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On the perturbation theory in spatially closed background

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Abstract In this article, we investigate some features of the perturbation theory in a spatially closed universe. We will show that the perturbative field equations in a spatially closed universe always have two independent adiabatic solutions provided that the wavelengths of perturbation modes are very much longer than the Hubble horizon. It will be revealed that these adiabatic solutions do not depend on the curvature directly. We also propose a new interpretation for the curvature perturbation in terms of the unperturbed background geometry.

1 Introduction

The theory of the linear perturbations is an important part of the modern cosmology which explains CMB anisotropies and the origin of structure formation. This theory has been investigated for a spatially flat universe to great extent [1–10]. However, observational data point out a universe with $\Omega_\Lambda \cong .68$ [11]. The existence of a positive cosmological constant necessitates a de Sitter spacetime for the vacuum background. From the different forms of the de Sitter spacetime with $K = 0, \pm 1$, merely $K = 1$ case, namely, a *Lorentzian de Sitter spacetime*, is maximally symmetric, maximally extended, and also geodesically complete [12]. So in the following we assume $\Lambda > 0$ and $K = 1$ for the vacuum background. Furthermore, it seems hard to believe that the total density of the universe has exactly been tuned in $\rho_{\mathrm{crit}0}$, because despite the fact that the observational data indicate $\Omega_K = 0$ [11], this fine-tuning seems somehow unlikely. Moreover, if Ω_{tot} equals $+1$ exactly, this cannot last forever because of the instability [13]. On the other hand, there are some reasons why the universe may have positive spatially curvature with non-trivial topology. In other words, some positive curvature models with non-trivial topology can solve the problem of the CMB quadrupole and octopole suppression and also the mystery of the missing fluctuations which appears in the concordance model of cosmology [14–18]. So these reasons augment the probability of a spatially closed case and it seems necessary to investigate the theory of small fluctuations in spatially closed universes.

The outline of this article is as follows. In Sect. 2 we derive the equations governing the linear perturbations in a FLRW universe without fixing K. In Sect. 3 we study the spectral and stochastic properties of these perturbations for the case $K = 1$ and in Sect. 4 the gauge problem will be discussed. Finally, in the last section we derive two independent adiabatic solutions for the obtained equations with $K = 1$, while the perturbations scales go outside of the Hubble horizon. It will be seen that one of these solutions is decaying, so it has no cosmological significance. We also deduce a new geometrical interpretation for the curvature perturbation as the *conformal factor* of the spatial section of the background spacetime. Furthermore, we will show that for the super-Hubble scales, curvature has no direct effect on the universe's evolution.

2 The perturbed spacetime

We assume that during most of the time the departures from homogeneity and isotropy have been very small, so that they can be treated as first order perturbations. The total perturbed metric is

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \tag{1}$$

where $\bar{g}_{\mu\nu}$ and $h_{\mu\nu}$ are the unperturbed metric and the first order perturbation, respectively. Note that $\bar{g}_{\mu\nu}$ is the FLRW metric which in the comoving quasi-Cartesian coordinates can be written as [2]

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$$g_{00}=-1,\quad g_{0i}=g_{i0}=0,$$
$$g_{ij}=a^2(t)\,\tilde{g}_{ij}=a^2(t)\left(\delta_{ij}+K\frac{x^ix^j}{1-K\mathbf{x}^2}\right).$$

A bar over any quantity denotes its unperturbed value. Perturbing the metric leads to perturbing the connection and Ricci tensor as [2]

$$\delta\Gamma^{\lambda}_{\mu\nu}=\frac{1}{2}\bar{g}^{\lambda\rho}(-2h_{\rho\eta}\bar{\Gamma}^{\eta}_{\mu\nu}+\partial_\mu h_{\nu\rho}+\partial_\nu h_{\mu\rho}-\partial_\rho h_{\mu\nu}), \tag{2}$$

and

$$\delta R_{\mu\nu}=\partial_\nu\delta\Gamma^{\lambda}_{\mu\lambda}-\partial_\lambda\delta\Gamma^{\lambda}_{\mu\nu}+\delta\Gamma^{\lambda}_{\mu\rho}\bar{\Gamma}^{\rho}_{\nu\lambda}+\delta\Gamma^{\lambda}_{\nu\rho}\bar{\Gamma}^{\rho}_{\mu\lambda}$$
$$-\delta\Gamma^{\lambda}_{\mu\nu}\bar{\Gamma}^{\rho}_{\rho\lambda}-\delta\Gamma^{\lambda}_{\lambda\rho}\bar{\Gamma}^{\rho}_{\mu\nu}. \tag{3}$$

The perturbative form of the Einstein field equations may be written as

$$\delta R_{\mu\nu}=-8\pi G\delta S_{\mu\nu}, \tag{4}$$

where

$$\delta S_{\mu\nu}=\delta T_{\mu\nu}-\frac{1}{2}\bar{g}_{\mu\nu}\delta T-\frac{1}{2}\bar{T}h_{\mu\nu}. \tag{5}$$

On the other hand, the perturbation of the energy-momentum conservation law gives

$$\partial_\mu\delta T^{\mu}{}_{\nu}-\delta\Gamma^{\lambda}_{\mu\nu}\bar{T}^{\mu}{}_{\lambda}-\bar{\Gamma}^{\lambda}_{\mu\nu}\delta T^{\mu}{}_{\lambda}+\bar{\Gamma}^{\mu}_{\mu\lambda}\delta T^{\lambda}{}_{\nu}+\delta\Gamma^{\mu}_{\mu\lambda}\bar{T}^{\lambda}{}_{\nu}=0. \tag{6}$$

Setting ν equal to 0 and i gives the equations of energy and momentum conservation, respectively. The explicit form of these equations is too lengthy and complicated, so we avoid expressing them here. Fortunately there is a mathematical technique, which simplifies these equations remarkably [3–5]. According to this technique we can decompose $h_{\mu\nu}$ into four scalars, two divergenceless, spatial vector and a symmetric, traceless, divergenceless spatial tensor as follows:

$$h_{00}=-E, \tag{7}$$
$$h_{i0}=a\left(\nabla_iF+G_i\right), \tag{8}$$
$$h_{ij}=a^2(A\tilde{g}_{ij}+H_{ij}B+\nabla_iC_j+\nabla_jC_i+D_{ij}), \tag{9}$$

where ∇_i is the covariant derivative with respect to the spatial unperturbed metric $\bar{g}_{ij}(=a^2\tilde{g}_{ij})$ and $H_{ij}=\nabla_i\nabla_j$ is the *covariant Hessian operator*. All the perturbations A, B, E, F, C_i, G_i and D_{ij} are functions of t and $\mathbf{x}$ which satisfy

$$\nabla^iC_i=\nabla^iG_i=0, \tag{10}$$
$$\tilde{g}^{ij}D_{ij}=0,\quad \nabla^iD_{ij}=0,\quad D_{ij}=D_{ji}. \tag{11}$$

Equation (8) is generalization of the Helmholtz decomposition theorem from $\mathbb{R}^3$ to Riemannian manifolds. Equation (9) is also a theorem in Riemannian geometry [19,20]. According to this theorem, every rank 2 symmetric tensor on a *compact* Riemannian manifold can be uniquely represented in form of Eq. (9). It is possible to carry out a similar decomposition of the energy-momentum tensor. One can show that [2]

$$\delta T_{00}=-\bar{\rho}h_{00}+\delta\rho,$$
$$\delta T_{i0}=\bar{p}h_{i0}-(\bar{\rho}+\bar{p})\delta u_i,$$
$$\delta T_{ij}=\bar{p}h_{ij}+a^2\tilde{g}_{ij}\delta p.$$

