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Lazer-Leach type condition for second order differential equations at resonance with impulsive effects via variational method

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Abstract

In this paper, we study the existence of periodic solutions of second order impulsive differential equations at resonance with impulsive effects. We prove the existence of periodic solutions under a generalized Lazer-Leach type condition by using variational method. The impulses can generate a periodic solution.

Keywords: impulsive differential equations; Lazer-Leach type condition; variational method

1 Introduction

We are concerned with the periodic boundary value problem of second order impulsive differential equations at resonance

$$\begin{cases} x''(t) + m^2x(t) + g(x(t)) = e(t), & \text{a.e. } t \in [0, 2\pi], \\ x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0, \\ x(t_j^+) = x(t_j^-), \\ \Delta x'(t_j) := x'(t_j^+) - x'(t_j^-) = I_j(t_j, x(t_j)), & j = 1, 2, \ldots, p, \end{cases} \tag{1.1}$$

where $m \in \mathbb{N}$, $g : \mathbb{R} \to \mathbb{R}$ is a continuous function, $e \in L^1(0, 2\pi)$, $0 < t_1 < t_2 < \cdots < t_p < 2\pi$, and $I_j : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$ is continuous for every j.

When $\Delta x'(t_j) \equiv 0$, problem (1.1) becomes the well-known periodic boundary value problem at resonance

$$\begin{cases} x''(t) + m^2x(t) + g(x(t)) = e(t), & \text{a.e. } t \in [0, 2\pi], \\ x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0. \end{cases} \tag{1.2}$$

Assume that

$$\lim_{x \to \pm\infty} g(x) = g(\pm\infty) \tag{g}$$

exist and are finite. Lazer and Leach [1] proved that (1.2) has at least one 2π-periodic solution provided that the following condition holds:

$$2\big[g(+\infty) - g(-\infty)\big] \neq \int_0^{2\pi} e(t) \sin(mt + \theta)\, dt, \quad \forall \theta \in \mathbb{R}. \tag{1.3}$$

From then on, a series of relevant resonant problems were studied (see [2–5] and the references cited therein) by some classical tools such as topological degree method, variational method, *etc.* Recently, the periodic problem of the second order differential equation with impulses has been widely studied because of its background in applied sciences (see [6–18] and the references cited therein). In this paper, we investigate problem (1.1) under a more general Lazer-Leach type condition. Define

$$G(x) = \int_0^x g(s)\,ds$$

and for $j = 1, 2, \dots, p$,

$$J_j(t,x) = \int_0^x I_j(t,s)\,ds.$$

Throughout this paper, we give the following fundamental assumptions.

(H_1) The limits

$$\lim_{x\to\pm\infty} \frac{G(x)}{x} = G(\pm\infty)$$

exist and are finite.

(H_2) There exist continuous, 2π-periodic functions $K_1(t), K_2(t), \dots, K_p(t)$ such that for $j = 1, 2, \dots, p$,

$$\lim_{|x|\to\infty} \frac{I_j(t,x)}{x} = K_j(t) \quad \text{uniformly for } t \in \mathbb{R}.$$

(H_3) For all $\theta \in \mathbb{R}$,

$$2\bigl[G(+\infty) - G(-\infty)\bigr] \neq \int_0^{2\pi} e(t)\sin(mt+\theta)\,dt + \sum_{j=1}^{p} K_j(t_j)\sin(mt_j+\theta).$$

For the sake of convenience, we decompose (H_3) into the following two conditions.

(H_3^+) For all $\theta \in \mathbb{R}$,

$$2\bigl[G(+\infty) - G(-\infty)\bigr] > \int_0^{2\pi} e(t)\sin(mt+\theta)\,dt + \sum_{j=1}^{p} K_j(t_j)\sin(mt_j+\theta).$$

(H_3^-) For all $\theta \in \mathbb{R}$,

$$2\bigl[G(+\infty) - G(-\infty)\bigr] < \int_0^{2\pi} e(t)\sin(mt+\theta)\,dt + \sum_{j=1}^{p} K_j(t_j)\sin(mt_j+\theta).$$

We now can state the main theorems of this paper.

Theorem 1.1 *Assume that conditions* (H_1), (H_2) *and* (H_3^+) *hold. Then problem* (1.1) *has at least one* 2π*-periodic solution.*

Theorem 1.2 *Assume that conditions* (H_1), (H_2) *and* (H_3^-) *hold. Then problem* (1.1) *has at least one* 2π*-periodic solution.*

From Theorem 1.1 and Theorem 1.2, we obtain the following theorem.

Theorem 1.3 *Assume that conditions* (H_1), (H_2) *and* (H_3) *hold. Then problem* (1.1) *has at least one* 2π*-periodic solution.*

Moreover, we have the following corollary.

Corollary 1.4 *Assume that conditions* (H_1) *and*

(H_3') *for all* $\theta \in \mathbb{R}$,

$$2\bigl[G(+\infty) - G(-\infty)\bigr] \neq \int_0^{2\pi} e(t)\sin(mt + \theta)\,dt \tag{1.4}$$

hold. Then problem (1.2) *has at least one* 2π*-periodic solution.*

Remark 1.5 It is easy to find a function $g(x)$ such that (g) is not satisfied and (H_1) holds. For example, we can take $g(x) = \cos x$. Hence, Corollary 1.4 improves the related results in the literature mentioned above. Moreover, since we consider the problem with impulses, Theorem 1.3 is also a complement of the pioneering works.

