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Highest weight generating functions for Hilbert series

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Abstract: We develop a new method for representing Hilbert series based on the highest weight Dynkin labels of their underlying symmetry groups. The method draws on plethystic functions and character generating functions along with Weyl integration. We give explicit examples showing how the use of such *highest weight generating functions* ("HWGs") permits an efficient encoding and analysis of the Hilbert series of the vacuum moduli spaces of classical and exceptional SQCD theories and also of the moduli spaces of instantons. We identify how the HWGs of gauge invariant operators of a selection of classical and exceptional SQCD theories result from the interaction under symmetrisation between a product group and the invariant tensors of its gauge group. In order to calculate HWGs, we derive and tabulate character generating functions for low rank groups by a variety of methods, including a general character generating function, based on the Weyl Character Formula, for any classical or exceptional group.

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Contents

1 Introduction

Hilbert series ("HS") provide an important tool in the analysis of the moduli spaces associated with various field theories. One key area of interest is the study of the moduli spaces of the gauge invariant operators ("GIOs") constructed from fields transforming in

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some combination of flavour, colour and/or other symmetry group representations. Such product group structures arise in supersymmetric ("SUSY") quiver gauge theories.

Considerable progress has been made in recent years under the auspices of the Plethystics Program [1, 2] in the systematic calculation and analysis of the Hilbert series of the moduli spaces of SUSY quiver gauge theories. The Plethystics Program deploys plethystic functions and Weyl integration to construct generating functions for the Hilbert series of objects transforming under various representations of classical or exceptional Lie groups or product groups.

The Hilbert series for these SUSY quiver gauge theories enumerate holomorphic operators, such as the chiral ring of BPS operators characterising the moduli space of a supersymmetric vacuum. Theories that have been well studied include SQCD for classical gauge groups [3–5], the moduli spaces of one and two instantons [6, 7] and the master space of $\mathcal{N} = 1$ SUSY gauge theories based on particular Calabi-Yau manifolds [8, 9]. In SQCD, the theories are without superpotentials ($W \equiv 0$) and the gauge group structures are based on those specified by the chiral scalar field definitions; in the other cases, such as SUSY quiver theories for instanton moduli spaces, the theories have non-trivial superpotentials W and the vacuum conditions can place F-flatness constraints on the field representations and/or give rise to hidden symmetry groups. In all cases, however, the Hilbert series associated with the moduli spaces are amenable to analysis by similar methods.

Key analytical tools within the Plethystics Program include the Plethystic Exponential ("PE"), the Fermionic Plethystic Exponential ("PEF"), Plethystic Logarithm ("PL") and Weyl integration. The PE symmetrises polynomials and can be used to generate symmetric tensor products of group representations described in terms of their characters. Weyl integration, combined with the orthogonality property of the inner products of characters, makes it possible to project out irreducible representations ("irreps") from such generating functions. The PL provides an inverse function. Such procedures can be used to identify GIOs, which are necessarily singlets of the gauge group, and the results can be arranged into Hilbert series and their generating functions that encode information about the GIOs at a given level of field counting.

Hilbert series can be expressed in refined form, with fields described in terms of class functions built from the characters of the irreps of the symmetry group. Alternatively, Hilbert series can be expressed in unrefined form, where they simply count representations according to their dimensions. Unrefined Hilbert series are the most straightforward to calculate, but their limitation to dimensional information entails that they do not fully encode moduli space data, for example, about the specific representations in which fields transform. On the other hand, while refined Hilbert series do fully encode moduli spaces, their generating functions can be cumbersome to deploy, with complicated plethystic procedures being necessary to extract character expansions.

Our aim in this paper is to outline a methodology for constructing highest weight generating functions ("HWGs") for the character expansions of Hilbert series, from which the latter can readily be extracted, either in refined or unrefined form. We wish to obtain HWGs that faithfully and efficiently encode the combinatorics of objects transforming in representations of the chosen symmetry group. We shall show that, just like Hilbert

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Generating Function	$g^{\text{Group}}\,(\textit{coordinates})$
HWG	$g^{\text{Group}}\,(t_j, m_i)$
Character	$g^{\text{Group}}\,(m_i, \mathcal{X})$
Refined HS *(character notation)*	$g^{\text{Group}}\,(t_j, \mathcal{X})$
Refined HS *(coordinates)*	$g^{\text{Group}}\,(t_j, x_i)$
Unrefined HS *(distinct counting)*	$g^{\text{Group}}\,(t_j)$
Unrefined HS	$g^{\text{Group}}\,(t)$

Table 1. Types of Generating Function.

series, HWGs can be constructed as polynomials and rational functions of polynomials. While the prime objective here is to facilitate the analysis of the moduli spaces of SUSY gauge theories, we shall also show how HWG techniques can in principle be applied to other aspects of the combinatorics of group and product group representations, such as the calculation of Littlewood-Richardson and similar coefficients.

Section 2 of this paper outlines the theory underlying the construction of HWGs. This draws upon the standard plethystic functions and their use in symmetrising and antisymmetrising characters, as summarised in appendix A. We show how Weyl integration, as summarised in appendix B, can be used together with the plethystic functions to construct generating functions for the characters of irreps of classical and exceptional groups. We also show how the well-known Weyl Character Formula can be recast as a generating function for the characters of irreps of any classical or exceptional group. These character generating functions are instrumental in deriving HWGs and can also be useful for other group theoretic analyses. In section 2.4, we show how the character generating functions can be used to construct HWGs that encode the Littlewood-Richardson and similar coefficients governing the decomposition of the tensor products of irreps. In section 2.5, we calculate the HWGs for symmetrisations of objects transforming in the basic irreps of lower rank SU(N) simple groups. In section 2.6 we examine the relationship between the invariant tensors of a group and the plethystics of product groups, since this will be helpful in understanding the structure of HWGs for the GIOs of product groups.

In sections 3 and 4 we apply the HWG methodology to explicate and extend some established results regarding SQCD and the moduli spaces of selected instanton theories, respectively. Section 5 summarises the key new results and perspectives and concludes with a comparison of some geometric properties of HWGs and Hilbert series.

Notation and terminology. Some preliminary comments on notation are in order. We present the characters of groups either in the generic form $\mathcal{X}_{\text{Group}}$ or, more specifically, using Dynkin labels such as $[n_1, \ldots, n_r]_{\text{Group}}$, where r is the rank of the group (dropping group subscripts if no ambiguities arise). We may refer somewhat interchangeably to *series*, such as $1 + f + f^2 + \ldots$, by their *generating functions* $1/(1-f)$. We rely on the use of coordinates/variables to distinguish the different types of generating functions, as indicated in table 1.

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These different types of generating function are related and can be considered as a hierarchy in which the HWG, character and refined HS generating functions fully encode the group theoretic information. We label Cartan subalgebra ("CSA") coordinates for weights within characters by x or y, using subscripts when necessary; these coordinates are unimodular. We generally label object counting variables with t, and Dynkin label counting variables with m, although we may also use other letters, where this is helpful. We choose such counting variables to have a complex modulus of less than unity, which is essential for the residue calculations involved in Weyl integration to be valid. We follow established practice in referring to both counting variables and coordinates as "fugacities".

2 Highest weight generators

2.1 Representation of Dynkin labels using monomials

The canonical classification of the irreps of a Lie group is carried out using Dynkin labels. These label the states or weights within an irrep; each irrep has a unique highest weight and can therefore be identified by the Dynkin labels of its highest weight state. We can map the Dynkin labels $[n_1, \ldots, n_r]$ of the irreps of a group of rank r onto the complex manifold $\mathbb{C}^r$ by introducing the Dynkin label fugacities $\{m_1, \ldots, m_r\}$ and establishing a correspondence:

$$[n_1, \ldots, n_r] \Leftrightarrow \prod_{i=1}^{r} m_i^{n_i}. \tag{2.1}$$

If we choose the fugacities $\{m_1, \ldots, m_r\}$ to have absolute values less than unity, then each point on the infinite lattice of Dynkin labels corresponds to a unique point lying inside the unit complex disk on $\mathbb{C}^r$. This lattice is freely generated since the Dynkin labels can be chosen independently of each other. Now consider a second lattice on $\mathbb{C}^N$, with its points representing the possible combinations of N objects that transform in some representations of the group. Each such combination of objects will transform in some (generally reducible) representation of the group and there will be a non-trivial relationship between these lattices. Such relationships can be described most simply in terms of the dimensions of the representations, and the resulting polynomials of fugacities on $\mathbb{C}^N$ are termed unrefined Hilbert series:

$$\mathrm{HS}\left(t_i\right) = \sum_{k_i} a_{k_1,\ldots,k_N}\ {t_1}^{k_1} \ldots {t_N}^{k_N}. \tag{2.2}$$

We can, however, also choose to represent such series in other ways. If instead of dimensions, we use characters composed of monomials of CSA coordinates, then we obtain a *refined* Hilbert series:

$$\begin{aligned} &\mathrm{HS}\left(x_i, t_j\right) = \sum_{k_i} c_{k_1,\ldots,k_N}\left(x_1, \ldots, x_r\right)\ {t_1}^{k_1} \ldots {t_N}^{k_N}, \\ &\text{where} \\ &\quad a_{k_1,\ldots,k_N} = c_{k_1,\ldots,k_N}\left(1, \ldots, 1\right). \end{aligned} \tag{2.3}$$

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Alternatively, if we use the Dynkin labels of irreps of the group, then we obtain series expansions in terms of monomials in the Dynkin label fugacities:

$$\mathrm{HWG}\left(m_i, t_j\right) = \sum_{n_i, k_j} b_{n_1,\ldots,n_r,k_1,\ldots,k_N}\ {m_1}^{n_1} \ldots {m_r}^{n_r}\ {t_1}^{k_1} \ldots {t_N}^{k_N}. \tag{2.4}$$

While such series expansions necessarily represent lattices on $\mathbb{C}^{N+r}$, they can often be obtained from highest weight generating functions ("HWGs") based on a small number of monomials that encode very concisely the relationships between the combinations of objects and the irreps in which they transform.

2.2 Character generating functions

As a further preliminary to developing the theory of HWGs, it is useful to recall the Peter Weyl Theorem [10]. This entails that, in addition to being orthonormal under Weyl integration, the characters of a compact group form a complete basis for the class functions of the group. Since the PEs and PEFs of characters, being functions of characters, belong to the class functions of the group, it is always possible, in principle, to decompose such a PE or PEF into a sum of characters, each with some polynomial coefficient.

As summarised in appendix A, we can use the PE and PEF functions to symmetrise or antisymmetrise the characters of representations. It is well known that any irrep of a group can be obtained from a small number of basic irreps by some combination of symmetrisations and antisymmetrisations. It is interesting to explore how this relationship can be encoded most succinctly.

We can illustrate this for classical A series Lie groups using the PEF. Thus, taking the PEF of the fundamental irrep gives, for example,

$$\begin{aligned} &\mathrm{SU}(2): \qquad \mathrm{PEF}[[1]t] = 1 + [1]t + t^2, \\ &\mathrm{SU}(3): \qquad \mathrm{PEF}[[1,0]t] = 1 + [1,0]t + [0,1]t^2 + t^3, \end{aligned} \tag{2.5}$$

where we have represented the characters of the singlets [0] and [0,0] by unity. The pattern generalises to higher rank groups within the A series, for which we can obtain all the basic irreps [0,...,1,...,0] either from the fundamental (or antifundamental) by antisymmetrisation. In the case of the C series, this role is played by the fundamental representation, while in the case of the B and D series, this role is played by the vector and spinor representations together. In all cases, including Exceptional groups, we can use the PEF to generate all the basic irreps carrying a single unit Dynkin label starting from a small subset of the irreps of lowest dimension.

Given such a set of basic irreps of a group, with Dynkin labels [1,0,...,0] through [0,...,0,1], we can construct irreps with higher Dynkin labels. These higher irreps can all be obtained by symmetrisation of the basic irreps, followed by taking tensor products between these series. So, we can usefully define character generating functions $g^G(m_i, \mathcal{X})$, which encode the relationship between the full set of irreps, as identified by their Dynkin labels, and the action of the PE on the basic irreps:

$$g^G\left(m_i, \mathcal{X}\right) \equiv \sum_{n_i=0}^{\infty} [n_1, \ldots, n_r]_G \prod_{j=1}^{r} m_j^{n_j} \equiv P^G\left(m_i, \mathcal{X}\right) \mathrm{PE}\left[\sum_{i=1}^{r} \underbrace{[0, \ldots, 1_i, \ldots, 0]}_{1 \text{ in ith slot}} m_i\right]. \tag{2.6}$$

The numerators $P^G(m_i, \mathcal{X})$, which have been defined *implicitly*, are necessarily class functions of characters, since both the sum on the l.h.s. of (2.6) and the PE on the r.h.s. are class functions. For low rank groups, $P^G(m_i, \mathcal{X})$ takes a simple form, although some considerable analytical work can be required to find it for higher rank groups. A general procedure for finding $P^G(m_i, \mathcal{X})$ can be outlined as follows.

Firstly, we rearrange (2.6) to obtain:

$$P^G(m_i, \mathcal{X}) = \sum_{n_i=0}^{\infty} \left([n_1, \ldots, n_r] \prod_{j=1}^{r} m_j^{n_j} \right) \mathrm{PE}\left[-\sum_{k=1}^{r} [0, \ldots, 1_k, \ldots, 0]\, m_k \right]. \tag{2.7}$$

The class function $P^G(m_i, \mathcal{X})$ is thus a product of two series, of which the first is infinite and the second is finite.

For SU(2), we can find $P^{\mathrm{SU}(2)}(m_i, \mathcal{X})$ directly: we take x as the CSA coordinate and use $[1] \equiv (x + 1/x)$ as the character of the fundamental. It then follows that:

$$\begin{aligned} P^{\mathrm{SU}(2)} &\equiv \mathrm{PE}\left[-[1]\, m \right] \sum_{n=0}^{\infty} [n]\, m^n \\ &= (1 - m/x)\,(1 - xm) \sum_{n=0}^{\infty} [n]\, m^n \\ &= [0] + \sum_{n=0}^{\infty} \underbrace{\left([n+2] - [n+1]\,[1] + [n] \right)}_{=0} m^{n+2} \\ &= 1. \end{aligned} \tag{2.8}$$

The sum in the r.h.s. term vanishes for all n as a consequence of the multiplication law for SU(2). Thus, we obtain the simple generating function for the series of characters of SU(2):

$$g^{\mathrm{SU}(2)}(m, \mathcal{X}) \equiv \sum_{n=0}^{\infty} [n] m^n = \mathrm{PE}\left[[1]\, m \right] = 1 + [1]\, m + [2]\, m^2 + \ldots. \tag{2.9}$$

For SU(3), we can follow a similar route:

$$\begin{aligned} P^{\mathrm{SU}(3)} &\equiv \mathrm{PE}\left[-[1,0]\, m_1 - [0,1]\, m_2 \right] \sum_{n_i=0}^{\infty} [n_1, n_2]\, {m_1}^{n_1} {m_2}^{n_2} \\ &= (1 - m_1 x)\,(1 - m_1 y)\,(1 - m_1/xy)\,(1 - m_2/x)\,(1 - m_2/y)\,(1 - m_2 xy) \sum_{n_i=0}^{\infty} [n_1, n_2]\, {m_1}^{n_1} {m_2}^{n_2} \\ &= \left(\begin{array}{r} \left(m_2^3 m_1^3 - m_1^3 + m_2^2 m_1^2 + m_2 m_1 - m_2^3 + 1 \right) [0,0] \\ + \left(m_2^2 - m_2^2 m_1^3 - m_2 m_1^2 + m_2^3 m_1 - m_1 \right) [1,0] \\ + \left(m_2 m_1^3 - m_2^3 m_1^2 + m_1^2 - m_2^2 m_1 - m_2 \right) [0,1] \\ + m_1 m_2 \left(m_1 m_2 + 1 \right) [1,1] \\ - m_1^2 m_2\, [0,2] \\ - m_1 m_2^2\, [2,0] \end{array} \right) \sum_{n_i=0}^{\infty} [n_1, n_2]\, {m_1}^{n_1} {m_2}^{n_2} \\ &= (1 - m_1 m_2)\, [0,0] + \underbrace{\sum_{n_1 > 0 \,||\, n_2 > 0}^{\infty} (\ldots, \ldots,)\, [\ldots, \ldots,]\, {m_1}^{n_1} {m_2}^{n_2}}_{\text{r.h.s.}} \\ &= 1 - m_1 m_2. \end{aligned} \tag{2.10}$$

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Group	$P^G\left(m_i,\mathcal{X}\right)$
$A_1 \cong B_1 \cong C_1$	1
A_2	$(1-m_1m_2)$
B_2	$\left(1-m_1^2+m_1m_2^2-m_1^3m_2^2\right)+m_1m_2\left(m_1-1\right)[0,1]$
C_2	*As* B_2 *with* $m_1 \Leftrightarrow m_2$ *and* $[1,0] \Leftrightarrow [0,1]$
$D_2 \cong A_1 \otimes A_1$	1
G_2	See appendix C
A_3	$\left(1-m_2^2\right)\left(1-m_1m_3+{m_2m_3}^2+{m_1}^2m_2-m_1{m_2}^2m_3+{m_1}^2{m_2}^2{m_3}^2\right)[0,0,0]$ $+m_2\left(-m_3+m_1m_2-{m_1}^2m_2m_3+m_1{m_2}^2{m_3}^2\right)[1,0,0]$ $+m_2\left(-m_1+m_2m_3-m_1{m_2m_3}^2+{m_1}^2{m_2}^2m_3\right)[0\,,0\,,1]$ $+m_1m_2m_3\left(1-m_2^2\right)[0,1,0]$
D_3	*As* A_3 *with* $m_1 \Leftrightarrow m_2$ *and* $[1,0,0] \Leftrightarrow [0,1,0]$
A_4	See appendix D

Table 2. P^G for Low Rank Classical and Exceptional Groups.

Once again, it can be shown that the r.h.s. term vanishes and so we obtain the generating function for the characters of SU(3):

$$g^{\mathrm{SU}(3)}\left(m_1,m_2,\mathcal{X}\right) \equiv \sum_{n_i=0}^{\infty}\left[n_1,n_2\right]{m_1}^{n_1}{m_2}^{n_2} = \left(1-m_1m_2\right)\mathrm{PE}\left[\left[1,0\right]m_1\right]\mathrm{PE}\left[\left[0,1\right]m_2\right]. \tag{2.11}$$

More generally, we use the completeness principle to expand $P^G(m_i,\mathcal{X})$ as a superposition of characters, each with polynomial coefficients $P^G_{[A]}\left(m_i\right)$, where $[A]$ denotes the collective Dynkin labels corresponding to an irrep:

$$P^G\left(m_i,\mathcal{X}\right) \equiv \sum_{[A]} P^G_{[A]}\left(m_i\right)[A]. \tag{2.12}$$

We can then use Weyl integration, as described in appendix B, to project out the polynomial coefficients $P^G_{[A]}\left(m_i\right)$, one at a time, from:

$$P^G_{[A]}\left(m_i\right) = \oint_G d\mu\,[A]^* \sum_{n_i=0}^{\infty}\left(\left[n_1,\ldots,n_r\right]\prod_{j=1}^{r} m_j^{n_j}\right)\mathrm{PE}\left[-\sum_{k=1}^{r}\left[0,\ldots,1_k,\ldots,0\right]m_k\right]. \tag{2.13}$$

Without digressing into the mechanics of efficient algorithms for computing these coefficients, using *Mathematica*, for example, we compile the results of such calculations in table 2.

The results up to B_2, C_2, D_2 and A_3 correspond to those obtained by other methods [11]. The numerators tabulated in appendix C and appendix D should be understood as sums of characters with the polynomial coefficients given, similar to the numerator for A_3 in table 2. Importantly, for all the Classical and Exceptional groups, the class function

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P^G is finite and has palindromic properties. The finite nature of P^G is a consequence of relations between the characters generated from each basic irrep and, indeed, follows as a corollary of the construction of character generating functions based on the Weyl Character Formula, as described later in this section.

A polynomial in $\{t_1, \ldots, t_r\}$ is defined as palindromic of degree $(d_1, \ldots, d_r)$ if each monomial term $t_1^{n_1} \ldots t_r^{n_r}$ in the polynomial is paired with a term $t_1^{d_1-n_1} \ldots t_r^{d_r-n_r}$ with the same coefficients [5]. The polynomial is anti-palindromic if the coefficients in each pair are of opposite sign. Under this definition, the class function P^{B_2} in table 2, for example, is anti-palindromic of degree (3,2).

The generating functions for characters of irreps can be simplified to give generating functions for the dimensions of irreps by replacing characters with dimensions. The general form of these relationships follows from (2.6) and (A.2) as:

$$g^G(m_i) = \sum_{n_i=0}^{\infty} Dim[n_1, \ldots, n_r] \prod_{i=1}^{r} m_i^{n_i} = P^G(m_i, Dim\mathcal{X}) \Big/ \prod_{i=1}^{r} (1-m_i)^{Dim[0,\ldots,1_i,\ldots,0]}. \tag{2.14}$$

We then obtain the generating functions for dimensions summarised in table 3, with P^{A4} as set out in appendix E.

The palindromic properties of $P^G(m_i, Dim\mathcal{X})$ can clearly be seen. Interestingly, the degrees of the palindromic numerators are in all cases equal to the degrees of the denominators minus two.[1] It can readily be verified that all the generating functions given in table 2, table 3 and appendix C through appendix E are consistent both with the usual formulae for group dimensions and also with characters obtained by other methods, such as applying Cartan matrices to highest weights and using the Freudenthal multiplicity formula [10].

A similar analysis can be carried out for unitary groups and we include for reference in table 4 the elements of the character generating functions for unitary groups U(1) through U(4). There is a clear relationship between the P^G functions for the U(N) and the SU(N) series, which can be seen by dropping the last Dynkin labels from the irreps within P^G for a given U(N) series and assigning the corresponding polynomials to the resulting (truncated) irreps of the SU(N) series.

Before concluding this section it is useful also to demonstrate how the Weyl Character Formula can be recast as a generating function. This provides an alternative method for deriving the character generating functions described above; it also makes it possible to replace complicated character generating functions by finite sums over simpler generating functions, which can be helpful for some calculations.

The Weyl Character Formula [10] is given by:

$$\mathcal{X}(\lambda) = \frac{\sum_{w \in W} \det[w]\, e^{w(\lambda+\rho)}}{e^{\rho} \prod_{\alpha \in \Phi+} (1-e^{-\alpha})}. \tag{2.15}$$

The conventional notation translates to the explicit character constructions developed in this note as follows. The weight λ corresponds to the desired irrep with Dynkin la-

[1] Observation by A. Thomson, Imperial College.

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Group	$g^G(m_i)$	*Degree of Palindromic Numerator*
$A_1 \cong B_1 \cong C_1$	$\frac{1}{(1-m)^2}$	(0)
A_2	$\frac{(1-m_1m_2)}{(1-m_1)^3(1-m_2)^3}$	(1,1)
B_2	$\frac{\left(1+m_1-4m_1m_2+m_1m_2^2+m_1^2m_2^2\right)}{(1-m_1)^4(1-m_2)^4}$	(2,2)
C_2	*As* B_2 *with* $m_1 \Leftrightarrow m_2$	(2,2)
D_2	$\frac{1}{(1-m_1)^2(1-m_2)^2}$	(0)
G_2	$\frac{\left(\begin{array}{c}1+m_1+8m_2+8m_2^2+m_2^3\\ -26m_1m_2-41m_1m_2^2-6m_1m_2^3+15m_1^2m_2+78m_1^2m_2^2\\ +15m_1^2m_2^3-6m_1^3m_2-41m_1^3m_2^2-26m_1^3m_2^3\\ +m_1^4m_2+8m_1^4m_2^2+8m_1^4m_2^3+m_1^3m_2^4+m_2^4m_1^4\end{array}\right)}{(1-m_1)^6(1-m_2)^6}$	(4,4)
A_3	$\frac{\left(\begin{array}{c}1+m_2-m_1m_3-4m_1m_2-4m_2m_3\\ +5m_1m_2m_3+m_1^2m_2+m_2m_3^2\\ +m_2^2m_3^2+m_1^2m_2^2+5m_1m_3m_2^2\\ -4m_1^2m_2^2m_3-4m_1m_2^2m_3^2-m_1m_2^3m_3+m_1^2m_2^2m_3^2+m_1^2m_2^3m_3^2\end{array}\right)}{(1-m_1)^4(1-m_2)^5(1-m_3)^4}$	(2,3,2)
D_3	*As* A_3 *with* $m_1 \Leftrightarrow m_2$	(3,2,2)
A_4	$\frac{P^{A4}(m_1,m_2,m_3,m_4)}{(1-m_1)^5(1-m_2)^7(1-m_3)^7(1-m_4)^5}$	(3,5,5,3)

Table 3. Generating Functions for Dimensions of Low Rank Classical and Exceptional Groups.

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Group	$P^G(m_i, \mathcal{X})$			PE *of basic irreps*
U (1)	1		×	PE $[m]$
U (2)	1		×	PE $[[1,0]\,m_1 + [0,1]\,m_2]$
U (3)	$[0,0,0] - m_1 m_2\,[0,0,1]$		×	PE $[[1,0,0]\,m_1 + [0,1,0]\,m_2 + [0,0,1]\,m_3]$
U (4)	1	$[0,0,0,0]$	×	PE $[[1,0,0,0]\,m_1 + [0,1,0,0]\,m_2 + [0,0,1,0]\,m_3 + [0,0,0,1]\,m_4]$
	$-m_1 m_2$	$[0,0,1,0]$		
	$+\left({m_1}^2 m_2 - m_1 m_3 - {m_2}^2\right)$	$[0,0,0,1]$		
	$+m_2(m_1 m_2 - m_3)$	$[1,0,0,1]$		
	$+m_1 m_2 m_3$	$[0,1,0,1]$		
	$+{m_2}^2 m_3$	$[0,0,1,1]$		
	$+m_2\left({m_3}^2 - {m_1}^2 {m_2}^2\right)$	$[0,0,0,2]$		
	$-{m_1}^2 {m_2}^2 m_3$	$[1,0,0,2]$		
	$-m_1 {m_2}^3 m_3$	$[0,1,0,2]$		
	$+m_1 {m_2}^2 m_3(m_1 m_2 - m_3)$	$[0,0,1,2]$		
	$+{m_2}^2 m_3\left({m_1}^2 m_3 + m_1 {m_2}^2 - m_2 m_3\right)$	$[0,0,0,3]$		
	$+m_1 {m_2}^3 {m_3}^2$	$[1,0,0,3]$		
	$-{m_1}^2 {m_2}^4 {m_3}^2$	$[0,0,0,4]$		

Table 4. Generating Functions for Characters of Low Rank Unitary Groups.

bels $[n_1, \ldots n_r]$. The weight ρ corresponds to the weight of the Weyl vector, which is $[1, 1, \ldots 1]$ and so the formal exponential e^{ρ} corresponds to the CSA coordinate monomial $x_1 x_2 \ldots x_r \equiv x$. The parameter α ranges over the weights of the positive root space, so that $e^{-\alpha}$ corresponds to the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_r^{\alpha_r}$ for positive roots $\alpha, \beta, \ldots$. The matrices w or w_{ij} are elements of the finite Weyl group W of the Lie algebra, which acts upon the weights; their signs are given by their determinants, which are all real and unimodular.

The generating functions that we seek are rational functions that generate series of characters, so we introduce the fugacities m_i for the Dynkin labels and use these to form the generating functions:

$$g^G(m_i, \mathcal{X}) = \sum_{\lambda=0}^{\infty} \mathcal{X}(\lambda)\, m^{\lambda} \equiv \sum_{n_i=0}^{\infty} \mathcal{X}[n_1, \ldots, n_r]\, m_1^{n_1} \ldots m_r^{n_r}. \tag{2.16}$$

We proceed by combining and rearranging (2.15) and (2.16):

$$g^G(m_i, \mathcal{X}) = \frac{1}{e^{\rho} \prod\limits_{\alpha \in \Phi+} (1 - e^{-\alpha})} \sum_{w \in W} \det[w]\, e^{w(\rho)} \sum_{\lambda=[0,\ldots,0]}^{\infty} e^{w(\lambda)} m^{\lambda}. \tag{2.17}$$

We focus on the right hand term and switch to monomial notation, obtaining the rational expression:

$$\sum_{\lambda=0}^{\infty} e^{w(\lambda)} m^{\lambda} = \sum_{n_i=0}^{\infty} x_1^{w(n_1)} \ldots x_r^{w(n_r)} m_1^{n_1} \ldots m_r^{n_r} = \prod_j \frac{1}{1 - m_j \prod\limits_i {x_i}^{w_{ij}}}. \tag{2.18}$$

We can now combine (2.17) and (2.18) into a generating function for the series of characters of any irreps of a group G:

$$g^G(m_i, \mathcal{X}) = \frac{1}{x \prod\limits_{\alpha \in \Phi+} (1 - x^{-\alpha})} \sum_{w \in W} \det[w] \prod_j \frac{\prod\limits_i {x_i}^{w_{ij}}}{1 - m_j \prod\limits_i {x_i}^{w_{ij}}}. \tag{2.19}$$

The generating function (2.19) has advantages over the Weyl Character Formula, since the summation over the Weyl group needs only to be carried once, for any number of characters; it is also a finite rational function and can therefore be used in Weyl integration to project out any rational class function into an HWG. The Weyl group matrices necessary for calculations can be obtained from *Mathematica* add-on programs such as LieArt [12].

2.3 Transforming a Plethystic Exponential into a highest weight generating function

Consider now some other class function, such as the PE of a given set of irreps:

$$\mathrm{PE}[[A]\, t_A + \ldots [D]\, t_D] = \sum_{n_i=0}^{\infty} g^G{}_{[n_1,\ldots,n_r]}(t_A, \ldots, t_D)\, [n_1, \ldots, n_r]. \tag{2.20}$$

In this expression, the collective Dynkin labels $\{[A], \ldots [D]\}$ range over some selected irreps of the group and the $\{t_A, \ldots, t_D\}$ are object fugacities. The polynomial coefficients

$g^G{}_{[n_1,\ldots,n_r]}(t_A,\ldots,t_D)$ can be obtained one at a time from (2.20) with the help of the character orthonormality and completeness relations and using Weyl integration:

$$g^G{}_{[n_1,\ldots,n_r]}(t_A,\ldots,t_D) = \oint_G d\mu\,[n_1,\ldots,n_r]^*\,\mathrm{PE}\,[[A]\,t_A+\ldots[D]\,t_D]. \tag{2.21}$$

We can aggregate the coefficients $g^G{}_{[n_1,\ldots,n_r]}(t_A,\ldots,t_D)$ into a polynomial series, which corresponds to an HWG series equivalent to (2.4):

$$g^G(t_A,\ldots,t_D,m_i) \equiv \sum_{n_i=0}^{\infty} g^G{}_{[n_1,\ldots,n_r]}(t_A,\ldots,t_D){m_1}^{n_1}\ldots{m_r}^{n_r}. \tag{2.22}$$

This HWG series (2.22) encodes all the group theoretic information relating the multiplicities of the objects being symmetrised and the irreps in which they transform. We can extract this information from the HWG in various different ways. For example:

1. Given some multiplicities $\{k_A,\ldots,k_D\}$ of the objects represented by the monomials $t_A^{k_A}\ldots t_D^{k_D}$, we can identify the linked exponents of highest weight variables $m_1^{n_1}\ldots m_r^{n_r}$ amongst the monomials of the HWG series and the integer coefficients of these monomial terms. These enumerate the irreps $[n_1,\ldots,n_r]$ in which the objects transform along with their multiplicities.