We can decompose the velocity perturbation δu_i into the gradient of a scalar (velocity potential) δu and a transverse vector δu_i^V,

$$\delta u_i=\nabla_i(\delta u)+\delta u_i^V,\quad \nabla^i\delta u_i^V=0. \tag{12}$$

We may account for the imperfectness of the cosmic fluid by adding a term Π_{ij} to δT_{ij}. Π_{ij} is known as the *anisotropic inertia* tensor field of the fluid and may be decomposed just like as h_{ij},

$$\Pi_{ij}=a^2(H_{ij}\Pi^S+\nabla_i\Pi_j^V+\nabla_j\Pi_i^V+\Pi_{ij}^T), \tag{13}$$

where Π_i^V and Π_{ij}^T satisfy conditions analogous to Eqs. (10) and (11), which are satisfied by C_i and D_{ij} in return. In Eq. (13) there is no term proportional to $\tilde{g}_{ij}$, because δT_{ij} itself contains such a term. Finally, we have

$$\delta T_{00}=-\bar{\rho}h_{00}+\delta\rho, \tag{14}$$
$$\delta T_{i0}=\bar{p}h_{i0}-(\bar{\rho}+\bar{p})(\nabla_i\delta u+\delta u_i^V), \tag{15}$$
$$\delta T_{ij}=\bar{p}h_{ij}+a^2(\tilde{g}_{ij}\delta p+H_{ij}\Pi^S+\nabla_i\Pi_j^V$$
$$+\nabla_j\Pi_i^V+\Pi_{ij}^T). \tag{16}$$

Now let us define the *Laplace–Beltrami operator*,

$$\nabla^2=\bar{g}^{ij}H_{ij}=\bar{g}^{ij}\nabla_i\nabla_j.$$

Thus, for scalar field $\mathcal{S}$ we have

$$a^2\nabla^2\mathcal{S}=\tilde{g}^{ij}\partial_i\partial_j\mathcal{S}-3K(\partial_i\mathcal{S})x^i. \tag{17}$$

Also for the vector field $\mathcal{V}_i$ and the tensor field $\mathcal{T}_{ij}$ we can write

$$a^2\nabla^2\mathcal{V}_i=\tilde{g}^{jk}\partial_j\partial_k\mathcal{V}_i-K\mathcal{V}_i-2K(\partial_i\mathcal{V}_j)x^j-3K\left(\partial_j\mathcal{V}_i\right)x^j, \tag{18}$$

$$a^2\nabla^2\mathcal{T}_{ij} = \tilde{g}^{kl}\partial_k\partial_l\mathcal{T}_{ij} - 2K\mathcal{T}_{ij} - 2K(\partial_i\mathcal{T}_{jk})x^k - 2K(\partial_j\mathcal{T}_{ik})x^k - 3K(\partial_k\mathcal{T}_{ij})x^k + 2K^2\tilde{g}_{ij}\mathcal{T}_{kl}x^kx^l. \quad (19)$$

Substituting Eqs. (7), (8), (9), (12), (14), (15), and (16) in the field and conservation equations namely Eqs. (4) and (6) and also separating the terms containing $\tilde{g}_{ij}$, ∇_i, and H_{ij}, accompanied by using Eqs. (17), (18), and (19) results in three independent sets of coupled equations.

2.1 Scalar mode equations

These equations involve just scalars:

$$2KA + \dot{a}a^2\nabla^2F - 3a\dot{a}\dot{A} - \frac{1}{2}\dot{a}a^3\nabla^2\dot{B} - \frac{1}{2}a^2\ddot{A} + (2\dot{a}^2 + a\ddot{a})E + \frac{1}{2}a\dot{a}\dot{E} + \frac{1}{2}a^2\nabla^2A = 4\pi Ga^2(-\delta\rho + \delta p + a^2\nabla^2\Pi^S), \quad (20)$$

$$4\dot{a}F - 3a\dot{a}\dot{B} + 2a\dot{F} - a^2\ddot{B} + E + A = -16\pi Ga^2\Pi^S, \quad (21)$$

$$a\dot{A} - \dot{a}E - Ka\dot{B} + 2KF = 8\pi Ga(\bar{\rho} + \bar{p})\delta u, \quad (22)$$

$$3\frac{\dot{a}}{a}\dot{A} + a\dot{a}\nabla^2\dot{B} + \frac{3}{2}\ddot{A} + \frac{1}{2}a^2\nabla^2\ddot{B} - \frac{3}{2}\frac{\dot{a}}{a}\dot{E} - \dot{a}\nabla^2F - a\nabla^2\dot{F} - \frac{1}{2}\nabla^2E - 3\frac{\ddot{a}}{a}E = -4\pi G(\delta\rho + 3\delta p + a^2\nabla^2\Pi^S), \quad (23)$$

$$\frac{\partial\delta\rho}{\partial t} + \nabla^2[-a(\bar{\rho} + \bar{p})F + (\bar{\rho} + \bar{p})\delta u + a\dot{a}\Pi^S] + \frac{1}{2}(\bar{\rho} + \bar{p})(3\dot{A} + a^2\nabla^2\dot{B}) + 3\frac{\dot{a}}{a}(\delta\rho + \delta p) = 0, \quad (24)$$

$$\dot{\bar{p}}\,\delta u + (\bar{\rho} + \bar{p})\frac{\partial\delta u}{\partial t} + \frac{1}{2}(\bar{\rho} + \bar{p})E + \delta p + a^2\nabla^2\Pi^S + 2K\Pi^S = 0. \quad (25)$$

2.2 Vector mode equations

We have

$$2\dot{a}G_i - 3a\dot{a}\dot{C}_i + a\dot{G}_i - a^2\ddot{C}_i = -16\pi Ga^2\Pi_i^V, \quad (26)$$

$$-\frac{1}{2}a^3\nabla^2\dot{C}_i + \frac{1}{2}a^2\nabla^2G_i - Ka\dot{C}_i + KG_i = 8\pi Ga\,(\bar{\rho} + \bar{p})\,\delta u_i^V, \quad (27)$$

$$\dot{\bar{p}}\,\delta u_i^V + (\bar{\rho} + \bar{p})\,\frac{\partial\delta u_i^V}{\partial t} + a^2\nabla^2\Pi_i^V + 2K\Pi_i^V = 0. \quad (28)$$

2.3 Tensor mode equation

We have

$$a^2\nabla^2D_{ij} - 3a\dot{a}\dot{D}_{ij} - a^2\ddot{D}_{ij} - 2KD_{ij} = -16\pi Ga^2\Pi_{ij}^T. \quad (29)$$

As previously mentioned, in linear perturbation theory, the scalar, vector, and tensor modes evolve independently. The vector and tensor modes are not important for structure formation because they produce no density perturbation, albeit they affect the CMB anisotropy.

3 Fourier decomposition and random fields

In this section, we study the spectral and stochastic properties of the perturbations for the case $K = +1$. Albeit the equations have been derived in Sect. 2 to describe the time evolution of the perturbative quantities, viewed as functions of position (at fixed time) they are considered as random fields on $S^3(a)$, because they are defined on a homogeneous and isotropic space [7,21]. Now we investigate the stochastic properties of perturbations for every mode separately.