Remark 1.6 When condition (H_3') is not satisfied, *i.e.*, there exists $\theta_0 \in \mathbb{R}$ such that

$$2\bigl[G(+\infty) - G(-\infty)\bigr] = \int_0^{2\pi} e(t)\sin(mt + \theta_0)\,dt,$$

problem (1.2) may have no solution. For example, we consider the resonant differential equation

$$x'' + m^2 x + \arctan x = 4\cos mt. \tag{1.5}$$

Obviously, $g(x) = \arctan x$, $e(t) = 4\cos mt$ and $G(+\infty) = \frac{\pi}{2}$, $G(-\infty) = -\frac{\pi}{2}$. We have

$$\begin{aligned}
&2\bigl[G(+\infty) - G(-\infty)\bigr] - \int_0^{2\pi} e(t)\sin(mt + \theta)\,dt \\
&\quad = 2\pi - 4\int_0^{2\pi} \cos mt \sin(mt + \theta)\,dt \\
&\quad = 2\pi - 4\pi \sin\theta.
\end{aligned}$$

We take $\theta_0 \in \mathbb{R}$ such that $\sin\theta_0 = \frac{1}{2}$. Then (H_3') is not satisfied. From now on, we prove that (1.5) has no 2π-periodic solution by contradiction. Assume that (1.5) has 2π-periodic solution. Multiplying both sides of (1.5) by $\cos mt$ and integrating over $[0, 2\pi]$, we get

$$4\pi = \int_0^{2\pi} \arctan x \cos mt\,dt \leq \int_0^{2\pi} |\arctan x \cos mt|\,dt \leq \frac{\pi}{2}\int_0^{2\pi} dt = \pi^2,$$

which is impossible. Hence, problem (1.2) may have no solution if condition (H_3') is not satisfied. Now, we give the following boundary value condition:

$$x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0 \tag{1.6}$$

and the impulsive condition

$$\Delta x'\left(\frac{\pi}{m}\right) = 3\pi. \tag{1.7}$$

Clearly, $p = 1$ and $K_1(\frac{\pi}{m}) = 3\pi$. Then

$$\begin{aligned} &2\bigl[G(+\infty) - G(-\infty)\bigr] - \int_0^{2\pi} e(t)\sin(mt+\theta)\,dt - K_1\left(\frac{\pi}{m}\right)\sin\left(m\cdot\frac{\pi}{m}+\theta\right) \\ &\quad = 2\pi - 4\pi\sin\theta + 3\pi\sin\theta \\ &\quad = 2\pi - \pi\sin\theta \neq 0 \quad \text{for } \forall\theta\in\mathbb{R}. \end{aligned}$$

Hence, (H_1), (H_2) and (H_3) hold. Equivalently, Eq. (1.5) with conditions (1.6) and (1.7) has at least one 2π-periodic solution. Therefore, the impulses in problem (1.1) can generate a periodic solution.

The rest of the paper is organized as follows. In Section 2, we shall state some notations, some necessary definitions and a saddle theorem due to Rabinowitz. In Section 3, we shall prove Theorem 1.1 and Theorem 1.2.

2 Preliminaries

In the following, we introduce some notations and some necessary definitions.

Define

$$H = \bigl\{x \in H^1(0,2\pi) : x(0) = x(2\pi)\bigr\},$$

with the norm

$$\|x\| = \left(\int_0^{2\pi}\bigl(x'^2(t) + x^2(t)\bigr)\,dt\right)^{\frac{1}{2}}.$$

Consider the functional $\varphi(x)$ defined on H by

$$\begin{aligned} \varphi(x) = {}& \frac{1}{2}\int_0^{2\pi} x'^2(t)\,dt - \frac{m^2}{2}\int_0^{2\pi} x^2(t)\,dt - \int_0^{2\pi} G\bigl(x(t)\bigr)\,dt \\ &+ \int_0^{2\pi} e(t)x(t)\,dt + \sum_{j=1}^{p} J_j\bigl(t_j, x(t_j)\bigr). \end{aligned} \tag{2.1}$$

Similarly as in [18], $\varphi(x)$ is continuously differentiable on H, and

$$\begin{aligned} \varphi'(x)v(t) = {}& \int_0^{2\pi} x'(t)v'(t)\,dt - m^2\int_0^{2\pi} x(t)v(t)\,dt - \int_0^{2\pi} g\bigl(x(t)\bigr)v(t)\,dt \\ &+ \int_0^{2\pi} e(t)v(t)\,dt + \sum_{j=1}^{p} I_j\bigl(t_j, x(t_j)\bigr)v(t_j) \quad \text{for } \forall v(t)\in H. \end{aligned} \tag{2.2}$$

Now, we have the following lemma.

Lemma 2.1 *If $x \in H$ is a critical point of φ, then x is a 2π-periodic solution of Eq.* (1.1).

The proof of Lemma 2.1 is similar to Lemma 2.1 in [9], so we omit it.

We say that φ satisfies (PS) if every sequence (x_n) for which $\varphi(x_n)$ is bounded in $\mathbb{R}$ and $\varphi'(x_n) \to 0$ (as $n \to \infty$) possesses a convergent subsequence.

