \infty$ *and let* $0 < \alpha_i < Q_i/p$, $i = 1,2$. *Then* I_{α_1,α_2} *is bounded from* $L^p_{\mathrm{dec},r}(G_1\times G_2)$ *to* $L^q(v,G_1\times G_2)$ *if and only if the following condition holds*:

$$B := \sup_{a_1,a_2>0} \left(\int_{B(e_1,a_1)} \int_{B(e_2,a_2)} v(x,y)\,dx\,dy \right)^{1/q} a_1^{\alpha_1-Q_1/p} a_2^{\alpha_2-Q_2/p} < \infty.$$

Proof Sufficiency is a consequence of the inequality $\max\{A_1,\dots,A_9\} \leq cB$, while necessity follows immediately by taking the test function $f_{a_1,a_2}(x,y) = \chi_{B(e_1,a_1)}(x)\chi_{B(e_2,a_2)}(y)$, $a_1, a_2 > 0$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
GM and MS established Propositions B, C, D, E and checked the proofs of the statements throughout the paper. All authors read and approved the final manuscript.

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