2. Conversely, given some set of irreps $[n_1,\ldots,n_r]$ with their associated monomials $m_1^{n_1}\ldots m_r^{n_r}$, we can identify the linked monomials $t_A^{k_A}\ldots t_D^{k_D}$ amongst the terms of the HWG. The exponents $\{k_A,\ldots,k_D\}$ correspond to the combinations of basic objects that transform in the given irreps.

3. We can replace the monomials in Dynkin label fugacities $m_1^{n_1}\ldots m_r^{n_r}$ within the HWG by the corresponding irrep dimensions and obtain information about the number of combinations of basic objects at any given multiplicity. These unrefined Hilbert series take the form:

$$g^G(t_A,\ldots,t_D) \equiv \sum_{n_i=0}^{\infty} g^G{}_{[n_1,\ldots,n_r]}(t_A,\ldots,t_D)Dim[n_1,\ldots,n_r]. \tag{2.23}$$

4. We can map distinct object fugacities in a Hilbert series onto a single fugacity $\{t_A,\ldots,t_D\}\to t$ to obtain an unrefined Hilbert series $g^G(t)$.

The Hilbert series of a theory can thus be presented in various ways. Importantly, the HWG captures all the group theoretic properties of the class functions of the theory; if we encode the information as a HWG series, we can always extract an unrefined HS, but not vice versa.

Clearly (2.22) and (2.23) are in the form of infinite series and it is desirable to obtain these from rational polynomial generating functions. It can often be a non-trivial exercise to find the polynomial generating function for the HWG series $g^G(t_A,m_i)$ for the PE of some given set of objects. This calculation is, however, facilitated for groups for which we

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have the character generating functions $g^G(m_i, \mathcal{X})$, as described above, available. In these cases, we can, in principle, calculate the HWG by using Weyl integration to project the PE (expressed as a class function in terms of CSA coordinates) onto the complete basis for class functions provided by the character generating function:

$$\begin{aligned} g^G(t_A, m_i) &= \sum_{n_i=0}^{\infty} {g^G}_{[n_1,\ldots,n_r]} {m_1}^{n_1} \ldots {m_r}^{n_r} \\ &= \sum_{n_i=0}^{\infty} \oint_G d\mu \; {\mathcal{X}^*}_{[n_1,\ldots,n_r]} {m_1}^{n_1} \ldots {m_r}^{n_r} \, \mathrm{PE}\left[\mathcal{X}_{[A]} t_A + \ldots \mathcal{X}_{[D]} t_D\right] \\ &= \oint_G d\mu \; g^G(m_i, \mathcal{X}^*) \, \mathrm{PE}\left[\mathcal{X}_{[A]} t_A + \ldots \mathcal{X}_{[D]} t_D\right] \\ &= \oint_G d\mu \; P^G(m_i, \mathcal{X}^*) \, \mathrm{PE}\left[\sum_{i=1}^{r} [0, \ldots, 1_i, \ldots, 0]^* \, m_i\right] \mathrm{PE}\left[\mathcal{X}_{[A]} t_A + \ldots \mathcal{X}_{[D]} t_D\right]. \end{aligned} \tag{2.24}$$

It will be clear from the foregoing that knowledge of the $P^G(m_i, \mathcal{X})$ function is necessary to construct such an HWG from first principles, although sometimes the HWG can be found by inspection, starting from a finite number of the ${g^G}_{[n_1,\ldots,n_r]}(t_A, \ldots, t_D)$ terms in the series expansion. There may also be situations where it is not necessary to use the full character generating function, for example, if the PE is known to generate only irreps with certain symmetry properties.

HWGs can be used to analyse the combinatorics of group representations quite generally and it is instructive to review some archetypal situations involving (a) finding coefficients for the decomposition of tensor products of irreps and (b) symmetrisations and antisymmetrisations of basic irreps of groups.

2.4 Decomposing tensor products using HWGs

Given two irreps of a group with composite Dynkin labels $[A]$ and $[B]$, we can ask how their tensor product decomposes into a sum of irreps [C] with integer multiplicities a_{AB}^C. These integer coefficients are similar to Clebsch-Gordan coefficients, although they relate irreps rather than states within irreps. For U(N) they correspond to Littlewood Richardson coefficients:

$$[A] \otimes [B] = \sum_C a_{AB}^C \, [C]. \tag{2.25}$$

We can transform the relationship (2.25) using Weyl integration and character completeness/orthonormality to obtain an expression for the individual coefficients:

$$a_{AB}^C = \oint_G d\mu \, [C]^* \, [A] \, [B] \, . \tag{2.26}$$

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	$\sum_{A,B,C} c^C_{A,B} m^A n^B t^C$
A_1	$\frac{1}{(1-mn)(1-mt)(1-nt)}$
A_2	$\frac{1-m_1m_2n_1n_2t_1t_2}{(1-m_2n_1)(1-m_1n_2)(1-m_1t_1)(1-n_1t_1)(1-m_2t_2)(1-n_2t_2)(1-m_2n_2t_1)(1-m_1n_1t_2)}$
A_3	$\frac{1+...152\ monomial\ terms...+m_1^3m_2^4m_3^3n_1^3n_2^4n_3^3t_1^3t_2^4t_3^3}{\left((1-m_3n_1)(1-m_2n_2)(1-m_1n_3)(1-m_1t_1)(1-m_2t_2)(1-m_3t_3)\right.}$ $\times(1-n_1t_1)(1-n_2t_2)(1-n_3t_3)$ $\times(1-m_3n_2t_1)(1-m_2n_3t_1)(1-m_1n_1t_2)$ $\times(1-m_3n_3t_2)(1-m_2n_1t_3)(1-m_1n_2t_3)$ $\left.\times(1-m_1m_3n_2t_2)(1-m_2n_1n_3t_2)(1-m_2n_2t_1t_3)\right)$

Table 5. HWG for SU(N) Tensor Products.

We now rearrange the problem by introducing three sets of fugacities $\{m_i, n_i, t_i\}$ and by using three character generating functions $g^G(m_i, \mathcal{X})$, etc.:

$$\begin{aligned}\sum_{A,B,C} a^C_{AB} m^A n^B t^C &= \oint_G d\mu \sum_{A,B,C} [C]^* [A] [B] m^A n^B t^C \\ &= \oint_G d\mu\ g\left(t^C, \mathcal{X}^*\right) g\left(m^A, \mathcal{X}\right) g\left(n^B, \mathcal{X}\right) \\ &\equiv g^G(m_i, n_j, t_k)\,.\end{aligned} \tag{2.27}$$

We have used the shorthand notation $m^A \equiv \prod_{i=1}^{r} (m_i)^{A_i}$ for the product of fugacities relating to irrep $[A]$, and so on [13]. Once calculated explicitly, the resulting HWG generating function $g^G(m_i, n_j, t_k)$ allows the values of a^C_{AB} to be read off from the integer coefficients of the monomials $m^A n^B t^C$. For SU(2) through SU(4) the tensor product generating functions are shown in table 5.

For example, to obtain the decomposition of $[1,0] \otimes [0,2]$, we identify the terms $m_1 n_2^2 \left(t_1 t_2^2 + t_2\right)$ in the Taylor series expansion for the A_2 generating function and obtain $[1,2] \oplus [0,1]$.

If we examine the individual denominator terms, we find that they obey the rule that the sum of the subscripts on the m_i and n_i less the sum of the subscripts on the t_i variables modulo the dimension of the fundamental is zero. This occurs since (i) the HWG is formed from singlets, (ii) the indices count antisymmetrisations (under the labelling system adopted) and, (iii) antisymmetrisation of the fundamental by the epsilon tensor,

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which has dimension equal to the fundamental, leads back to a singlet. Put another way, the sum of central charges counting the degree of antisymmetrisation of each irrep is a conserved quantity under a tensor product.

Also, since (2.27) is symmetric between the irreps A,B and C*, and the irreps of SU(N) are conjugate under reversal of Dynkin labels, we have the consquence that the HWGs in table 5 are symmetric under interchange of any two fugacities $\{m_i, n_i, t_{N-i}\}$.

The numerators of these HWGs for tensor products of A series groups are all palindromic. We can also observe that in all three A series cases, the degree of each variable in the denominator exceeds that in the numerator by two. The palindromic numerator for A_3 is somewhat lengthy containing 154 monomial terms in total and we do not present it here.

This approach to calculating the coefficients for decomposing tensor products of irreps from Dynkin label fugacties can, in principle, be generalised to unitary groups, yielding generators for Littlewood Richardson Coefficients, as well as to other classical and exceptional groups.

2.5 HWGs for symmetrisation of basic irreps of SU(N) Groups

We can also use the character generating functions $g^G(m_i, \mathcal{X})$ to obtain HWGs for symmetrisations of objects transforming in the basic irreps of a SU(N) group by evaluating Weyl integrals:

$$\begin{aligned} g^G(t_i, m_j) &= \oint_G d\mu\; g^G(m_j, \mathcal{X}^*)\, \mathrm{PE}\left[\sum_{i=1}^{r} [0,\ldots,1_i,\ldots,0]\, t_i\right] \\ &= \oint_G d\mu P^G(m_j, \mathcal{X}^*) \mathrm{PE}\left[\sum_{j=1}^{r} [0,\ldots,1_j,\ldots,0]^*\, m_j\right] \mathrm{PE}\left[\sum_{i=1}^{r} [0,\ldots,1_i,\ldots,0]\, t_i\right]. \end{aligned} \tag{2.28}$$

The t_i count the objects being symmetrised, while the m_i count the Dynkin labels of the resulting irreps. Explicit evaluations of $g^G(t_i, m_j)$ using generating functions based on table 2 are set out in table 6 for SU(2) through SU(4). These can alternatively be written more compactly by taking their PLs with respect to the fugacities, as shown in table 7. In all the cases calculated, the HWGs of the basic irreps lead to finite PLs with all positive terms and are said to be *freely generated*. If we choose to restrict the series by selecting a subset of the basic irreps for symmetrisation (by setting some of the t_i to zero) this simply has the effect of dropping monomials from the PL, and so the subseries of the HWGs are also *freely generated*.

These polynomial generating functions $g^G(t_i, m_j)$ encode, through the exponents of the fugacities, the combinations of irreps formed by the symmetrising action of the PE on the basic irreps. We can in principle carry out similar procedures for other classical or exceptional Lie algebras and also obtain polynomial generating functions that encode the antisymmetrising action of the PEF. The combinatorics of subsets of the basic irreps, when the t_i do not span the rank of the group, can be read off from such HWGs. Alternatively, HWG series can in principle be obtained for any chosen subset of fugacities by following similar procedures to those set out above.

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A_r	HWG of PE $\left[\sum_{i=1}^{r}[0,\ldots,1_i,\ldots,0]\,t_i\right]$
A_1	$\frac{1}{(1-mt)}$
A_2	$\frac{1}{(1-t_1t_2)(1-m_1t_1)(1-m_2t_2)}$
A_3	$\frac{1}{(1-t_1t_3)\left(1-t_2^2\right)(1-m_1t_1)(1-m_2t_2)(1-m_3t_3)(1-m_3t_1t_2)(1-m_1t_2t_3)(1-m_2t_1t_2t_3)}$

Table 6. HWG of SU(N) basic irreps.

A_r	PL [HWG]
A_1	mt
A_2	$t_1t_2+m_1t_1+m_2t_2$
A_3	$t_1t_3+t_2^2+m_1t_1+m_2t_2+m_3t_3+m_3t_1t_2+m_1t_2t_3+m_2t_1t_2t_3$

Table 7. PLs of HWG of SU(N) basic irreps.

2.6 HWGs for invariant tensors

Any group has various invariant tensors, such as delta or epsilon tensors, structure constants and intertwiners, and these can be combined in many ways. For example, the invariant tensors can be contracted with vector or spinor fields to yield fields transforming in other irreps. Also, the invariant tensors of the adjoint representation can be combined with the generators of group transformations to create scalar Casimir operators whose eigenvalues identify the irreps in which fields transform [14]. The totally symmetric and totally antisymmetric invariant tensors for the defining representations of various Classical and Exceptional Lie Groups are summarised in table 8.

As can be seen from table 8, each classical and exceptional group has a unique signature in terms of the invariant tensors of its defining representation [14]. Within these, there is a minimal set of tensors in terms of which the other invariant tensors can be expressed, termed primitive tensors. If these primitive tensors are symmetric, they can be symmetrised into symmetric tensors of higher degree; if they are antisymmetric, they are forms over the co-cycles of the group manifold [15], and can only be antisymmetrised up to the length of the overall volume form (or epsilon tensor), as determined by the dimension of the defining representation. The symmetric and antisymmetric tensors can also be combined into tensors of mixed symmetry and the number of such possible combinations compounds with increasing rank and dimension; the exceptional groups in particular posses a very complicated set of defining representation invariants when invariant tensors of mixed symmetry are included [16].

Corresponding tables can also be generated for other representations, such as the adjoint, in which cases the tensors correspond to the Casimirs of the group. There are also many invariant tensors that can be formed from combinations of representations.

It is therefore interesting to consider how all the invariant tensors of some representation(s) might most effectively be enumerated. This question is closely related to the

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Group	*Defining Representation* $\mathcal{X}$	$\mathrm{PL}\left[\oint d\mu\, \mathrm{PE}\left[\mathcal{X}\ t\right]\right]$	$\oint d\mu\, \mathrm{PEF}\left[\mathcal{X}\ t\right]$	*Degrees of primitive Sym tensors*	*Degrees of primitive AS tensors*
A_r	$[1,0,\ldots,0]$	0	$1+t^{r+1}$	–	$r+1$
B_r	$\begin{cases}[2]\, for\ r=1\\ [1,0,\ldots,0]\, for\ r>1\end{cases}$	t^2	$1+t^{2r+1}$	2	$2r+1$
C_r	$[1,0,\ldots,0]$	0	$1+t^2+t^4+\ldots,t^{2r}$	–	$2,4,\ldots,2r$[(*)]
D_r	$\begin{cases}[1,1]\, for\ r=2\\ [1,0,\ldots,0]\, for\ r>2\end{cases}$	t^2	$1+t^{2r}$	2	$2r$
G_2	$[0,1]$	t^2	$\left(1+t^3\right)\left(1+t^4\right)$	2	$3,4$
F_4	$[0,0,0,1]$	t^2+t^3	$\left(1+t^9\right)\left(1+t^{17}\right)$	$2,3$	$9,17$
E_6	$[0,0,0,0,1,0]$	t^3	$1+t^{27}$	3	27
E_7	$[0,0,0,0,0,1,0]$	t^4	$1+t^2+t^4\ldots,+t^{56}$	4	$2,4,\ldots,56$[(*)]
E_8	$[0,0,0,0,0,0,1,0]$	$t^2+t^8+t^{12}+t^{14}$ $+t^{18}+t^{20}+t^{24}+t^{30}$	$\left(1+t^3\right)\left(1+t^{15}\right)\left(1+t^{23}\right)\left(1+t^{27}\right)$ $\times\left(1+t^{35}\right)\left(1+t^{39}\right)\left(1+t^{47}\right)\left(1+t^{59}\right)$	$2,8,12,14,$ $18,20,24,30$	$3,15,23,27,$ $35,39,47,59$

[(*)] Full list of AS tensors rather than primitives.

Table 8. Invariants of Defining Representations of Classical and Exceptional Lie Groups.

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problem of identification of GIOs in SQCD, which is the subject of the next section. In the case of symmetric and antisymmetric tensors, respectively, these can be enumerated in a straightforward manner by using Weyl integration to project out the singlets of characters that have been symetrised using the PE, or antisymmetrised using the PEF. Thus we find the degrees of primitive symmetric tensors are given by $t^{d_i^S}$ from the Hilbert series:

$$\sum_i t^{d_i^S} = \mathrm{PL}\left[\oint d\mu \ \mathrm{PE}\left[\mathcal{X}\ t\right]\right], \tag{2.29}$$

and the degrees of antisymmetric tensors given by $t^{d_i^A}$ from the series:

$$1 + \sum_i t^{d_i^A} = \oint d\mu \ \mathrm{PEF}\left[\mathcal{X}\ t\right]. \tag{2.30}$$

The identification of invariant tensors of mixed symmetry is not so straightforward; in particular, it is necessary to have some way of describing the symmetry properties of each tensor, in addition to tracking the number of its indices. One solution is provided by encoding the symmetry properties of the tensor as a HWG of a unitary representation of sufficiently high dimension. This method makes use of the pattern of symmetrisations and antisymmetrisations that arises when a product group is symmetrised, as will be explained.

Consider the symmetrisation of an irrep transforming in a product group $A \otimes B$:

$$Sym^2(A \otimes B) = Sym^2 A \otimes Sym^2 B + \Lambda^2 A \otimes \Lambda^2 B. \tag{2.31}$$

The resulting irreps are either symmetrisations of both constituent groups, or antisymmetrisations of both constituent groups. This contrasts with the situation under overall antisymmetrisation, when the constituent group irreps are of opposite symmetry:

$$\Lambda^2(A \otimes B) = Sym^2 A \otimes \Lambda^2 B + \Lambda^2 A \otimes Sym^2 B. \tag{2.32}$$

This behaviour of product groups under symmetrisation generalises to any order of symmetrisation n to give all possible combinations, such that the symmetry properties of $Sym^n(A \otimes B)$, which can be described in terms of all the possible Young tableaux at order n, match between A and B [16]. Whenever the symmetry properties of B correspond to one of its invariant tensors or singlets, this yields a GIO of B that transforms in some representation of A; and we can use a HWG to encode the symmetry properties of this representation of A.

As a first step we express the series of GIOs of B as a class function in terms of the characters of $\mathrm{U}(A)$ using the fugacity t:

$$\sum_{n_i=0}^{\infty} a_{n_1,\ldots,n_A}(t)\,[n_1,\ldots,n_A] = \oint_B d\mu \ \mathrm{PE}\left[\mathcal{X}\left[U(A) \otimes B\right] t\right]. \tag{2.33}$$

As a second step we use a generating function for the characters of $\mathrm{U}(A)$ with the Dynkin label fugacities m_i to transform the class function (2.33) into an HWG:

$$g^{U(A)}(m_i, t) = \oint_{\mathrm{U}(A)} d\mu \ g^{U(A)}(m_i, \mathcal{X}^*) \oint_B d\mu \ \mathrm{PE}\left[\mathcal{X}\left[\mathrm{U}(A) \otimes B\right] t\right]. \tag{2.34}$$

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	Gauge/Colour	Flavour	Flavour				
	$SU(N_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_Q$	$U(1)_{\bar{Q}}$	$U(1)_B$	$U(1)_R$
Q_a^i	$[0,\ldots,1]$	$[1,0\ldots]$	1	1	0	1	$(N_f-N_c)/N_f$
$\bar{Q}_i^a$	$[1,0\ldots]$	1	$[0,\ldots,1]$	0	- 1	- 1	$(N_f-N_c)/N_f$

Table 9. SQCD Charge Assignments for $SU(N_f) \times SU(N_c)$.

The HWG series (2.34) can be presented in terms of Young tableaux for $U(N)$, with the total number of boxes being given by the exponents of the fugacities t in a partition described by the fugacities m_i. Similar results also can be obtained by decomposition of tensor products [16], however, the HWG approach has the potential advantage of generating the complete infinite series of GIOs, thereby resolving uncertainties about the multiplicities of distinct invariants and/or their appearance at higher orders. It can be noted that equivalent series can also be obtained by using the $SU(N)$ group in place of $U(N)$, since the rightmost Dynkin label of a $U(N)$ representation can be calculated from the Dynkin labels of the corresponding $SU(N)$ representation, together with the total number of Young boxes given by the fugacity t. This provides an alternative calculation schema, which can be more efficient.

Thus, HWGs neatly illustrate how the invariant tensors of the group representations comprised within a product group theory determine the Hilbert series of GIOs. The HWG monomials identify both the order at which such GIOs are formed and the representations in which these objects transform. The above series for product groups incorporating $U(N)$ are closely related to the series for SQCD, which are the subject of detailed examples in the next section.

3 HWGs of gauge invariant objects of SQCD

We now apply the tools and methods developed in the previous sections to study the HWGs of product groups associated with SQCD quiver theories. These are theories with no superpotential, i.e. $W \equiv 0$, whose Hilbert series have been calculated within the Plethystics Program [3–5]. We now wish to construct HWG's for such theories. While we do not aim to give an exhaustive treatment, we set out the principles and provide representative examples.

3.1 GIOs of $SU(N_f) \times SU(N_c)$

It is helpful to start with an explicit statement of the symmetry transformation properties of the objects within the theory and these are set out in table 9, based on [4].

The theory consists of chiral quarks Q_a^i and antiquarks $\bar{Q}_i^a$ transforming in the $SU(N_f)_L \times SU(N_f)_R \times SU(N_c)$ product group, where the indices i range over the fundamentals/anti-fundamentals of the $SU(N_f)_L$ and $SU(N_f)_R$ flavour groups and the indices a range over the $SU(N_c)$ colour group. The GIOs of the theory consist of $SU(N_c)$ colour singlets composed of quarks and/or antiquarks, with some combination of paired and unpaired $SU(N_f)_L \times SU(N_f)_R$ indices determining the flavour representation in which they transform. The

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Theory	HWG
$\mathrm{SU}(2)_f \times \mathrm{SU}(2)_c$	$1/(1-t^2)$
$\mathrm{SU}(2N_f \geq 4) \times \mathrm{SU}(2)_c$	$1/(1-l_2t^2)$
$\mathrm{U}(1)_{f,L} \times \bar{\mathrm{U}}(1)_{f,R} \times \mathrm{SU}(2)_c$	$1/(1-t_1t_2)$
$\mathrm{SU}(2)_{f,L} \times \mathrm{SU}(2)_{f,R} \times \mathrm{SU}(2)_c$	$1/(1-{t_1}^2)(1-{t_2}^2)(1-l_1r_1t_1t_2)$
$\mathrm{SU}(N_f \geq 3)_L \times \mathrm{SU}(N_f \geq 3)_R \times \mathrm{SU}(2)_c$	$1/(1-l_2{t_1}^2)(1-r_2{t_2}^2)(1-l_1r_1t_1t_2)$
$\mathrm{U}(1)_{f,L} \times \bar{\mathrm{U}}(1)_{f,R} \times \mathrm{SU}(3)_c$	$1/(1-t_1t_2)$
$\mathrm{SU}(2)_{f,L} \times \mathrm{SU}(2)_{f,R} \times \mathrm{SU}(3)_c$	$1/(1-l_1r_1t_1t_2)(1-{t_1}^2{t_2}^2)$
$\mathrm{SU}(3)_{f,L} \times \mathrm{SU}(3)_{f,R} \times \mathrm{SU}(3)_c$	$1/(1-{t_1}^3)(1-{t_2}^3)(1-l_1r_1t_1t_2)(1-l_2r_2{t_1}^2{t_2}^2)$
$\mathrm{SU}(N_f \geq 4)_L \times \mathrm{SU}(N_f \geq 4)_R \times \mathrm{SU}(3)_c$	$1/(1-l_3{t_1}^3)(1-r_3{t_2}^3)(1-l_1r_1t_1t_2)(1-l_2r_2{t_1}^2{t_2}^2)$

Table 10. HWGs of $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{SU}(N_c)$ colour singlets.

fields also carry U(1) charges. The GIOs are restricted to symmetrised combinations of the chiral quarks and antiquarks.

To obtain HWGs for these theories, we first form the characters of the fields for a given $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{SU}(N_c)$ product group. The characters are given by:

$$\begin{aligned} \mathcal{X}_Q &= [1,0,\ldots,0]_{(N_f)_L}[0,\ldots,0,1]_{N_c}, \\ \mathcal{X}_{\bar{Q}} &= [1,0,\ldots,0]_{N_c}[0,\ldots,0,1]_{(N_f)_R}. \end{aligned} \tag{3.1}$$

While the colour group CSA coordinates are shared by the quarks and antiquarks, it is necessary to use different CSA coordinates to distinguish the L and R flavour groups. We introduce the fugacities t_1 and t_2 for the quarks and anti-quarks respectively, reflecting their different U(1) charges, and symmetrise their characters by applying the PE. A first Weyl integration over the colour group projects out the $\mathrm{SU}(N_c)$ singlets. We then introduce the character generating functions $g^{(N_f)_L}(l_i, \mathcal{X}_L^*)$ and $g^{(N_f)_R}(r_i, \mathcal{X}_R^*)$, where the l_i and r_i Dynkin label fugacities apply to the L and R flavour groups, respectively. We adopt a particular labelling system for the Dynkin label slots by counting $[l_1,\ldots,l_r]$ from the left and $[r_r,\ldots,r_1]$ from the right. Finally, we apply second and third Weyl integrations over the L and R flavour groups. This procedure, which yields the HWG generating functions that enumerate the GIOs formed from the quarks and antiquarks, can be encapsulated as:

$$\begin{aligned} g^{((N_f)_L,(N_f)_R,N_c)}(l_i,r_i,t_i) = \oint_{\mathrm{SU}(N_f)_L} \oint_{\mathrm{SU}(N_f)_R} \oint_{\mathrm{SU}(N_c)} d\mu g^{\mathrm{SU}(N_f)_L}(l_i,\mathcal{X}_L{}^*)\, g^{\mathrm{SU}(N_f)_R}(r_i,\mathcal{X}_R{}^*) \\ \times \mathrm{PE}\left[\mathcal{X}_Q t_1 + \mathcal{X}_{\bar{Q}} t_2\right]. \end{aligned} \tag{3.2}$$

The results of the calculation of (3.2) are summarised as HWGs in table 10. For comparison with established results [4] (and for reasons to be explained shortly) this table also includes cases where the symmetry group is $\mathrm{SU}(2N_f) \times \mathrm{SU}(2)_c$, with just one flavour group and a single fugacity labelled by t.

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The terms in the HWG encode the structure of the colour singlets of the theory and the flavour group representations to which they belong. Thus, for example, the monomials $\{{t_1}^3, {t_2}^3, l_1 r_1 t_1 t_2, l_2 r_2 {t_1}^2 {t_2}^2\}$ for $\mathrm{SU}(3)_{f,L} \times \mathrm{SU}(3)_{f,R} \times \mathrm{SU}(3)_c$ correspond, respectively, to three quarks and three antiquarks transforming as flavour singlets, a quark antiquark pair (meson) transforming in $[1,0]_L \otimes [0,1]_R$ and a combination of two quarks and two antiquarks (tetraquark) transforming in $[0,1]_L \otimes [1,0]_R$.

Interestingly, all the HWGs constructed are the same for all flavour groups of fundamental dimension exceeding the fundamental dimension of the colour group. This arises because the antisymmetrisations of the fundamental of the flavour group generated by the PE are limited by the length of the colour group epsilon tensor, which is the most antisymmetric invariant tensor available in the defining representation. The differences in the unrefined HS as the rank of the flavour group is increased are simply due to the different dimensions of the flavour group irreps. Thus, the HWGs of this SQCD theory are the same for all $\mathrm{SU}(N)$ flavour groups of fundamental dimension exceeding that of the $\mathrm{SU}(N)$ colour group.

A further interesting observation [4] is that with a SU(2) colour group, the left and right flavour groups can be combined into a $\mathrm{SU}(2N_f)$ global symmetry. This particular feature arises for a SU(2) colour group because SU(2) quarks and antiquarks share the same character. Thus, we can set up a fugacity map in which the CSA flavour coordinates plus fugacity degrees of freedom match between $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ and $\mathrm{SU}(2N_f)_L$, equalling $2N_f$ on both sides.

The only proviso is that it is necessary to choose all the fugacities to differ only by a U(1) phase. Such alternative ways of analysing the same problem give rise to correspondences between the various Hilbert series as tabulated. Thus, for example, the unrefined Hilbert series for $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ in table 11 is the same as the Hilbert series for $\mathrm{SU}(4) \times \mathrm{SU}(2)$, (which we can obtain by taking the Hilbert series with distinct counting for $\mathrm{SU}(4) \times \mathrm{SU}(4) \times \mathrm{SU}(2)$ and applying the mapping $t_1 \to t, t_2 \to 0$).

If we are primarily interested in counting dimensions of flavour irreps, the HWGs can be reduced to unrefined Hilbert series by replacing the monomial terms in $\{l_i, r_j\}$ by the dimensions of the irreps to which they refer. This is equivalent to replacing the coordinates within the characters of the flavour group by unity. We then have:

$$g^{((N_f)_L,(N_f)_R,N_c)}(t_i) = \oint_{\mathrm{SU}(N_c)} d\mu \ \mathrm{PE}^{N_f}\left[[0,\ldots,0,1]_{\mathrm{SU}(N_c)} t_1 + [1,0,\ldots,0]_{\mathrm{SU}(N_c)} t_2\right]. \quad (3.3)$$

These unrefined Hilbert series, shown in tables 11 and 12, can be further simplified, as shown, by equating the quark and antiquark fugacities. The results match those given in [4]. We defer a comparison of the descriptions of the vacuum moduli spaces given by the HWGs with those provided by Hilbert series to the concluding section.