3.1 Scalar perturbations and scalar random fields

An important class of random fields are described by their Fourier transformations. There are many different Fourier transform conventions; however, here our intention is the expansion of each mode of the perturbation fields in terms of the corresponding eigenfunctions of the Laplace–Beltrami operator. Thus, we have to find the eigenfunctions of ∇^2 on $S^3(a)$. For the scalar mode we have

$$\nabla^2\Phi = \Xi\Phi, \quad (30)$$

where $\nabla^2 = \bar{g}^{ij}H_{ij}$. In pseudo-spherical coordinates with the line element

$$\mathrm{d}s^2 = a^2(\mathrm{d}\chi^2 + \sin^2\chi\,\mathrm{d}\theta^2 + \sin^2\chi\sin^2\theta\,\mathrm{d}\varphi^2), \quad (31)$$

Eq. (30) gives

$$\frac{1}{a^2}\left(\frac{\partial^2\Phi}{\partial\chi^2} + \frac{1}{\sin^2\chi}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{1}{\sin^2\chi\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2} + 2\cot\chi\frac{\partial\Phi}{\partial\chi} + \frac{\cot\theta}{\sin^2\chi}\frac{\partial\Phi}{\partial\theta}\right) = \Xi\Phi. \quad (32)$$

Solving Eq. (32) one gets the following eigenvalues and eigenfunctions [22–25]:

$$\Xi = \Xi_n = \frac{1-n^2}{a^2}, \quad n = 1, 2, \ldots \quad (33)$$

$$\Phi = \mathcal{Y}_{nlm}(\chi, \theta, \varphi) = \Pi_{nl}(\chi)\,Y_{lm}(\theta, \varphi), \quad (34)$$

$$n = 1, 2, \ldots, \quad l \le n-1, \quad |m| \le l,$$

where

$$\Pi_{nl}(\chi) = \frac{(2l)!!}{\sqrt{a^3}} \sqrt{\frac{2}{\pi} \frac{n(n-l-1)!}{(n+l)!}} \sin^l \chi \, C^{l+1}_{n-l-1}(\cos \chi) \tag{35}$$

are known as *Fock harmonics* [22,25]. Also Y_{lm} and C^{λ}_n are scalar spherical harmonics on S^2 and Gegenbauer (ultraspherical) polynomials, respectively. It can be shown that

$$\int_{S^3(a)} \mathrm{d}\mu \mathcal{Y}_{nlm}(\chi, \theta, \varphi) \mathcal{Y}^*_{n'l'm'}(\chi, \theta, \varphi) = \delta_{nn'} \delta_{ll'} \delta_{mm'}, \tag{36}$$

where $\mathrm{d}\mu = a^3 \sin^2 \chi \sin \theta \mathrm{d}\chi \mathrm{d}\theta \mathrm{d}\varphi$ is the invariant volume element on $S^3(a)$. Scalar harmonics on $S^3(a)$ also can be expressed in terms of *Jacobi polynomials* or associated Legendre functions [26,27]. Furthermore the $\mathcal{Y}_{nlm}$ constitute a complete orthonormal set for the expansion of any scalar field on $S^3(a)$. Thus, for the scalar perturbative quantity $A(t, \mathbf{x})$ at some instant (which thereafter will be denoted by $A(\mathbf{x})$) we can write

$$A(\mathbf{x}) = \sum_{nlm} A_{nlm} \mathcal{Y}_{nlm}(\chi, \theta, \varphi). \tag{37}$$

A_{nlm} just like $A(\mathbf{x})$ is a scalar random field. Apart from the distribution function of A_{nlm}, its simplest statistics are described by the mean value and two-point covariance function, and the latter is defined by $\langle A_{nlm} A^*_{n'l'm'} \rangle$. Here $\langle \ \rangle$ means the ensemble average which equals the spatial average according to the ergodic theorem [7].
The homogeneity of $S^3(a)$ implies for any pair of scalar random fields A and B

$$\langle A(\mathbf{x}) B^*(\mathbf{x}') \rangle = \langle A(\mathbf{x} + \mathbf{R}) B^*(\mathbf{x}' + \mathbf{R}) \rangle. \tag{38}$$

($\mathbf{R}$ is an arbitrary 3-vector in $\mathbb{R}^3$.) Thus $\langle A(\mathbf{x}) B^*(\mathbf{x}') \rangle$ must be just a function of $\mathbf{x} - \mathbf{x}'$. This implies that

$$\langle A_{nlm} A^*_{n'l'm'} \rangle \propto \delta_{nn'} \delta_{ll'} \delta_{mm'}. \tag{39}$$

It means that A_{nlm} and $A_{n'l'm'}$ are uncorrelated random fields for different indices (indeed it results from the homogeneity of the universe). The homogeneity also implies that the coefficient of proportionality in Eq. (39) is just a function of n i.e.

$$\langle A_{nlm} A^*_{n'l'm'} \rangle = P^0_A(n) \delta_{nn'} \delta_{ll'} \delta_{mm'}. \tag{40}$$

$P^0_A(n)$ is a power spectrum or spectral density of A (the superscript “0” over P states corresponding to the spin of the random field) which depends on the distribution function governing A. Moreover, we have

$$\langle A_{nlm} B^*_{n'l'm'} \rangle = P^0_{A,B}(n) \delta_{nn'} \delta_{ll'} \delta_{mm'}, \tag{41}$$

in which $P^0_{A,B}(n)$ is a joint power (cross-correlation) spectrum of A and B [28,29]. One may define the correlation coefficient between A and B:

$$\Delta_{A,B}(n) = \frac{P^0_{A,B}(n)}{\sqrt{P^0_A(n) P^0_B(n)}}. \tag{42}$$

$-1 \le \Delta_{A,B}(n) \le 1$ and the two extreme values $\Delta_{A,B}(n) = +1$ and $\Delta_{A,B}(n) = -1$ correspond, respectively, to full correlation and full anti-correlation [29].

Finally, let us define the spectral index of the random field A as

$$\mathfrak{N}_A = 4 + \frac{n}{P^0_A(n)} \frac{d P^0_A(n)}{dn}. \tag{43}$$

Now we prove that the homogeneity of the universe yields Eq. (41). First, let us calculate $\langle A(\mathbf{x}) B^*(\mathbf{x}') \rangle$

$$\begin{aligned} &\langle A(\mathbf{x}) B^*(\mathbf{x}') \rangle \\ &= \sum_{nlm} \sum_{n'l'm'} \langle A_{nlm} B^*_{n'l'm'} \rangle \mathcal{Y}_{nlm}(\mathbf{x}) \mathcal{Y}^*_{n'l'm'}(\mathbf{x}') \\ &= \sum_{nlm} P^0_{A,B}(n) \mathcal{Y}_{nlm}(\mathbf{x}) \mathcal{Y}^*_{n'l'm'}(\mathbf{x}') \\ &= \sum_{nl} \frac{2l+1}{4\pi} P^0_{A,B}(n) \Pi_{nl}(\chi) \Pi_{nl}(\chi') P_l(\hat{\mathbf{x}}.\hat{\mathbf{x}}'). \end{aligned} \tag{44}$$

On the other hand, according to the addition formula of Gegenbauer polynomials (Fock harmonics) [30] we have

$$\frac{\sin n\gamma}{\sin \gamma} = \frac{\pi}{2} \frac{a^3}{n} \sum_{l=0}^{n-1} (2l+1) \Pi_{nl}(\chi) \Pi_{nl}(\chi') P_l(\hat{\mathbf{x}}.\hat{\mathbf{x}}'), \tag{45}$$

where $\cos \gamma = \cos \chi \cos \chi' + \sin \chi \sin \chi' \left(\hat{\mathbf{x}}.\hat{\mathbf{x}}'\right)$. Consequently

$$\langle A(\mathbf{x}) B^*(\mathbf{x}') \rangle = \frac{1}{2\pi^2 a^3} \frac{1}{\sin \gamma} \sum_{n=1}^{\infty} n P^0_{A,B}(n) \sin n\gamma, \tag{46}$$

which is obviously invariant under the following transformations:

$$\begin{cases} \varphi \to \varphi + \delta \\ \varphi' \to \varphi' + \delta \end{cases}, \quad \begin{cases} \varphi = \varphi' \\ \theta \to \theta + \delta \\ \theta' \to \theta' + \delta \end{cases}, \quad \begin{cases} \varphi = \varphi' \\ \theta = \theta' \\ \chi \to \chi + \delta \\ \chi' \to \chi' + \delta \end{cases} \tag{47}$$

Moreover, one can show that

$$\cos \gamma = 1 - \frac{1}{2}(\cos \chi - \cos \chi')^2 - \frac{1}{2} |\mathbf{x} - \mathbf{x}'|^2. \tag{48}$$

This shows for $\chi = \chi'$, that the $\cos\gamma$ is a function of $|\mathbf{x}-\mathbf{x}'|$, and thus we conclude that Eq. (41) depends merely on $|\mathbf{x}-\mathbf{x}'|$. Now let us turn to the vector mode.