To prove the main result, we will use the following saddle point theorem due to Rabinowitz [19] (or see [20]).

Theorem 2.2 *Let $\varphi \in C^1(H, \mathbb{R})$ and $H = H^- \oplus H^+$, $\dim(H^-) < \infty$, $\dim(H^+) = \infty$. We suppose that*:

(a) *there exist a bounded neighborhood D of* 0 *in H^- and a constant α such that* $\varphi|_{\partial D} \leq \alpha$;
(b) *there exists a constant $\beta > \alpha$ such that $\varphi|_{H^+} \geq \beta$*;
(c) *φ satisfies* (PS).

Then the functional φ has a critical point in H.

3 The proof of the main results

In this section, we first show that the functional φ satisfies the Palais-Smale condition.

Lemma 3.1 *Assume that conditions* (H_1), (H_2) *and* (H_3) *hold. Then φ defined by* (2.1) *satisfies* (PS).

Proof Let $M > 0$ be a constant and $\{x_n\} \subset H$ be a sequence satisfying

$$\begin{aligned}\left|\varphi(x_n)\right| &= \left|\frac{1}{2}\int_0^{2\pi} x_n'^2\,dt - \frac{m^2}{2}\int_0^{2\pi} x_n^2\,dt - \int_0^{2\pi} G(x_n)\,dt \right.\\ &\quad \left. + \int_0^{2\pi} e(t)x_n(t)\,dt + \sum_{j=1}^{p} J_j\bigl(t_j, x_n(t_j)\bigr)\right|\\ &\leq M \end{aligned} \tag{3.1}$$

and

$$\lim_{n\to\infty}\left\|\varphi'(x_n)\right\| = 0. \tag{3.2}$$

We first prove that $\{x_n\}$ is bounded in H by contradiction. Assume that $\{x_n\}$ is unbounded. Let $\{z_k\}$ be an arbitrary sequence bounded in H. It follows from (3.2) that, for any $k \in \mathbb{N}$,

$$\lim_{n\to\infty}\left|\varphi'(x_n)z_k\right| \leq \lim_{n\to\infty}\left\|\varphi'(x_n)\right\|\|z_k\| = 0.$$

Thus

$$\lim_{n\to\infty}\varphi'(x_n)z_k = 0 \quad \text{uniformly for } k \in \mathbb{N}.$$

Hence

$$\lim_{n\to\infty}\left(\int_0^{2\pi}\left(x_n'z_k'-m^2x_nz_k\right)dt-\int_0^{2\pi}\left(g(x_n)z_k-e(t)z_k\right)dt\right.$$
$$\left.+\sum_{j=1}^{p}I_j\left(t_j,x_n(t_j)\right)z_k(t_j)\right)=0. \tag{3.3}$$

By (H_1) and (H_2), we have

$$\lim_{n\to\infty}\left(\int_0^{2\pi}\frac{g(x_n)z_k-e(t)z_k}{\|x_n\|}\,dt-\frac{\sum_{j=1}^{p}I_j(t_j,x_n(t_j))z_k(t_j)}{\|x_n\|}\right)=0. \tag{3.4}$$

From (3.3) and (3.4), we obtain

$$\lim_{n\to\infty}\int_0^{2\pi}\left(\frac{x_n'}{\|x_n\|}z_k'-m^2\frac{x_n}{\|x_n\|}z_k\right)dt=0. \tag{3.5}$$

Set

$$y_n=\frac{x_n}{\|x_n\|}.$$

Then we have

$$\lim_{n\to\infty}\int_0^{2\pi}\left(y_n'z_k'-m^2y_nz_k\right)dt=0,$$

and furthermore,

$$\lim_{\substack{n\to\infty\\ i\to\infty}}\int_0^{2\pi}\left[(y_n-y_i)'z_k'-m^2(y_n-y_i)z_k\right]dt=0. \tag{3.6}$$

Replacing z_k in (3.6) by (y_n-y_i), we get

$$\lim_{\substack{n\to\infty\\ i\to\infty}}\left(\|y_n-y_i\|^2-\left(m^2+1\right)\|y_n-y_i\|_2^2\right)=0.$$

Due to the compact imbedding $H\hookrightarrow L^2(0,2\pi)$, going to a subsequence,

$$y_n\rightharpoonup y_0\quad\text{weakly in } H,\qquad y_n\to y_0\quad\text{in } L^2(0,2\pi).$$

Therefore,

$$\lim_{\substack{n\to\infty\\ i\to\infty}}\|y_n-y_i\|_2^2=0.$$

Furthermore, we have

$$\lim_{\substack{n\to\infty\\ i\to\infty}}\|y_n-y_i\|^2=0,$$

which implies that $\{y_n\}$ is a Cauchy sequence in H. Thus, $y_n \to y_0$ in H. It follows from (3.5) and the usual regularity argument for ordinary differential equations (see [21]) that

$$y_0 = k_1 \sin mt + k_2 \cos mt, \tag{3.7}$$

where $k_1^2 + k_2^2 = \frac{1}{(m^2+1)\pi}$ ($\|y_0\| = 1$). (Different subsequences of $\{y_n\}$ correspond to different k_1 and k_2.)