3.2 GIOs of $\mathrm{SU}(N_f) \times \mathrm{SO}(N_c)$

We can apply a similar methodology to the analysis of the SQCD gauge theories that arise when the colour group is a member of the B, C or D series of classical groups. The

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Theory	*Unrefined Hilbert Series* (t_1, t_2)	*Unrefined Hilbert Series* (t)
$\mathrm{SU}(2)_f \times \mathrm{SU}(2)_c$	$\frac{1}{(1-t^2)}$	$\frac{1}{(1-t^2)}$
$\mathrm{SU}(2)_{f,L} \times \mathrm{SU}(2)_{f,R} \times \mathrm{SU}(2)_c$	$\frac{(1+t_1t_2)}{(1-t_1{}^2)(1-t_2{}^2)(1-t_1t_2)^3}$	$\frac{1+t^2}{(1-t^2)^5}$
$\mathrm{SU}(3)_{f,L} \times \mathrm{SU}(3)_{f,R} \times \mathrm{SU}(2)_c$	$\frac{(1+t_1t_2)(1+3t_1t_2-3t_1^3t_2-2t_1^2t_2^2-3t_1t_2^3+3t_1^3t_2^3+t_1^4t_2^4)}{(1-t_1^2)^3(1-t_2^2)^3(1-t_1t_2)^5}$	$\frac{(1+t^2)(1+5t^2+t^4)}{(1-t^2)^9}$
$\mathrm{SU}(4)_{f,L} \times \mathrm{SU}(4)_{f,R} \times \mathrm{SU}(2)_c$	$\frac{(1+t_1t_2)\left(\begin{array}{l}1+t_1^2+8t_1t_2-8t_1^3t_2+t_2^2\\+2t_1^2t_2^2-21t_1^4t_2^2+10t_1^6t_2^2-8t_1t_2^3-8t_1^3t_2^3\\+16t_1^5t_2^3-21t_1^2t_2^4+54t_1^4t_2^4-21t_1^6t_2^4+16t_1^3t_2^5\\-8t_1^5t_2^5-8t_1^7t_2^5+10t_1^2t_2^6-21t_1^4t_2^6+2t_1^6t_2^6\\+t_1^8t_2^6-8t_1^5t_2^7+8t_1^7t_2^7+t_1^6t_2^8+t_1^8t_2^8\end{array}\right)}{(1-t_1^2)^5(1-t_2^2)^5(1-t_1t_2)^7}$	$\frac{(1+t^2)(1+14t^2+36t^4+14t^6+t^8)}{(1-t^2)^{13}}$
$\mathrm{SU}(5)_{f,L} \times \mathrm{SU}(5)_{f,R} \times \mathrm{SU}(2)_c$	$\frac{(1+t_1t_2)\left(\begin{array}{l}1+3t_1^2+t_1^4+15t_1t_2-5t_1^3t_2\\-10t_1^5t_2+3t_2^2+30t_1^2t_2^2-109t_1^4t_2^2+46t_1^6t_2^2\\-5t_1t_2^3-70t_1^3t_2^3-5t_1^5t_2^3+115t_1^7t_2^3-35t_1^9t_2^3\\+t_2^4-109t_1^2t_2^4+171t_1^4t_2^4+106t_1^6t_2^4-94t_1^8t_2^4\\-10t_1t_2^5-5t_1^3t_2^5+405t_1^5t_2^5-505t_1^7t_2^5+115t_1^9t_2^5\\+46t_1^2t_2^6+106t_1^4t_2^6-404t_1^6t_2^6+106t_1^8t_2^6+46t_1^{10}t_2^6\\+115t_1^3t_2^7-505t_1^5t_2^7+405t_1^7t_2^7-5t_1^9t_2^7-10t_1^{11}t_2^7\\-94t_1^4t_2^8+106t_1^6t_2^8+171t_1^8t_2^8-109t_1^{10}t_2^8+t_1^{12}t_2^8\\-35t_1^3t_2^9+115t_1^5t_2^9-5t_1^7t_2^9-70t_1^9t_2^9-5t_1^{11}t_2^9\\+46t_1^6t_2^{10}-109t_1^8t_2^{10}+30t_1^{10}t_2^{10}+3t_1^{12}t_2^{10}-10t_1^7t_2^{11}\\-5t_1^9t_2^{11}+15t_1^{11}t_2^{11}+t_1^8t_2^{12}+3t_1^{10}t_2^{12}+t_1^{12}t_2^{12}\end{array}\right)}{(1-t_1^2)^7(1-t_2^2)^7(1-t_1t_2)^9}$	$\frac{(1+t^2)(1+27t^2+169t^4+321t^6+169t^8+27t^{10}+t^{12})}{(1-t^2)^{17}}$

Table 11. Unrefined HS of $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{SU}(2)_c$ colour singlets.

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Theory	*Unrefined Hilbert Series* (t_1, t_2)	*Unrefined Hilbert Series* (t)
$\mathrm{U}(1)_{f,L} \times \bar{\mathrm{U}}(1)_{f,R} \times \mathrm{SU}(3)_c$	$\frac{1}{(1-t_1t_2)}$	$\frac{1}{(1-t^2)}$
$\mathrm{SU}(2)_{f,L} \times \mathrm{SU}(2)_{f,R} \times \mathrm{SU}(3)_c$	$\frac{1}{(1-t_1t_2)^4}$	$\frac{1}{(1-t^2)^4}$
$\mathrm{SU}(3)_{f,L} \times \mathrm{SU}(3)_{f,R} \times \mathrm{SU}(3)_c$	$\frac{\left(1-t_1^3t_2^3\right)}{\left(1-t_1^3\right)\left(1-t_2^3\right)(1-t_1t_2)^9}$	$\frac{t^2-t+1}{(1-t)(1-t^2)^8(1-t^3)}$
$\mathrm{SU}(4)_{f,L} \times \mathrm{SU}(4)_{f,R} \times \mathrm{SU}(3)_c$	$\frac{\left(\begin{array}{l} 1+4t_1t_2-4t_1^4t_2+10t_1^2t_2^2-16t_1^5t_2^2+6t_1^8t_2^2+4t_1^3t_2^3 \\ -16t_1^6t_2^3+8t_1^9t_2^3-4t_1t_2^4+2t_1^4t_2^4-4t_1^7t_2^4+6t_1^{10}t_2^4-16t_1^2t_2^5 \\ +20t_1^5t_2^5-4t_1^8t_2^5-16t_1^3t_2^6+38t_1^6t_2^6-16t_1^9t_2^6-4t_1^4t_2^7+20t_1^7t_2^7 \\ -16t_1^{10}t_2^7+6t_1^2t_2^8-4t_1^5t_2^8+2t_1^8t_2^8-4t_1^{11}t_2^8+8t_1^3t_2^9-16t_1^6t_2^9 \\ +4t_1^9t_2^9+6t_1^4t_2^{10}-16t_1^7t_2^{10}+10t_1^{10}t_2^{10}-4t_1^8t_2^{11}+4t_1^{11}t_2^{11}+t_1^{12}t_2^{12} \end{array}\right)}{\left(1-t_1^3\right)^4\left(1-t_2^3\right)^4(1-t_1t_2)^{12}}$	$\frac{\left(1+t^2\right)\left(1+3t^2+4t^3+7t^4+4t^5+7t^6+4t^7+3t^8+t^{10}\right)}{(1-t^2)^{12}(1-t^3)^4}$
$\mathrm{SU}(5)_{f,L} \times \mathrm{SU}(5)_{f,R} \times \mathrm{SU}(3)_c$	$\frac{\left(\begin{array}{l} 1+3t_1^3+t_1^6+9t_1t_2+2t_1^4t_2-16t_1^7t_2+45t_1^2t_2^2-75t_1^5t_2^2 \\ +15t_1^8t_2^2+15t_1^{11}t_2^2+3t_2^3+74t_1^3t_2^3-292t_1^6t_2^3+245t_1^9t_2^3 \\ -65t_1^{12}t_2^3+2t_1t_2^4+t_1^4t_2^4-313t_1^7t_2^4+595t_1^{10}t_2^4-300t_1^{13}t_2^4 \\ +50t_1^{16}t_2^4-75t_1^2t_2^5-45t_1^5t_2^5+150t_1^8t_2^5+210t_1^{11}t_2^5-315t_1^{14}t_2^5 \\ +75t_1^{17}t_2^5+t_2^6-292t_1^3t_2^6+731t_1^6t_2^6-210t_1^9t_2^6-140t_1^{12}t_2^6 \\ -35t_1^{15}t_2^6+50t_1^{18}t_2^6-16t_1t_2^7-313t_1^4t_2^7+1634t_1^7t_2^7-2090t_1^{10}t_2^7 \\ +715t_1^{13}t_2^7-35t_1^{16}t_2^7+15t_1^2t_2^8+150t_1^5t_2^8+675t_1^8t_2^8-2175t_1^{11}t_2^8 \\ +1650t_1^{14}t_2^8-315t_1^{17}t_2^8+245t_1^3t_2^9-210t_1^6t_2^9-725t_1^9t_2^9+100t_1^{12}t_2^9 \\ +715t_1^{15}t_2^9-300t_1^{18}t_2^9+595t_1^4t_2^{10}-2090t_1^7t_2^{10}+1775t_1^{10}t_2^{10} \\ +100t_1^{13}t_2^{10}-140t_1^{16}t_2^{10}-65t_1^{19}t_2^{10}+15t_1^2t_2^{11}+210t_1^5t_2^{11} \\ -2175t_1^8t_2^{11}+3900t_1^{11}t_2^{11}+palindrome\ldots+t_1^{22}t_2^{22} \end{array}\right)}{\left(1-t_1^3\right)^7\left(1-t_2^3\right)^7(1-t_1t_2)^{16}}$	$(1-t)\frac{\left(\begin{array}{l} 1+t+10t^2+23t^3+68t^4+135t^5+281t^6 \\ +446t^7+695t^8+895t^9+1090t^{10} \\ +1115t^{11}+1090t^{12}+895t^{13}+695t^{14} \\ +446t^{15}+281t^{16}+135t^{17}+68t^{18} \\ +23t^{19}+10t^{20}+t^{21}+t^{22} \end{array}\right)}{(1-t^2)^{16}(1-t^3)^7}$

Table 12. Unrefined Hilbert Series of $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{SU}(3)_c$ colour singlets.

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unrefined Hilbert series of these theories have been studied extensively [3]. We can revisit these analyses using HWG methodology.

In the case of the $\mathrm{SU}(N_f) \times \mathrm{SO}(N_c)$ product groups, the symmetrisations are carried out on an object transforming in the fundamental representation of the flavour group and in the vector representation of the colour group, with charge assignments as in table 13.

The theory consists of quarks Q_a^i transforming in the $\mathrm{SU}(N_f) \times \mathrm{SO}(N_c)$ product group, where the indices i and a range over the fundamental dimension of the $\mathrm{SU}(N_f)$ flavour groups and the vector representation of the $\mathrm{SO}(N_c)$ colour group, respectively. The GIOs of the theory consist of colour singlets composed of quarks, with some combination of paired and unpaired $\mathrm{SU}(N_f)$ indices determining the flavour irrep in which they transform. The fields also carry U(1) charges, although these are not central to the analysis. The GIOs are restricted to symmetrised combinations of the quarks.

The characters of the quarks are given by:

$$\mathcal{X}_Q = \begin{cases} [1,0\ldots,0]_{\mathrm{SU}}\,[2]_{\mathrm{SO}} & for\ N_c = 3 \\ [1,0\ldots,0]_{\mathrm{SU}}\,[1,1]_{\mathrm{SO}} & for\ N_c = 4 \\ [1,0\ldots,0]_{\mathrm{SU}}\,[1,0,\ldots,0]_{\mathrm{SO}} & for\ N_c > 4. \end{cases} \tag{3.4}$$

Proceeding as before, the HWG generating function is given by:

$$g^{(N_f,N_c)}\left(m_i,t\right) = \oint_{\mathrm{SU}(N_f)} \oint_{\mathrm{SO}(N_c)} d\mu\ g^{N_f}\left(m_i,\mathcal{X}^*\right) \mathrm{PE}\left[\mathcal{X}_Q t\right], \tag{3.5}$$

where we use the Dynkin label fugacities m_i. The unrefined Hilbert series generating function is given by:

$$g^{(N_f,N_c)}\left(t\right) = \oint_{\mathrm{SO}(N_c)} d\mu\ \mathrm{PE}^{N_f}\left[\mathcal{X}_{\mathrm{SO}(N_c)} t\right]. \tag{3.6}$$

The HWGs and unrefined Hilbert series for the $\mathrm{SU}(N_f) \times \mathrm{SO}(N_c)$ product groups are set out in table 14, table 15 and table 16. The unrefined HS replicate established results [3].

As before, the HWGs of this SQCD theory are the same for all flavour $\mathrm{SU}(N_f)$ groups of fundamental dimension exceeding the vector dimension of the $\mathrm{SO}(N_c)$ colour group.

3.3 GIOs of $\mathrm{SU}(N_f) \times \mathrm{USp}(2n_c)$

In the case of $\mathrm{SU}(N_f) \times \mathrm{USp}(2n_c)$ product groups, the symmetrisations are carried out on an object transforming in the fundamental representation of the flavour group and in the defining $2n$ dimensional fundamental representation of the symplectic colour group. We use the charge assignments in table 17 [3].

The theory consists of quarks Q_a^i transforming in the $\mathrm{SU}(2N_f) \times \mathrm{USp}(2n_c)$ product group, where the indices i and a range over the fundamental dimension of the flavour group and the $2n$ dimensional fundamental representation of the colour group, respectively. The GIOs of the theory are colour singlets composed of quarks, with some combination of paired and unpaired flavour indices determining the flavour irrep in which they transform. The

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	Gauge/Colour	Flavour		
	$\mathrm{SO}(N_c)$	$\mathrm{SU}(N_f)$	$\mathrm{U}(1)_B$	$\mathrm{U}(1)_R$
Q_a^i	vector	fundamental	1	$(N_f+2-N_c)/N_f$

Table 13. SQCD Charge Assignments: $\mathrm{SU}(N_f) \times \mathrm{SO}(N_c)$.

Theory	HWG	*Unrefined Hilbert Series*
$\mathrm{SU}(2) \times \mathrm{SO}(3)$	$\frac{1}{(1-t^2m^2)(1-t^4)}$	$\frac{1}{(1-t^2)^3}$
$\mathrm{SU}(3) \times \mathrm{SO}(3)$	$\frac{1}{(1-t^2{m_1}^2)(1-t^3)(1-t^4{m_2}^2)}$	$\frac{1+t^3}{(1-t^2)^6}$
$\mathrm{SU}(4) \times \mathrm{SO}(3)$	$\frac{1}{(1-t^2{m_1}^2)(1-t^3m_3)(1-t^4{m_2}^2)}$	$\frac{1+t^2+4t^3+t^4+t^6}{(1-t^2)^9}$
$\mathrm{SU}(5) \times \mathrm{SO}(3)$	*as above*	$\frac{1+3t^2+10t^3+6t^4+6t^5+10t^6+3t^7+t^9}{(1-t^2)^{12}}$
$\mathrm{SU}(6) \times \mathrm{SO}(3)$	*as above*	$\frac{1+6t^2+20t^3+21t^4+36t^5+56t^6+36t^7+21t^8+20t^9+6t^{10}+t^{12}}{(1-t^2)^{15}}$
$\mathrm{SU}(7) \times \mathrm{SO}(3)$	*as above*	$\frac{\left(\begin{array}{l}1+10t^2+35t^3+55t^4+126t^5+220t^6+225t^7\\+225t^8+220t^9+126t^{10}+55t^{11}+35t^{12}+10t^{13}+t^{15}\end{array}\right)}{(1-t^2)^{18}}$

Table 14. Hilbert Series of $\mathrm{SU}(N_f) \times \mathrm{SO}(3)_c$ product groups.

Theory	HWG	*Unrefined Hilbert Series*
$\mathrm{SU}(2) \times \mathrm{SO}(4)$	$\frac{1}{(1-t^2m^2)(1-t^4)}$	$\frac{1}{(1-t^2)^3}$
$\mathrm{SU}(3) \times \mathrm{SO}(4)$	$\frac{1}{(1-t^2{m_1}^2)\left(1-t^4m_2^2\right)(1-t^6)}$	$\frac{1}{(1-t^2)^6}$
$\mathrm{SU}(4) \times \mathrm{SO}(4)$	$\frac{1}{(1-t^2{m_1}^2)\left(1-t^4m_2^2\right)(1-t^4)\left(1-t^6m_3^2\right)}$	$\frac{1+t^4}{(1-t^2)^{10}}$
$\mathrm{SU}(5) \times \mathrm{SO}(4)$	$\frac{1}{(1-t^2{m_1}^2)\left(1-t^4m_2^2\right)(1-t^4m_4)\left(1-t^6m_3^2\right)}$	$\frac{1+t^2+6t^4+t^6+t^8}{(1-t^2)^{14}}$
$\mathrm{SU}(6) \times \mathrm{SO}(4)$	*as above*	$\frac{1+3t^2+21t^4+20t^6+21t^8+3t^{10}+t^{12}}{(1-t^2)^{18}}$
$\mathrm{SU}(7) \times \mathrm{SO}(4)$	*as above*	$\frac{1+6t^2+56t^4+126t^6+210t^8+126t^{10}+56t^{12}+6t^{14}+t^{16}}{(1-t^2)^{22}}$
$\mathrm{SU}(8) \times \mathrm{SO}(4)$	*as above*	$\frac{\left(\begin{array}{l}1+10t^2+125t^4+500t^6+1310t^8+1652t^{10}\\+1310t^{12}+500t^{14}+125t^{16}+10t^{18}+t^{20}\end{array}\right)}{(1-t^2)^{26}}$

Table 15. Hilbert Series of $\mathrm{SU}(N_f) \times \mathrm{SO}(4)_c$ product groups.

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Theory	HWG	*Unrefined Hilbert Series*
$\mathrm{SU}(2)\times\mathrm{SO}(5)$	$\frac{1}{(1-t^2m^2)(1-t^4)}$	$\frac{1}{(1-t^2)^3}$
$\mathrm{SU}(3)\times\mathrm{SO}(5)$	$\frac{1}{(1-t^2{m_1}^2)(1-t^4{m_2}^2)(1-t^6)}$	$\frac{1}{(1-t^2)^6}$
$\mathrm{SU}(4)\times\mathrm{SO}(5)$	$\frac{1}{(1-t^2{m_1}^2)(1-t^4{m_2}^2)(1-t^6{m_3}^2)(1-t^8)}$	$\frac{1}{(1-t^2)^{10}}$
$\mathrm{SU}(5)\times\mathrm{SO}(5)$	$\frac{1}{(1-t^2{m_1}^2)(1-t^4{m_2}^2)(1-t^5)(1-t^6{m_3}^2)(1-t^8{m_4}^2)}$	$\frac{1+t^5}{(1-t^2)^{15}}$
$\mathrm{SU}(6)\times\mathrm{SO}(5)$	$\frac{1}{(1-t^2{m_1}^2)(1-t^4{m_2}^2)(1-t^5m_5)(1-t^6{m_3}^2)(1-t^8{m_4}^2)}$	$\frac{1+t^2+t^4+6t^5+t^6+t^8+t^{10}}{(1-t^2)^{20}}$
$\mathrm{SU}(7)\times\mathrm{SO}(5)$	*as above*	$\frac{\left(\begin{array}{l}1+3t^2+6t^4+21t^5+10t^6+15t^7\\ +palindrome\ \ldots\ +t^{15}\end{array}\right)}{(1-t^2)^{25}}$
$\mathrm{SU}(8)\times\mathrm{SO}(5)$	*as above*	$\frac{\left(\begin{array}{l}1+6t^2+21t^4+56t^5+56t^6+120t^7\\ +126t^8+160t^9+252t^{10}+160t^{11}+126t^{12}\\ +120t^{13}+56t^{14}+56t^{15}+21t^{16}+6t^{18}+t^{20}\end{array}\right)}{(1-t^2)^{30}}$
$\mathrm{SU}(9)\times\mathrm{SO}(5)$	*as above*	$\frac{\left(\begin{array}{l}1+10t^2+55t^4+126t^5+220t^6+540t^7\\ +715t^8+1270t^9+2002t^{10}+2080t^{11}+2485t^{12}\\ +palindrome\ \ldots\ +t^{25}\end{array}\right)}{(1-t^2)^{35}}$

Table 16. Unrefined Hilbert Series of $\mathrm{SU}(N_f)\times\mathrm{SO}(5)_c$ colour singlets.

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fields also carry U(1) charges, although these are not central to the analysis. The GIOs are restricted to symmetrised combinations of quarks. The further restriction in table 17 to flavour groups of even fundamental dimension follows [17]. For greater generality, we derive below the HWGs for any flavour group $\mathrm{SU}(N_f)$; these results can readily be specialised to flavour groups of even dimension, as desired.

The characters of the quarks are given by:

$$\mathcal{X}_Q = [1,\ldots,0]_{\mathrm{SU}}[1,\ldots,0]_{\mathrm{USp}}. \tag{3.7}$$

Proceeding as before, the HWG generating function is given by:

$$g^{(n_c,N_f)}(m_i,t) = \oint_{\mathrm{SU}(N_f)} \oint_{\mathrm{USp}(2n)} d\mu\; g^{N_f}\left(m_i,\mathcal{X}^*\right)\mathrm{PE}\left[\mathcal{X}_Q t\right], \tag{3.8}$$

where we use the Dynkin label fugacities m_i, and the unrefined Hilbert series generating function is given by:

$$g^{(n_c,N_f)}\left(t\right) = \oint_{\mathrm{USp}(2n)} d\mu\;\; \mathrm{PE}^{N_f}\left[[1,\ldots,0]_{\mathrm{USp}}t\right]. \tag{3.9}$$

The HWGs and unrefined Hilbert series for the $\mathrm{SU}(N_f)\times\mathrm{USp}(2n_c)$ product groups are set out in table 18 and table 19. The unrefined Hilbert series for flavour groups of even fundamental dimension match established results [3, 4].

Once again, the HWGs are the same for all flavour $\mathrm{SU}(N_f)$ groups of fundamental dimension exceeding that of the colour group.

3.4 GIOs of $\mathrm{SU}(N_f)\times G_2$

Finally, it is interesting to examine the situation where the SQCD colour group is taken as an exceptional group, of which G_2 is the lowest rank example. We use the charge assignments in table 20.

The theory consists of quarks Q_a^i transforming in the $\mathrm{SU}(N_f)\times G_2$ product group, where the indices i and a range over the fundamental representation $[1,0,\ldots,0]$ of the $\mathrm{SU}(N_f)$ flavour group and fundamental representation $[0,1]$ of the G_2 colour group, respectively. The GIOs of the theory consist of colour singlets composed of quarks, with some combination of paired and unpaired flavour indices determining the irrep in which they transform. U(1) charges are not shown. The GIOs are restricted to symmetrised combinations of quarks.

The characters of the quarks are given by:

$$\mathcal{X}_Q = [1,0,\ldots,0]_{\mathrm{SU}\left(N_f\right)}[0,1]_{G_2}. \tag{3.10}$$

The HWG generating function is given by:

$$g^{(\mathrm{SU}(N_f),G_2)}\left(m_i,t\right) = \oint_{\mathrm{SU}\left(N_f\right)} \oint_{G_2} d\mu\; g^{N_f}\left(m_i,\mathcal{X}^*\right)\mathrm{PE}\left[\mathcal{X}_Q t\right], \tag{3.11}$$

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	Gauge/Colour	Flavour		
	$\mathrm{USp}(2n_c)$	$\mathrm{SU}(2N_f)$	$\mathrm{U}(1)_B$	$\mathrm{U}(1)_R$
Q_a^i	vector	fundamental	1	$(N_f-1-n)/N_f$

Table 17. SQCD Charge Assignments: $\mathrm{SU}(2N_f) \times \mathrm{USp}(2n_c)$.

Theory	HWG	*Unrefined Hilbert series*
$\mathrm{SU}(2) \times \mathrm{USp}(2)$	$\frac{1}{1-t^2}$	$\frac{1}{1-t^2}$
$\mathrm{SU}(3) \times \mathrm{USp}(2)$	$\frac{1}{1-t^2m_2}$	$\frac{1}{(1-t^2)^3}$
$\mathrm{SU}(4) \times \mathrm{USp}(2)$	*as above*	$\frac{1+t^2}{(1-t^2)^5}$
$\mathrm{SU}(5) \times \mathrm{USp}(2)$	*as above*	$\frac{1+3t^2+t^4}{(1-t^2)^7}$
$\mathrm{SU}(6) \times \mathrm{USp}(2)$	*as above*	$\frac{(1+t^2)(1+5t^2+t^4)}{(1-t^2)^9}$

Table 18. Hilbert Series of $\mathrm{SU}(N_f) \times \mathrm{USp}(2)_c$ colour singlets.

Theory	HWG	*Unrefined Hilbert Series*
$\mathrm{SU}(2) \times \mathrm{USp}(4)$	$\frac{1}{(1-t^2)}$	$\frac{1}{(1-t^2)}$
$\mathrm{SU}(3) \times \mathrm{USp}(4)$	$\frac{1}{(1-t^2m_2)}$	$\frac{1}{(1-t^2)^3}$
$\mathrm{SU}(4) \times \mathrm{USp}(4)$	$\frac{1}{(1-t^2m_2)(1-t^4)}$	$\frac{1}{(1-t^2)^6}$
$\mathrm{SU}(5) \times \mathrm{USp}(4)$	$\frac{1}{(1-t^2m_2)(1-t^4m_4)}$	$\frac{1}{(1-t^2)^{10}}$
$\mathrm{SU}(6) \times \mathrm{USp}(4)$	*as above*	$\frac{1-t^6}{(1-t^2)^{15}}$
$\mathrm{SU}(7) \times \mathrm{USp}(4)$	*as above*	$\frac{1+3t^2+6t^4+3t^6+t^8}{(1-t^2)^{18}}$
$\mathrm{SU}(8) \times \mathrm{USp}(4)$	*as above*	$\frac{1+6t^2+21t^4+28t^6+21t^8+6t^{10}+t^{12}}{(1-t^2)^{22}}$
$\mathrm{SU}(9) \times \mathrm{USp}(4)$	*as above*	$\frac{\left(\begin{array}{l}1+10t^2+55t^4+136t^6+190t^8\\+136t^{10}+55t^{12}+10t^{14}+t^{16}\end{array}\right)}{(1-t^2)^{26}}$
$\mathrm{SU}(10) \times \mathrm{USp}(4)$	*as above*	$\frac{\left(\begin{array}{l}1+15t^2+120t^4+470t^6+1065t^8+1377t^{10}+\\1065t^{12}+470t^{14}+120t^{16}+15t^{18}+t^{20}\end{array}\right)}{(1-t^2)^{30}}$

Table 19. Hilbert Series of $\mathrm{SU}(N_f) \times \mathrm{USp}(4)_c$ colour singlets.

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	Gauge/Colour	Flavour
	G_2	$\mathrm{SU}(N_f)$
Q_a^i	$[0,1]$	$[1,0,\ldots,0]$

Table 20. SQCD Charge Assignments: $\mathrm{SU}(N_f) \times G_2$.

Theory	HWG
$\mathrm{SU}(2) \times G_2$	$\frac{1}{(1-m^2t^2)(1-t^4)}$
$\mathrm{SU}(3) \times G_2$	$\frac{1}{(1-m_1^2t^2)(1-t^3)(1-m_2^2t^4)(1-t^6)}$
$\mathrm{SU}(4) \times G_2$	$\frac{(1+m_1m_2t^7)(1+m_2m_3t^9)}{(1-m_1^2t^2)(1-m_3t^3)(1-t^4)(1-m_2^2t^4)(1-m_1t^5)(1-m_3^2t^6)(1-t^8)(1-m_2^2t^{12})}$

Table 21. HWGs of $\mathrm{SU}(N_f) \times G_2$ colour singlets.

Theory	*Unrefined Hilbert Series*
$\mathrm{SU}(2) \times G_2$	$\frac{1}{(1-t^2)^3}$
$\mathrm{SU}(3) \times G_2$	$\frac{1}{(1-t^3)(1-t^2)^6}$
$\mathrm{SU}(4) \times G_2$	$\frac{1+t^4}{(1-t^3)^4(1-t^2)^{10}}$
$\mathrm{SU}(5) \times G_2$	$\frac{(1+t^2+3t^3+6t^4+3t^5+7t^6+8t^7+7t^8+3t^9+6t^{10}+3t^{11}+t^{12}+t^{14})}{(1-t^2)^{14}(1-t^3)^7}$

Table 22. Unrefined Hilbert series of $\mathrm{SU}(N_f) \times G_2$ colour singlets.

where we use the $\mathrm{SU}(N_f)$ Dynkin label fugacities m_i. The unrefined Hilbert series generating function is given by:

$$g^{(\mathrm{SU}(N_f),G_2)}(t) = \oint_{G_2} d\mu \ \mathrm{PE}^{N_f}\left[[0,1]_{G_2}t\right]. \tag{3.12}$$

The HWGs and unrefined Hilbert series for the $\mathrm{SU}(N_f) \times G_2$ product groups are set out in tables 21 and 22.

We can identify the roles of the G_2 symmetric invariant tensors of order 2, 4, 6 and of the antisymmetric invariant tensors of order 3 and 4, which appear in the HWGs for SU(2), SU(3) and SU(4) flavour groups. There are also other invariant tensors that appear. In the case of an SU(5) flavour group, proceeding as before, we can obtain the refined Hilbert series in the form of a class function:

$$\begin{aligned} g^{(\mathrm{SU}(5),G_2)}(\mathcal{X},t) = (1-t^5) \Bigg(& \left(1+t^5-t^{15}-t^{20}\right)[0,0,0,0] \\ & +t^4\left(1+t^5+t^{10}\right)[0,0,0,1] \\ & +t^{13}[0,0,1,0] \\ & -t^7[0,1,0,0] \\ & -t^6\left(1+t^5+t^{10}\right)[1,0,0,0] \Bigg) \\ & \times \mathrm{PE}\left[[2,0,0,0]\,t^2\right] \mathrm{PE}\left[[0,0,1,0]\,t^3\right]. \end{aligned} \tag{3.13}$$

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To illustrate the rich nature of the series of GIOs arising in G_2 gauge theories, we give below the terms of the refined Hilbert series expansion up to order 15 in the fugacity t, using HWG notation. The exact rational HWG corresponding to this series, however, remains to be found:[2]

$$\begin{aligned} g^{(\mathrm{SU}(5),G_2)}(m_i,t) = 1 &+ m_1^2t^2 + m_3t^3 + m_1^4t^4 + m_2^2t^4 + m_4t^4 + m_1^2m_3t^5 + m_1m_4t^5 \\ &+ m_1t^6 + m_1^6t^6 + m_1^2m_2^2t^6 + 2m_3^2t^6 + m_1^2m_4t^6 \\ &+ m_2t^7 + m_1^4m_3t^7 + m_2^2m_3t^7 + m_1^3m_4t^7 + m_1m_2m_4t^7 + m_3m_4t^7 \\ &+ m_1^3t^8 + m_1^8t^8 + m_1m_2t^8 + m_1^4m_2^2t^8 + m_2^4t^8 + 2m_1^2m_3^2t^8 \\ &+ m_1^4m_4t^8 + m_2^2m_4t^8 + m_1m_3m_4t^8 + 2m_4^2t^8 \\ &+ m_1^2m_2t^9 + 2m_1m_3t^9 + m_1^6m_3t^9 + m_1^2m_2^2m_3t^9 + 2m_3^3t^9 \\ &+ m_1^5m_4t^9 + m_1^3m_2m_4t^9 + m_1m_2^2m_4t^9 + m_1^2m_3m_4t^9 \\ &+ m_2m_3m_4t^9 + m_1m_4^2t^9 + t^{10} + m_1^5t^{10} + m_1^{10}t^{10} \\ &+ m_1^3m_2t^{10} + m_1m_2^2t^{10} + m_1^6m_2^2t^{10} + m_1^2m_2^4t^{10} \\ &+ 2m_2m_3t^{10} + 2m_1^4m_3^2t^{10} + 2m_2^2m_3^2t^{10} + m_1m_4t^{10} \\ &+ m_1^6m_4t^{10} + m_1^2m_2^2m_4t^{10} + m_1^3m_3m_4t^{10} \\ &+ m_1m_2m_3m_4t^{10} + 2m_3^2m_4t^{10} + 3m_1^2m_4^2t^{10} + \ldots \end{aligned} \tag{3.14}$$

Moreover, it is clear that the HWGs calculated have not yet converged in the manner noted earlier for classical colour groups (of defining dimension lower than G_2). Noting the seven dimensional epsilon tensor of G_2, we can expect that such convergence of the HWG should occur for flavour groups SU(8) and above.