3.2 Vector perturbations and vector random fields

In order to investigate the vector perturbation we should find the vector spherical harmonics on $S^3(a)$ first. They are solutions of the following equation:

$$\nabla^2 V_i = \Upsilon V_i, \quad \nabla^i V_i = 0 \tag{49}$$

The transversality condition is added as a constraint, because every vector perturbation in cosmology (C_i, G_i, Π_i^V) is divergenceless. It can be shown that the vector spectrum of $S^3(a)$ is [22–25]

$$\Upsilon = \Upsilon_n = \frac{2-n^2}{a^2}, \quad n = 2, 3, ..., \tag{50}$$

and there are two independent eigenfunctions, which in pseudo-spherical coordinates are

$$\begin{aligned}
(V_1^o)_{nlm} &= 0,\\
(V_2^o)_{nlm} &= -\frac{a}{\sqrt{l\,(l+1)}}\sin\chi\,\Pi_{nl}\,(\chi)\,\frac{1}{\sin\theta}\frac{\partial Y_{lm}}{\partial\varphi},\\
(V_3^o)_{nlm} &= \frac{a}{\sqrt{l\,(l+1)}}\sin\chi\,\Pi_{nl}\,(\chi)\sin\theta\frac{\partial Y_{lm}}{\partial\theta},
\end{aligned} \tag{51}$$

and the other

$$\begin{aligned}
(V_1^e)_{nlm} &= a\frac{\sqrt{l\,(l+1)}}{n}\frac{\Pi_{nl}\,(\chi)}{\sin\chi}Y_{lm}\,(\theta,\varphi),\\
(V_2^e)_{nlm} &= \frac{a}{n\sqrt{l\,(l+1)}}[(l+1)\cos\chi\,\Pi_{nl}\,(\chi)\\
&\quad -\sqrt{n^2-(l+1)^2}\sin\chi\,\Pi_{nl+1}\,(\chi)]\frac{\partial Y_{lm}}{\partial\theta},\\
(V_3^e)_{nlm} &= \frac{a}{n\sqrt{l\,(l+1)}}[(l+1)\cos\chi\,\Pi_{nl}\,(\chi)\\
&\quad -\sqrt{n^2-(l+1)^2}\sin\chi\,\Pi_{nl+1}\,(\chi)]\frac{\partial Y_{lm}}{\partial\varphi}.
\end{aligned} \tag{52}$$

One can show that

$$\int_{S^3(a)} \mathrm{d}\mu\bar{g}^{ij}(V_i^o)_{nlm}(V_j^o)^*_{n'l'm'} = \int_{S^3(a)} \mathrm{d}\mu\bar{g}^{ij}(V_i^e)_{nlm}(V_j^e)^*_{n'l'm'} = \delta_{nn'}\delta_{ll'}\delta_{mm'} \tag{53}$$

and

$$\bar{g}^{ij}(V_i^o)_{nlm}(V_j^e)_{n'l'm'} = 0. \tag{54}$$

These vector harmonics constitute a complete orthonormal set for the expansion of any transverse vector field on $S^3(a)$. Thus, for the vector perturbation $A_i(\mathbf{x})$ we can write

$$A_i\,(\mathbf{x}) = \sum_{nlm}[A^o_{nlm}(V_i^o)_{nlm} + A^e_{nlm}(V_i^e)_{nlm}], \tag{55}$$

where A^o_{nlm} and A^e_{nlm} are two random fields and like scalar perturbations we have

$$\begin{aligned}
\langle A^o_{nlm}A^{o*}_{n'l'm'}\rangle &= P^o_A\,(n)\,\delta_{nn'}\delta_{ll'}\delta_{mm'},\\
\langle A^e_{nlm}A^{e*}_{n'l'm'}\rangle &= P^e_A\,(n)\,\delta_{nn'}\delta_{ll'}\delta_{mm'},\\
\langle A^o_{nlm}A^{e*}_{n'l'm'}\rangle &= P^{oe}_A\,(n)\,\delta_{nn'}\delta_{ll'}\delta_{mm'}.
\end{aligned}$$

It yields

$$\begin{aligned}
\langle A_i(\mathbf{x})A_j^*(\mathbf{x})\rangle &= \sum_{nlm}[P^o_A(n)(V_i^o(\mathbf{x}))_{nlm}(V_i^{o*}(\mathbf{x}))_{nlm}\\
&\quad + P^e_A(n)(V_i^e(\mathbf{x}))_{nlm}(V_i^{e*}(\mathbf{x}))_{nlm}\\
&\quad + P^{oe}_A(n)(V_i^o(\mathbf{x}))_{nlm}(V_i^{e*}(\mathbf{x}))_{nlm}\\
&\quad + P^{oe}_A(n)(V_i^e(\mathbf{x}))_{nlm}(V_i^{o*}(\mathbf{x}))_{nlm}].
\end{aligned} \tag{56}$$

On the other hand, $\langle A_i(\mathbf{x})A_j^*(\mathbf{x})\rangle$ must not change under a parity transformation, because the probability distribution function is invariant under a spatial inversion, and we have $P^{oe}_A(n) = 0$. Furthermore,

$$P^o_A(n) = P^e_A(n) = P^{+1}_A(n). \tag{57}$$

Because the power spectrum just depends on the probability distribution function it cannot be a function of parity. Thus,

$$\langle A^o_{nlm}A^{o*}_{nlm}\rangle = \langle A^e_{nlm}A^{e*}_{nlm}\rangle = P^{+1}_A(n)\delta_{nn'}\delta_{ll'}\delta_{mm'} \tag{58}$$

$$\langle A^o_{nlm}A^{e*}_{nlm}\rangle = 0. \tag{59}$$

The last relation means that A^o_{nlm} and A^e_{nlm} are statistically independent random fields; however, they have the same spectrum.

3.3 Tensor perturbations and tensor random fields

Every symmetric, traceless, and transverse covariant tensor of rank 2 on $S^3(a)$ can be expanded in terms of t–t tensor spherical harmonics [22]. These harmonics can be classified into two groups.