Write (3.7) as

$$y_0 = \frac{1}{\sqrt{(m^2+1)\pi}} \sin(mt + \theta),$$

where θ satisfies $\sin\theta = \frac{k_2}{\sqrt{k_1^2+k_2^2}}$ and $\cos\theta = \frac{k_1}{\sqrt{k_1^2+k_2^2}}$.

Taking $z_k = \frac{1}{\sqrt{(m^2+1)\pi}} \sin(mt+\theta)$, we get, for any $n \in \mathbb{N}$,

$$\int_0^{2\pi} \bigl(x_n' z_k' - m^2 x_n z_k\bigr)\, dt = 0. \tag{3.8}$$

Thus, it follows from (3.3) and (3.8) that

$$\begin{aligned} \lim_{n\to\infty} \Biggl[&\int_0^{2\pi} \bigl(g(x_n) - e(t)\bigr) \frac{1}{\sqrt{(m^2+1)\pi}} \sin(mt+\theta)\, dt \\ &- \sum_{j=1}^{p} I_j\bigl(t_j, x_n(t_j)\bigr) \frac{1}{\sqrt{(m^2+1)\pi}} \sin(mt_j + \theta) \Biggr] = 0. \end{aligned} \tag{3.9}$$

By (H_1) and (H_2), we obtain

$$\begin{aligned} \lim_{n\to\infty} \Biggl[&\int_0^{2\pi} \bigl(g(x_n) - e(t)\bigr) \biggl(\frac{1}{\sqrt{(m^2+1)\pi}} \sin(mt+\theta) - y_n \biggr)\, dt \\ &- \sum_{j=1}^{p} I_j\bigl(t_j, x_n(t_j)\bigr) \biggl(\frac{1}{\sqrt{(m^2+1)\pi}} \sin(mt_j + \theta) - y_n(t_j) \biggr) \Biggr] = 0. \end{aligned} \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$\lim_{n\to\infty} \Biggl[\int_0^{2\pi} \bigl(g(x_n) - e(t)\bigr) y_n\, dt - \sum_{j=1}^{p} I_j\bigl(t_j, x_n(t_j)\bigr) y_n(t_j) \Biggr] = 0.$$

Hence, replacing z_k in (3.3) by y_n, we have

$$\lim_{n\to\infty} \int_0^{2\pi} \biggl(x_n' \frac{x_n'}{\|x_n\|} - m^2 x_n \frac{x_n}{\|x_n\|} \biggr)\, dt = 0. \tag{3.11}$$

Now, dividing (3.1) by $\|x_n\|$, we get

$$\left| \frac{1}{2} \int_0^{2\pi} \biggl(\frac{x_n'^2}{\|x_n\|} - \frac{m^2 x_n^2}{\|x_n\|} \biggr)\, dt - \int_0^{2\pi} \frac{G(x_n) - e(t) x_n}{\|x_n\|}\, dt + \frac{\sum_{j=1}^{p} J_j(t_j, x_n(t_j))}{\|x_n\|} \right| \le \frac{M}{\|x_n\|}.$$

Passing to the limits, we have

$$\begin{aligned}
0 &= \lim_{n\to\infty} \frac{\int_0^{2\pi} (G(x_n) - e(t)x_n)\,dt}{\|x_n\|} - \lim_{n\to\infty} \frac{\sum_{j=1}^{p} J_j(t_j, x_n(t_j))}{\|x_n\|} \\
&= \lim_{n\to\infty} \int_0^{2\pi} \frac{G(x_n)}{x_n} \cdot \frac{x_n}{\|x_n\|}\,dt - \lim_{n\to\infty} \int_0^{2\pi} e(t) \cdot \frac{x_n}{\|x_n\|}\,dt \\
&\quad - \lim_{n\to\infty} \sum_{j=1}^{p} \frac{J_j(t_j, x_n(t_j))}{x_n(t_j)} \cdot \frac{x_n(t_j)}{\|x_n\|}.
\end{aligned}$$

Noting that $\frac{x_n}{\|x_n\|} \to \frac{1}{\sqrt{(m^2+1)\pi}} \sin(mt+\theta)$ in H as $n \to \infty$ and

$$\lim_{n\to\infty} x_n(t) = \begin{cases} +\infty, & \forall t \in I_+, \\ -\infty, & \forall t \in I_-, \end{cases}$$

where $I_+ := \{t \in [0, 2\pi] \mid \sin(mt+\theta) > 0\}$, $I_- := \{t \in [0, 2\pi] \mid \sin(mt+\theta) < 0\}$, we get from the Lebesgue domain convergence theorem that

$$\begin{aligned}
0 &= \int_{I_+} G(+\infty) \frac{1}{\sqrt{(m^2+1)\pi}} \sin^+(mt+\theta)\,dt - \int_{I_+} G(-\infty) \frac{1}{\sqrt{(m^2+1)\pi}} \sin^-(mt+\theta)\,dt \\
&\quad - \int_0^{2\pi} e(t) \frac{1}{\sqrt{(m^2+1)\pi}} \sin(mt+\theta)\,dt - \sum_{j=1}^{p} K_j(t_j) \frac{1}{\sqrt{(m^2+1)\pi}} \sin(mt_j+\theta),
\end{aligned}$$

i.e.,

$$0 = 2\bigl[G(+\infty) - G(-\infty)\bigr] - \int_0^{2\pi} e(t)\sin(mt+\theta)\,dt - \sum_{j=1}^{p} K_j(t_j)\sin(mt_j+\theta),$$

which contradicts (H_3). This implies that the sequence $\{x_n\}$ is bounded. Thus, there exists $x_0 \in H$ such that $x_n \rightharpoonup x_0$ weakly in H. Due to the compact imbedding $H \hookrightarrow L^2(0, 2\pi)$ and $H \hookrightarrow C(0, 2\pi)$, going to a subsequence,

$$x_n \to x_0 \quad \text{in } L^2(0, 2\pi), \qquad x_n \to x_0 \quad \text{in } C(0, 2\pi).$$