4 HWGs of instanton moduli spaces

Instantons continue to attract considerable interest since their discovery as self-dual solutions of Yang-Mills field equations in 1975 [18]. Many studies have been carried out on different aspects of instantons, such as [19], which gives an ADHM construction for the instantons of Yang-Mills fields transforming in Classical gauge groups. This paper focuses just on the analysis of the moduli spaces of instantons, in terms of their Hilbert series and character expansions. In this context, the Hilbert series for the moduli spaces of one and two instantons of Classical gauge groups have been constructed from SUSY quiver theories [6, 7]. Efforts have been made to construct the moduli spaces of two instantons of Exceptional gauge groups [20]. Our aim here is to construct the HWGs for the moduli spaces of some low rank SU(N) instantons on $\mathbb{C}^2$ and to show how HWGs can be used to study the structure of these moduli spaces.

We set out in table 23 the field content of a quiver theory for the moduli space of k SU(N) instantons on $\mathbb{C}^2$, following [6]. The instanton moduli space is identified with the Higgs branch of the quiver theory [21]. The fields in the quiver theory transform in some representation of a quiver gauge group determined by the number of instantons, in addition

[2] Due to computing constraints.

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Brane perspective	*k D3 branes*		*N D7 branes*	$\mathbb{C}^2$		
	Quiver Gauge Group :	U(k)	*Yang-Mills Gauge Group*	*Global Symmetry* : U(2)		*Fugacity*
Field	SU(k)	U(1)	SU(N)	SU(2)$_g$	U(1)$_g$	
Φ	$[1,0,\ldots,0,1]+1$	0	$[0,\ldots,0]$	[0]	0	–
$\phi^{(\alpha)}$	$[1,0,\ldots,0,1]+1$	0	$[0,\ldots,0]$	[1]	(1)	t
X_{12}	$[1,0,\ldots,0]$	(1)	$[0,\ldots,0,1]$	[0]	(1)	t
X_{21}	$[0,\ldots,0,1]$	(-1)	$[1,0\ldots,0]$	[0]	(1)	t
CSA Coordinates	$y_1,\ldots,y_{k-1}$	w	$x_1,\ldots,x_{N-1}$	x		

Table 23. Field Content of Quiver Theory for Moduli Space of k SU(N) Instantons on $\mathbb{C}^2$.

to transforming under the Yang-Mills gauge group (also referred to in the literature as a *flavour* group). The instanton moduli spaces are given by field combinations that are singlets of the quiver gauge group.

Since the representations of unitary groups do not contain antiparticles [14], it is helpful to decompose the unitary representations within the quiver into special unitary representations, by extracting overall U(1) charges. Then the X_{12} and X_{21} bi-fundamental fields transform in conjugate representations with respect to the SU(k) quiver gauge and SU(N) Yang-Mills gauge groups and also carry conjugate U(1) charges. The fields $\{\Phi, \phi^{(a)}\}$ transform in the U(k) adjoint, formed from the product of the fundamental U(k) representation with its conjugate, which can be decomposed as a SU(k) adjoint plus a singlet. The U(1) symmetry associated with the Yang-Mills gauge group has been absorbed into the local U(1) symmetry of the quiver gauge group. We use a fugacity t, corresponding to the global U(1) charge,[3] to count the fields.

The theory is defined not only by its basic fields, but also by its superpotential [6]:

$$\mathcal{W} = Tr\left(X_{21}\Phi X_{12} + \varepsilon_{\alpha\beta}\phi^{(\alpha)}\Phi\phi^{(\beta)}\right). \tag{4.1}$$

The trace is taken over all unpaired gauge group indices. We apply variational principles, requiring that the superpotential should be extremised with respect to the field Φ:

$$\delta\mathcal{W} = \frac{\partial\mathcal{W}}{\partial\Phi}\delta\Phi = 0. \tag{4.2}$$

This in turn leads to the following F-term constraints or selection rules between SU(N) singlets formed from the X fields and the global singlets of the $\phi^{(\alpha)}$ fields, respectively, where we denote quiver gauge indices by $(a, b, \ldots)$ and Yang-Mills gauge indices by $(i, j, \ldots)$:

$$(X_{12})_a^{\ i}\,(X_{21})_i^{\ b} = \varepsilon_{\alpha\beta}\left(\phi^{(\alpha)}\right)_a^{\ c}\left(\phi^{(\beta)}\right)_c^{\ b}. \tag{4.3}$$

[3]Technically this global U(1) counts the highest weights of an SU(2) R-symmetry.

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For the gauge group U(k), there are k^2 relations in total. For a U(1) gauge symmetry the commutator of the $\phi^{(\alpha)}$ fields vanishes and this imposes the F-term constraint that there can be no SU(N) singlets formed from pairs of X fields. For $k > 1$, the commutator does not vanish, and so the SU(N) singlets formed from pairs of X fields become identical to global SU(2) singlets formed by contracting pairs of $\phi^{(\alpha)}$ fields.

If we specialise these relations to quiver gauge singlets, we obtain the general constraint:

$$(X_{12})_a^{\ i}\,(X_{21})_i^{\ a} = \varepsilon_{\alpha\beta}\left(\phi^{(\alpha)}\right)_a^{\ c}\left(\phi^{(\beta)}\right)_c^{\ a} = 0. \tag{4.4}$$

We are interested in finding symmetrised combinations of the fields $\{\phi^{(a)}, X_{12}, X_{21}\}$ that are also singlets of the U(k) gauge group. We can generate such combinations of the fields using the PE followed by Weyl integration, however, the fields are also subject to the relations (4.3). When we construct Hilbert series for the various theories we need to incorporate the appropriate F-term constraints into our generating functions to exclude any disallowed combinations and to avoid over-counting.

Following Plethystic Program methodology, we start by identifying the irreps of the fields with their characters, which transform in the given product groups. Thus:

$$\begin{aligned}
\mathcal{X}(\phi^{(\alpha)}) &= [1,0\ldots,0,1]_{\mathrm{SU}(k)}[1]_{\mathrm{SU}(2)} + [1]_{\mathrm{SU}(2)},\\
\mathcal{X}(X_{12}) &= [1,0\ldots,0,0]_{\mathrm{SU}(k)}[0,0\ldots,1]_{\mathrm{SU}(N)}(1)_{\mathrm{U}(1)},\\
\mathcal{X}(X_{21}) &= [0,0\ldots,0,1]_{\mathrm{SU}(k)}[1,0\ldots,0]_{\mathrm{SU}(N)}(1)_{\mathrm{U}(1)}.
\end{aligned} \tag{4.5}$$

We can choose to incorporate F-term constraints at this point by subtracting the PE of the character of field combinations made redundant by the relations (4.3).

Next we extract the quiver gauge group singlets. This projection onto singlets is carried out by Weyl integration over both the U(1) and SU(k) quiver gauge groups. The first integration imposes the constraint that the overall U(1) quiver gauge charge is zero and so the X fields within the generating functions always consist of pairs of SU(k) fundamentals and anti-fundamentals, and, similarly, of pairs of SU(N) fundamentals and anti-fundamentals.

HWGs can be calculated by projecting these generating functions onto the irreps of the SU(N) Yang-Mills symmetry and the global SU(2) symmetry groups. The HWGs faithfully encode the structure of the representation space and this can facilitate the application of F-term constraints.

Instanton moduli spaces invariably contain a component generated by the fundamental of the global SU(2) symmetry. This component represents the position of the instanton on the on $\mathbb{C}^2$ manifold. For multiple instanton theories, this corresponds to the centre of mass. Instanton moduli spaces can be presented in reduced form by taking a quotient of the full moduli space by this SU(2) symmetry. This can lead to significant simplifications in the HWGs for the moduli spaces, as will be shown.

The Hilbert series of an instanton moduli space can also be summarised in an unrefined form by replacing the characters of the SU(N) and global SU(2) symmetry groups by their dimensions.

We set out in the following sections the results of such calculations for one SU(3) instanton on $\mathbb{C}^2$ and also for two and three SU(2) instantons on $\mathbb{C}^2$.

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4.1 Moduli space of one SU(3) instanton

Using table 23 and noting that, for one instanton, the adjoint of U(k) becomes the adjoint of U(1), we obtain the following characters in terms of the CSA coordinates $\{x, x_1, x_2, w\}$:

$$\begin{aligned}\mathcal{X}(\phi^{(\alpha)}) &= 1/x + x,\\ \mathcal{X}(X_{12}) &= (x_1 + x_2 + 1/x_1x_2)\, w,\\ \mathcal{X}(X_{21}) &= (1/x_1 + 1/x_2 + x_1x_2)\, w^{-1}.\end{aligned} \tag{4.6}$$

In addition to symmetrising the various fields using the PE, we need to correct for the relation (4.3). Since the commutator of two ϕ fields transforming in the Abelian U(1) gauge group is zero, this relation entails that SU(3) singlets composed of pairs of X fields should vanish and so the resulting F-term constraint is that $t^2 m_1^0 m_2^0 = t^2 \to 0$. Singlets of the Yang-Mills gauge group composed of pairs of X fields are therefore excluded. We can achieve this by taking a *hyper-Kahler quotient*, which incorporates the redundancy of the t^2 SU(3) singlets into the PE before the projection onto the quiver gauge singlets as:

$$g^{1,\mathrm{SU}(3)}(t, x, x_1, x_2) \equiv \oint_{\mathrm{U}(1)} d\mu \ \mathrm{PE}[\mathcal{X}(\phi^{(\alpha)})t + \mathcal{X}(X_{12})t + \mathcal{X}(X_{21})t - \underbrace{\mathcal{X}([0,0;0])}_{1} t^2]. \tag{4.7}$$

The hyper-Kahler quotient follows from the last term of (4.7) (with negative sign) and the Weyl integral evaluates as:

$$\begin{aligned}g^{1,\mathrm{SU}(3)}(t, x, x_1, x_2) = (1-t^2)^2((1 + 2t^2 + 2t^4 + 2t^6 + t^8)[0,0] - t^4[1,1])\\ \times\ \mathrm{PE}[[1,1]\ t^2]\ \mathrm{PE}[[1]\ t],\end{aligned} \tag{4.8}$$

where we are using character notation for the class functions of the SU(3) Yang-Mills gauge and SU(2) global symmetry groups, for brevity.

To obtain a HWG we use the generating functions for the characters of SU(2) and SU(3) as per (2.6) and table 2, using Dynkin label fugacities $\{m_1, m_2\}$ for the Yang-Mills symmetry and m for the global symmetry. We then apply Weyl integration over the SU(3) and SU(2) symmetry groups to obtain the generating function:

$$\begin{aligned}\mathrm{HWG}^{1,\mathrm{SU}(3)}(t, m, m_1, m_2) &\equiv \oint_{\mathrm{SU}(2)} \oint_{\mathrm{SU}(3)} d\mu\ (1 - m_1m_2)\, \mathrm{PE}[\mathcal{X}^*[1,0]m_1 + \mathcal{X}^*[0,1]m_2]\\ &\qquad \times \mathrm{PE}[\mathcal{X}[1]m]\ g^{1,\mathrm{SU}(3)}(t, x, x_1, x_2)\\ &= \frac{1}{(1-t^2m_1m_2)(1-tm)}.\end{aligned} \tag{4.9}$$

The HWG analysis explicates the basic objects that generate the moduli space. We can identify the combinations of the fields within the HWG (4.9) as shown in table 24.

The Dynkin label fugacities m_1m_2 always appear paired and so this is an example of the general result [6] that, for one instanton, the resulting tensor products between SU(N) particles and antiparticles always transform in a real representation that is a symmetrisation $[n, 0, \ldots, 0, n]$ of the SU(N) adjoint representation. We can also see that the t^2 singlets have been excluded, as intended, by the hyper-Kahler quotient.

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$SU(3)_{YM}; SU(2)_{global}$	HWG *Terms*	*Basic Objects*
$[1,1;0]$	$t^2 m_1 m_2$	SU(3) *adjoint from* X_{12} *and* X_{21}
$[0,0;1]$	tm	SU(2) *fundamental from* $\phi^{(\alpha)}$

Table 24. Basic Objects of HWG for One SU(3) Instanton Moduli Space.

The physical interpretation of the HWG is that the term tm enumerates the representations of the global SU(2) symmetry that describe the position of the instanton on $\mathbb{C}^2$, while the $t^2 m_1 m_2$ term enumerates the holomorphic operators in the reduced moduli space. Thus the reduced moduli space for one SU(3) instanton is given by the one dimensional HWG:

$$\mathrm{HWG}^{1,\mathrm{SU}(3)}_{\mathrm{reduced}}(t, m_1, m_2) = \mathrm{PE}[m_1 m_2 t^2]. \tag{4.10}$$

We can unrefine the series given by the HWG (4.9), by Taylor expansion followed by replacement of the monomials in the Dynkin label fugacities by the dimensions of the corresponding irreducible representations using the schema:

$$m_1^{n_1} m_2^{n_2} m^n \to Dim[n_1, n_2] Dim[n] = \frac{1}{2}(n_1+1)(n_2+1)(n_1+n_2+2)\,(n+1)\,. \tag{4.11}$$

The resulting unrefined Hilbert series matches that given by [6]:

$$\mathrm{HS}^{1,\mathrm{SU}(3)}_{\mathrm{unrefined}}(t) = \frac{1+4t^2+t^4}{(1-t)^2(1-t^2)^4}. \tag{4.12}$$

4.2 Moduli space of two SU(2) instantons

The analysis for multiple instantons is complicated by the enlarged gauge group symmetry; the characters combine the three separate non-Abelian product groups; quiver gauge U($k=2$), Yang-Mills SU($N=2$) and the global SU(2), in addition to their U(1) charges. We use the CSA coordinates $\{w, x, x_1, y_1\}$ and apply table 23 to obtain the following characters for the various fields:

$$\begin{aligned}
\mathcal{X}(\phi^{(\alpha)}) &= \left(y_1^2 + 1 + \frac{1}{y_1^2}\right)\left(x + \frac{1}{x}\right) + \left(x + \frac{1}{x}\right), \\
\mathcal{X}(X_{12}) &= \left(y_1 + \frac{1}{y_1}\right)\left(x_1 + \frac{1}{x_1}\right) w, \\
\mathcal{X}(X_{21}) &= \left(y_1 + \frac{1}{y_1}\right)\left(x_1 + \frac{1}{x_1}\right) w^{-1}.
\end{aligned} \tag{4.13}$$

In addition to symmetrising the fields using the PE, we need to incorporate the F-term constraints that follow from (4.3) and (4.4), corresponding to the tensor relations:

$$(X_{12})_a{}^i (X_{21})_i{}^b = \varepsilon_{\alpha\beta} (\phi^{(\alpha)})_a{}^c (\phi^{(\beta)})_c{}^b, \tag{4.14}$$

and

$$(X_{12})_a{}^i (X_{21})_i{}^a = \varepsilon_{\alpha\beta} (\phi^{(\alpha)})_a{}^c (\phi^{(\beta)})_c{}^a = 0. \tag{4.15}$$

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Firstly, (4.15) entails that singlets under all three symmetries (quiver gauge, Yang-Mills and global), which are composed just of pairs of ϕ fields (or pairs of X fields), should vanish. Indeed, since the only Casimir of SU(2) is of degree 2, it follows that all singlets formed from chains of odd numbers of pairs of ϕ fields (or X fields) should vanish. Thus:

$$\underbrace{(X_{12})_a{}^i(X_{21})_i{}^b\ldots\ldots(X_{12})_c{}^k(X_{21})_k{}^a}_{odd\ number\ of\ X_{12}X_{21}\ pairs}=0. \tag{4.16}$$

This F-term constraint corresponds to adding the following term to the generating function:

$$\mathrm{PE}[-[0;0;0]t^2]=\mathrm{PE}[-t^2]. \tag{4.17}$$

Secondly, (4.14) entails that simple symmetrisation of the characters (4.13) over-counts combinations of pairs of ϕ fields (or pairs of X fields) transforming in the adjoint of the SU(2) quiver gauge group. The relation does not apply generally, but only between pairs that are Yang-Mills gauge singlets and global singlets. We can correct for this duplication by incorporating a generating function term containing the character for the adjoint of the SU(2) quiver gauge group:

$$\mathrm{PE}[-[2;0;0]t^2]=\mathrm{PE}\left[-\left(y_1^2+1+\frac{1}{y_1^2}\right)t^2\right]. \tag{4.18}$$

We combine the characters (4.13) and the F-term constraints (4.17) and (4.18) into a hyper-Kahler quotient and then apply Weyl integration, first over the U(1) and then over the SU(2) of the quiver gauge group to obtain the generating function:

$$g^{2,\mathrm{SU}(2)}(t,x,x_1)\equiv\oint_{\mathrm{SU}(2)}\oint_{\mathrm{U}(1)}d\mu\ \mathrm{PE}[\mathcal{X}(\phi^{(\alpha)})t+\mathcal{X}(X_{12})t+\mathcal{X}(X_{21})t-t^2-\mathcal{X}[2;0;0]t^2]. \tag{4.19}$$

We evaluate (4.19) and then rearrange the resulting functions of the $\{x,x_1\}$ CSA coordinates into characters, with Dynkin labels ordered as [Yang-Mills; global], to give the refined HS generating function:

$$\begin{aligned}g^{2,\mathrm{SU}(2)}(t,\mathcal{X})\ =\left(1-t^2\right)&\left(\begin{array}{r}\left(1+t^2-t^{14}-t^{16}\right)[0;0]\\+t^3\left(1-t^{10}\right)[0;1]\\-t^6\left(1-t^4\right)[2;0]\\-t^5\left(1-t^6\right)[2;1]\end{array}\right)\\&\times\mathrm{PE}\left[[0;1]t+[2;0]t^2+[0;2]t^2+[2;1]t^3-[0;1]t^3\right].\end{aligned} \tag{4.20}$$

We now introduce the Dynkin label fugacities m_1 and m to track the Yang-Mills SU(2) and global SU(2) irreps respectively and use Weyl integration to project the generating function (4.20) onto the irreps of the SU(2) groups to give the HWG:

$$\mathrm{HWG}^{2,\mathrm{SU}(2)}(t,m,m_1)\equiv\oint_{\mathrm{SU}(2)}\oint_{\mathrm{SU}(2)}d\mu\ \mathrm{PE}[[1]m_1+[1]_{\mathrm{global}}m]g^{2,\mathrm{SU}(2)}(t,\mathcal{X}). \tag{4.21}$$

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$SU(2)_{YM}; SU(2)_{global}$	HWG *Terms*	*Basic Objects*
[0; 0]	t^4	*Singlets from* $\phi^{(\alpha)}$
[0; 1]	mt	*Global* SU(2) *fundamental from* $\phi^{(\alpha)}$
[0; 2]	m^2t^2	*Global* SU(2) *adjoint from* $\phi^{(\alpha)}$
[2; 0]	$m_1^2t^2$	*Yang-Mills* SU(2) *adjoint from* X_{12} *and* X_{21}
[2; 0]	$m_1^2t^4$	*Yang-Mills* SU(2) *adjoint from* $\phi^{(\alpha)}, X_{12}$ *and* X_{21}
[2; 1]	$mm_1^2t^3$	*Yang-Mills* SU(2) *adjoint & global* SU(2) *fund. from* $\phi^{(\alpha)}, X_{12}$ *and* X_{21}

Table 25. Basic Objects of HWG for Two SU(2) Instanton Moduli Space.

Evaluation yields the result:

$$\mathrm{HWG}^{2,\mathrm{SU}(2)}\left(t,m,m_1\right)=\frac{Num(t,m,m_1)}{\left(1-mt\right)\left(1-m^2t^2\right)\left(1-m_1^2t^2\right)\left(1-mm_1^2t^3\right)\left(1-t^4\right)^2\left(1-m_1^2t^4\right)}, \tag{4.22}$$

where

$$Num\left(t,m,m_1\right)=1+mt^3+mm_1^2t^5+m_1^2t^6-m^2m_1^2t^6-mm_1^2t^7-mm_1^4t^9-m^2m_1^4t^{12}. \tag{4.23}$$

Interestingly, the polynomial $Num(t, m, m_1)$ is anti-palindromic of degree (12,2,4) in the variables (t, m, m_1). We can identify combinations of the fields giving rise to the HWG denominator terms as shown in table 25.

We read off the exponents of the fugacities $\{m, m_1\}$, which give the Dynkin labels of the global SU(2) and Yang-Mills SU(2) representations respectively, and identify the building blocks of the theory according to the irreps in which they transform. In particular, while the global SU(2) irreps include the fundamental, the only Yang-Mills SU(2) irreps that occur are the adjoint and symmetrisations of the adjoint. We can identify the term $mm_1^2t^3$, which is responsible for the mixing that occurs between the global and Yang-Mills symmetries in multiple instanton theories. $Num(t, m, m_1)$ also contains the monomial term mt^3, which is not just a product of the other building blocks. Some of these basic objects can be identified from the PE in (4.20), however, the HWG gives a complete enumeration.

We can verify that the addition of the terms (4.17) and (4.18) has had the desired effect of excluding Yang-Mills gauge singlets formed from pairs of X fields from the Hilbert series. Thus, if we specialise the series (4.22) to Yang-Mills singlets, by setting m_1 to zero, we obtain:

$$\mathrm{HWG}^{2,\mathrm{SU}(2)}(t,m,0)=\frac{1+mt^3}{(1-t^4)^2(1-mt)(1-m^2t^2)}. \tag{4.24}$$

This series does not contain any t^2 monomials, confirming that all Yang-Mills singlet pairs of X or ϕ fields have been excluded as intended by applying the F-term constraint terms to the Hilbert series. Also, the only global and Yang-Mills singlets are at orders of t^4, confirming also that the singlets only contain even numbers of pairs of X or ϕ fields.

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Returning to the HWG for the two SU(2) instanton moduli space given by (4.22) and (4.23), we can see that the global symmetry only appears in its generating monomials as m or m^2, corresponding to the [1] and [2] global irreps after Taylor expansion. This appears to be part of a more general pattern, where the global symmetry appears amongst the HWG generating monomials at orders up to m^k, where k is the instanton number and equals the maximum degree of the Casimirs of the U(k) quiver gauge group.

Importantly, the [1] irrep of the global SU(2) arises from the PE[[0; 1]t] term within the refined HS generating function (4.20) and we can simplify the HWG considerably by factoring out the tensor products that result from this term. This gives us the *reduced* moduli space. Physically, the [1] irrep of the global SU(2) corresponds to the centre of mass of a system of instantons and so working with the reduced moduli space corresponds to an analysis in the instanton rest frame.

If we reduce the HS generating function (4.20) by taking a quotient by this PE term, but otherwise proceed as before, the HWG (4.22) simplifies to the reduced two SU(2) instanton moduli space:

$$\mathrm{HWG}^{2,\mathrm{SU}(2)}_{\mathrm{reduced}}(t,m,m_1) = \frac{1+mm_1^2t^5}{(1-m^2t^2)\left(1-m_1^2t^2\right)\left(1-mm_1^2t^3\right)(1-t^4)}. \tag{4.25}$$

This result is equivalent to a series expansion of the HWG presented in [7], but uses the more concise HWG notation, which clarifies the structure of the basic objects in the moduli space. Unlike the HWG (4.22) for the full moduli space, the HWG (4.25) for the reduced moduli space constitutes a complete intersection. We can simplify an expansion of (4.25) to give an unrefined series by replacing monomial terms in the $\{m, m_1\}$ fugacities by the corresponding irrep dimensions using the mapping:

$$m^n {m_1}^{n_1} \to Dim[n]Dim[n_1] = (n+1)(n_1+1). \tag{4.26}$$

We obtain:

$$\mathrm{HS}^{2,\mathrm{SU}(2)}_{\mathrm{reduced,\,unrefined}}(t) = \frac{(1-t)\left(1+t+3t^2+6t^3+8t^4+6t^5+8t^6+6t^7+3t^8+t^9+t^{10}\right)}{(1-t^2)^4(1-t^3)^3}. \tag{4.27}$$

The numerator of the generating function (4.27) is a palindromic polynomial, and the result matches that given in [7].

4.3 Moduli space of three SU(2) instantons

The presence of three instantons gives rise to a U(3) quiver gauge symmetry, and the field characters combine three separate non-Abelian product groups: quiver gauge SU($k = 3$), Yang-Mills SU($N = 2$) and the global SU(2), in addition to the local and global U(1) symmetries. We express the characters using the CSA coordinates $\{y_1, y_2, x, x_1, w\}$ in accordance with table 23:

$$\begin{aligned}
\mathcal{X}(\phi^{(\alpha)}) &= \left(y_2y_1^2 + y_2^2y_1 + \frac{y_1}{y_2} + \frac{1}{y_1^2y_2} + \frac{1}{y_1y_2^2} + \frac{y_2}{y_1} + 2\right)\left(x+\frac{1}{x}\right) + \left(x+\frac{1}{x}\right), \\
\mathcal{X}(X_{12}) &= \left(y_1+y_2+\frac{1}{y_1y_2}\right) w \left(x_1+\frac{1}{x_1}\right), \\
\mathcal{X}(X_{21}) &= \left(y_1y_2+\frac{1}{y_1}+\frac{1}{y_2}\right) w^{-1} \left(x_1+\frac{1}{x_1}\right).
\end{aligned} \tag{4.28}$$

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We also need to identify the F-term constraints that follow from (4.3) and (4.4). As before, (4.4) entails that singlets under all three symmetries (quiver gauge, Yang-Mills and global), which are composed just of pairs of ϕ fields (or pairs of X fields), should vanish, and this constraint corresponds to the generating function term:

$$\mathrm{PE}[-[0,0;0;0]t^2] = \mathrm{PE}[-t^2]. \tag{4.29}$$

Secondly, (4.3) entails that to avoid duplication of pairs of fields we should also incorporate a generating function term to eliminate symmetrised characters from the adjoint of SU(3):

$$\mathrm{PE}[-[1,1;0;0]t^2] = \mathrm{PE}\left[-\left(y_2y_1^2 + y_2^2y_1 + \frac{y_1}{y_2} + \frac{1}{y_1^2y_2} + \frac{1}{y_1y_2^2} + \frac{y_2}{y_1} + 2\right)t^2\right]. \tag{4.30}$$

Combining the PEs of the characters (4.28) with the hyper-Kahler quotient terms (4.29) and (4.30), we obtain a refined generating function by Weyl integration over the $\{w, y_1, y_2\}$ CSA coordinates:

$$g^{3,\mathrm{SU}(2)}(t,x,x_1) = \oint_{\mathrm{SU}(3)} \oint_{\mathrm{U}(1)} d\mu\ \mathrm{PE}[\mathcal{X}(\phi^{(\alpha)})t + \mathcal{X}(X_{12})t + \mathcal{X}(X_{21})t - t^2 - [1,1;0;0]t^2]. \tag{4.31}$$

We shall not give the explicit evaluation of the generating function in this form since it is extremely unwieldy. Instead we simplify the analysis by taking a quotient by the global symmetry and working with the reduced three instanton moduli space:

$$g^{3,\mathrm{SU}(2)}_{\mathrm{reduced}}(t,x,x_1) = g^{3,\mathrm{SU}(2)}(t,x,x_1)\ \ \mathrm{PE}[-[0,0;0;1]t]. \tag{4.32}$$

As before, we introduce character generating functions and the Dynkin label fugacities m_1 and m to track the Yang-Mills SU(2) and global SU(2) irreps respectively and apply Weyl integration over the remaining $\{x_1, x_2\}$ CSA coordinates to obtain an HWG:

$$\mathrm{HWG}^{3,\mathrm{SU}(2)}_{\mathrm{reduced}}(m_1,m,t) \equiv \oint_{\mathrm{SU}(2)} \oint_{\mathrm{SU}(2)} d\mu\ \ \mathrm{PE}\left[[1]_{\mathrm{YM}}m_1 + [1]_{\mathrm{global}}m\right] g^{3,\mathrm{SU}(2)}_{\mathrm{reduced}}(t,x,x_1). \tag{4.33}$$

Evaluation yields:

$$\mathrm{HWG}^{3,\mathrm{SU}(2)}_{\mathrm{reduced}}(m_1,m,t) = \frac{Num(m_1,m,t)}{Den(m_1,m,t)}, \tag{4.34}$$

where the denominator is given by:

$$\begin{aligned} Den(m_1,m,t) = &\left(1-t^4\right)^2\left(1-t^6\right)\left(1-t^8\right) \\ &\times \left(1-m^2t^2\right)\left(1-mt^3\right)\left(1-m^3t^3\right) \\ &\times \left(1-m_1^2t^2\right)\left(1-m_1^2t^4\right)\left(1-m_1^6t^{10}\right) \\ &\times \left(1-mm_1^2t^3\right)\left(1-m^2m_1^2t^4\right), \end{aligned} \tag{4.35}$$

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$SU(2)_{YM}; SU(2)_{global}$	HWG *Terms*	*Basic Objects*
[0; 0]	t^4, t^6, t^8	*Singlets from* $\phi^{(\alpha)}$
[0; 1]	mt^3	*Global* SU(2)
[0; 2]	m^2t^2	*irreps*
[0; 3]	m^3t^3	*from* $\phi^{(\alpha)}$
[2; 1]	$m_1^2mt^3$	SU(2) *adjoint & global* SU(2) *irreps*
[2; 2]	$m_1^2m^2t^4$	*from* $\phi^{(\alpha)}, X_{12}$ *and* X_{21}
[2; 0]	$m_1^2t^2$	SU(2) *adjoint from* X_{12} *and* X_{21}
[2; 0]	$m_1^2t^4$	SU(2) *adjoint from* $\phi^{(\alpha)}, X_{12}$ *and* X_{21}
[6; 0]	$m_1^6t^{10}$	SU(2) *symmetrised adjoint from* $\phi^{(\alpha)}, X_{12}$ *and* X_{21}

Table 26. Basic Objects of HWG for Three SU(2) Instanton Reduced Moduli Space.