Odd parity

$$
\begin{aligned}
(T^o_{11})_{nlm} &= 0,\\
(T^o_{22})_{nlm} &= -\frac{a^2}{\sqrt{2(n^2-1)l(l-1)(l+1)(l+2)}}\\
&\quad\times \sin\chi\,[(l+2)\cos\chi\,\Pi_{nl}(\chi)\\
&\quad-\sqrt{n^2-(l+1)^2}\sin\chi\,\Pi_{nl+1}(\chi)]\frac{X_{lm}(\theta,\varphi)}{\sin\theta},\\
(T^o_{33})_{nlm} &= \frac{a^2}{\sqrt{2\left(n^2-1\right)l(l-1)(l+1)(l+2)}}\\
&\quad\times \sin\chi\,[(l+2)\cos\chi\,\Pi_{nl}(\chi)\\
&\quad-\sqrt{n^2-(l+1)^2}\sin\chi\,\Pi_{nl+1}(\chi)]\\
&\quad\times \sin\theta\, X_{lm}(\theta,\varphi),\\
(T^o_{12})_{nlm} &= -a^2\sqrt{\frac{(l-1)(l+2)}{2\left(n^2-1\right)l(l+1)}}\\
&\quad\times \Pi_{nl}(\chi)\frac{1}{\sin\theta}\frac{\partial Y_{lm}}{\partial\varphi},\\
(T^o_{13})_{nlm} &= a^2\sqrt{\frac{(l-1)(l+2)}{2\left(n^2-1\right)l(l+1)}}\\
&\quad\times \Pi_{nl}(\chi)\sin\theta\frac{\partial Y_{lm}}{\partial\theta},\\
(T^o_{23})_{nlm} &= \frac{a^2}{\sqrt{2\left(n^2-1\right)l(l-1)(l+1)(l+2)}}\\
&\quad\times \sin\chi\,[(l+2)\cos\chi\,\Pi_{nl}(\chi)\\
&\quad-\sqrt{n^2-(l+1)^2}\sin\chi\,\Pi_{nl+1}(\chi)]\\
&\quad\times \sin\theta\, W_{lm}(\theta,\varphi),
\end{aligned}
\tag{60}
$$

where

$$
X_{lm}(\theta,\varphi) = 2\left(\frac{\partial^2 Y_{lm}}{\partial\theta\partial\varphi} - \cot\theta\frac{\partial Y_{lm}}{\partial\varphi}\right),
\tag{61}
$$

$$
W_{lm}(\theta,\varphi) = 2\frac{\partial^2 Y_{lm}}{\partial\theta^2} + l(l+1)Y_{lm}(\theta,\varphi).
\tag{62}
$$

Even parity

$$
\begin{aligned}
(T^e_{11})_{nlm} &= \frac{a^2}{n}\sqrt{\frac{l(l-1)(l+1)(l+2)}{2(n^2-1)}}\frac{\Pi_{nl}(\chi)}{\sin^2\chi}Y_{lm}(\theta,\varphi),\\
(T^e_{22})_{nlm} &= -\frac{a^2}{2n}\sqrt{\frac{l(l-1)(l+1)(l+2)}{2(n^2-1)}}\Pi_{nl}(\chi)Y_{lm}(\theta,\varphi)\\
&\quad+\frac{a^2}{n\sqrt{2(n^2-1)l(l-1)(l+1)(l+2)}}\\
&\quad\times G_{nl}(\chi)W_{lm}(\theta,\varphi),\\
(T^e_{33})_{nlm} &= -\frac{a^2}{2n}\sqrt{\frac{l(l-1)(l+1)(l+2)}{2(n^2-1)}}\\
&\quad\times \Pi_{nl}(\chi)\sin^2\theta\, Y_{lm}(\theta,\varphi)\\
&\quad-\frac{a^2}{n\sqrt{2(n^2-1)l(l-1)(l+1)(l+2)}}\\
&\quad\times G_{nl}(\chi)\sin^2\theta\, W_{lm}(\theta,\varphi),\\
(T^e_{12})_{nlm} &= \frac{a^2}{n}\sqrt{\frac{(l-1)(l+2)}{2(n^2-1)l(l+1)}}\\
&\quad\times [(l+1)\cot\chi\,\Pi_{nl}(\chi)\\
&\quad-\sqrt{n^2-(l+1)^2}\Pi_{nl+1}(\chi)]\frac{\partial Y_{lm}}{\partial\theta},\\
(T^e_{13})_{nlm} &= \frac{a^2}{n}\sqrt{\frac{(l-1)(l+2)}{2(n^2-1)l(l+1)}}\\
&\quad\times [(l+1)\cot\chi\,\Pi_{nl}(\chi)\\
&\quad-\sqrt{n^2-(l+1)^2}\Pi_{nl+1}(\chi)]\frac{\partial Y_{lm}}{\partial\varphi},\\
(T^e_{23})_{nlm} &= \frac{a^2}{n\sqrt{2(n^2-1)l(l-1)(l+1)(l+2)}}\\
&\quad\times G_{nl}(\chi)X_{lm}(\theta,\varphi),
\end{aligned}
\tag{63}
$$

where

$$
\begin{aligned}
G_{nl}(\chi) &= (l+2)\cos^2\chi\,\Pi_{nl}(\chi) - (n^2-1)\sin^2\chi\,\Pi_{nl}(\chi)\\
&\quad+\frac{1}{2}(l-1)(l+2)\Pi_{nl}(\chi) - \sqrt{n^2-(l+1)^2}\\
&\quad\times \sin\chi\cos\chi\,\Pi_{nl+1}(\chi).
\end{aligned}
\tag{64}
$$

It is also possible to express the tensor spherical harmonics in terms of the *Chebyshev polynomials of the first kind* [31]. It can be shown that

$$
\nabla^2(T^o_{ij})_{nlm} = \frac{3-n^2}{a^2}(T^o_{ij})_{nlm}, \quad n = 3, 4, \ldots
\tag{65}
$$

$$
\nabla^2(T^e_{ij})_{nlm} = \frac{3-n^2}{a^2}(T^e_{ij})_{nlm}, \quad n = 3, 4, \ldots
\tag{66}
$$

and also

$$
\begin{aligned}
&\int_{S^3(a)} d\mu\,\bar{g}^{ik}\bar{g}^{jl}(T^o_{ij})_{nlm}(T^o_{kl})^*_{n'l'm'}\\
&\quad= \int_{S^3(a)} d\mu\,\bar{g}^{ik}\bar{g}^{jl}(T^e_{ij})_{nlm}(T^e_{kl})^*_{n'l'm'} = \delta_{nn'}\delta_{ll'}\delta_{mm'}.
\end{aligned}
\tag{67}
$$

The set $\{(T^o_{ij})_{nlm}, (T^e_{ij})_{nlm}\}$ constitutes a complete orthonormal basis for the expansion of any symmetric traceless-divergence-free covariant tensor field of rank 2 on $S^3(a)$. On the other hand, the tensor mode is completely characterized by two traceless-transverse symmetric tensors $D_{ij}(t, \mathbf{x})$ and $\Pi^T_{ij}(t, \mathbf{x})$. We can expand them in terms of t–t tensor spherical harmonics on $S^3(a)$:

$$D_{ij}(\mathbf{x}) = \sum_{nlm} [D^o_{nlm}(T^o_{ij})_{nlm} + D^e_{nlm}(T^e_{ij})_{nlm}]. \tag{68}$$

There is a similar expansion for $\Pi^T_{ij}(t, \mathbf{x})$. (Note that we drop t here, because all quantities are considered at a fixed instant.) D^o_{nlm} and D^e_{nlm} just like $D_{ij}(t, \mathbf{x})$ are two random fields, so

$$\begin{aligned}\langle D^o_{nlm} D^{o*}_{n'l'm'}\rangle &= \langle D^e_{nlm} D^{e*}_{n'l'm'}\rangle \\ &= P^{+2}_D(n)\,\delta_{nn'}\delta_{ll'}\delta_{mm'},\end{aligned} \tag{69}$$

where $P^{+2}_D(n)$ is the power spectrum of the gravitational wave D_{ij} [32]. The probability distribution is independent of parity, so we cannot expect $\langle D^o_{nlm} D^{o*}_{n'l'm'}\rangle$ and $\langle D^e_{nlm} D^{e*}_{n'l'm'}\rangle$ to have different values. In addition, because the scalar, vector, and tensor modes are independent, their joint power spectra vanish.