From (3.3), we obtain

$$\begin{aligned}
&\lim_{\substack{n\to\infty \\ i\to\infty}} \Biggl(\int_0^{2\pi} \bigl((x_n' - x_i')z_k' - m^2(x_n - x_i)z_k\bigr)\,dt - \int_0^{2\pi} \bigl(g(x_n) - g(x_i)\bigr)z_k\,dt \\
&\quad + \sum_{j=1}^{p} \bigl(I_j\bigl(t_j, x_n(t_j)\bigr) - I_j\bigl(t_j, x_i(t_j)\bigr)\bigr)z_k(t_j) \Biggr) = 0.
\end{aligned}$$

Replacing z_k by $x_n - x_i$ in the above equality, we get

$$\begin{aligned}
&\lim_{\substack{n\to\infty \\ i\to\infty}} \Biggl(\int_0^{2\pi} \bigl((x_n' - x_i')^2 - m^2(x_n - x_i)^2\bigr)\,dt - \int_0^{2\pi} \bigl(g(x_n) - g(x_i)\bigr)(x_n - x_i)\,dt \\
&\quad + \sum_{j=1}^{p} \bigl(I_j\bigl(t_j, x_n(t_j)\bigr) - I_j\bigl(t_j, x_i(t_j)\bigr)\bigr)\bigl(x_n(t_j) - x_i(t_j)\bigr) \Biggr) = 0. \qquad (3.12)
\end{aligned}$$

By (H_1) and (H_2), we have

$$\lim_{\substack{n\to\infty\\ i\to\infty}} \int_0^{2\pi} \big(g(x_n) - g(x_i)\big)(x_n - x_i)\,dt = 0 \tag{3.13}$$

and

$$\lim_{\substack{n\to\infty\\ i\to\infty}} \sum_{j=1}^{p} \big(I_j\big(t_j, x_n(t_j)\big) - I_j\big(t_j, x_i(t_j)\big)\big)\big(x_n(t_j) - x_i(t_j)\big) = 0. \tag{3.14}$$

Thus, it follows from (3.12), (3.13) and (3.14) that

$$\lim_{\substack{n\to\infty\\ i\to\infty}} \int_0^{2\pi} \big[\big(x_n' - x_i'\big)^2 - m^2(x_n - x_i)^2\big]\,dt = 0.$$

Therefore,

$$\lim_{\substack{n\to\infty\\ i\to\infty}} \|x_n - x_i\|^2 = 0,$$

which implies $x_n \to x_0$ in H. It shows that φ satisfies (PS). $\square$

Remark 3.2 If conditions (H_1), (H_2) and (H_3^+) (or (H_3^-)), φ defined by (2.1) still satisfies (PS).

Now, we can give the proof of Theorem 1.1.

Proof of Theorem 1.1 Denote

$$H^- = \mathbb{R} \oplus \operatorname{span}\{\sin t, \cos t, \sin 2t, \cos 2t, \dots, \sin mt, \cos mt\}$$

and

$$H^+ = \operatorname{span}\big\{\sin(m+1)t, \cos(m+1)t, \dots\big\}.$$

We first prove that

$$\liminf_{\|x\|\to\infty} \varphi(x) = -\infty \quad \text{for } x \in H^- \tag{3.15}$$

by contradiction. Assume that there exists a sequence $(x_n) \subset H^-$ such that $\|x_n\| \to \infty$ (as $n \to \infty$) and there exists a constant c_- satisfying

$$\liminf_{n\to\infty} \varphi(x_n) \geq c_-. \tag{3.16}$$

By (H_1), we have

$$\lim_{n\to\infty} \int_0^{2\pi} \frac{G(x_n) - e(t)x_n}{\|x_n\|^2}\,dt = 0. \tag{3.17}$$

By (H_2), we get

$$\lim_{n\to\infty}\sum_{j=1}^{p}\frac{J_j(t_j,x_n(t_j))}{\|x_n\|^2}=0. \tag{3.18}$$

From (3.16) and the definition of φ, we obtain

$$\liminf_{n\to\infty}\left[\frac{1}{2}\int_0^{2\pi}\frac{x_n'^2-m^2x_n^2}{\|x_n\|^2}\,dt-\int_0^{2\pi}\frac{G(x_n)-e(t)x_n}{\|x_n\|^2}\,dt+\sum_{j=1}^{p}\frac{J_j(t_j,x_n(t_j))}{\|x_n\|^2}\right]\geq 0. \tag{3.19}$$