and the numerator $Num(m_1, m, t)$ consists of unity followed by 248 monomial terms, being palindromic of degree (12,7,43) in the variables (m_1, m, t):

$$\left(\begin{array}{l}
1 - mt^3 + mt^5 + m^3t^5 - t^6 + m^2t^6 + mt^7 - m^4t^8 + mt^9 - m^3t^{11} + t^{12} - m^3t^{13} - m^2t^{14} + m^4t^{14} - mt^{15} - m^3t^{15} + \\
m^3t^{17} - m^4t^{20} - m_1^2t^4 + 2mm_1^2t^5 + m_1^2t^6 + 2m^2m_1^2t^6 + 2mm_1^2t^7 + m^3m_1^2t^7 + m_1^2t^8 - m^2m_1^2t^8 - m^4m_1^2t^8 - \\
mm_1^2t^9 - 3m^3m_1^2t^9 - m^5m_1^2t^9 + 2m_1^2t^{10} - m^2m_1^2t^{10} - m^4m_1^2t^{10} - 2mm_1^2t^{11} - m^3m_1^2t^{11} + m^5m_1^2t^{11} - 2m^2m_1^2t^{12} + m^4m_1^2t^{12} + m^6m_1^2t^{12} - \\
2mm_1^2t^{13} - 2m^3m_1^2t^{13} - m_1^2t^{14} - m^2m_1^2t^{14} - mm_1^2t^{15} + m^3m_1^2t^{15} + m^5m_1^2t^{15} - m_1^2t^{16} - m^2m_1^2t^{16} + mm_1^2t^{17} + 2m^3m_1^2t^{17} + \\
2m^2m_1^2t^{18} - m^6m_1^2t^{18} + mm_1^2t^{19} + m^3m_1^2t^{19} + m^5m_1^2t^{19} - m^4m_1^2t^{20} - m^3m_1^2t^{21} - m^5m_1^2t^{21} + m^4m_1^2t^{22} + m^5m_1^2t^{23} + m^4m_1^2t^{24} + m^6m_1^2t^{24} + \\
mm_1^4t^7 + m_1^4t^8 - 2m^3m_1^4t^9 - 3m^2m_1^4t^{10} - m^4m_1^4t^{10} - 3m^3m_1^4t^{11} + m^4m_1^4t^{12} + m^6m_1^4t^{12} - mm_1^4t^{13} + m^3m_1^4t^{13} + 2m^5m_1^4t^{13} - \\
2m^2m_1^4t^{14} - mm_1^4t^{15} + m^3m_1^4t^{15} - m^5m_1^4t^{15} - m^7m_1^4t^{15} + m^2m_1^4t^{16} - m^6m_1^4t^{16} - mm_1^4t^{17} + 2m^3m_1^4t^{17} + 2m^5m_1^4t^{17} + \\
m^2m_1^4t^{18} + m^3m_1^4t^{19} - m_1^4t^{20} + m^2m_1^4t^{20} + m^4m_1^4t^{20} - m^3m_1^4t^{21} + m^7m_1^4t^{21} + m^6m_1^4t^{22} + mm_1^4t^{23} + m^3m_1^4t^{23} + \\
2m^4m_1^4t^{24} + m^6m_1^4t^{24} - m^4m_1^4t^{26} - m^6m_1^4t^{26} - m^5m_1^4t^{27} - m^7m_1^4t^{27} - m^6m_1^4t^{28} - mm_1^6t^{11} + m_1^6t^{12} - m^2m_1^6t^{12} + \\
m^3m_1^6t^{13} + m_1^6t^{14} + m^4m_1^6t^{14} - mm_1^6t^{15} - m^3m_1^6t^{15} + m^5m_1^6t^{15} - 2m^2m_1^6t^{16} - m^4m_1^6t^{16} - m^6m_1^6t^{16} - m^3m_1^6t^{17} - \\
2m_1^6t^{18} + 2m^4m_1^6t^{18} + mm_1^6t^{19} - m^3m_1^6t^{19} + 2m^5m_1^6t^{19} + m^7m_1^6t^{19} - m_1^6t^{20} + m^4m_1^6t^{20} + mm_1^6t^{21} + m^3m_1^6t^{21} - m^7m_1^6t^{21} - \\
m_1^6t^{22} + m^4m_1^6t^{22} + m^6m_1^6t^{22} + m^3m_1^6t^{23} - m^7m_1^6t^{23} + m_1^6t^{24} + 2m^2m_1^6t^{24} - m^4m_1^6t^{24} + m^6m_1^6t^{24} + 2m^3m_1^6t^{25} - 2m^7m_1^6t^{25} - m^4m_1^6t^{26} - \\
mm_1^6t^{27} - m^3m_1^6t^{27} - 2m^5m_1^6t^{27} + m^2m_1^6t^{28} - m^4m_1^6t^{28} - m^6m_1^6t^{28} + m^3m_1^6t^{29} + m^7m_1^6t^{29} + m^4m_1^6t^{30} - m^5m_1^6t^{31} + m^7m_1^6t^{31} - \\
m^6m_1^6t^{32} - mm_1^8t^{15} - m_1^8t^{16} - m^2m_1^8t^{16} - mm_1^8t^{17} - m^3m_1^8t^{17} + mm_1^8t^{19} + 2m^3m_1^8t^{19} + m^4m_1^8t^{20} + m^6m_1^8t^{20} + mm_1^8t^{21} + \\
m_1^8t^{22} - m^4m_1^8t^{22} + m^3m_1^8t^{23} + m^5m_1^8t^{23} - m^7m_1^8t^{23} + m^4m_1^8t^{24} + m^5m_1^8t^{25} + 2m^2m_1^8t^{26} + 2m^4m_1^8t^{26} - m^6m_1^8t^{26} - \\
mm_1^8t^{27} + m^5m_1^8t^{27} - m_1^8t^{28} - m^2m_1^8t^{28} + m^4m_1^8t^{28} - m^6m_1^8t^{28} - 2m^5m_1^8t^{29} + 2m^2m_1^8t^{30} + m^4m_1^8t^{30} - m^6m_1^8t^{30} + mm_1^8t^{31} + m^3m_1^8t^{31} - \\
3m^4m_1^8t^{32} - m^3m_1^8t^{33} - 3m^5m_1^8t^{33} - 2m^4m_1^8t^{34} + m^7m_1^8t^{35} + m^6m_1^8t^{36} + mm_1^{10}t^{19} + m^3m_1^{10}t^{19} + m^2m_1^{10}t^{20} + m^3m_1^{10}t^{21} - \\
m^2m_1^{10}t^{22} - m^4m_1^{10}t^{22} - m^3m_1^{10}t^{23} + m^2m_1^{10}t^{24} + m^4m_1^{10}t^{24} + m^6m_1^{10}t^{24} - mm_1^{10}t^{25} + 2m^5m_1^{10}t^{25} + 2m^4m_1^{10}t^{26} + m^6m_1^{10}t^{26} - \\
m^5m_1^{10}t^{27} - m^7m_1^{10}t^{27} + m^2m_1^{10}t^{28} + m^4m_1^{10}t^{28} - m^6m_1^{10}t^{28} - m^5m_1^{10}t^{29} - m^7m_1^{10}t^{29} - 2m^4m_1^{10}t^{30} - 2m^6m_1^{10}t^{30} + \\
mm_1^{10}t^{31} + m^3m_1^{10}t^{31} - 2m^5m_1^{10}t^{31} + m^2m_1^{10}t^{32} - m^4m_1^{10}t^{32} - 2m^6m_1^{10}t^{32} - m^3m_1^{10}t^{33} - m^5m_1^{10}t^{33} + 2m^7m_1^{10}t^{33} - \\
m^2m_1^{10}t^{34} - 3m^4m_1^{10}t^{34} - m^6m_1^{10}t^{34} - m^3m_1^{10}t^{35} - m^5m_1^{10}t^{35} + m^7m_1^{10}t^{35} + m^4m_1^{10}t^{36} + 2m^6m_1^{10}t^{36} + 2m^5m_1^{10}t^{37} + m^7m_1^{10}t^{37} + \\
2m^6m_1^{10}t^{38} - m^7m_1^{10}t^{39} - m^3m_1^{12}t^{23} + m^4m_1^{12}t^{26} - m^4m_1^{12}t^{28} - m^6m_1^{12}t^{28} + m^3m_1^{12}t^{29} - m^5m_1^{12}t^{29} - m^4m_1^{12}t^{30} + m^7m_1^{12}t^{31} - \\
m^4m_1^{12}t^{32} + m^6m_1^{12}t^{34} - m^3m_1^{12}t^{35} + m^6m_1^{12}t^{36} + m^5m_1^{12}t^{37} - m^7m_1^{12}t^{37} + m^4m_1^{12}t^{38} + m^6m_1^{12}t^{38} - m^6m_1^{12}t^{40} + m^7m_1^{12}t^{43}
\end{array}\right) . \tag{4.36}$$

We identify in table 26 the combinations of the fields in the denominator that contribute to the HWG. We read off the exponents of the Dynkin label fugacities m_1 and m, which label the Yang-Mills SU(2) and global SU(2) representations respectively, and identify the building blocks of the theory according to the irreps in which they transform. In particular, while the global SU(2) irreps include the fundamental, the only Yang-Mills SU(2) irreps that occur in the PL are the adjoint and symmetrisations of the adjoint. The objects do not include Yang-Mills singlets comprised of X fields, so we can verify that the F-term constraints have been implemented as intended by the terms (4.29) and (4.30).

As before, we can also simplify the HWG into an unrefined version by either (a) setting the Yang-Mills gauge and global SU(2) CSA coordinates in (4.32) to unity or (b) replacing monomial terms in the m and m_1 Dynkin label fugacities in a Taylor expansion of (4.34)

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Theory	PL[HWG]	PL[*Unrefined* HS]
$SU(2)_f \times SU(2)_c$	t^2	t^2
$SU(4)_f \times SU(2)_c$	$l_2 t^2$	$6t^2 - t^4$
$SU(2N_f \geq 6)_f \times SU(2)_c$	$l_2 t^2$	*infinite series*
$SU(2)_{f,L} \times SU(2)_{f,R} \times SU(2)_c$	${t_1}^2 + {t_2}^2 + l r t_1 t_2$	${t_1}^2 + {t_2}^2 + 4t_1 t_2 - t_1^2 t_2^2$
$SU(N_f \geq 3)_L \times SU(\geq 3)_R \times SU(2)_c$	$l_2 {t_1}^2 + r_2 {t_2}^2 + l_1 r_1 t_1 t_2$	*infinite series*
$SU(2)_{f,L} \times SU(2)_{f,R} \times SU(3)_c$	$l r t_1 t_2 + {t_1}^2 {t_2}^2$	$4t_1 t_2$
$SU(3)_{f,L} \times SU(3)_{f,R} \times SU(3)_c$	${t_1}^3 + {t_2}^3 + l_1 r_1 t_1 t_2 + l_2 r_2 {t_1}^2 {t_2}^2$	$9t_1 t_2 + {t_1}^3 + {t_2}^3 - {t_1}^3 {t_2}^3$
$SU(N_f \geq 4)_L \times SU(N_f \geq 4)_R \times SU(3)_c$	$l_3 {t_1}^3 + r_3 {t_2}^3 + l_1 r_1 t_1 t_2 + l_2 r_2 {t_1}^2 {t_2}^2$	*infinite series*

Table 27. PLs of HWGs and Hilbert Series of $SU(N_f)_L \times SU(N_f)_R \times SU(N_c)$ colour singlets.

by the corresponding irrep dimensions using the mapping:

$$m^n {m_1}^{n_1} \to Dim[n] Dim[n_1] = (n+1)(n_1+1). \tag{4.37}$$

In either case we obtain the quotient of palindromic polynomials:[4]

$$\mathrm{HS}^{3,SU(2)}_{\text{reduced, unrefined}}(t) = \frac{\left(\begin{array}{l} 1+3t^2+6t^3+12t^4+16t^5+31t^6+36t^7+55t^8+54t^9+60t^{10} \\ +54t^{11}+55t^{12}+36t^{13}+31t^{14}+16t^{15}+12t^{16}+6t^{17}+3t^{18}+t^{20} \end{array}\right)}{(1-t^2)^3(1-t^3)^4(1-t^4)^3}. \tag{4.38}$$

5 Discussion and conclusions

We have shown how the HWG methodology takes the often complicated plethystic class functions of refined Hilbert series and transforms them into corresponding generating functions for the coefficients of the irreps in the series, identified by their Dynkin label fugacities. Both refined and unrefined Hilbert series can be recovered in a straightforward manner from these HWGs. It is instructive, therefore, to compare the properties of HWGs with those of HS.[5] The geometric properties of refined Hilbert series largely follow from those of unrefined Hilbert series and so we do not comment on them separately.

We use PLs for the purpose of comparison since these encode the geometric nature of the series most concisely. Tables 27 through 29 set out the PLs of the HWG and unrefined HS for the colour singlets of the various SQCD theories examined herein.

The PLs of the unrefined Hilbert series for SQCD theories correspond to established results [3, 4]. The PLs of the HWGs differ from the unrefined Hilbert series for all but the simplest series and explicate the structure of the GIOs of the theory. Thus, for example, the PL $t^2 m_2 + t^4$ in table 29 indicates that the highest weight basis for all GIO colour singlets formed from quarks transforming in a $SU(4) \times USp(4)$ product group consists of a contraction of two quarks transforming in the [0, 2, 0] irrep of the SU(4) flavour group

[4] This is consistent with the series obtained by using instanton counting methods set out in [22].

[5] It is worth commenting that both palindromic properties and the use of unrefined Hilbert series can be extremely useful as a check on the overall counting of dimensions during the calculation of HWGs.

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Theory	PL[HWG]	PL[*Unrefined* HS]
SU(2) × SO(3)	$t^2m^2 + t^4$	$3t^2$
SU(3) × SO(3)	$t^2{m_1}^2 + t^3 + t^4{m_2}^2$	$6t^2 + t^3 - t^6$
SU(≥ 4) × SO(3)	$t^2{m_1}^2 + t^3m_3 + t^4{m_2}^2$	*infinite series*
SU(2) × SO(4)	$t^2m^2 + t^4$	$3t^2$
SU(3) × SO(4)	$t^2{m_1}^2 + t^4{m_2}^2 + t^6$	$6t^2$
SU(4) × SO(4)	$t^2{m_1}^2 + t^4{m_2}^2 + t^6m_3^2 + t^4$	$10t^2 + t^4 - t^8$
SU(≥ 5) × SO(4)	$t^2{m_1}^2 + t^4{m_2}^2 + t^4m_4 + t^6m_3^2$	*infinite series*
SU(2) × SO(5)	$t^2m^2 + t^4$	$3t^2$
SU(3) × SO(5)	$t^2{m_1}^2 + t^4{m_2}^2 + t^6$	$6t^2$
SU(4) × SO(5)	$t^2{m_1}^2 + t^4{m_2}^2 + t^6m_3^2 + t^8$	$10t^2$
SU(5) × SO(5)	$t^2{m_1}^2 + t^4{m_2}^2 + t^5 + t^6m_3^2 + t^8m_4^2$	$15t^2 + t^5 - t^{10}$
SU(≥ 6) × SO(5)	$t^2{m_1}^2 + t^4{m_2}^2 + t^5m_5 + t^6m_3^2 + t^8m_4^2$	*infinite series*

Table 28. PLs of HWGs and Hilbert Series of SU(N_f) × SO(N_c) colour singlets.

Theory	PL[HWG]	PL[*Unrefined Hilbert Series*]
SU(2) × USp(2)	t^2	t^2
SU(3) × USp(2)	t^2m_2	$3t^2$
SU(4) × USp(2)	t^2m_2	$6t^2 - t^4$
SU(≥ 5) × USp(2)	t^2m_2	*infinite series*
SU(2) × USp(4)	t^2	t^2
SU(3) × USp(4)	t^2m_2	$3t^2$
SU(4) × USp(4)	$t^2m_2 + t^4$	$6t^2$
SU(5) × USp(4)	$t^2m_2 + t^4m_4$	$10t^2$
SU(6) × USp(4)	$t^2m_2 + t^4m_4$	$15t^2 - t^6$
SU(≥ 7) × USp(4)	$t^2m_2 + t^4m_4$	*infinite series*

Table 29. PLs of HWGs and Hilbert Series of SU(N_f) × USp($2n_c$) colour singlets.

and a contraction of 4 quarks transforming as a [0, 0, 0] flavour singlet. This HWG has all positive terms in the PL and is a *freely generated* series. Negative terms in a PL indicate that a series is a *complete intersection* or the quotient of two freely generated series. When a PL does not terminate, this shows that the GIOs of the theory cannot be reduced to symmetrisations of a finite basis set of GIOs, or their quotients, and the series is termed a *non-complete intersection* [4]. In the case of SQCD, tables 27 through 29 illustrate how the HWGs can have moduli spaces that are freely generated or complete intersections, when the unrefined HS do not have finite PLs.

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Theory	PL [HWG]
$\mathrm{SU}(N_f) \times \mathrm{SU}(N_c)$	$m_{N_c} t^{N_c}$
$\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{SU}(N_c)$	$\sum_{i=1}^{\min(N_f,N_c)-1} l_i r_i t_1^i t_2^i + \begin{cases} if\ N_f < N_c: & t_1^{N_f} t_2^{N_f} \\ if\ N_f = N_c: & t_1^{N_c} + t_2^{N_c} \\ if\ N_f > N_c: & l_{N_c} t_1^{N_c} + r_{N_c} t_2^{N_c} \end{cases}$
$\mathrm{SU}(N_f) \times \mathrm{SO}(N_c)$	$\sum_{i=1}^{\min(N_f,N_c)-1} m_i^2 t^{2i} + \begin{cases} if\ N_f < N_c: & t^{2N_f} \\ if\ N_f = N_c: & t^{N_f} \\ if\ N_f > N_c: & m_{N_c} t^{N_c} \end{cases}$
$\mathrm{SU}(N_f) \times \mathrm{USp}(N_c)$	$\sum_{i=1}^{\min\left(\left\lfloor \frac{N_f-1}{2} \right\rfloor, \frac{N_c}{2}\right)} m_{2i} t^{2i} + \begin{cases} if\ N_f \leq N_c\ \&\ N_f\ is\ even: & t^{N_f} \\ if\ N_f > N_c: & m_{N_c} t^{N_c} \end{cases}$

Table 30. Generalised PLs of HWGs for SQCD with Classical Colour Groups.

As noted in section 3, providing one adopts a labelling system that reflects group symmetries, the HWG generating functions are identical for all the SU(N) flavour groups once their fundamental dimension exceeds the defining space dimension of the colour group. This arises because the antisymmetrisations of the fundamental of the flavour group generated by the PE are limited by the length of the colour group epsilon tensor. The differences in the unrefined Hilbert series as the rank of the flavour group is increased are simply due to the different dimensions of the flavour group irreps. Thus, we can reason that the HWGs of SQCD are the same for all SU(N) flavour groups of fundamental dimension exceeding that of the colour group. This important insight makes it possible to calculate the Hilbert series for GIOs of SQCD theories with high rank flavour groups, for which a direct calculation using Weyl integration might not be feasible. The result corresponds to observations within [3, 4].

Following similar tensorial reasoning, it is possible to relate all the HWGs for classical SQCD to the primitive invariant tensors of the classical groups, as set out in table 8, with each HWG monomial corresponding to a contraction of delta and epsilon tensors with the numbers of quarks identified by the fugacities. Using this knowledge of the structure of the primitive invariant tensors, we can write down expressions for the HWGs describing the SQCD theory for a given classical product group as in table 30 .

The HWGs typically contain a different number of generators and relations to the HS. The generators can be identified by PL terms with positive coefficients, while the relations between the generators are given by PL terms with negative coefficients. The overall dimension of a moduli space is given by the number of generators less relations [4]. We summarise the HWG and HS descriptions of the moduli spaces of a selection of classical SQCD theories in table 31. The dimensions are obtained by summing the PLs in tables 27 through 29 with all coordinate fugacities set to unity, or, in the case of the non-terminating PLs of unrefined Hilbert series, by summing the orders of the poles calculated in section 3.

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Theory	HWG *Dimension* (a)	HWG *Irrep*	HWG *Irrep Degree* (b)	*Moduli Space Dimension* (HS) $(a)+(b)$
$\mathrm{SU}(2)\times\mathrm{SU}(2)$	1	$[0]$	0	1
$\mathrm{SU}(4)\times\mathrm{SU}(2)$	1	$[0,n,0]$	4	5
$\mathrm{SU}(6)\times\mathrm{SU}(2)$	1	$[0,n,0,0,0]$	8	9
$\mathrm{SU}(8)\times\mathrm{SU}(2)$	1	$[0,n,0,0,0,0,0]$	12	13
$\mathrm{SU}(10)\times\mathrm{SU}(2)$	1	$[0,n,0,0,0,0,0,0,0]$	16	17
$\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(3)$	2	$[n][n]$	2	4
$\mathrm{SU}(3)\times\mathrm{SU}(3)\times\mathrm{SU}(3)$	4	$[n_1,n_2][n_1,n_2]$	6	10
$\mathrm{SU}(4)\times\mathrm{SU}(4)\times\mathrm{SU}(3)$	4	$[n_1,n_2,n_3][n_1,n_2,n_4]$	12	16
$\mathrm{SU}(2)\times\mathrm{SO}(3)$	2	$[2n]$	1	3
$\mathrm{SU}(3)\times\mathrm{SO}(3)$	3	$[2n_1,2n_2]$	3	6
$\mathrm{SU}(4)\times\mathrm{SO}(3)$	3	$[2n_1,2n_2,n_3]$	6	9
$\mathrm{SU}(2)\times\mathrm{SO}(4)$	2	$[2n]$	1	3
$\mathrm{SU}(3)\times\mathrm{SO}(4)$	3	$[2n_1,2n_2]$	3	6
$\mathrm{SU}(4)\times\mathrm{SO}(4)$	4	$[2n_1,2n_2,2n_3]$	6	10
$\mathrm{SU}(5)\times\mathrm{SO}(4)$	4	$[2n_1,2n_2,2n_3,n_4]$	10	14
$\mathrm{SU}(2)\times\mathrm{SO}(5)$	2	$[2n]$	1	3
$\mathrm{SU}(3)\times\mathrm{SO}(5)$	3	$[2n_1,2n_2]$	3	6
$\mathrm{SU}(4)\times\mathrm{SO}(5)$	4	$[2n_1,2n_2,2n_3]$	6	10
$\mathrm{SU}(5)\times\mathrm{SO}(5)$	5	$[2n_1,2n_2,2n_3,2n_4]$	10	15
$\mathrm{SU}(6)\times\mathrm{SO}(5)$	5	$[2n_1,2n_2,2n_3,2n_4,n_5]$	15	20
$\mathrm{SU}(2)\times\mathrm{USp}(2)$	1	$[0]$	0	1
$\mathrm{SU}(3)\times\mathrm{USp}(2)$	1	$[0,n]$	2	3
$\mathrm{SU}(4)\times\mathrm{USp}(2)$	1	$[0,n,0]$	4	5
$\mathrm{SU}(5)\times\mathrm{USp}(2)$	1	$[0,n,0,0]$	6	7
$\mathrm{SU}(2)\times\mathrm{USp}(4)$	1	$[0]$	0	1
$\mathrm{SU}(3)\times\mathrm{USp}(4)$	1	$[0,n]$	2	3
$\mathrm{SU}(4)\times\mathrm{USp}(4)$	2	$[0,n,0]$	4	6
$\mathrm{SU}(5)\times\mathrm{USp}(4)$	2	$[0,n_1,0,n_2]$	8	10
$\mathrm{SU}(6)\times\mathrm{USp}(4)$	2	$[0,n_1,0,n_2,0]$	12	14
$\mathrm{SU}(7)\times\mathrm{USp}(4)$	2	$[0,n_1,0,n_2,0,0]$	16	18

Table 31. Dimensions of Moduli Spaces of Classical SQCD Theories.

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Theory	HWG *Dimension* (a)	HWG *Irrep Degree* (b)	*Moduli Space Dimension* (HS) $(a)+(b)$
$\mathrm{SU}(2N_f)\times\mathrm{SU}(2)$	1	$4N_f-4$	$4N_f-3$
$\mathrm{SU}(N_f)\times\mathrm{SU}(N_f)\times\mathrm{SU}(N_c)$	N_c+1	$2N_fN_c-N_c^2-N_c$	$2N_fN_c-N_c^2+1$
$\mathrm{SU}(N_f)\times\mathrm{SO}(N_c)$	N_c	$N_fN_c-N_c(N_c+1)/2$	$N_fN_c-N_c(N_c-1)/2$
$\mathrm{SU}(N_f)\times\mathrm{USp}(N_c)$	$N_c/2$	$N_fN_c-N_c(N_c+2)/2$	$N_fN_c-N_c(N_c+1)/2$

Table 32. Dimensions of Moduli Spaces of Classical SQCD Theories for $N_f \geq N_c$.

As can be seen, the Hilbert series typically describe moduli spaces with a higher dimension than those of the HWGs. We can develop a systematic account of the relationship between the dimensions of a Hilbert series and those of its underlying HWG by expanding the latter and analysing the dimensional structure of its irreps. For example, the expansion for $\mathrm{SU}(4)_{f,L}\times\mathrm{SU}(4)_{f,R}\times\mathrm{SU}(3)_c$ in table 27 takes the form:

$$\mathrm{PE}\left[l_3t_1^3+r_3t_2^3+l_1r_1t_1t_2+l_2r_2t_1^2t_2^2\right]\Leftrightarrow\sum_{n_1,n_2,n_3,n_4=0}^{\infty}[n_1,n_2,n_3]\,[n_1,n_2,n_4]t_1^{n_1+2n_2+3n_3}\,t_2^{n_1+2n_2+3n_4}. \tag{5.1}$$

The Dynkin labels in this HWG series expansion are described by four different parameters $\{n_1,n_2,n_3,n_4\}$ corresponding to the four generators $\{l_1r_1t_1t_2,\ l_2r_2t_1^2t_2^2,\ l_3t_1^3,\ r_3t_2^3\}$ respectively. These parameters define the sub-lattice of the group spanned by the irreps of the HWG. The dimensions of the irreps are given by a polynomial function of the parameters and the degree of this polynomial accounts for the extra dimensions of the Hilbert series compared to the HWG.

We now define the *HWG Irrep Degree* as the total degree of the polynomial that gives the dimensions of the HWG irreps.[6] Example (5.1) is built from irreps of SU(4), for which the dimension formula is:

$$Dim[n_1,n_2,n_3]=\frac{1}{12}(n_1+1)(n_2+1)(n_3+1)(n_1+n_2+2)(n_2+n_3+2)(n_1+n_2+n_3+3). \tag{5.2}$$

The degree of this polynomial is six and thus the HWG Irrep Degree is 12, being the sum of the degrees for the two SU(4) sub-groups. In HWGs where some Dynkin labels are fixed at zero, the HWG Irrep Degree is reduced. The HWG Irrep Degree, as defined, exactly accounts for the difference between the dimensions of the HWG and the Hilbert series. This analysis can be repeated for all the SQCD theories studied and is summarised in table 31.

Noting the constant nature of the HWG dimension for $N_f \geq N_c$ we can, by inspection, generalise the dimensions of the moduli spaces for large N_f as in table 32.

In all these cases, for $N_f \geq N_c$ the gauge group is completely broken, and the dimension of the Hilbert series is given by the dimension of the underlying product group representation less the group dimension of the colour group. This Hilbert series dimension

[6] This definition leads to HWG Irrep Degrees that are consistent with the degrees of dimensional polynomials given in [7].

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Theory	PL[HWG]	PL[*Unrefined Hilbert Series*]
$\mathrm{SU}(2) \times G_2$	$m^2t^2 + t^4$	$3t^2$
$\mathrm{SU}(3) \times G_2$	$t^3 + t^6 + m_1^2t^2 + m_2^2t^4$	$t^3 + 6t^2$
$\mathrm{SU}(4) \times G_2$	$m_1^2t^2 + m_3t^3 + t^4$ $+m_2^2t^4 + m_1t^5 + m_3^2t^6 + m_1m_2t^7 + t^8$ $+m_2m_3t^9 + m_2^2t^{12} - m_1^2m_2^2t^{14} - m_2^2m_3^2t^{18}$	$10t^2 + 4t^3 + t^4 - t^8$
$\mathrm{SU}(5) \times G_2$	*to be calculated*	*infinite series*

Table 33. PLs of HWGs and Hilbert Series of $\mathrm{SU}(N_f) \times G_2$ colour singlets.

Theory	HWG *Dimension* (a)	HWG *Irrep*	HWG *Irrep Degree* (b)	*Moduli Space Dimension* (HS) $(a) + (b)$
$\mathrm{SU}(2) \times G_2$	2	$[n]$	1	3
$\mathrm{SU}(3) \times G_2$	4	$[n_1, n_2]$	3	7
$\mathrm{SU}(4) \times G_2$	8	$[n_1, n_2, n_3]$	6	14
$\mathrm{SU}(5) \times G_2$	*t.b.c.*	$[n_1, n_2, n_3, n_4]$	10	21

Table 34. Dimensions of Moduli Spaces of $\mathrm{SU}(N_f) \times G_2$ colour SQCD Theories.

reduces into the dimension of the HWG and the degree of the dimensional polynomial for its irreps.

The Hilbert series of the GIOs of exceptional gauge groups are considerably more complicated than those of classical gauge groups. This complicated structure can be seen from table 33, which contains some Hilbert series and HWGs for G_2. We can identify basic GIOs built on the G_2 primitive symmetric invariant tensor of rank two and the G_2 primitive antisymmetric invariant tensors of rank three and four, being $m_1^2t^2$, m_3t^3 and t^4 respectively, corresponding to the [2,0,0], [0,0,1] and [0,0,0] irreps of SU(4). The HWG identifies, in addition, the complicated pattern of GIOs in the many other SU(N) irreps that can be formed from combinations of these basic objects, taking account of the relations amongst them. The dimensions of the Hilbert series and the HWGs are related to each other in a similar manner to those of the classical SQCD theories analysed above, as can be seen from table 34.

It is argued that the Hilbert series of SQCD are always palindromic, which entails that the moduli spaces of the fugacities are Calabi-Yau [4]. All the Hilbert series, and also all the HWGs for SQCD calculated herein, are palindromic, considering that the numerators of freely generated series and complete intersections are also simple palindromes, and therefore Calabi-Yau. This palindromic property of many generating functions for Hilbert series is shared with the character generating functions discussed in section 2 that are used to derive the HWGs and Hilbert series.

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Instanton Theory	PL[HWG$_{\text{reduced}}$]	PL[HWG]	PL[HS$_{\text{unrefined}}$]
One SU(2)	$m_1{}^2t^2$	$mt + m_1{}^2t^2$	*infinite series*
Two SU(2)	$m^2t^2 + m_1^2t^2 + mm_1^2t^3 + t^4$ $+mm_1^2t^5 - m^2m_1^4t^{10}$	*infinite series*	*infinite series*
Three SU(2)	*infinite series*	*infinite series*	*infinite series*
One SU(3)	$m_1m_2t^2$	$mt + m_1m_2t^2$	*infinite series*

Table 35. PLs of Moduli Spaces of Selected Instanton Theories.

An important demonstration from the HWG analysis is that, in all cases, the unrefined HS are reducible to sums of series associated with individual flavour group irreps. This contrasts with the conjecture regarding SQCD [4] (Observation 3.11) that:

> "[...] we find in all case studies that $\mathcal{M}(N_f, N_c)$ is irreducible using primary decomposition and conjecture this to hold in general."

The group theoretic reducibility of the unrefined Hilbert series follows from the facts that the PE and PEF map class functions into other class functions and that these can always be decomposed in terms of characters. This reducibility corresponds to the precise description of the group structures underlying the Hilbert series in terms of the HWG generating functions.