4 The gauge problem

In this section, we investigate the behavior of the perturbations under the gauge transformations. The equations derived in Sect. 2 may have physically equivalent solutions. This problem is called gauge freedom. Similar to the Einstein field equations this gauge freedom may be fixed by choosing a coordinate system. For this purpose, let us consider a spacetime coordinate transformation,

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x), \tag{70}$$

with small $\epsilon^\mu(x)$ in the same sense as that $h_{\mu\nu}$ and the other perturbations are small. In cosmology, we call Eq. (70) a gauge transformation if it affects only the field perturbations and preserves the unperturbed metric [2,33]. Under such a gauge transformation, the metric of the spacetime changes as

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\lambda}{\partial x'^\nu} g_{\rho\lambda}(x), \tag{71}$$

equivalently

$$g_{\mu\nu}(x) = \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\lambda}{\partial x^\nu} g'_{\rho\lambda}(x+\epsilon). \tag{72}$$

It yields

$$\begin{aligned}\bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x) = (\delta^\rho{}_\mu + \partial_\mu\epsilon^\rho)(\delta^\lambda{}_\nu + \partial_\nu\epsilon^\lambda) \times \\ \times[\bar{g}_{\rho\lambda}(x+\epsilon) + h'_{\rho\lambda}(x)].\end{aligned} \tag{73}$$

After simplification we have

$$h'_{\mu\nu}(x) = h_{\mu\nu}(x) - \epsilon^\lambda(\partial_\lambda \bar{g}_{\mu\nu}) - \bar{g}_{\mu\lambda}(\partial_\nu\epsilon^\lambda) - \bar{g}_{\nu\lambda}(\partial_\mu\epsilon^\lambda). \tag{74}$$

Thus

$$\Delta h_{\mu\nu}(x) = h'_{\mu\nu}(x) - h_{\mu\nu}(x) = -\boldsymbol{\nabla}_\mu\epsilon_\nu - \boldsymbol{\nabla}_\nu\epsilon_\mu, \tag{75}$$

where $\boldsymbol{\nabla}_\mu$ is the covariant derivative corresponding to $\bar{g}_{\mu\nu}$. Consequently

$$\Delta h_{00} = -2\dot{\epsilon}_0, \tag{76}$$

$$\Delta h_{i0} = \Delta h_{0i} = -\dot{\epsilon}_i - \partial_i\epsilon_0 + 2\frac{\dot{a}}{a}\epsilon_i, \tag{77}$$

$$\Delta h_{ij} = -\nabla_i\epsilon_j - \nabla_j\epsilon_i + 2a\dot{a}\tilde{g}_{ij}\epsilon_0, \tag{78}$$

where ∇_i is the covariant derivative respect to $\tilde{g}_{ij}$. Similarly we can derive the effect of gauge transformation Eq. (70) on the energy-momentum tensor:

$$\Delta(\delta T_{\mu\nu}) = -\epsilon^\lambda(\partial_\lambda \bar{T}_{\mu\nu}) - \bar{T}_{\mu\lambda}(\partial_\nu\epsilon^\lambda) - \bar{T}_{\nu\lambda}(\partial_\mu\epsilon^\lambda), \tag{79}$$

or in more detail

$$\Delta(\delta T_{00}) = 2\bar{\rho}\dot{\epsilon}_0 + \dot{\bar{\rho}}\,\epsilon_0, \tag{80}$$

$$\Delta(\delta T_{i0}) = \Delta(\delta T_{0i}) = 2\bar{p}\frac{\dot{a}}{a}\epsilon_i - \bar{p}\dot{\epsilon}_i + \bar{\rho}\partial_i\epsilon_0, \tag{81}$$

$$\Delta(\delta T_{ij}) = -\bar{p}(\nabla_i\epsilon_j + \nabla_j\epsilon_i) + \frac{\mathrm{d}}{\mathrm{d}t}(a^2\bar{p})\tilde{g}_{ij}\epsilon_0. \tag{82}$$

In order to derive the gauge transformations of the scalar, vector, and tensor parts of $h_{\mu\nu}$ and $T_{\mu\nu}$, it is necessary to decompose the spatial part of ϵ^μ as follows:

$$\epsilon_i = \nabla_i\epsilon^S + \epsilon^V_i, \quad \nabla^i\epsilon^V_i = 0. \tag{83}$$

Now with substitution of Eq. (83) into Eqs. (76), (77), (78), (80), (81), and (82), we find

$$\begin{aligned}&\Delta A = 2\tfrac{\dot{a}}{a}\epsilon_0, \quad \Delta B = -\tfrac{2}{a^2}\epsilon^S, \\ &\Delta E = 2\dot{\epsilon}_0, \quad \Delta F = \tfrac{1}{a}\left(-\dot{\epsilon}^S - \epsilon_0 + 2\tfrac{\dot{a}}{a}\epsilon^S\right), \\ &\Delta C_i = -\tfrac{1}{a^2}\epsilon^V_i, \quad \Delta G_i = \tfrac{1}{a}\left(-\dot{\epsilon}^V_i + 2\tfrac{\dot{a}}{a}\epsilon^V_i\right), \\ &\Delta D_{ij} = 0, \quad \Delta\Pi^S = \Delta\Pi^V_i = \Delta\Pi^T_{ij} = 0, \\ &\Delta\delta u = -\epsilon_0, \quad \Delta\delta u^V_i = 0, \\ &\Delta\delta\rho = \dot{\bar{\rho}}\,\epsilon_0, \quad \Delta\delta p = \dot{\bar{p}}\,\epsilon_0.\end{aligned} \tag{84}$$

Obviously Π^S, Π_i^V, Π_{ij}^T, D_{ij} and δu_i^V are gauge invariant quantities. Besides, one can construct more gauge invariant quantities by combination of the perturbative quantities, e.g. $\zeta = \frac{A}{2} - H\frac{\delta\rho}{\dot{\bar{\rho}}}$ $(H = \frac{\dot{a}}{a})$, which is known as the *curvature perturbation on the uniform density slices* [34,35]. Note in particular that ζ is a pivotal quantity in cosmology, which is related to the fluctuations of inflation as well as the CMB angular power spectrum [32,36] and consequently connects the primordial perturbations to the present observational data. All of the tensor quantities are gauge invariant and as a result gauge fixing is not required. On the other hand, for the vector mode, we can fix a gauge by choosing ϵ_i^V so that either C_i or G_i vanishes. For the scalar perturbations, fixing a gauge means choosing ϵ_0 and ϵ^S, so there are several ways to fix a gauge [5], but here we concentrate on a special gauge which was introduced by Mukhanov et al. [37] and is known as the *Newtonian gauge*. In this gauge we choose ϵ_0 and ϵ^S by setting $B = F = 0$. It is convenient to write E and A in this gauge as

$$E = 2\Phi, \quad A = -2\Psi. \tag{85}$$

Φ and Ψ are known as *Bardeen's potentials* [34]. This gauge eliminates the gauge freedom completely, in contrast to the synchronous gauge [2,38], which was introduced by Lifshitz [3]. In the Newtonian gauge the line element of the universe takes the form

$$ds^2 = -(1+2\Phi)\,dt^2 + a^2\tilde{g}_{ij}(1-2\Psi)\,dx^i dx^j, \tag{86}$$

and the gravitational field and conservation equations become

$$-\frac{4}{a^2}\Psi + 6H\dot{\Psi} + \ddot{\Psi} + 2(3H^2+\dot{H})\Phi + H\dot{\Phi} - \nabla^2\Psi$$
$$= 4\pi G(-\delta\rho + \delta p + a^2\nabla^2\Pi^S), \tag{87}$$

$$\Psi - \Phi = 8\pi G a^2 \Pi^S, \tag{88}$$

$$\dot{\Psi} + H\Phi = -4\pi G(\bar{\rho} + \bar{p})\,\delta u, \tag{89}$$

$$3\ddot{\Psi} + 6H\dot{\Psi} + 3H\dot{\Phi} + \nabla^2\Phi + 6(H^2+\dot{H})\Phi$$
$$= 4\pi G(\delta\rho + 3\delta p + a^2\nabla^2\Pi^S), \tag{90}$$

$$3(\bar{\rho}+\bar{p})\,\dot{\Psi} = \frac{\partial\delta\rho}{\partial t} + 3H(\delta\rho + \delta p)$$
$$+ \nabla^2[(\bar{\rho}+\bar{p})\,\delta u + a^2 H\Pi^S] \tag{91}$$

$$(\bar{\rho}+\bar{p})\,\Phi = -\dot{\bar{p}}\,\delta u - (\bar{\rho}+\bar{p})\frac{\partial\delta u}{\partial t} - \delta p - a^2\nabla^2\Pi^S - 2\Pi^S. \tag{92}$$

In the next section we shall show that this system of equations has two independent adiabatic solutions.