For $x\in H^-$, we get that there exist constants $a_0,a_1,\dots,a_m$, $b_1,b_2,\dots,b_m$ such that

$$x(t)=\sum_{j=0}^{m}a_j\cos jt+\sum_{j=1}^{m}b_j\sin jt.$$

Since $\int_0^{2\pi}\sin^2 jt\,dt=\int_0^{2\pi}\cos^2 jt\,dt$ for $j=1,2,\dots$, we have, for $x\in H^-$,

$$\begin{aligned}\int_0^{2\pi}x'^2\,dt&=\int_0^{2\pi}\sum_{j=1}^{m}j^2a_j^2\sin^2 jt\,dt+\int_0^{2\pi}\sum_{j=1}^{m}j^2b_j^2\cos^2 jt\,dt\\&\leq m^2\left[\int_0^{2\pi}\sum_{j=1}^{m}a_j^2\sin^2 jt\,dt+\int_0^{2\pi}\sum_{j=1}^{m}b_j^2\cos^2 jt\,dt\right]\\&=m^2\left[\int_0^{2\pi}\sum_{j=1}^{m}a_j^2\cos^2 jt\,dt+\int_0^{2\pi}\sum_{j=1}^{m}b_j^2\sin^2 jt\,dt\right]\\&\leq m^2\int_0^{2\pi}x^2\,dt.\end{aligned}$$

Hence, for $x\in H^-$,

$$\int_0^{2\pi}\left(x'^2-m^2x^2\right)dt\leq 0. \tag{3.20}$$

The equality in (3.20) holds only for

$$x=\frac{1}{\sqrt{(m^2+1)\pi}}\sin(mt+\theta),\quad \theta\in\mathbb{R}.$$

Set $y_n=\frac{x_n}{\|x_n\|}$. Since $\dim H^-<\infty$, going to a subsequence, there exists $y_0\in H^-$ such that $y_n\to y_0$ in H and $y_n\to y_0$ in $L^2(0,2\pi)$. Then (3.17), (3.18), (3.19) and (3.20) imply that

$$y_0=\frac{1}{\sqrt{(m^2+1)\pi}}\sin(mt+\theta),\quad \theta\in\mathbb{R}.$$

By (3.16), we have, for n large enough,

$$\frac{1}{2}\int_0^{2\pi}\frac{x_n'^2-m^2x_n^2}{\|x_n\|}\,dt-\int_0^{2\pi}\frac{G(x_n)-e(t)x_n}{\|x_n\|}\,dt+\sum_{j=1}^{p}\frac{J_j(t_j,x_n(t_j))}{\|x_n\|}\geq\frac{c_-}{\|x_n\|}. \tag{3.21}$$

It follows from $x_n \in H^-$ that

$$\int_0^{2\pi} \frac{x_n'^2 - m^2 x_n^2}{\|x_n\|} \le 0. \tag{3.22}$$

From (3.21) and (3.22), we get, for n large enough,

$$\frac{c_-}{\|x_n\|} \le -\int_0^{2\pi} \frac{G(x_n) - e(t)x_n}{\|x_n\|}\,dt + \sum_{j=1}^{p} \frac{J_j(t_j, x_n(t_j))}{\|x_n\|}.$$

Passing to the limits and using an argument similarly as in the proof of Lemma 3.1, we get

$$2\bigl[G(+\infty) - G(-\infty)\bigr] \le \int_0^{2\pi} e(t)\sin(mt+\theta)\,dt + \sum_{j=1}^{p} K_j(t_j)\sin(mt_j+\theta),$$

which is a contradiction to (H_3^+).

Then (3.15) holds.

Next, we prove that

$$\lim_{\|x\|\to\infty} \varphi(x) = \infty \quad \text{for all } x \in H^+,$$

and φ is bounded on bounded sets.

Because of the compact imbedding of $H \hookrightarrow C(0,2\pi)$ and $H \hookrightarrow L^2(0,2\pi)$, there exist constants m_1, m_2 such that

$$\|x\|_\infty \le m_1\|x\|, \qquad \|x\|_2 \le m_2\|x\|.$$

Then by (H_1) and (H_2), one has that there exist positive constants $c_g, c_1, c_2, \dots, c_p$ such that

$$\begin{aligned}
\bigl|\varphi(x)\bigr| &= \left|\frac{1}{2}\int_0^{2\pi} x'^2\,dt - \frac{m^2}{2}\int_0^{2\pi} x^2\,dt - \int_0^{2\pi}\bigl[G(x) - e(t)x\bigr]\,dt \right.\\
&\quad \left. + \sum_{j=1}^{p} J_j\bigl(t_j, x(t_j)\bigr)\right| \\
&\le \frac{1}{2}\|x\|^2 + \frac{m^2}{2}m_2^2\|x\|^2 + \int_0^{2\pi}\bigl(c_g|x| + \bigl|e(t)\bigr||x|\bigr)\,dt \\
&\quad + \sum_{j=1}^{p} c_j\bigl|x(t_j)\bigr| \\
&\le \frac{1+m^2m_2^2}{2}\|x\|^2 + m_1\bigl(c_g + \|e\|_1\bigr)\|x\| + \sum_{j=1}^{p} c_j m_1\|x\|.
\end{aligned} \tag{3.23}$$

Hence, φ is bounded on the bounded sets of H.