Turning to instanton moduli spaces, we have set out in section 4, the HWG for one SU(3) instanton on $\mathbb{C}^2$ and for two and three SU(2) instantons on $\mathbb{C}^2$. We have shown that all the operators in the moduli space of three SU(2) instantons on $\mathbb{C}^2$ transform in some symmetrised irrep of the adjoint, with Dynkin labels $[\underbrace{0,0}_{\text{gauge}}; \underbrace{2n_1}_{\text{flavour}}; \underbrace{n}_{\text{global}}]$, for non-negative integers n and n_1. This is similar to the established result for one instanton Hilbert series [6]. It results from the initial choice of gauge group as U(k), since this has no epsilon tensor, and its singlets can only be formed from equal numbers of U(k) representations and conjugate $\bar{\text{U}}(k)$ representations.

The HWGs for instanton moduli spaces are generally considerably more complicated than those of SQCD, since they involve symmetrisations of the adjoint in addition to those of basic irreps. This can be seen from table 35.

While the generating functions for the HWGs and Hilbert series of instanton moduli spaces are all palindromic, only some of these spaces turn out to be freely generated. In particular, the HWGs for single instanton theories are freely generated. We have also shown that the *reduced* moduli space for two SU(N) instantons is a complete intersection, although this is the case neither for the higher multiple instanton theories examined, nor for any of the unrefined Hilbert series.

We should mention also, without giving details, that the moduli spaces of multiple instanton theories all contain the moduli spaces of one instanton theories and so an alternative approach to multiple instanton theories is to study the quotient space between the

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Theory	HWG *Dimension* (a)	HWG *Irrep*	HWG *Irrep Degree* (b)	*Moduli Space Dimension* (HS) (a) + (b)
One SU(2) *Instanton*	2	$[2n_1, n]$	2	4
Two SU(2) *Instantons*	6	$[2n_1; n]$	2	8
Three SU(2) *Instantons*	10	$[2n_1; n]$	2	12
One SU(3) *Instanton*	2	$[n_1, n_1; n]$	4	6

Table 36. Dimensions of Moduli Spaces of Selected Instanton Theories.

multiple instanton theory and its underlying one instanton theory. Such quotient spaces lend themselves naturally to analysis in terms of HWGs.

Finally, the dimensions of the instanton moduli spaces decompose into the dimension of their HWGs and the degrees of the dimensional polynomial of the HWG irreps in a similar manner to the SQCD theories, as can be seen from table 36.

The instanton moduli spaces calculated in table 36 all include a contribution from global SU(2) symmetries. If this contribution is excluded, we obtain reduced instanton moduli spaces, as discussed earlier. The HWGs of these reduced instanton moduli spaces of one instanton theories based on simple Lie groups are all one dimensional. For example, the HWG of the reduced moduli space for one SU(3) instanton is just $m_1 m_2 t^2$. This leads to an elegant decomposition of the moduli spaces of one instanton theories into one dimensional HWGs and corresponding HWG dimensional polynomials, calculated in the same manner as above, and these are set out in table 37. All the reduced one instanton moduli spaces have a dimension equal to the sum of the Dual Coxeter labels of the group [6].

Conclusion. In conclusion, the HWG approach provides an efficient means of encoding, calculating and decomposing Hilbert series and opens up many avenues for further investigation. These could include explication of the general relationship between the invariant tensors of groups and the structures of GIOs that arise within product groups. Specific theories that could merit further study include SQCD with fields transforming in various representations of classical and/or exceptional gauge groups and multiple instanton theories generally. Finally, it could prove interesting to understand more fully the geometric nature of the HWG manifolds, which all appear to be palindromic and therefore Calabi-Yau in nature, and to relate this more precisely to the geometry of the Hilbert series manifolds.

It should be added that these investigations make extensive use of contour integration and face the challenge of implementing algorithms in *Mathematica* to combine and simplify large numbers of residues within the computing constraints of memory limits and a practical timescale. The development of more effective algorithms would therefore facilitate the extension of the results herein to a wider range of theories built from other group or product group representations.

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Series	*Adjoint Representation*	*Instanton* HWG	*Dim* HWG (a)	*Degree of Dimensional Polynomial* (b)	*Dim Instanton* $(a)+(b)$	*Sum Dual Coxeter Labels* $\frac{1}{2}((a)+(b))$
A_n	$A_1:[2]$	m^2t	1	1	2	1
	$A_2:[1,1]$	$m_1m_2t^2$	1	3	4	2
	$A_{\geq 3}:[1,0,\ldots,1]$	$m_1m_nt^2$	1	$2n-1$	$2n$	n
B_n	$B_1:[2]$	m^2t	1	1	2	1
	$B_2:[0,2]$	${m_2}^2t$	1	3	4	2
	$B_{\geq 3}:[0,1,\ldots,0]$	m_2t	1	$4n-5$	$4n-4$	$2n-2$
C_n	$C_1:[2]$	m^2t	1	1	2	1
	$C_2:[2,0]$	${m_1}^2t$	1	3	4	2
	$C_{\geq 3}:[2,0,\ldots,0]$	m_1^2t	1	$2n-1$	$2n$	n
D_n	$D_2:[2,0]\oplus[0,2]$	$m_1^2t\oplus m_2^2t$	$1\oplus 1$	$1\oplus 1$	$2\oplus 2$	$1\oplus 1$
	$D_3:[0,1,1]$	$m_2m_3t^2$	1	5	6	3
	$D_{\geq 4}:[0,1,\ldots,0]$	m_2t	1	$4n-7$	$4n-6$	$2n-3$
E_6	$[0,0,0,0,0,1]$	m_6t	1	21	22	11
E_7	$[1,0,0,0,0,0,0]$	m_1t	1	33	34	17
E_8	$[0,0,0,0,0,0,1,0]$	m_7t	1	57	58	29
F_4	$[1,0,0,0]$	m_1t	1	15	16	8
G_2	$[1,0]$	m_1t	1	5	6	3

Table 37. Dimensional Analysis of Reduced One Instanton Moduli Spaces.

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A Plethystic exponential and logarithm

Consider a function in some variable t, which can be expressed as a power series:

$$f(t) \equiv \sum_{n=0}^{\infty} a_n t^n. \tag{A.1}$$

The Plethystic Exponential ("PE") for such a function is defined [4] as:

$$\begin{aligned} \mathrm{PE}[f(t),t] &\equiv \exp\left(\sum_{k=1}^{\infty}\frac{f(t^k)-f(0)}{k}\right) \\ &= \prod_{n=1}^{\infty}\frac{1}{(1-t^n)^{a_n}}. \end{aligned} \tag{A.2}$$

The PE can be generalised for power series of more than one variable, so that for:

$$f(t_1,\ldots,t_N) \equiv \sum_{n=0}^{\infty}\sum_{i=1}^{N} a_{n_i} t_i^n, \tag{A.3}$$

we obtain the PE:

$$\begin{aligned}\mathrm{PE}[f(t_1,\ldots,t_N),\ (t_1,\ldots,t_N)] &\equiv \exp\left(\sum_{k=1}^{\infty}\frac{f(t_1^k,\ldots,t_N^k)-f(0,\ldots,0)}{k}\right)\\ &= \prod_{n=1}^{\infty}\prod_{i=1}^{N}\frac{1}{(1-t_i^n)^{a_{ni}}}.\end{aligned} \tag{A.4}$$

In order to avoid ambiguities, we shall, where necessary, use the notation:

$$\mathrm{PE}[f(t_1,\ldots,t_N),\ (t_1,\ldots,t_N)], \tag{A.5}$$

to clarify the variables with respect to which the PE is being taken (and similarly for the PL).

The Plethystic Logarithm ("PL") can be used to invert the PE. The PL makes use of the Mobius function $\mu(k)$, which is defined as $(-1)^n$ for an integer that is the product of n distinct primes other than unity, and zero otherwise, such that $\mu(1)=1, \mu(2)=\mu(3)=-1,\ldots$ etc. For the general case, the PL is defined as:

$$\mathrm{PL}[g(t_1\ldots,t_N),\ (t_1\ldots,t_N)] \equiv \sum_{k=1}^{\infty}\frac{1}{k}\mu(k)\log g(t^k{}_1,\ldots,t^k{}_N). \tag{A.6}$$

If we set $g(t_1\ldots,t_N)=\mathrm{PE}[f(t_1\ldots,t_N)]$, we then obtain $f(t_1\ldots,t_N)=\mathrm{PL}[g(t_1,\ldots,t_N),(t_1,\ldots,t_N)]$, as required. The identity can be proved by manipulation of the various series using the properties of the Mobius function [2], which include the key simplifying identity:

$$\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\frac{\mu(l)}{lm}t^{klm} = t^k. \tag{A.7}$$

The PE, which is a symmetrising function, has a related antisymmetrising function called the Fermionic Plethystic Exponential ("PEF"). This is defined as:

$$\begin{aligned}\mathrm{PEF}[f(t_1,\ldots,t_N),\ (t_1,\ldots,t_N)] &\equiv \exp\left(\sum_{k=1}^{\infty}(-1)^{k+1}\frac{f(t_1^k,\ldots,t_N^k)-f(0,\ldots,0)}{k}\right)\\ &= \prod_{n=1}^{\infty}\prod_{i=1}^{N}(1+t_i^n)^{a_{ni}}.\end{aligned} \tag{A.8}$$

Like the PE, the PEF also has an inverse function, which we term the Fermionic Plethystic Logarithm (or "PLF"), given by:

$$\begin{aligned}\mathrm{PLF}[g(t_1,\ldots,t_N),\ (t_1,\ldots,t_N)] &= \sum_{m=0}^{\infty}\mathrm{PL}[g(t_1^{2^m},\ldots,t_N^{2^m}),\ (t_1^{2^m},\ldots,t_N^{2^m})]\\ &= \sum_{m=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k}\mu(k)\log g(t_1^{(2^m)k}\ldots,t_N^{(2^m)k}).\end{aligned} \tag{A.9}$$

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The PE and PEF have the useful properties that:

$$\begin{aligned} \mathrm{PE}[f_1 + f_2] &= \mathrm{PE}[f_1]\,\mathrm{PE}[f_2] \\ \mathrm{PEF}[f_1 + f_2] &= \mathrm{PEF}[f_1]\,\mathrm{PEF}[f_2] \end{aligned} \tag{A.10}$$

and the PL and PLF have the related properties that:

$$\begin{aligned} \mathrm{PL}[g_1 g_2] &= \mathrm{PL}[g_1] + \mathrm{PL}[g_2] \\ \mathrm{PLF}[g_1 g_2] &= \mathrm{PLF}[g_1] + \mathrm{PLF}[g_2]. \end{aligned} \tag{A.11}$$

All the above results are exact, providing that the series are convergent.

We can use the Plethystic Exponential to symmetrise the character of an irrep of some group G as follows. Suppose the character $\mathcal{X}$ of the irrep is composed of monomials $A_i(x_1, \ldots, x_r)$, where the x_j are CSA coordinates ranging over the rank r of the group and the index i ranges over the dimension $Dim(\mathcal{X})$ of the irrep:

$$\mathcal{X} = \sum_{i=1}^{Dim(\mathcal{X})} A_i(x_1, \ldots, x_r). \tag{A.12}$$

We form a generating function $g^G(t, \mathcal{X})$ by taking the PE of the sum of fugacities $f_i \equiv tA_i$, which are given by the products of each coordinate monomial with a fugacity t, where $0 < |t| < 1$:

$$\begin{aligned} g^G(t, \mathcal{X}) &\equiv \mathrm{PE}[\mathcal{X}t] \\ &\equiv \mathrm{PE}\left[\sum_{i=1}^{Dim(\mathcal{X})} f_i, (f_1, \ldots, f_{Dim(\mathcal{X})})\right] \\ &= \prod_{i=1}^{Dim(\mathcal{X})} \frac{1}{(1 - tA_i)}. \end{aligned} \tag{A.13}$$

The Taylor expansion of $g^G(t, \mathcal{X})$ generates an infinite polynomial in the fugacity t, whose coefficients are functions of the coordinate monomials. Importantly, the PE of a character is a class function and the Peter Weyl Theorem [10] entails that the characters of a compact group form a complete basis for its class functions, so this Taylor expansion can be decomposed as a sum of characters of irreps, each of which is related to the initial character by symmetrisation, and each of which has a distinct coefficient in the form of a polynomial in the fugacity t.

$$\mathrm{PE}[\mathcal{X}t] = \sum_{\text{irreps}} g_{\text{irrep}}(t)\mathcal{X}_{\text{irrep}}(A_i). \tag{A.14}$$

The symmetrising PE is complemented by the PEF, which we can use in a similar manner to antisymmetrise the character of an irrep.

$$\begin{aligned} g^G_F(t, \mathcal{X}) &\equiv \mathrm{PEF}[\mathcal{X}t] \\ &\equiv \mathrm{PEF}\left[\sum_{i=1}^{Dim(X)} f_i, (f_1, \ldots, f_{Dim(X)})\right] \\ &= \prod_{i=1}^{Dim(X)} (1 + tA_i). \end{aligned} \tag{A.15}$$

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Following similar reasoning to that above, the PEF can also be expanded as a sum of characters:

$$\mathrm{PEF}[\mathcal{X}t] = \sum_{\text{irreps}} g_F^G{}_{\text{irrep}}(t)\mathcal{X}_{\text{irrep}}(A_i). \tag{A.16}$$

In this case, the sum of characters is finite.

B Weyl integration

Weyl integration (also known as Molien-Weyl integration) provides a helpful method for integrating the class functions of a group G, such as characters, which depend only upon the identity of a chosen irrep, over the volume of the group. Normally, group integration of a function $f[\gamma]$, where $\gamma \in G$, requires taking the integral over all the dimensions $Dim(G)$ of the group:

$$I = \int_G d\mu(\gamma) f(\gamma), \tag{B.1}$$

where $d\mu(\gamma)$ is the Haar measure. In Weyl integration the integral is simplified to one over the maximal torus of the group (as generated by its Cartan subalgebra), by conjugating the class function with other elements of the Group, such that it is always represented by an element of the maximal torus. This is permissible since the integral is effectively a trace over the group. This conjugation reduces the number of integrations required from the dimension of the group to the rank of the group. To do this consistently, the Haar measure, which is effectively a volume element, has to be modified by scaling to reflect the projection of the entire group onto its maximal torus [10, 23].

Without digressing further into the technical details, we can usefully tabulate the modified Haar measures for U(r) and the classical groups [24] as in table 38. In this table, the rank of a group is always labelled by r and $\{\theta_1, \ldots, \theta_r\}$ are coordinates of periodicity 2π on its maximal torus. The Haar measure is normalised such that it integrates to unity: $\int_{\theta_i=0}^{2\pi} d\mu = 1$.

It can be helpful to express the Weyl integral and Haar measures as unimodular contour integrals. If we make the coordinate substitution:

$$x_j = e^{i\theta_j}, \tag{B.2}$$

we can then rewrite the Weyl integrals in table 38 as in table 39. This form of the Weyl integral readily lends itself to application of the residue theorem.

Importantly, the characters of irreps are orthonormal under the Weyl integral [10]); if we consider the characters of two irreps of a group G labelled by $\mathcal{X}_{[A]}$ and $\mathcal{X}_{[B]}$, their inner product (appropriately normalised) is given by:

$$\int_G d\mu \mathcal{X}^*_{[A]}\mathcal{X}_{[B]} = \delta_{AB}. \tag{B.3}$$

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Thus, the Weyl integral can be used to form an inner product that projects out the singlet content of products of characters (or functions of characters) and, as a corollary, the Weyl integral of a single character is zero for any irrep other than the singlet itself.

We can obtain the characters of irreps by the usual methods from Cartan matrices and by using the Freudenthal recursion formula to find the correct multiplicities of weights [10, 25]. The modified Haar measures do however depend on the CSA coordinate system used. The modified Haar measures in table 39 are correct for the characters of defining representations as set out in table 40. If a different choice of weights is used for the coordinate monomials of the defining representations, then the modified Haar measures must be transformed to the new coordinate system to ensure that the orthonormality relations (B.3) remain satisfied.

A simpler form of the Haar measure is noted in [5], which is based on characters whose coordinate monomials carry canonical weights derived from the Cartan matrix and gives the Haar measure in terms of the positive root space only. This generalises to exceptional groups and can produce simpler expressions that can be evaluated more quickly.

$$\oint_G d\mu = \frac{1}{(2\pi i)^r} \oint_{|x_i|=1} \prod_{i=1}^{r} \frac{dx_i}{x_i} \prod_{\alpha\in\Phi+} (1 - A_\alpha(x_1, \ldots, x_r)). \tag{B.4}$$

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Group	*Haar Measure on Maximal Torus*
U(r)	$\frac{1}{(2\pi)^r r!}\int_{\theta_i=0}^{2\pi}\prod_{1\le j<k\le r}\left\|e^{i\theta_j}-e^{i\theta_k}\right\|^2 d\theta_1\ldots,d\theta_r$
SU($r+1$)	$\frac{1}{(2\pi)^r (r+1)!}\int_{\theta_i=0}^{2\pi}\prod_{1\le j<k\le r+1}\left\|e^{i\theta_j}-e^{i\theta_k}\right\|^2 d\theta_1\ldots,d\theta_r$ *where* $\theta_{r+1}\equiv-\sum_{j=1}^{r}\theta_j$
SO($2r+1$)	$\frac{2^{r^2}}{(2\pi)^r r!}\int_{\theta_i=0}^{2\pi}\prod_{1\le j<k\le r}\left(\cos(\theta_j)-\cos(\theta_k)\right)^2\prod_{n=1}^{r}\sin^2\left(\frac{1}{2}\theta_n\right)d\theta_1\ldots,d\theta_r$
USp($2r$)	$\frac{2^{r^2}}{(2\pi)^r r!}\int_{\theta_i=0}^{2\pi}\prod_{1\le j<k\le r}\left(\cos(\theta_j)-\cos(\theta_k)\right)^2\prod_{n=1}^{r}\sin^2(\theta_n)d\theta_1\ldots,d\theta_r$
SO($2r$)	$\frac{2^{(r-1)^2}}{(2\pi)^r r!}\int_{\theta_i=0}^{2\pi}\prod_{1\le j<k\le r}\left(\cos(\theta_j)-\cos(\theta_k)\right)^2 d\theta_1\ldots,d\theta_r$

Table 38. Modified Haar Measures on the Maximal Torus.

Group	*Haar Measure on Maximal Torus*
U(r)	$\frac{1}{(2\pi i)^r r!}\oint_{\lvert x_i\rvert=1}\frac{dx_i}{x_i}\prod_{1\le j<k\le r}\lvert x_j-x_k\rvert^2$
SU($r+1$)	$\frac{1}{(2\pi i)^r (r+1)!}\prod_{i=1}^{r}\oint_{\lvert x_i\rvert=1}\frac{dx_i}{x_i}\prod_{1\le j<k\le r+1}\lvert x_j-x_k\rvert^2$ $x_{r+1}\equiv\prod_{j=1}^{r}\frac{1}{x_j}$
SO($2r+1$)	$\frac{1}{2^r(2\pi i)^r r!}\prod_{i=1}^{r}\oint_{\lvert x_i\rvert=1}\frac{dx_i}{x_i}\prod_{1\le j<k\le r}\frac{(x_j-x_k)^2(1-x_jx_k)^2}{x^2{}_j x^2{}_k}\prod_{m=1}^{r}\frac{(1-x_m)(x_m-1)}{x_m}$
USp($2r$)	$\frac{1}{2^r(2\pi i)^r r!}\prod_{i=1}^{r}\oint_{\lvert x_i\rvert=1}\frac{dx_i}{x_i}\prod_{1\le j<k\le r}\frac{(x_j-x_k)^2(1-x_jx_k)^2}{x^2{}_j x^2{}_k}\prod_{m=1}^{r}\frac{(1-x^2{}_m)(x^2{}_m-1)}{x^2{}_m}$
SO($2r$)	$\frac{1}{2^{r-1}(2\pi i)^r r!}\prod_{i=1}^{r}\oint_{\lvert x_i\rvert=1}\frac{dx_i}{x_i}\prod_{1\le j<k\le r}\frac{(x_j-x_k)^2(1-x_jx_k)^2}{x^2{}_j x^2{}_k}$

Table 39. Modified Haar Measures on the Maximal Torus: Contour Integrals.

Group Series	*Defining Representation*	*Defining Character used by Haar Measure*
A_r	$[1,\ldots,0]$	$\sum_{i=1}^{r}x_i+\prod_{i=1}^{r}x_i^{-1}$
B_r	$[2]$ $r=1$ $[1,\ldots,0]$ $r>1$	$\sum_{i=1}^{r}x_i+1+\sum_{i=1}^{r}x_i^{-1}$
C_r	$[1,\ldots,0]$	$\sum_{i=1}^{r}x_i+\sum_{i=1}^{r}x_i^{-1}$
D_r	$[1,1]$ $r=2$ $[1,\ldots,0]$ $r>2$	$\sum_{i=1}^{r}x_i+\sum_{i=1}^{r}x_i^{-1}$

Table 40. Characters used by Haar Measures.

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C Numerator of generating function for G_2 characters

$P_{G2}(t_i)$	Character
$(1-t_1)\,(-1+t_2)^2\,(1+t_2)$ $\left(1+t_1+t_2+t_1\,t_2+t_1^2\,t_2+t_1^3\,t_2+t_1^4\,t_2+t_2^2+t_1\,t_2^2+t_1^3\,t_2^2+t_1^4\,t_2^2+t_2^3+\right.$ $t_1\,t_2^3+t_1^2\,t_2^3+t_1^3\,t_2^3+t_1^4\,t_2^3+t_2^4+2\,t_1\,t_2^4+t_1^2\,t_2^4+t_1^3\,t_2^4+t_1^4\,t_2^4+t_2^5+$ $t_1\,t_2^5+t_1^2\,t_2^5+2\,t_1^3\,t_2^5+t_1^4\,t_2^5+t_2^6+t_1\,t_2^6+t_1^2\,t_2^6+t_1^3\,t_2^6+t_1^4\,t_2^6+t_2^7+$ $\left.t_1\,t_2^7+t_1^3\,t_2^7+t_1^4\,t_2^7+t_2^8+t_1\,t_2^8+t_1^2\,t_2^8+t_1^3\,t_2^8+t_1^4\,t_2^8+t_1^3\,t_2^9+t_1^4\,t_2^9\right)$	{0, 0}
$(-1+t_1)\,(-1+t_2)^2\,t_2$ $\left(t_1+t_1^3+t_1^3\,t_2+t_1\,t_2^2+t_2^3+2\,t_1\,t_2^3+t_1^3\,t_2^3+t_2^4+t_1\,t_2^4+2\,t_1^2\,t_2^4+\right.$ $\left.t_1^3\,t_2^4+t_1^4\,t_2^4+t_1\,t_2^5+2\,t_1^3\,t_2^5+t_1^4\,t_2^5+t_1^3\,t_2^6+t_1\,t_2^7+t_1\,t_2^8+t_1^3\,t_2^8\right)$	{0, 1}
$(1-t_1)\,(-1+t_2)^2\,t_2$ $\left(t_1^2+t_1\,t_2+t_1^2\,t_2+t_2^2+t_1\,t_2^2+t_1^2\,t_2^2+t_1^3\,t_2^2+t_2^3+t_1\,t_2^3+2\,t_1^2\,t_2^3+\right.$ $t_1^3\,t_2^3+t_1^4\,t_2^3+t_2^4+2\,t_1\,t_2^4+2\,t_1^3\,t_2^4+t_1^4\,t_2^4+t_2^5+t_1\,t_2^5+2\,t_1^2\,t_2^5+$ $\left.t_1^3\,t_2^5+t_1^4\,t_2^5+t_1\,t_2^6+t_1^2\,t_2^6+t_1^3\,t_2^6+t_1^4\,t_2^6+t_1^2\,t_2^7+t_1^3\,t_2^7+t_1^2\,t_2^8\right)$	{1, 0}
$(-1+t_1)\,(-1+t_2)^2\,t_2$ $\left(t_1+t_2+t_1\,t_2+t_1^2\,t_2+t_2^2+2\,t_1\,t_2^2+2\,t_1^2\,t_2^2+t_1^3\,t_2^2+t_1^4\,t_2^2+t_2^3+t_1\,t_2^3+\right.$ $3\,t_1^3\,t_2^3+t_1^4\,t_2^3+t_2^4+2\,t_1^2\,t_2^4+t_1^4\,t_2^4+t_2^5+3\,t_1\,t_2^5+t_1^3\,t_2^5+t_1^4\,t_2^5+$ $\left.t_2^6+t_1\,t_2^6+2\,t_1^2\,t_2^6+2\,t_1^3\,t_2^6+t_1^4\,t_2^6+t_1^2\,t_2^7+t_1^3\,t_2^7+t_1^4\,t_2^7+t_1^3\,t_2^8\right)$	{0, 2}
$(1-t_1)\,(-1+t_2)^2\,t_2^2$ $\left(t_1+t_2+t_1\,t_2+t_1^3\,t_2+t_2^2+t_1\,t_2^2+2\,t_1^2\,t_2^2+t_1^4\,t_2^2+t_2^3+2\,t_1\,t_2^3-t_1^2\,t_2^3+\right.$ $\left.2\,t_1^3\,t_2^3+t_1^4\,t_2^3+t_2^4+2\,t_1^2\,t_2^4+t_1^3\,t_2^4+t_1^4\,t_2^4+t_1\,t_2^5+t_1^3\,t_2^5+t_1^4\,t_2^5+t_1^3\,t_2^6\right)$	{1, 1}
$(-1+t_1)^2\,t_1\,(-1+t_2)^2\,t_2^4\,\left(t_1-t_2^2\right)$	{2, 0}
$(-1+t_1)\,(-1+t_2)^2\,t_2^2\,\left(-t_1-t_1^3\,t_2+t_2^2+t_1\,t_2^2-t_1^2\,t_2^2+\right.$ $\left.t_2^3+2\,t_1^2\,t_2^3+t_1^4\,t_2^3-t_1^2\,t_2^4+t_1^3\,t_2^4+t_1^4\,t_2^4-t_1\,t_2^5-t_1^3\,t_2^6\right)$	{0, 3}
$(-1+t_1)\,t_1\,(-1+t_2)^2\,t_2^3$ $\left(1+t_1+t_2+t_1^2\,t_2+2\,t_1\,t_2^2+t_2^3+t_1^2\,t_2^3+t_1\,t_2^4+t_1^2\,t_2^4\right)$	{1, 2}
$(1-t_1)\,t_1^2\,(-1+t_2)^2\,t_2^4\,\left(1+t_2^2\right)$	{2, 1}
$(1-t_1)\,t_1\,(-1+t_2)^2\,t_2^4\,\left(1+t_1+t_2+t_1^2\,t_2+t_1\,t_2^2+t_1^2\,t_2^2\right)$	{0, 4}
$(-1+t_1)\,t_1^2\,(-1+t_2)^2\,t_2^5$	{1, 3}