5 Adiabatic modes in a spatially closed universe

In this section, we want to generalize the Weinberg theorem [2,39], which has been proved for a spatially flat universe to the spatially closed case. According to this theorem whatever the contents of the universe, the perturbative field equations have two independent *adiabatic* solutions in the time intervals when the perturbation scales are often very longer than the Hubble horizon of the universe. These two solutions in the Newtonian gauge are

$$\begin{cases} \Psi(t,\mathbf{x}) = \Phi(t,\mathbf{x}) = \zeta(\mathbf{x})\left[\frac{H}{a}\int\limits_{t_0}^{t} a(\tau)\,d\tau - 1\right], \\ \frac{\delta\rho(t,\mathbf{x})}{\dot{\bar{\rho}}} = \frac{\delta p(t,\mathbf{x})}{\dot{\bar{p}}} = -\delta u(t,\mathbf{x}) = -\frac{\zeta(\mathbf{x})}{a}\int\limits_{t_0}^{t} a(\tau)\,d\tau, \\ \Pi^S(t,\mathbf{x}) = 0, \end{cases} \tag{93}$$

and

$$\begin{cases} \Psi(t,\mathbf{x}) = \Phi(t,\mathbf{x}) = \chi(\mathbf{x})\,\frac{H}{a}, \\ \frac{\delta\rho(t,\mathbf{x})}{\dot{\bar{\rho}}} = \frac{\delta p(t,\mathbf{x})}{\dot{\bar{p}}} = -\delta u(t,\mathbf{x}) = -\frac{\chi(\mathbf{x})}{a}, \\ \Pi^S(t,\mathbf{x}) = 0, \end{cases} \tag{94}$$

in which $\zeta(\mathbf{x})$ is the curvature perturbation on the uniform density slices when the perturbations are outside of the Hubble horizon or equivalently the *conformal factor* of S^3 and $\chi(\mathbf{x})$ is an arbitrary function of position.
In order to prove this, initially we put $\Pi^S = 0$, because the cosmic fluid is approximately perfect; thus, from Eq. (88) we have

$$\Psi = \Phi, \tag{95}$$

Now take the gauge transformation

$$x^\mu \to x^\mu + \epsilon^\mu(x), \tag{96}$$

which converts the present Newtonian gauge to another Newtonian gauge. Consequently

$$\Delta h_{00} = -2\dot{\epsilon}_0 \Rightarrow \Delta\Phi = \dot{\epsilon}_0, \tag{97}$$

$$\Delta h_{i0} = 0 \Rightarrow -\dot{\epsilon}_i - \partial_i\epsilon_0 + 2\frac{\dot{a}}{a}\epsilon_i = 0, \tag{98}$$

$$\Delta h_{ij} = -2a^2\tilde{g}_{ij}\Delta\Psi \Rightarrow -\nabla_i\epsilon_j - \nabla_j\epsilon_i + 2a\dot{a}\tilde{g}_{ij}\epsilon_0$$
$$= -2a^2\tilde{g}_{ij}\Delta\Psi. \tag{99}$$

Equation (98) results in

$$\epsilon_i(t,\mathbf{x}) = -a^2\int\limits_{t_0}^{t}\frac{\partial_i\epsilon_0(\tau,\mathbf{x})}{a^2(\tau)}d\tau + a^2\eta_i(\mathbf{x}), \tag{100}$$

in which t_0 and $\eta_i(\mathbf{x})$, respectively, are arbitrary time and an arbitrary 3-vector field on S^3. Substituting Eq. (100) into Eq. (99) yields

$$2\int_{t_0}^{t}\frac{H_{ij}\epsilon_0(\tau,\mathbf{x})}{a^2(\tau)}\mathrm{d}\tau-(\nabla_i\eta_j+\nabla_j\eta_i)+2H\tilde{g}_{ij}\epsilon_0(t,\mathbf{x})=-2\tilde{g}_{ij}\Delta\Psi. \tag{101}$$

Now suppose that $\eta_i(\mathbf{x})$ is a *conformal Killing vector* of S^3

$$\nabla_i\eta_j+\nabla_j\eta_i=2\gamma(\mathbf{x})\,\tilde{g}_{ij}, \tag{102}$$

where $\gamma(\mathbf{x})=\frac{1}{3}\nabla_i\eta^i$ is a function on S^3, the so-called *conformal factor* of S^3 [40]. Note that S^3 has no *homothetic Killing vector* [40,41], but due to its conformal symmetry, it has a conformal Killing vector. Indeed in [42] it has been proved that S^3 has a four-gradient conformal Killing vector. For instance, $\eta_i={\delta^m}_i(m=1,2,3)$ is a conformal Killing vector of S^3 with conformal factor $-x^m$:

$$\nabla_i{\delta^m}_j+\nabla_j{\delta^m}_i=-2x^m\tilde{g}_{ij}.$$

On the other hand, on the super-Hubble scales we can ignore the first term on the left side of Eq. (101), because $2\int_{t_0}^{t}\frac{H_{ij}\epsilon_0(\tau,\mathbf{x})}{a^2(\tau)}\mathrm{d}\tau$ is of the order of $2\int_{t_0}^{t}\nabla^2\epsilon_0(\tau,\mathbf{x})\,\mathrm{d}\tau$, so its Fourier transform has same order of $2\int_{t_0}^{t}\frac{1-n^2}{a^2(\tau)}\epsilon_{0_{nlm}}(\tau)\,\mathrm{d}\tau$, which is negligible for super-Hubble scales. Thus Eq. (101), on the time intervals when the perturbation scales are very longer than the Hubble horizon, turns to

$$-(\nabla_i\eta_j+\nabla_j\eta_i)+2H\tilde{g}_{ij}\epsilon_0(t,\mathbf{x})=-2\tilde{g}_{ij}\Delta\Psi,$$

or

$$\Delta\Psi=\gamma(\mathbf{x})-H\epsilon_0(t,\mathbf{x}). \tag{103}$$

Besides, in the Newtonian gauge both Ψ and $\Psi+\Delta\Psi$ are solutions, so that it results from the linearity of equations that $\Delta\Psi$ is another solution of the Newtonian field equations too. It is also true for other perturbations. Consequently, we have a set of solutions of the Newtonian gauge field equations:

$$\Psi=\gamma(\mathbf{x})-H\epsilon_0(t,\mathbf{x}), \tag{104}$$
$$\Phi=\dot{\epsilon}_0(t,\mathbf{x}), \tag{105}$$
$$\delta\rho=\dot{\bar{\rho}}\,\epsilon_0(t,\mathbf{x}), \tag{106}$$
$$\delta p=\dot{\bar{p}}\,\epsilon_0(t,\mathbf{x}), \tag{107}$$
$$\delta u=-\epsilon_0(t,\mathbf{x}). \tag{108}$$