For $x \in H^+$, using an argument similar to the case $x \in H^-$, we have

$$\|x\|^2 \ge \bigl((m+1)^2 + 1\bigr)\|x\|_2^2. \tag{3.24}$$

Thus, from (3.23) and (3.24), we obtain

$$\begin{aligned}\varphi(x) &= \frac{1}{2}\int_0^{2\pi} x'^2\,dt - \frac{m^2}{2}\int_0^{2\pi} x^2\,dt - \int_0^{2\pi}\bigl[G(x) - e(t)x\bigr]\,dt + \sum_{j=1}^{p} J_j\bigl(t_j, x(t_j)\bigr)\\ &\geq \frac{2m+1}{2((m+1)^2+1)}\|x\|^2 - m_1\left(c_g + \|e\|_1 + \sum_{j=1}^{p} c_j\right)\|x\|,\end{aligned}$$

which implies

$$\lim_{\|x\|\to\infty}\varphi(x) = \infty \quad \text{for all } x \in H^+.$$

Up to now, the conditions (a) and (b) of Theorem 2.2 are satisfied. According to Remark 3.2, (c) is also satisfied. Hence, by Theorem 2.2, problem (1.1) has at least one solution. This completes the proof. □

Next, we prove Theorem 1.2 slightly differently from Theorem 1.1.

Proof of Theorem 1.2 Denote

$$H^- = \mathbb{R} \oplus \operatorname{span}\bigl\{\sin t, \cos t, \sin 2t, \cos 2t, \ldots, \sin(m-1)t, \cos(m-1)t\bigr\}$$

and

$$H^+ = \operatorname{span}\{\sin mt, \cos mt, \ldots\}.$$

We first prove that

$$\liminf_{\|x\|\to\infty}\varphi(x) = -\infty \quad \text{for } x \in H^-. \tag{3.25}$$

For $x \in H^-$, we get that there exist constants $a_0, a_1, \ldots, a_{m-1}$, $b_1, b_2, \ldots, b_{m-1}$ such that

$$x(t) = \sum_{j=0}^{m-1} a_j \cos jt + \sum_{j=1}^{m-1} b_j \sin jt.$$

Since $\int_0^{2\pi} \sin^2 jt\,dt = \int_0^{2\pi} \cos^2 jt\,dt$ for $j = 1, 2, \ldots$, we have, for $x \in H^-$,

$$\begin{aligned}\int_0^{2\pi} x'^2\,dt &= \int_0^{2\pi} \sum_{j=1}^{m-1} j^2 a_j^2 \sin^2 jt\,dt + \int_0^{2\pi} \sum_{j=1}^{m-1} j^2 b_j^2 \cos^2 jt\,dt\\ &\leq (m-1)^2\left[\int_0^{2\pi} \sum_{j=1}^{m-1} a_j^2 \sin^2 jt\,dt + \int_0^{2\pi} \sum_{j=1}^{m-1} b_j^2 \cos^2 jt\,dt\right]\\ &= (m-1)^2\left[\int_0^{2\pi} \sum_{j=1}^{m-1} a_j^2 \cos^2 jt\,dt + \int_0^{2\pi} \sum_{j=1}^{m-1} b_j^2 \sin^2 jt\,dt\right]\\ &\leq (m-1)^2 \int_0^{2\pi} x^2\,dt.\end{aligned}$$

Hence, for $x \in H^-$,

$$\|x\|^2 = \int_0^{2\pi} \left(x'^2 + x^2\right) dt \le \left[(m-1)^2 + 1\right] \int_0^{2\pi} x^2 \, dt = \left[(m-1)^2 + 1\right] \|x\|_2^2.$$

The equality holds only for

$$x = \frac{1}{\sqrt{((m-1)^2+1)\pi}} \sin\bigl((m-1)t + \theta\bigr), \quad \theta \in \mathbb{R}.$$

If $x \in H^-$ and $\|x\| \to \infty$, then

$$\|x\|_2 \to \infty.$$

For $x \in H^-$, we have

$$\int_0^{2\pi} \left(x'^2 - m^2 x^2\right) dt \le (m-1)^2 \int_0^{2\pi} x^2 \, dt - m^2 \int_0^{2\pi} x^2 \, dt = -(2m-1)\|x\|_2^2.$$

By (H_1) and (H_2), we get that there exists a constant $c_0 > 0$ such that

$$\left| -\int_0^{2\pi} \left[G(x) - e(t)x\right] dt + \sum_{j=1}^{p} J_j\bigl(t_j, x(t_j)\bigr) \right| \le c_0 \|x\|_2.$$

Hence, for $x \in H^-$, we obtain

$$\begin{aligned}\varphi(x) &= \frac{1}{2} \int_0^{2\pi} \left(x'^2 - m^2 x^2\right) dt - \int_0^{2\pi} \left[G(x) - e(t)x\right] dt + \sum_{j=1}^{p} J_j\bigl(t_j, x(t_j)\bigr) \\ &\le -\frac{1}{2}(2m-1)\|x\|_2^2 + c_0 \|x\|_2 \to -\infty \quad \text{as } \|x\| \to \infty.\end{aligned}$$

Therefore, (3.25) holds.

Next, we prove that

$$\lim_{\|x\| \to \infty} \varphi(x) = \infty \quad \text{for all } x \in H^+,$$

and φ is bounded on bounded sets.