D Numerator of generating function for A_4 characters

$P_{A4}(t_i)$	Character
$1 - t_1^3 t_2 + t_1 t_2^2 + t_1^2 t_2^4 - t_2^5 + t_1^3 t_2^6 + t_1^2 t_3 - t_2 t_3 + t_1^3 t_2^2 t_3 - t_1 t_2^3 t_3 + t_1 t_2^8 t_3 -$ $t_1^2 t_2 t_3^2 - t_1 t_2^4 t_3^2 - t_2^7 t_3^2 + t_1^3 t_2^8 t_3^2 - t_1^2 t_2^2 t_3^3 - 2 t_1^2 t_2^7 t_3^3 + 2 t_1 t_2 t_3^4 + t_1 t_2^6 t_3^4 -$ $t_3^5 + t_1^3 t_2 t_3^5 + t_1^2 t_2^4 t_3^5 + t_1 t_2^7 t_3^5 - t_1^2 t_3^6 + t_1^2 t_2^5 t_3^6 - t_2^6 t_3^6 + t_1^3 t_2^7 t_3^6 - t_1 t_2^8 t_3^6 -$ $t_2^2 t_3^7 + t_1^3 t_2^3 t_3^7 - t_1 t_2^4 t_3^7 - t_1^2 t_2^6 t_3^7 + t_2^7 t_3^7 - t_1^3 t_2^8 t_3^7 - t_1 t_4 - t_1^2 t_2^7 t_4 - t_1 t_2 t_3 t_4 -$ $2 t_1^2 t_2^3 t_3 t_4 + 2 t_2^4 t_3 t_4 - 2 t_1^3 t_2^5 t_3 t_4 + t_1 t_2^6 t_3 t_4 - t_1^2 t_2^8 t_3 t_4 + t_3^2 t_4 -$ $t_1^3 t_2 t_3^2 t_4 + 3 t_1 t_2^2 t_3^2 t_4 + t_1^2 t_2^4 t_3^2 t_4 + t_1 t_2^7 t_3^2 t_4 + t_1^2 t_3^3 t_4 - t_2 t_3^3 t_4 + t_1^3 t_2^2 t_3^3 t_4 +$ $2 t_1 t_2^3 t_3^3 t_4 + 2 t_1^2 t_2^5 t_3^3 t_4 + t_1 t_2^8 t_3^3 t_4 - t_1^2 t_2 t_3^4 t_4 - t_2^2 t_3^4 t_4 + t_1^3 t_2^3 t_3^4 t_4 -$ $2 t_1 t_2^4 t_3^4 t_4 + t_1^2 t_2^6 t_3^4 t_4 - t_2^7 t_3^4 t_4 + t_1^3 t_2^8 t_3^4 t_4 - 2 t_1^2 t_2^2 t_3^5 t_4 - 2 t_1 t_2^5 t_3^5 t_4 -$ $2 t_1^2 t_2^7 t_3^5 t_4 + t_1 t_2 t_3^6 t_4 - 2 t_1^2 t_2^3 t_3^6 t_4 + t_2^4 t_3^6 t_4 - t_1^3 t_2^5 t_3^6 t_4 + t_1 t_2^2 t_3^7 t_4 -$ $t_1^2 t_2^4 t_3^7 t_4 + t_2^5 t_3^7 t_4 - t_1^3 t_2^6 t_3^7 t_4 + t_1 t_2^7 t_3^7 t_4 + t_2 t_3^8 t_4 - t_1^3 t_2^2 t_3^8 t_4 + t_1 t_2^3 t_3^8 t_4 +$ $t_1^2 t_2^5 t_3^8 t_4 - t_2^6 t_3^8 t_4 + t_1^3 t_2^7 t_3^8 t_4 + t_2 t_4^2 - t_1^3 t_2^2 t_4^2 + t_1 t_2^3 t_4^2 + t_1^2 t_2^5 t_4^2 - t_2^6 t_4^2 +$ $t_1^3 t_2^7 t_4^2 + t_1^2 t_2 t_3 t_4^2 - t_2^2 t_3 t_4^2 + t_1^3 t_2^3 t_3 t_4^2 - t_1 t_2^4 t_3 t_4^2 + t_1^2 t_2^6 t_3 t_4^2 - t_2^3 t_3^2 t_4^2 +$ $t_1^3 t_2^4 t_3^2 t_4^2 - 2 t_1 t_2^5 t_3^2 t_4^2 + t_1^2 t_2^7 t_3^2 t_4^2 - 2 t_1 t_2 t_3^3 t_4^2 - 2 t_1^2 t_2^3 t_3^3 t_4^2 - 2 t_1 t_2^6 t_3^3 t_4^2 +$ $t_3^4 t_4^2 - t_1^3 t_2 t_3^4 t_4^2 + t_1 t_2^2 t_3^4 t_4^2 - 2 t_1^2 t_2^4 t_3^4 t_4^2 + t_2^5 t_3^4 t_4^2 - t_1^3 t_2^6 t_3^4 t_4^2 - t_1 t_2^7 t_3^4 t_4^2 +$ $t_1^2 t_3^5 t_4^2 + 2 t_1 t_2^3 t_3^5 t_4^2 + 2 t_1^2 t_2^5 t_3^5 t_4^2 + t_2^6 t_3^5 t_4^2 - t_1^3 t_2^7 t_3^5 t_4^2 + t_1 t_2^8 t_3^5 t_4^2 +$ $t_1^2 t_2 t_3^6 t_4^2 + t_1 t_2^4 t_3^6 t_4^2 + 3 t_1^2 t_2^6 t_3^6 t_4^2 - t_2^7 t_3^6 t_4^2 + t_1^3 t_2^8 t_3^6 t_4^2 - t_1 t_3^7 t_4^2 + t_1^2 t_2^2 t_3^7 t_4^2 -$ $2 t_2^3 t_3^7 t_4^2 + 2 t_1^3 t_2^4 t_3^7 t_4^2 - 2 t_1 t_2^5 t_3^7 t_4^2 - t_1^2 t_2^7 t_3^7 t_4^2 - t_1 t_2 t_3^8 t_4^2 - t_1^2 t_2^8 t_3^8 t_4^2 -$ $t_3 t_4^3 + t_1^3 t_2 t_3 t_4^3 - t_1 t_2^2 t_3 t_4^3 - t_1^2 t_2^4 t_3 t_4^3 + t_2^5 t_3 t_4^3 - t_1^3 t_2^6 t_3 t_4^3 - t_1^2 t_2^7 t_3^3 t_4^3 +$ $t_2 t_3^2 t_4^3 - t_1^3 t_2^2 t_3^2 t_4^3 + t_1 t_2^3 t_3^2 t_4^3 - t_1 t_2^8 t_3^2 t_4^3 + t_1^2 t_2 t_3^3 t_4^3 + t_1 t_2^4 t_3^3 t_4^3 + t_2^7 t_3^3 t_4^3 -$ $t_1^3 t_2^8 t_3^3 t_4^3 + t_1^2 t_2^2 t_3^4 t_4^3 + 2 t_1^2 t_2^7 t_3^4 t_4^3 - 2 t_1 t_2 t_3^5 t_4^3 - t_1 t_2^6 t_3^5 t_4^3 + t_3^6 t_4^3 -$ $t_1^3 t_2 t_3^6 t_4^3 - t_1^2 t_2^4 t_3^6 t_4^3 - t_1 t_2^7 t_3^6 t_4^3 + t_1^2 t_3^7 t_4^3 - t_1^2 t_2^5 t_3^7 t_4^3 + t_2^6 t_3^7 t_4^3 - t_1^3 t_2^7 t_3^7 t_4^3 +$ $t_1 t_2^8 t_3^7 t_4^3 + t_2^2 t_3^8 t_4^3 - t_1^3 t_2^3 t_3^8 t_4^3 + t_1 t_2^4 t_3^8 t_4^3 + t_1^2 t_2^6 t_3^8 t_4^3 - t_2^7 t_3^8 t_4^3 + t_1^3 t_2^8 t_3^8 t_4^3$	{0, 0, 0, 0}
$t_1^2 t_2 - t_2^2 + t_1^3 t_2^3 - t_1 t_2^4 - t_1 t_3 - t_1^2 t_2^2 t_3 - t_1 t_2^5 t_3 - t_1 t_2 t_3^2 - 2 t_1^2 t_2^3 t_3^2 + t_2^4 t_3^2 - t_1^3 t_2^5 t_3^2 -$ $t_1 t_2^6 t_3^2 - t_1^2 t_2^8 t_3^2 + t_3^3 - t_1^3 t_2 t_3^3 + 3 t_1 t_2^2 t_3^3 + t_2^5 t_3^3 - t_1^3 t_2^6 t_3^3 + t_1 t_2^7 t_3^3 + t_1^2 t_3^4 -$ $t_2 t_3^4 + t_1^3 t_2^2 t_3^4 + t_1 t_2^3 t_3^4 + 3 t_1^2 t_2^5 t_3^4 + t_1 t_2^8 t_3^4 - t_1^2 t_2 t_3^5 - t_1 t_2^4 t_3^5 + t_1^2 t_2^6 t_3^5 -$ $t_2^7 t_3^5 + t_1^3 t_2^8 t_3^5 - t_2^3 t_3^6 + t_1^3 t_2^4 t_3^6 - 2 t_1 t_2^5 t_3^6 - 2 t_1^2 t_2^7 t_3^6 - t_1^2 t_2^3 t_3^7 + t_2^4 t_3^7 - t_1^3 t_2^5 t_3^7 +$ $t_1 t_2^6 t_3^7 - t_1^2 t_2^4 t_4 + t_2^5 t_4 - t_1^3 t_2^6 t_4 + 2 t_2 t_3 t_4 - 2 t_1^3 t_2^2 t_3 t_4 + 4 t_1 t_2^3 t_3 t_4 +$ $2 t_1^2 t_2^5 t_3 t_4 + 3 t_1^2 t_2 t_3^2 t_4 - t_2^2 t_3^2 t_4 + t_1^3 t_2^3 t_3^2 t_4 + 2 t_1 t_2^4 t_3^2 t_4 + 4 t_1^2 t_2^6 t_3^2 t_4 -$ $t_1 t_3^3 t_4 + t_1^2 t_2^2 t_3^3 t_4 - 3 t_2^3 t_3^3 t_4 + 3 t_1^3 t_2^4 t_3^3 t_4 - 4 t_1 t_2^5 t_3^3 t_4 + 2 t_1^2 t_2^7 t_3^3 t_4 -$ $3 t_1 t_2 t_3^4 t_4 - 4 t_1^2 t_2^3 t_3^4 t_4 - 4 t_1 t_2^6 t_3^4 t_4 - t_1^2 t_2^8 t_3^4 t_4 + t_3^5 t_4 - t_1^3 t_2 t_3^5 t_4 +$ $t_1 t_2^2 t_3^5 t_4 - 4 t_1^2 t_2^4 t_3^5 t_4 + 2 t_2^5 t_3^5 t_4 - 2 t_1^3 t_2^6 t_3^5 t_4 + t_1^2 t_3^6 t_4 + 3 t_1 t_2^3 t_3^6 t_4 +$ $t_1^2 t_2^5 t_3^6 t_4 + t_2^6 t_3^6 t_4 - t_1^3 t_2^7 t_3^6 t_4 + t_1 t_2^8 t_3^6 t_4 + t_2^2 t_3^7 t_4 - t_1^3 t_2^3 t_3^7 t_4 + 2 t_1 t_2^4 t_3^7 t_4 +$ $3 t_1^2 t_2^6 t_3^7 t_4 - t_2^7 t_3^7 t_4 + t_1^3 t_2^8 t_3^7 t_4 + t_1^2 t_2^2 t_3^8 t_4 - t_2^3 t_3^8 t_4 + t_1^3 t_2^4 t_3^8 t_4 - t_1 t_2^5 t_3^8 t_4 +$ $t_1^2 t_2^2 t_4^2 - t_2^3 t_4^2 + t_1^3 t_2^4 t_4^2 - t_1 t_2^5 t_4^2 - t_1 t_2 t_3 t_4^2 - t_2^4 t_3 t_4^2 + t_1^3 t_2^5 t_3 t_4^2 - t_1 t_2^6 t_3 t_4^2 -$ $t_3^2 t_4^2 + t_1^3 t_2 t_3^2 t_4^2 - 4 t_1 t_2^2 t_3^2 t_4^2 - 4 t_1^2 t_2^4 t_3^2 t_4^2 + t_2^5 t_3^2 t_4^2 - t_1^3 t_2^6 t_3^2 t_4^2 - t_1 t_2^7 t_3^2 t_4^2 -$ $t_1^2 t_3^3 t_4^2 + 2 t_2 t_3^3 t_4^2 - 2 t_1^3 t_2^2 t_3^3 t_4^2 + 2 t_1 t_2^3 t_3^3 t_4^2 - 4 t_1^2 t_2^5 t_3^3 t_4^2 + t_2^6 t_3^3 t_4^2 -$ $t_1^3 t_2^7 t_3^3 t_4^2 - t_1 t_2^8 t_3^3 t_4^2 + 2 t_1^2 t_2 t_3^4 t_4^2 + t_2^2 t_3^4 t_4^2 - t_1^3 t_2^3 t_3^4 t_4^2 + 6 t_1 t_2^4 t_3^4 t_4^2 +$ $2 t_1^2 t_2^6 t_3^4 t_4^2 + t_2^7 t_3^4 t_4^2 - t_1^3 t_2^8 t_3^4 t_4^2 + 3 t_1^2 t_2^2 t_3^5 t_4^2 - t_2^3 t_3^5 t_4^2 + t_1^3 t_2^4 t_3^5 t_4^2 +$ $2 t_1 t_2^5 t_3^5 t_4^2 + 4 t_1^2 t_2^7 t_3^5 t_4^2 - 2 t_1 t_2 t_3^6 t_4^2 + t_1^2 t_2^3 t_3^6 t_4^2 - 3 t_2^4 t_3^6 t_4^2 + 3 t_1^3 t_2^5 t_3^6 t_4^2 -$ $4 t_1 t_2^6 t_3^6 t_4^2 - 2 t_1 t_2^2 t_3^7 t_4^2 - 3 t_1^2 t_2^4 t_3^7 t_4^2 + t_2^5 t_3^7 t_4^2 - t_1^3 t_2^6 t_3^7 t_4^2 - t_1 t_2^7 t_3^7 t_4^2 -$ $t_1^2 t_2^5 t_3^8 t_4^2 + t_2^6 t_3^8 t_4^2 - t_1^3 t_2^7 t_3^8 t_4^2 - t_1^2 t_2 t_3 t_4^3 + t_2^2 t_3 t_4^3 - t_1^3 t_2^3 t_3 t_4^3 + t_1 t_2^4 t_3 t_4^3 +$ $t_1 t_3^2 t_4^3 + t_1^2 t_2^2 t_3^2 t_4^3 + t_1 t_2^5 t_3^2 t_4^3 + t_1 t_2 t_3^3 t_4^3 + 2 t_1^2 t_2^3 t_3^3 t_4^3 - t_2^4 t_3^3 t_4^3 + t_1^3 t_2^5 t_3^3 t_4^3 +$ $t_1 t_2^6 t_3^3 t_4^3 + t_1^2 t_2^8 t_3^3 t_4^3 - t_3^4 t_4^3 + t_1^3 t_2 t_3^4 t_4^3 - 3 t_1 t_2^2 t_3^4 t_4^3 - t_2^5 t_3^4 t_4^3 + t_1^3 t_2^6 t_3^4 t_4^3 -$ $t_1 t_2^7 t_3^4 t_4^3 - t_1^2 t_3^5 t_4^3 + t_2 t_3^5 t_4^3 - t_1^3 t_2^2 t_3^5 t_4^3 - t_1 t_2^3 t_3^5 t_4^3 - 3 t_1^2 t_2^5 t_3^5 t_4^3 - t_1 t_2^8 t_3^5 t_4^3 +$ $t_1^2 t_2 t_3^6 t_4^3 + t_1 t_2^4 t_3^6 t_4^3 - t_1^2 t_2^6 t_3^6 t_4^3 + t_2^7 t_3^6 t_4^3 - t_1^3 t_2^8 t_3^6 t_4^3 + t_2^3 t_3^7 t_4^3 - t_1^3 t_2^4 t_3^7 t_4^3 +$ $2 t_1 t_2^5 t_3^7 t_4^3 + 2 t_1^2 t_2^7 t_3^7 t_4^3 + t_1^2 t_2^3 t_3^8 t_4^3 - t_2^4 t_3^8 t_4^3 + t_1^3 t_2^5 t_3^8 t_4^3 - t_1 t_2^6 t_3^8 t_4^3$	{0, 0, 0, 1}

$P_{A4}(t_i)$	Character
$-t_1\,t_2-t_1\,t_2^2\,t_3-2\,t_1^2\,t_2^4\,t_3+t_2^5\,t_3-t_1^3\,t_2^6\,t_3-t_1\,t_2^7\,t_3+t_2\,t_3^2-t_1^3\,t_2^2\,t_3^2+2\,t_1\,t_2^3\,t_3^2+$ $t_2^6\,t_3^2-t_1^3\,t_2^7\,t_3^2+2\,t_1^2\,t_2\,t_3^3+3\,t_1\,t_2^4\,t_3^3+3\,t_1^2\,t_2^6\,t_3^3-t_1\,t_3^4+t_1^2\,t_2^2\,t_3^4-2\,t_2^3\,t_3^4+$ $2\,t_1^3\,t_2^4\,t_3^4-2\,t_1\,t_2^5\,t_3^4+t_1^2\,t_2^7\,t_3^4-t_1\,t_2\,t_3^5-2\,t_1^2\,t_2^3\,t_3^5-2\,t_1\,t_2^6\,t_3^5-t_1^2\,t_2^8\,t_3^5-$ $2\,t_1^2\,t_2^4\,t_3^6+t_2^5\,t_3^6-t_1^3\,t_2^6\,t_3^6+t_1\,t_2^7\,t_3^6+t_1\,t_2^3\,t_3^7+t_2^2\,t_4-t_1^3\,t_2^3\,t_4+t_1\,t_2^4\,t_4+t_1^2\,t_2^6\,t_4+$ $t_1\,t_3\,t_4+3\,t_1^2\,t_2^2\,t_3\,t_4-t_2^3\,t_3\,t_4+t_1^3\,t_2^4\,t_3\,t_4+2\,t_1^2\,t_2^7\,t_3\,t_4-2\,t_1\,t_2\,t_3^2\,t_4+$ $3\,t_1^2\,t_2^3\,t_3^2\,t_4-3\,t_2^4\,t_3^2\,t_4+3\,t_1^3\,t_2^5\,t_3^2\,t_4-2\,t_1\,t_2^6\,t_3^2\,t_4+t_1^2\,t_2^8\,t_3^2\,t_4-6\,t_1\,t_2^2\,t_3^3\,t_4-$ $6\,t_1^2\,t_2^4\,t_3^3\,t_4-3\,t_1\,t_2^7\,t_3^3\,t_4+2\,t_2\,t_3^4\,t_4-2\,t_1^3\,t_2^2\,t_3^4\,t_4+t_1\,t_2^3\,t_3^4\,t_4-6\,t_1^2\,t_2^5\,t_3^4\,t_4+$ $2\,t_2^6\,t_3^4\,t_4-2\,t_1^3\,t_2^7\,t_3^4\,t_4+2\,t_1^2\,t_2\,t_3^5\,t_4+2\,t_2^2\,t_3^5\,t_4-2\,t_1^3\,t_2^3\,t_3^5\,t_4+7\,t_1\,t_2^4\,t_3^5\,t_4+$ $3\,t_1^2\,t_2^6\,t_3^5\,t_4+2\,t_1^2\,t_2^2\,t_3^6\,t_4-t_2^3\,t_3^6\,t_4+t_1^3\,t_2^4\,t_3^6\,t_4+2\,t_1\,t_2^5\,t_3^6\,t_4+3\,t_1^2\,t_2^7\,t_3^6\,t_4+$ $2\,t_1^2\,t_2^3\,t_3^7\,t_4-2\,t_2^4\,t_3^7\,t_4+2\,t_1^3\,t_2^5\,t_3^7\,t_4-2\,t_1\,t_2^6\,t_3^7\,t_4-t_1\,t_2^2\,t_3^8\,t_4-t_1\,t_2^2\,t_4^2-$ $t_2\,t_3\,t_4^2+t_1^3\,t_2^2\,t_3\,t_4^2-2\,t_1\,t_2^3\,t_3\,t_4^2-3\,t_1^2\,t_2^5\,t_3\,t_4^2+t_2^6\,t_3\,t_4^2-t_1^3\,t_2^7\,t_3\,t_4^2-$ $2\,t_1^2\,t_2\,t_3^2\,t_4^2+2\,t_2^2\,t_3^2\,t_4^2-2\,t_1^3\,t_2^3\,t_3^2\,t_4^2+2\,t_1\,t_2^4\,t_3^2\,t_4^2-2\,t_1^2\,t_2^6\,t_3^2\,t_4^2+t_1\,t_3^3\,t_4^2+$ $2\,t_1^2\,t_2^2\,t_3^3\,t_4^2+2\,t_2^3\,t_3^3\,t_4^2-2\,t_1^3\,t_2^4\,t_3^3\,t_4^2+6\,t_1\,t_2^5\,t_3^3\,t_4^2+t_1\,t_2\,t_3^4\,t_4^2+7\,t_1^2\,t_2^3\,t_3^4\,t_4^2-$ $3\,t_2^4\,t_3^4\,t_4^2+3\,t_1^3\,t_2^5\,t_3^4\,t_4^2+2\,t_1\,t_2^6\,t_3^4\,t_4^2+t_1^2\,t_2^8\,t_3^4\,t_4^2-t_2^5\,t_4^2+t_1^3\,t_2\,t_3^5\,t_4^2-$ $5\,t_1\,t_2^2\,t_3^5\,t_4^2+t_1^2\,t_2^4\,t_3^5\,t_4^2-2\,t_2^5\,t_3^5\,t_4^2+2\,t_1^3\,t_2^6\,t_3^5\,t_4^2-2\,t_1\,t_2^7\,t_3^5\,t_4^2-t_1^2\,t_2^6\,t_4^2-$ $5\,t_1\,t_2^3\,t_3^6\,t_4^2-6\,t_1^2\,t_2^5\,t_3^6\,t_4^2+t_2^6\,t_3^6\,t_4^2-t_1^3\,t_2^7\,t_3^6\,t_4^2-t_1\,t_2^8\,t_3^6\,t_4^2+t_2^2\,t_3^7\,t_4^2-$ $t_1^3\,t_2^3\,t_3^7\,t_4^2+t_1\,t_2^4\,t_3^7\,t_4^2-2\,t_1^2\,t_2^6\,t_3^7\,t_4^2+t_2^7\,t_3^7\,t_4^2-t_1^3\,t_2^8\,t_3^7\,t_4^2+t_2^3\,t_3^8\,t_4^2-t_1^3\,t_2^4\,t_3^8\,t_4^2+$ $t_1\,t_2^5\,t_3^8\,t_4^2+t_1^2\,t_2^7\,t_3^8\,t_4^2+t_1\,t_2\,t_3\,t_4^3+t_1\,t_2^2\,t_3^2\,t_4^3+2\,t_1^2\,t_2^4\,t_3^2\,t_4^3-t_2^5\,t_3^2\,t_4^3+$ $t_1^3\,t_2^6\,t_3^2\,t_4^3+t_1\,t_2^7\,t_3^2\,t_4^3-t_2\,t_3^3\,t_4^3+t_1^3\,t_2^2\,t_3^3\,t_4^3-2\,t_1\,t_2^3\,t_3^3\,t_4^3-t_2^6\,t_3^3\,t_4^3+t_1^3\,t_2^7\,t_3^3\,t_4^3-$ $2\,t_1^2\,t_2\,t_3^4\,t_4^3-3\,t_1\,t_2^4\,t_3^4\,t_4^3-3\,t_1^2\,t_2^6\,t_3^4\,t_4^3+t_1\,t_3^5\,t_4^3-t_1^2\,t_2^2\,t_3^5\,t_4^3+2\,t_2^3\,t_3^5\,t_4^3-$ $2\,t_1^3\,t_2^4\,t_3^5\,t_4^3+2\,t_1\,t_2^5\,t_3^5\,t_4^3-t_1^2\,t_2^7\,t_3^5\,t_4^3+t_1\,t_2\,t_3^6\,t_4^3+2\,t_1^2\,t_2^3\,t_3^6\,t_4^3+2\,t_1\,t_2^6\,t_3^6\,t_4^3+$ $t_1^2\,t_2^8\,t_3^6\,t_4^3+2\,t_1^2\,t_2^4\,t_3^7\,t_4^3-t_2^5\,t_3^7\,t_4^3+t_1^3\,t_2^6\,t_3^7\,t_4^3-t_1\,t_2^7\,t_3^7\,t_4^3-t_1\,t_2^3\,t_3^8\,t_4^3$	$\{0,\,0,\,1,\,0\}$
$-t_1^2\,t_2^5-t_1^2\,t_2\,t_3+t_2^2\,t_3-t_1^3\,t_2^3\,t_3+2\,t_1\,t_2^4\,t_3+t_1\,t_2^2+2\,t_1^2\,t_2^2\,t_3^2+2\,t_1\,t_2^5\,t_3^2+$ $t_1^2\,t_2^7\,t_3^2-t_1\,t_2\,t_3^3+2\,t_1^2\,t_2^3\,t_3^3-2\,t_2^4\,t_3^3+2\,t_1^3\,t_2^5\,t_3^3-t_1\,t_2^6\,t_3^3+t_1^2\,t_2^8\,t_3^3-3\,t_1\,t_2^2\,t_3^4-$ $3\,t_1^2\,t_2^4\,t_3^4-2\,t_1\,t_2^7\,t_3^4+t_2\,t_3^5-t_1^3\,t_2^2\,t_3^5-2\,t_1^2\,t_2^5\,t_3^5+t_2^6\,t_3^5-t_1^3\,t_2^7\,t_3^5+t_1^2\,t_2\,t_3^6+$ $t_2^2\,t_3^6-t_1^3\,t_2^3\,t_3^6+2\,t_1\,t_2^4\,t_3^6+t_1^2\,t_2^6\,t_3^6+t_1^2\,t_2^7\,t_3^7+t_1\,t_2\,t_4+t_1^2\,t_2^3\,t_4-t_2^4\,t_4+$ $t_1^3\,t_2^5\,t_4-t_3\,t_4+t_1^3\,t_2\,t_3\,t_4-2\,t_1\,t_2^2\,t_3\,t_4+t_1^2\,t_2^4\,t_3\,t_4-t_2^5\,t_3\,t_4+t_1^3\,t_2^6\,t_3\,t_4-$ $t_1^2\,t_2^3\,t_4-t_2\,t_3^2\,t_4+t_1^3\,t_2^2\,t_3^2\,t_4-6\,t_1\,t_2^3\,t_3^2\,t_4-5\,t_1^2\,t_2^5\,t_3^2\,t_4-t_1\,t_2^8\,t_3^2\,t_4-$ $2\,t_1^2\,t_2\,t_3^3\,t_4+2\,t_2^2\,t_3^3\,t_4-2\,t_1^3\,t_2^3\,t_3^3\,t_4+t_1\,t_2^4\,t_3^3\,t_4-5\,t_1^2\,t_2^6\,t_3^3\,t_4+t_2^7\,t_3^3\,t_4-$ $t_1^3\,t_2^8\,t_3^3\,t_4+t_1\,t_3^4\,t_4+2\,t_1^2\,t_2^2\,t_3^4\,t_4+3\,t_2^3\,t_3^4\,t_4-3\,t_1^3\,t_2^4\,t_3^4\,t_4+7\,t_1\,t_2^5\,t_3^4\,t_4+$ $t_1^2\,t_2^7\,t_3^4\,t_4+6\,t_1^2\,t_2^3\,t_3^5\,t_4-2\,t_2^4\,t_3^5\,t_4+2\,t_1^3\,t_2^5\,t_3^5\,t_4+2\,t_1\,t_2^6\,t_3^5\,t_4+t_1^2\,t_2^8\,t_3^5\,t_4-$ $2\,t_1\,t_2^2\,t_3^6\,t_4+2\,t_1^2\,t_2^4\,t_3^6\,t_4-2\,t_2^5\,t_3^6\,t_4+2\,t_1^3\,t_2^6\,t_3^6\,t_4-2\,t_1\,t_2^7\,t_3^6\,t_4-t_2\,t_3^7\,t_4+$ $t_1^3\,t_2^2\,t_3^7\,t_4-3\,t_1\,t_2^3\,t_3^7\,t_4-2\,t_1^2\,t_2^5\,t_3^7\,t_4+t_2^6\,t_3^7\,t_4-t_1^3\,t_2^7\,t_3^7\,t_4-t_1^2\,t_2^6\,t_3^8\,t_4-$ $t_1^2\,t_2^6\,t_4^2-2\,t_1^2\,t_2^2\,t_3\,t_4^2+2\,t_2^3\,t_3\,t_4^2-2\,t_1^3\,t_2^4\,t_3\,t_4^2+2\,t_1\,t_2^5\,t_3\,t_4^2+3\,t_1\,t_2\,t_3^2\,t_4^2+$ $2\,t_1^2\,t_2^3\,t_3^2\,t_4^2+t_2^4\,t_3^2\,t_4^2-t_1^3\,t_2^5\,t_3^2\,t_4^2+2\,t_1\,t_2^6\,t_3^2\,t_4^2+3\,t_1^2\,t_2^4\,t_3^3\,t_4^2+7\,t_1^2\,t_2^4\,t_3^3\,t_4^2-$ $2\,t_2^5\,t_3^3\,t_4^2+2\,t_1^3\,t_2^6\,t_3^3\,t_4^2+2\,t_1\,t_2^7\,t_3^3\,t_4^2-2\,t_2\,t_3^4\,t_4^2+2\,t_1^3\,t_2^2\,t_3^4\,t_4^2-6\,t_1\,t_2^3\,t_3^4\,t_4^2+$ $t_1^2\,t_2^5\,t_3^4\,t_4^2-2\,t_2^6\,t_3^4\,t_4^2+2\,t_1^3\,t_2^7\,t_3^4\,t_4^2-3\,t_1^2\,t_2\,t_3^5\,t_4^2-6\,t_1\,t_2^4\,t_3^5\,t_4^2-6\,t_1^2\,t_2^6\,t_3^5\,t_4^2+$ $t_1\,t_3^6\,t_4^2-2\,t_1^2\,t_2^2\,t_3^6\,t_4^2+3\,t_2^3\,t_3^6\,t_4^2-3\,t_1^3\,t_2^4\,t_3^6\,t_4^2+3\,t_1\,t_2^5\,t_3^6\,t_4^2-2\,t_1^2\,t_2^7\,t_3^6\,t_4^2+$ $2\,t_1\,t_2\,t_3^7\,t_4^2+t_2^4\,t_3^7\,t_4^2-t_1^3\,t_2^5\,t_3^7\,t_4^2+3\,t_1\,t_2^6\,t_3^7\,t_4^2+t_1^2\,t_2^8\,t_3^7\,t_4^2+t_1\,t_2^2\,t_3^8\,t_4^2+$ $t_1^2\,t_2^4\,t_3^8\,t_4^2-t_2^5\,t_3^8\,t_4^2+t_1^3\,t_2^6\,t_3^8\,t_4^2+t_1^2\,t_2^5\,t_3\,t_4^3+t_1^2\,t_2\,t_3^2\,t_4^3-t_2^2\,t_3^2\,t_4^3+t_1^3\,t_2^3\,t_3^2\,t_4^3-$ $2\,t_1\,t_2^4\,t_3^2\,t_4^3-t_1\,t_3^3\,t_4^3-2\,t_1^2\,t_2^2\,t_3^3\,t_4^3-2\,t_1\,t_2^5\,t_3^3\,t_4^3-t_1^2\,t_2^7\,t_3^3\,t_4^3+t_1\,t_2\,t_3^4\,t_4^3-$ $2\,t_1^2\,t_2^3\,t_3^4\,t_4^3+2\,t_2^4\,t_3^4\,t_4^3-2\,t_1^3\,t_2^5\,t_3^4\,t_4^3+t_1\,t_2^6\,t_3^4\,t_4^3-t_1^2\,t_2^8\,t_3^4\,t_4^3+3\,t_1\,t_2^2\,t_3^5\,t_4^3+$ $3\,t_1^2\,t_2^4\,t_3^5\,t_4^3+2\,t_1\,t_2^7\,t_3^5\,t_4^3-t_2\,t_3^6\,t_4^3+t_1^3\,t_2^2\,t_3^6\,t_4^3+2\,t_1^2\,t_2^5\,t_3^6\,t_4^3-t_2^6\,t_3^6\,t_4^3+$ $t_1^3\,t_2^7\,t_3^6\,t_4^3-t_1^2\,t_2\,t_3^7\,t_4^3-t_2^2\,t_3^7\,t_4^3+t_1^3\,t_2^3\,t_3^7\,t_4^3-2\,t_1\,t_2^4\,t_3^7\,t_4^3-t_1^2\,t_2^6\,t_3^7\,t_4^3-t_1^2\,t_2^7\,t_3^8\,t_4^3$	$\{0,\,1,\,0,\,0\}$