Furthermore,

$$\zeta=-\Psi-H\frac{\delta\rho}{\dot{\bar{\rho}}}=-\gamma(\mathbf{x}). \tag{109}$$

It can be concluded from Eq. (109) that ζ is conserved i.e. it does not depend on the time, so that the above solutions are appropriate for a period when the perturbations are outside of the Hubble horizon. In order to see the conservation of ζ on the super-Hubbles scales, it is sufficient to write the Fourier transformation of Eq. (24),

$$\frac{\partial\delta\rho_n}{\partial t}+\frac{1-n^2}{a^2}[-a(\bar{\rho}+\bar{p})F_n+(\bar{\rho}+\bar{p})\delta u_n+a\dot{a}\Pi_n^S]+\frac{3}{2}(\bar{\rho}+\bar{p})\dot{A}_n+\frac{1}{2}(\bar{\rho}+\bar{p})(1-n^2)\dot{B}_n+3\frac{\dot{a}}{a}(\delta\rho_n+\delta p_n)=0; \tag{110}$$

for simplicity we drop l and m indices. On the super-Hubble scales ($n<<aH$) we can approximate this equation as follows:

$$\frac{\partial\delta\rho_n}{\partial t}+\frac{3}{2}(\bar{\rho}+\bar{p})\dot{A}_n+3\frac{\dot{a}}{a}(\delta\rho_n+\delta p_n)=0. \tag{111}$$

On the other hand, we have

$$A_n=2\zeta_n-\frac{2}{3}\frac{\delta\rho_n}{\bar{\rho}+\bar{p}}. \tag{112}$$

By substituting Eq. (112) in Eq. (111) and using the conservation law of energy in an unperturbed universe we can write

$$\dot{\zeta}_n=\frac{\dot{\bar{p}}\,\delta\rho_n-\dot{\bar{\rho}}\,\delta p_n}{3(\bar{\rho}+\bar{p})^2}. \tag{113}$$

Thus for adiabatic perturbations for which $\frac{\delta\rho_n}{\dot{\bar{\rho}}}=\frac{\delta p_n}{\dot{\bar{p}}}$, we have

$$\dot{\zeta}_n=0. \tag{114}$$

Consequently, if the perturbations are adiabatic[1], ζ is conserved of course in the epoch when the wavelength of most perturbations are very much longer than the Hubble radius. Indeed, the conservation of ζ is a general theorem in cosmology which has been proved even for a nonlinear generalization of ζ [43]. Note that ignoring the first term of the left hand side of Eq. (101) causes ζ to be independent of time, which is equivalent to going outside of the Hubble horizon. From the combination of Eqs. (95), (104), (105), and also Eq. (109) we may write

$$\dot{\epsilon}_0(t,\mathbf{x})+H\epsilon_0(t,\mathbf{x})=-\zeta(\mathbf{x}). \tag{115}$$

Equation (115) is a first order differential equation for $\epsilon_0(t,\mathbf{x})$ and we solve it in two different cases: First we assume $\zeta(\mathbf{x})\neq 0$, and consequently, Eq. (115) results in

[1] Strictly speaking, the adiabatic condition is $\frac{\delta\rho_\alpha}{\dot{\bar{\rho}}_\alpha}=\frac{\delta\rho_\beta}{\dot{\bar{\rho}}_\beta}$ where α and β stand for every two different species of cosmic fluid elements whereas the condition $\frac{\delta\rho_n}{\dot{\bar{\rho}}}=\frac{\delta p_n}{\dot{\bar{p}}}$ is known as the *generalized* adiabatic condition.

$$\epsilon_0(t,\mathbf{x}) = -\frac{\zeta(\mathbf{x})}{a}\int_{t_0}^{t} a(\tau)\,\mathrm{d}\tau. \tag{116}$$

By inserting Eq. (116) in Eqs. (104)–(108) we have

$$\Psi(t,\mathbf{x}) = \Phi(t,\mathbf{x}) = \zeta(\mathbf{x})\left[\frac{H}{a}\int_{t_0}^{t} a(\tau)\,\mathrm{d}\tau - 1\right], \tag{117}$$

$$\delta\rho(t,\mathbf{x}) = -\zeta(\mathbf{x})\frac{\dot{\bar{\rho}}}{a}\int_{t_0}^{t} a(\tau)\,\mathrm{d}\tau, \tag{118}$$

$$\delta p(t,\mathbf{x}) = -\zeta(\mathbf{x})\frac{\dot{\bar{p}}}{a}\int_{t_0}^{t} a(\tau)\,\mathrm{d}\tau, \tag{119}$$

$$\delta u(t,\mathbf{x}) = \frac{\zeta(\mathbf{x})}{a}\int_{t_0}^{t} a(\tau)\,\mathrm{d}\tau. \tag{120}$$

On the other hand, if we take $\zeta = 0$ Eq. (115) gives

$$\epsilon_0(t,\mathbf{x}) = -\frac{\chi(\mathbf{x})}{a}, \tag{121}$$

where $\chi(\mathbf{x})$ is an arbitrary function on the $S^3(a)$. Note that in this case η_i is a *Killing vector* of S^3. By substituting Eq. (121) in Eqs. (104)–(108) we derive the second set of solutions as follows:

$$\Psi(t,\mathbf{x}) = \Phi(t,\mathbf{x}) = \chi(\mathbf{x})\frac{H}{a}, \tag{122}$$

$$\delta\rho(t,\mathbf{x}) = -\chi(\mathbf{x})\frac{\dot{\bar{\rho}}}{a}, \tag{123}$$

$$\delta p(t,\mathbf{x}) = -\chi(\mathbf{x})\frac{\dot{\bar{p}}}{a}, \tag{124}$$

$$\delta u(t,\mathbf{x}) = \frac{\chi(\mathbf{x})}{a}. \tag{125}$$

Unlike the first solution, this solution is a decaying mode, so it can be neglected at late times and its existence is significant just for counting of adiabatic solutions. In both solutions $\frac{\delta\rho(t,\mathbf{x})}{\dot{\bar{\rho}}} = \frac{\delta p(t,\mathbf{x})}{\dot{\bar{p}}}$, which means they are adiabatic solutions.

In general, S^3 has four independent gradient conformal Killing vectors and six independent Killing vectors, however, we have totally two independent solutions for perturbations equations in super-Hubble scales.

It can be shown that whatever would happen during inflation, if the universe subsequently spends sufficient time in a state of local thermal equilibrium with conserved quantities, then the perturbations become adiabatic and they remain adiabatic, even when the conditions of local thermal equilibrium are no longer satisfied [44].

6 Conclusion and summary

The de Sitter background is maximally extended and also maximally symmetric if and only if $K = 1$, i.e. its spatial section is closed. For this purpose, we obtained the required linear perturbation field equations and then proved the existence of two independent adiabatic solutions for these equations in the time interval when perturbations scales go outside of the Hubble horizon. We showed the curvature perturbation on the uniform density slices in a spatially closed universe is proportional to the divergence of the conformal Killing vector of S^3. This indicates some perturbative cosmological potentials in the time intervals when the scales of the majority of perturbative modes become longer than the Hubble horizon and reduce to the geometrical properties of the background. In comparison with the adiabatic solutions in the spatially flat background, it seems that the curvature has no direct role when $aH \gg 1$, but the dependence of $\zeta(\mathbf{x})$ on the background geometry manifests itself even outside the horizon where the curvature is significant. We also investigate the stochastic properties of the perturbation fields in a spatially closed background and show that their spectra are discrete due to the compactness of $S^3(a)$.

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