Because of the compact imbedding of $H \hookrightarrow C(0, 2\pi)$ and $H \hookrightarrow L^2(0, 2\pi)$, there exist constants m_1, m_2 such that

$$\|x\|_\infty \le m_1 \|x\|, \qquad \|x\|_2 \le m_2 \|x\|.$$

Then, by (H_1) and (H_2), one has that there exist positive constants $c_g, c_1, c_2, \ldots, c_p$ such that

$$\begin{aligned}\left|\varphi(x)\right| = &\left| \frac{1}{2} \int_0^{2\pi} x'^2 \, dt - \frac{m^2}{2} \int_0^{2\pi} x^2 \, dt - \int_0^{2\pi} \left[G(x) - e(t)x\right] dt \right. \\ &\left. + \sum_{j=1}^{p} J_j\bigl(t_j, x(t_j)\bigr) \right|\end{aligned}$$

$$\begin{aligned}
&\le \frac{1}{2}\|x\|^2 + \frac{m^2}{2}m_2^2\|x\|^2 + \int_0^{2\pi} \bigl(c_g|x| + \bigl|e(t)\bigr||x|\bigr)\,dt \\
&\quad + \sum_{j=1}^{p} c_j\bigl|x(t_j)\bigr| \\
&\le \frac{1+m^2m_2^2}{2}\|x\|^2 + m_1\bigl(c_g + \|e\|_1\bigr)\|x\| + \sum_{j=1}^{p} c_j m_1\|x\|.
\end{aligned}$$

Hence, φ is bounded on the bounded sets of H.

In what follows, we prove that

$$\lim_{\|x\|\to\infty} \varphi(x) = +\infty \quad \text{for } x \in H^+$$

by contradiction. Assume that there exists a sequence $(x_n) \subset H^-$ such that $\|x_n\| \to \infty$ (as $n \to \infty$), and there exists a constant c_+ satisfying

$$\limsup_{n\to\infty} \varphi(x_n) \le c_+. \tag{3.26}$$

By (H_1), we have

$$\lim_{n\to\infty} \int_0^{2\pi} \frac{G(x_n) - e(t)x_n}{\|x_n\|^2}\,dt = 0. \tag{3.27}$$

By (H_2), we get

$$\lim_{n\to\infty} \sum_{j=1}^{p} \frac{J_j(t_j, x_n(t_j))}{\|x_n\|^2} = 0. \tag{3.28}$$

From (3.26) and the definition of φ, we obtain

$$\limsup_{n\to\infty}\left[\frac{1}{2}\int_0^{2\pi} \frac{x_n'^2 - m^2x_n^2}{\|x_n\|^2}\,dt - \int_0^{2\pi} \frac{G(x_n) - e(t)x_n}{\|x_n\|^2}\,dt + \sum_{j=1}^{p} \frac{J_j(t_j, x_n(t_j))}{\|x_n\|^2}\right] \le 0. \tag{3.29}$$

For $x \in H^+$, we get

$$\int_0^{2\pi} x'^2\,dt \ge m^2 \int_0^{2\pi} x^2\,dt.$$

Hence, for $x \in H^+$, we have

$$\int_0^{2\pi} \bigl(x'^2 - m^2x^2\bigr)\,dt \ge 0. \tag{3.30}$$

The equality in (3.30) holds only for

$$x = \frac{1}{\sqrt{(m^2+1)\pi}}\sin(mt + \theta), \quad \theta \in \mathbb{R}.$$

Set $y_n = \frac{x_n}{\|x_n\|}$. There exists $y_0 \in H^+$ such that $y_n \rightharpoonup y_0$ weakly in H. Due to the compact imbedding $H \hookrightarrow L^2(0,2\pi)$, going to a subsequence, $y_n \to y_0$ in $L^2(0,2\pi)$. Then (3.27), (3.28), (3.29) and (3.30) imply that

$$y_0 = \frac{1}{\sqrt{(m^2+1)\pi}} \sin(mt+\theta), \quad \theta \in \mathbb{R}.$$

By (3.26), we have, for n large enough,

$$\frac{1}{2}\int_0^{2\pi} \frac{x_n'^2 - m^2 x_n^2}{\|x_n\|}\,dt - \int_0^{2\pi} \frac{G(x_n) - e(t)x_n}{\|x_n\|}\,dt + \sum_{j=1}^{p} \frac{J_j(t_j, x_n(t_j))}{\|x_n\|} \le \frac{c_+}{\|x_n\|}. \tag{3.31}$$

It follows from $x_n \in H^+$ that

$$\int_0^{2\pi} \frac{x_n'^2 - m^2 x_n^2}{\|x_n\|}\,dt \ge 0. \tag{3.32}$$

From (3.31) and (3.32), we get, for n large enough,

$$\frac{c_+}{\|x_n\|} \ge -\int_0^{2\pi} \frac{G(x_n) - e(t)x_n}{\|x_n\|}\,dt + \sum_{j=1}^{p} \frac{J_j(t_j, x_n(t_j))}{\|x_n\|}.$$

Passing to the limits and using an argument similarly as in the proof of Lemma 3.1, we get

$$2\big[G(+\infty) - G(-\infty)\big] \ge \int_0^{2\pi} e(t)\sin(mt+\theta)\,dt + \sum_{j=1}^{p} K_j(t_j)\sin(mt_j+\theta),$$

which is a contradiction to (H_3^-). This completes the proof. □

Competing interests
The author declares that they have no competing interests.

Author's contributions
The author has contributed in obtaining new results and written the whole article.

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