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$P_{A4}(t_i)$	Character
$-t_1^2 t_2^2 + t_2^3 - t_1^3 t_2^4 + t_1 t_2^5 + 2 t_1 t_2 t_3 + 2 t_1^2 t_2^3 t_3 - t_2^4 t_3 + t_1^3 t_2^5 t_3 - t_2^2 + t_1^3 t_2 t_3^2 -$ $t_1 t_2^2 t_3^2 + t_1^2 t_2^4 t_3^2 + t_1 t_2^7 t_3^2 - t_1^2 t_3^3 - 3 t_1 t_2^3 t_3^3 - t_1^2 t_2^5 t_3^3 - t_2^6 t_3^3 + t_1^3 t_2^7 t_3^3 -$ $t_1 t_2^8 t_3^3 - t_1^2 t_2 t_3^4 + t_2^2 t_3^4 - t_1^3 t_2^3 t_3^4 - 3 t_1^2 t_2^6 t_3^4 + t_2^7 t_3^4 - t_1^3 t_2^8 t_3^4 + t_1 t_3^5 + t_1^2 t_2^2 t_3^5 +$ $t_2^3 t_3^5 - t_1^3 t_2^4 t_3^5 + 2 t_1 t_2^5 t_3^5 + t_1^2 t_2^7 t_3^5 + t_1^2 t_2^3 t_3^6 + t_1 t_2^6 t_3^6 + t_1^2 t_2^8 t_3^6 + t_1^2 t_2^4 t_3^7 -$ $t_2^5 t_3^7 + t_1^3 t_2^6 t_3^7 - t_1 t_2^7 t_3^7 - t_2 t_4 + t_1^3 t_2^2 t_4 - t_1 t_2^3 t_4 - t_1^2 t_2 t_3 t_4 - t_2^2 t_3 t_4 +$ $t_1^3 t_2^3 t_3 t_4 - 3 t_1 t_2^4 t_3 t_4 - 2 t_1^2 t_2^6 t_3 t_4 - 4 t_1^2 t_2^2 t_3^2 t_4 + 3 t_2^3 t_3^2 t_4 - 3 t_1^3 t_2^4 t_3^2 t_4 +$ $t_1 t_2^5 t_3^2 t_4 - 2 t_1^2 t_2^7 t_3^2 t_4 + 4 t_1 t_2 t_3^3 t_4 + 2 t_1^2 t_2^3 t_3^3 t_4 + t_2^4 t_3^3 t_4 - t_1^3 t_2^5 t_3^3 t_4 +$ $3 t_1 t_2^6 t_3^3 t_4 - t_3^4 t_4 + t_1^3 t_2 t_3^4 t_4 + 2 t_1 t_2^2 t_3^4 t_4 + 6 t_1^2 t_2^4 t_3^4 t_4 - t_2^5 t_3^4 t_4 + t_1^3 t_2^6 t_3^4 t_4 +$ $2 t_1 t_2^7 t_3^4 t_4 - t_1^2 t_3^5 t_4 - t_2 t_3^5 t_4 + t_1^3 t_2^2 t_3^5 t_4 - 4 t_1 t_2^3 t_3^5 t_4 + 2 t_1^2 t_2^5 t_3^5 t_4 -$ $2 t_2^6 t_3^5 t_4 + 2 t_1^3 t_2^7 t_3^5 t_4 - t_1 t_2^8 t_3^5 t_4 - t_1^2 t_2 t_3^6 t_4 - t_2^2 t_3^6 t_4 + t_1^3 t_2^3 t_3^6 t_4 -$ $4 t_1 t_2^4 t_3^6 t_4 - 4 t_1^2 t_2^6 t_3^6 t_4 + t_2^7 t_3^6 t_4 - t_1^3 t_2^8 t_3^6 t_4 - t_1^2 t_2^2 t_3^7 t_4 + t_2^3 t_3^7 t_4 - t_1^3 t_2^4 t_3^7 t_4 -$ $t_1^2 t_2^7 t_3^7 t_4 - t_1^2 t_2^3 t_3^8 t_4 + t_2^4 t_3^8 t_4 - t_1^3 t_2^5 t_3^8 t_4 + t_1 t_2^6 t_3^8 t_4 - t_1^2 t_2^3 t_4^2 + t_2^4 t_4^2 -$ $t_1^3 t_2^5 t_4^2 + t_1 t_2^6 t_4^2 + t_3 t_4^2 - t_1^3 t_2 t_3 t_4^2 + 3 t_1 t_2^2 t_3 t_4^2 + 2 t_1^2 t_2^4 t_3 t_4^2 - t_2^5 t_3 t_4^2 +$ $t_1^3 t_2^6 t_3 t_4^2 + t_1^2 t_2^3 t_4^2 - t_2 t_3^2 t_4^2 + t_1^3 t_2^2 t_3^2 t_4^2 + t_1 t_2^3 t_3^2 t_4^2 + 3 t_1^2 t_2^5 t_3^2 t_4^2 + t_1 t_2^8 t_3^2 t_4^2 -$ $2 t_2^2 t_3^3 t_4^2 + 2 t_1^3 t_2^3 t_3^3 t_4^2 - 4 t_1 t_2^4 t_3^3 t_4^2 + t_1^2 t_2^6 t_3^3 t_4^2 - t_2^7 t_3^3 t_4^2 + t_1^3 t_2^8 t_3^3 t_4^2 -$ $t_1 t_3^4 t_4^2 - 4 t_1^2 t_2^2 t_3^4 t_4^2 - 4 t_1 t_2^5 t_3^4 t_4^2 - 3 t_1^2 t_2^7 t_3^4 t_4^2 + 2 t_1 t_2 t_3^5 t_4^2 - 4 t_1^2 t_2^3 t_3^5 t_4^2 +$ $3 t_2^4 t_3^5 t_4^2 - 3 t_1^3 t_2^5 t_3^5 t_4^2 + t_1 t_2^6 t_3^5 t_4^2 - t_1^2 t_2^8 t_3^5 t_4^2 + 4 t_1 t_2^2 t_3^6 t_4^2 + 2 t_1^2 t_2^4 t_3^6 t_4^2 +$ $t_2^5 t_3^6 t_4^2 - t_1^3 t_2^6 t_3^6 t_4^2 + 3 t_1 t_2^7 t_3^6 t_4^2 + 2 t_1 t_2^3 t_3^7 t_4^2 + 4 t_1^2 t_2^5 t_3^7 t_4^2 - 2 t_2^6 t_3^7 t_4^2 +$ $2 t_1^3 t_2^7 t_3^7 t_4^2 - t_2^2 t_3^8 t_4^2 + t_1^3 t_2^3 t_3^8 t_4^2 - t_1 t_2^4 t_3^8 t_4^2 + t_1^2 t_2^2 t_3 t_4^3 - t_2^3 t_3 t_4^3 + t_1^3 t_2^4 t_3 t_4^3 -$ $t_1 t_2^5 t_3 t_4^3 - 2 t_1 t_2 t_3^2 t_4^3 - 2 t_1^2 t_2^3 t_3^2 t_4^3 + t_2^4 t_3^2 t_4^3 - t_1^3 t_2^5 t_3^2 t_4^3 + t_3^3 t_4^3 - t_1^3 t_2 t_3^3 t_4^3 +$ $t_1 t_2^2 t_3^3 t_4^3 - t_1^2 t_2^4 t_3^3 t_4^3 - t_1 t_2^7 t_3^3 t_4^3 + t_1^2 t_3^4 t_4^3 + 3 t_1 t_2^3 t_3^4 t_4^3 + t_1^2 t_2^5 t_3^4 t_4^3 + t_2^6 t_3^4 t_4^3 -$ $t_1^3 t_2^7 t_3^4 t_4^3 + t_1 t_2^8 t_3^4 t_4^3 + t_1^2 t_2 t_3^5 t_4^3 - t_2^2 t_3^5 t_4^3 + t_1^3 t_2^3 t_3^5 t_4^3 + 3 t_1^2 t_2^6 t_3^5 t_4^3 - t_2^7 t_3^5 t_4^3 +$ $t_1^3 t_2^8 t_3^5 t_4^3 - t_1 t_3^6 t_4^3 - t_1^2 t_2^2 t_3^6 t_4^3 - t_2^3 t_3^6 t_4^3 + t_1^3 t_2^4 t_3^6 t_4^3 - 2 t_1 t_2^5 t_3^6 t_4^3 - t_1^2 t_2^7 t_3^6 t_4^3 -$ $t_1^2 t_2^3 t_3^7 t_4^3 - t_1 t_2^6 t_3^7 t_4^3 - t_1^2 t_2^8 t_3^7 t_4^3 - t_1^2 t_2^4 t_3^8 t_4^3 + t_2^5 t_3^8 t_4^3 - t_1^3 t_2^6 t_3^8 t_4^3 + t_1 t_2^7 t_3^8 t_4^3$	{1, 0, 0, 0}
$-t_1^2 t_2^3 - t_1^2 t_2^4 t_3 + t_2 t_3^2 - t_1^3 t_2^2 t_3^2 + 3 t_1 t_2^3 t_3^2 + t_1^2 t_2 t_3^3 - t_2^2 t_3^3 + t_1^3 t_2^3 t_3^3 + t_1 t_2^4 t_3^3 +$ $2 t_1^2 t_2^6 t_3^3 - t_1^2 t_2^2 t_3^4 - t_1 t_2^5 t_3^4 + t_1^2 t_2^7 t_3^4 - t_2^4 t_3^5 + t_1^3 t_2^5 t_3^5 - 2 t_1 t_2^6 t_3^5 - t_1^2 t_2^4 t_3^6 +$ $t_2^5 t_3^6 - t_1^3 t_2^6 t_3^6 + t_1^2 t_2^5 t_3^7 + t_1 t_2^4 t_4 + 2 t_1^2 t_2^2 t_3 t_4 - t_2^3 t_3 t_4 + t_1^3 t_2^4 t_3 t_4 +$ $t_1 t_2^5 t_3 t_4 - t_1 t_2 t_3^2 t_4 - 2 t_2^4 t_3^2 t_4 + 2 t_1^3 t_2^5 t_3^2 t_4 - t_1 t_2^6 t_3^2 t_4 - t_3^3 t_4 + t_1^3 t_2 t_3^3 t_4 -$ $4 t_1 t_2^2 t_3^3 t_4 - 5 t_1^2 t_2^4 t_3^3 t_4 - t_1 t_2^7 t_3^3 t_4 - t_1^2 t_3^4 t_4 + t_2 t_3^4 t_4 - t_1^3 t_2^2 t_3^4 t_4 +$ $t_1 t_2^3 t_3^4 t_4 - 5 t_1^2 t_2^5 t_3^4 t_4 + t_2^6 t_3^4 t_4 - t_1^3 t_2^7 t_3^4 t_4 - t_1 t_2^8 t_3^4 t_4 + t_1^2 t_2 t_3^5 t_4 +$ $4 t_1 t_2^4 t_3^5 t_4 + t_2^7 t_3^5 t_4 - t_1^3 t_2^8 t_3^5 t_4 + t_2^3 t_3^6 t_4 - t_1^3 t_2^4 t_3^6 t_4 + 2 t_1 t_2^5 t_3^6 t_4 +$ $2 t_1^2 t_2^7 t_3^6 t_4 + t_1^2 t_2^3 t_3^7 t_4 - t_2^4 t_3^7 t_4 + t_1^3 t_2^5 t_3^7 t_4 - t_1 t_2^6 t_3^7 t_4 - t_1^2 t_2^4 t_3^8 t_4 -$ $t_1^2 t_2^4 t_4^2 - t_1 t_2^3 t_3 t_4^2 - t_1^2 t_2^5 t_3 t_4^2 - t_1^2 t_2 t_3^2 t_4^2 + 2 t_2^2 t_3^2 t_4^2 - 2 t_1^3 t_2^3 t_3^2 t_4^2 +$ $2 t_1 t_2^4 t_3^2 t_4^2 - t_1^2 t_2^6 t_3^2 t_4^2 + t_1 t_3^3 t_4^2 + 2 t_1^2 t_2^2 t_3^3 t_4^2 + t_2^3 t_3^3 t_4^2 - t_1^3 t_2^4 t_3^3 t_4^2 +$ $4 t_1 t_2^5 t_3^3 t_4^2 + t_1^2 t_2^7 t_3^3 t_4^2 + t_1 t_2 t_3^4 t_4^2 + 4 t_1^2 t_2^3 t_3^4 t_4^2 - t_2^4 t_3^4 t_4^2 + t_1^3 t_2^5 t_3^4 t_4^2 +$ $2 t_1 t_2^6 t_3^4 t_4^2 + t_1^2 t_2^8 t_3^4 t_4^2 - 2 t_1 t_2^2 t_3^5 t_4^2 + t_1^2 t_2^4 t_3^5 t_4^2 - 3 t_2^5 t_3^5 t_4^2 + 3 t_1^3 t_2^6 t_3^5 t_4^2 -$ $t_1 t_2^7 t_3^5 t_4^2 - 2 t_1 t_2^3 t_3^6 t_4^2 - 4 t_1^2 t_2^5 t_3^6 t_4^2 + t_1 t_2^4 t_3^7 t_4^2 - t_1^2 t_2^6 t_3^7 t_4^2 + t_1 t_2^5 t_3^8 t_4^2 +$ $t_1^2 t_2^3 t_3 t_4^3 + t_1^2 t_2^4 t_3^2 t_4^3 - t_2 t_3^3 t_4^3 + t_1^3 t_2^2 t_3^3 t_4^3 - 3 t_1 t_2^3 t_3^3 t_4^3 - t_1^2 t_2 t_3^4 t_4^3 + t_2^2 t_3^4 t_4^3 -$ $t_1^3 t_2^4 t_3^4 t_4^3 - t_1 t_2^4 t_3^5 t_4^3 - 2 t_1^2 t_2^6 t_3^4 t_4^3 + t_1^2 t_2^2 t_3^5 t_4^3 + t_2^3 t_3^5 t_4^3 - t_1^3 t_2^5 t_3^5 t_4^3 +$ $t_2^4 t_3^6 t_4^3 - t_1^3 t_2^5 t_3^6 t_4^3 + 2 t_1 t_2^6 t_3^6 t_4^3 + t_1^2 t_2^4 t_3^7 t_4^3 - t_2^5 t_3^7 t_4^3 + t_1^3 t_2^6 t_3^7 t_4^3 - t_1^2 t_2^5 t_3^8 t_4^3$	{0, 0, 0, 2}

$P_{A4}(t_i)$	Character
$t_2^2 t_3 - t_1^3 t_2^3 t_3 + t_1 t_2^4 t_3 + 2 t_1^2 t_2^2 t_3^2 + 2 t_1 t_2^5 t_3^2 + t_1^2 t_2^7 t_3^2 - t_1 t_2 t_3^3 + 2 t_1^2 t_2^3 t_3^3 - 2 t_2^4 t_3^3 +$ $2 t_1^3 t_2^5 t_3^3 - t_1 t_2^6 t_3^3 - 2 t_1 t_2^2 t_3^4 - 3 t_1^2 t_2^4 t_3^4 - t_1 t_2^7 t_3^4 - 3 t_1^2 t_2^5 t_3^5 + t_2^6 t_3^5 - t_1^3 t_2^7 t_3^5 +$ $2 t_1 t_2^4 t_3^6 + t_1^2 t_2^6 t_3^6 + t_1^2 t_2^3 t_4 - 2 t_1 t_2^2 t_3 t_4 + t_1^2 t_2^4 t_3 t_4 - t_2^5 t_3 t_4 + t_1^3 t_2^6 t_3 t_4 -$ $t_2 t_3^2 t_4 + t_1^3 t_2^2 t_3^2 t_4 - 6 t_1 t_2^3 t_3^2 t_4 - 5 t_1^2 t_2^5 t_3^2 t_4 - 2 t_1^2 t_2 t_3^3 t_4 + 3 t_2^2 t_3^3 t_4 -$ $3 t_1^3 t_2^3 t_3^3 t_4 + t_1 t_2^4 t_3^3 t_4 - 5 t_1^2 t_2^6 t_3^3 t_4 + t_1 t_3^4 t_4 + 2 t_1^2 t_2^2 t_3^4 t_4 + 3 t_2^3 t_3^4 t_4 -$ $3 t_1^3 t_2^4 t_3^4 t_4 + 8 t_1 t_2^5 t_3^4 t_4 + t_1^2 t_2^7 t_3^4 t_4 + t_1 t_2 t_3^5 t_4 + 6 t_1^2 t_2^3 t_3^5 t_4 - 2 t_2^4 t_3^5 t_4 +$ $2 t_1^3 t_2^5 t_3^5 t_4 + 2 t_1 t_2^6 t_3^5 t_4 + t_1^2 t_2^8 t_3^5 t_4 - t_1 t_2^2 t_3^6 t_4 + 2 t_1^2 t_2^4 t_3^6 t_4 - 2 t_2^5 t_3^6 t_4 +$ $2 t_1^3 t_2^6 t_3^6 t_4 - t_1 t_2^7 t_3^6 t_4 - 2 t_1 t_2^3 t_3^7 t_4 - 2 t_1^2 t_2^5 t_3^7 t_4 - t_1^2 t_2^2 t_3 t_4^2 + t_2^3 t_3 t_4^2 -$ $t_1^3 t_2^4 t_3 t_4^2 + t_1 t_2^5 t_3 t_4^2 + 2 t_1 t_2 t_3^2 t_4^2 + 2 t_1^2 t_2^3 t_3^2 t_4^2 + t_2^4 t_3^2 t_4^2 - t_1^3 t_2^5 t_3^2 t_4^2 +$ $2 t_1 t_2^6 t_3^2 t_4^2 + 2 t_1 t_2^2 t_3^3 t_4^2 + 8 t_1^2 t_2^4 t_3^3 t_4^2 - 2 t_2^5 t_3^3 t_4^2 + 2 t_1^3 t_2^6 t_3^3 t_4^2 + t_1 t_2^7 t_3^3 t_4^2 -$ $2 t_2 t_3^4 t_4^2 + 2 t_1^3 t_2^2 t_3^4 t_4^2 - 7 t_1 t_2^3 t_3^4 t_4^2 + t_1^2 t_2^5 t_3^4 t_4^2 - t_2^6 t_3^4 t_4^2 + t_1^3 t_2^7 t_3^4 t_4^2 - 2 t_1^2 t_2 t_3^5 t_4^2 -$ $7 t_1 t_2^4 t_3^5 t_4^2 - 6 t_1^2 t_2^6 t_3^5 t_4^2 - t_1^2 t_2^2 t_3^6 t_4^2 + 2 t_2^3 t_3^6 t_4^2 - 2 t_1^3 t_2^4 t_3^6 t_4^2 + 2 t_1 t_2^5 t_3^6 t_4^2 -$ $2 t_1^2 t_2^7 t_3^6 t_4^2 + t_1^2 t_2^3 t_3^7 t_4^2 + t_2^4 t_3^7 t_4^2 - t_1^3 t_2^5 t_3^7 t_4^2 + 2 t_1 t_2^6 t_3^7 t_4^2 + t_1^2 t_2^4 t_3^8 t_4^2 - t_2^2 t_3^2 t_4^3 +$ $t_1^3 t_2^3 t_3^2 t_4^3 - t_1 t_2^4 t_3^2 t_4^3 - 2 t_1^2 t_2^2 t_3^3 t_4^3 - 2 t_1 t_2^5 t_3^3 t_4^3 - t_1^2 t_2^7 t_3^3 t_4^3 + t_1 t_2 t_3^4 t_4^3 -$ $2 t_1^2 t_2^3 t_3^4 t_4^3 + 2 t_2^4 t_3^4 t_4^3 - 2 t_1^3 t_2^5 t_3^4 t_4^3 + t_1 t_2^6 t_3^4 t_4^3 + 2 t_1 t_2^2 t_3^5 t_4^3 + 3 t_1^2 t_2^4 t_3^5 t_4^3 +$ $t_1 t_2^7 t_3^5 t_4^3 + 3 t_1^2 t_2^5 t_3^6 t_4^3 - t_2^6 t_3^6 t_4^3 + t_1^3 t_2^7 t_3^6 t_4^3 - 2 t_1 t_2^4 t_3^7 t_4^3 - t_1^2 t_2^6 t_3^7 t_4^3$	{0, 0, 1, 1}
$t_2 t_3$ $\left(t_1^2 t_2^3 t_3 - 2 t_1 t_2^2 t_3^2 - t_1^2 t_2^4 t_3^2 + t_2 t_3^3 - t_1^3 t_2^2 t_3^3 - t_1^2 t_2^5 t_3^3 + t_2^2 t_3^4 - t_1^3 t_2^3 t_3^4 + t_1 t_2^4 t_3^4 +\right.$ $t_1^2 t_2^6 t_3^4 + t_1^2 t_2^2 t_3^5 - t_2 t_4 + t_1^3 t_2^2 t_4 - t_1 t_2^3 t_4 - t_1^2 t_2^5 t_4 - 2 t_1^2 t_2 t_3 t_4 + t_2^2 t_3 t_4 -$ $t_1^3 t_2^3 t_3 t_4 - t_1^2 t_2^6 t_3 t_4 + t_1 t_2^3 t_4 + t_1^2 t_2^2 t_3^2 t_4 + t_2^3 t_3^2 t_4 - t_1^3 t_2^4 t_3^2 t_4 +$ $2 t_1 t_2^5 t_3^2 t_4 + t_1 t_2 t_3^3 t_4 + 6 t_1^2 t_2^3 t_3^3 t_4 - t_2^4 t_3^3 t_4 + t_1^3 t_2^5 t_3^3 t_4 - t_3^4 t_4 +$ $t_1^3 t_2 t_3^4 t_4 - 4 t_1 t_2^2 t_3^4 t_4 + t_1^2 t_2^4 t_3^4 t_4 - 2 t_1 t_2^3 t_3^5 t_4 - 2 t_1^2 t_2^5 t_3^5 t_4 +$ $t_1 t_2 t_4^2 + t_1 t_2^2 t_3 t_4^2 + 2 t_1^2 t_2^4 t_3 t_4^2 - t_2 t_3^2 t_4^2 + t_1^3 t_2^2 t_3^2 t_4^2 - 2 t_1 t_2^3 t_3^2 t_4^2 -$ $2 t_1^2 t_2 t_3^3 t_4^2 - 2 t_1 t_2^4 t_3^3 t_4^2 - t_1^2 t_2^6 t_3^3 t_4^2 + t_1 t_3^4 t_4^2 - 4 t_1^2 t_2^2 t_3^4 t_4^2 + 2 t_2^3 t_3^4 t_4^2 -$ $2 t_1^3 t_2^4 t_3^4 t_4^2 + t_1 t_2^5 t_3^4 t_4^2 + 2 t_1^2 t_2^3 t_3^5 t_4^2 + t_1^2 t_2^5 t_3^6 t_4^2 + t_1 t_2^6 t_3^5 t_4^2 + t_1 t_2^2 t_3^6 t_4^2 +$ $t_1^2 t_2^4 t_3^6 t_4^2 - t_1^2 t_2^2 t_3^3 t_4^3 + 2 t_1 t_2^3 t_3^3 t_4^3 + t_1^2 t_2^5 t_3^3 t_4^3 - t_2 t_3^4 t_4^3 + t_1^3 t_2^2 t_3^4 t_4^3 +$ $\left.t_1^2 t_2^5 t_3^4 t_4^3 - t_2^2 t_3^5 t_4^3 + t_1^3 t_2^3 t_3^5 t_4^3 - t_1 t_2^4 t_3^5 t_4^3 - t_1^2 t_2^6 t_3^5 t_4^3 - t_1^2 t_2^2 t_3^6 t_4^3\right)$	{0, 0, 2, 0}
$t_1 t_2 t_3 + 2 t_1^2 t_2^3 t_3 + t_1 t_2^6 t_3 - t_1 t_2^2 t_3^2 + 3 t_1^2 t_2^4 t_3^2 - 2 t_2^5 t_3^2 + 2 t_1^3 t_2^6 t_3^2 - t_2 t_3^3 +$ $t_1^3 t_2^2 t_3^3 - 5 t_1 t_2^3 t_3^3 - 4 t_1^2 t_2^5 t_3^3 - t_1^2 t_2 t_3^4 + t_2^2 t_3^4 - t_1^3 t_2^3 t_3^4 - 3 t_1^2 t_2^6 t_3^4 + t_1^2 t_2^2 t_3^5 +$ $t_2^3 t_3^5 - t_1^3 t_2^4 t_3^5 + 4 t_1 t_2^5 t_3^5 + t_1^2 t_2^3 t_3^6 - t_2^4 t_3^6 + t_1^3 t_2^5 t_3^6 - t_1 t_2^3 t_4 - t_1^2 t_2 t_3 t_4 -$ $3 t_1 t_2^4 t_3 t_4 - 2 t_1^2 t_2^6 t_3 t_4 - 4 t_1^2 t_2^2 t_3^2 t_4 + 3 t_2^3 t_3^2 t_4 - 3 t_1^3 t_2^4 t_3^2 t_4 + t_1 t_2^5 t_3^2 t_4 -$ $2 t_1^2 t_2^7 t_3^2 t_4 + 4 t_1 t_2 t_3^3 t_4 + 2 t_1^2 t_2^3 t_3^3 t_4 + 4 t_2^4 t_3^3 t_4 - 4 t_1^3 t_2^5 t_3^3 t_4 + 6 t_1 t_2^6 t_3^3 t_4 +$ $4 t_1 t_2^2 t_3^4 t_4 + 12 t_1^2 t_2^4 t_3^4 t_4 - 3 t_2^5 t_3^4 t_4 + 3 t_1^3 t_2^6 t_3^4 t_4 + 2 t_1 t_2^7 t_3^4 t_4 - t_2 t_3^5 t_4 +$ $t_1^3 t_2^2 t_3^5 t_4 - 5 t_1 t_2^3 t_3^5 t_4 + 2 t_1^2 t_2^5 t_3^5 t_4 - 2 t_2^6 t_3^5 t_4 + 2 t_1^3 t_2^7 t_3^5 t_4 - t_1^2 t_2 t_3^6 t_4 -$ $t_2^2 t_3^6 t_4 + t_1^3 t_2^3 t_3^6 t_4 - 6 t_1 t_2^4 t_3^6 t_4 - 4 t_1^2 t_2^6 t_3^6 t_4 - t_1^2 t_2^2 t_3^7 t_4 + t_2^3 t_3^7 t_4 - t_1^3 t_2^4 t_3^7 t_4 +$ $t_1 t_2^5 t_3^7 t_4 - t_1^2 t_2^7 t_3^7 t_4 + 2 t_1 t_2^2 t_3 t_4^2 + 2 t_1^2 t_2^4 t_3 t_4^2 + 2 t_1 t_2^3 t_3^2 t_4^2 + 6 t_1^2 t_2^5 t_3^2 t_4^2 -$ $t_2^6 t_3^2 t_4^2 + t_1^3 t_2^7 t_3^2 t_4^2 + t_1^2 t_2 t_3^3 t_4^2 - 4 t_2^2 t_3^3 t_4^2 + 4 t_1^3 t_2^3 t_3^3 t_4^2 - 9 t_1 t_2^4 t_3^3 t_4^2 +$ $t_1^2 t_2^6 t_3^3 t_4^2 - t_1 t_3^4 t_4^2 - 6 t_1^2 t_2^2 t_3^4 t_4^2 - 9 t_1 t_2^5 t_3^4 t_4^2 - 3 t_1^2 t_2^7 t_3^4 t_4^2 + t_1 t_2 t_3^5 t_4^2 -$ $5 t_1^2 t_2^3 t_3^5 t_4^2 + 5 t_2^4 t_3^5 t_4^2 - 5 t_1^3 t_2^5 t_3^5 t_4^2 + 2 t_1 t_2^6 t_3^5 t_4^2 - t_1^2 t_2^8 t_3^5 t_4^2 + 4 t_1 t_2^2 t_3^6 t_4^2 +$ $4 t_1^2 t_2^4 t_3^6 t_4^2 + t_2^5 t_3^6 t_4^2 - t_1^3 t_2^6 t_3^6 t_4^2 + 2 t_1 t_2^7 t_3^6 t_4^2 + t_1 t_2^3 t_3^7 t_4^2 + 4 t_1^2 t_2^5 t_3^7 t_4^2 -$ $t_2^6 t_3^7 t_4^2 + t_1^3 t_2^7 t_3^7 t_4^2 - t_1 t_2^4 t_3^8 t_4^2 - t_1 t_2 t_3^2 t_4^3 - 2 t_1^2 t_2^3 t_3^2 t_4^3 - t_1 t_2^6 t_3^2 t_4^3 +$ $t_1 t_2^2 t_3^3 t_4^3 - 3 t_1^2 t_2^4 t_3^3 t_4^3 + 2 t_2^5 t_3^3 t_4^3 - 2 t_1^3 t_2^6 t_3^3 t_4^3 + t_2 t_3^4 t_4^3 - t_1^3 t_2^2 t_3^4 t_4^3 +$ $5 t_1 t_2^3 t_3^4 t_4^3 + 4 t_1^2 t_2^5 t_3^4 t_4^3 + t_1^2 t_2 t_3^5 t_4^3 - t_2^2 t_3^5 t_4^3 + t_1^3 t_2^3 t_3^5 t_4^3 + 3 t_1^2 t_2^6 t_3^5 t_4^3 -$ $t_1^2 t_2^2 t_3^6 t_4^3 - t_2^3 t_3^6 t_4^3 + t_1^3 t_2^4 t_3^6 t_4^3 - 4 t_1 t_2^5 t_3^6 t_4^3 - t_1^2 t_2^3 t_3^7 t_4^3 + t_2^4 t_3^7 t_4^3 - t_1^3 t_2^5 t_3^7 t_4^3$	{0, 1, 0, 1}

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$P_{A4}(t_i)$	Character
$t_2\,t_3\,\big(t_1^2\,t_2^4 - t_2\,t_3 + t_1^3\,t_2^2\,t_3 - 3\,t_1\,t_2^3\,t_3 - t_1^2\,t_2^5\,t_3 - 2\,t_1^2\,t_2\,t_3^2 + t_2^2\,t_3^2 - t_1^3\,t_2^3\,t_3^2 - t_1^2\,t_2^6\,t_3^2 + t_1\,t_3^3 + t_2^3\,t_3^3 - t_1^3\,t_2^4\,t_3^3 + 2\,t_1\,t_2^5\,t_3^3 + t_1\,t_2\,t_3^4 + 3\,t_1^2\,t_2^3\,t_3^4 - t_2^4\,t_3^4 + t_1^3\,t_2^5\,t_3^4 - t_1\,t_2^2\,t_3^5 - 2\,t_1^2\,t_2^2\,t_4 + t_2^3\,t_4 - t_1^3\,t_2^4\,t_4 + 4\,t_1\,t_2\,t_3\,t_4 + t_1^2\,t_2^3\,t_3\,t_4 + t_2^4\,t_3\,t_4 - t_1^3\,t_2^5\,t_3\,t_4 + t_1\,t_2^6\,t_3\,t_4 + 5\,t_1\,t_2^2\,t_3^2\,t_4 + 8\,t_1^2\,t_2^4\,t_3^2\,t_4 - t_2^5\,t_3^2\,t_4 + t_1^3\,t_2^6\,t_3^2\,t_4 - 3\,t_2\,t_3^3\,t_4 + 3\,t_1^3\,t_2^2\,t_3^3\,t_4 - 7\,t_1\,t_2^3\,t_3^3\,t_4 + t_1^2\,t_2^5\,t_3^3\,t_4 - 3\,t_1^2\,t_2\,t_3^4\,t_4 - 5\,t_1\,t_2^4\,t_3^4\,t_4 - 2\,t_1^2\,t_2^6\,t_3^4\,t_4 - 3\,t_1^2\,t_2^2\,t_3^5\,t_4 + 2\,t_2^3\,t_3^5\,t_4 - 2\,t_1^3\,t_2^4\,t_3^5\,t_4 + t_1\,t_2^5\,t_3^5\,t_4 + t_1\,t_2\,t_3^6\,t_4 + t_1^2\,t_2^5\,t_4^2 + t_1^2\,t_2\,t_3\,t_4^2 - 2\,t_2^2\,t_3\,t_4^2 + 2\,t_1^3\,t_2^3\,t_3\,t_4^2 - 3\,t_1\,t_2^4\,t_3\,t_4^2 - 2\,t_1\,t_2^2\,t_3^2\,t_4^2 - 5\,t_1^2\,t_2^2\,t_3^2\,t_4^2 - 3\,t_1\,t_2^5\,t_3^2\,t_4^2 + t_1\,t_2\,t_3^3\,t_4^2 - 7\,t_1^2\,t_2^3\,t_3^3\,t_4^2 + 3\,t_2^4\,t_3^3\,t_4^2 - 3\,t_1^3\,t_2^5\,t_3^3\,t_4^2 + t_3^4\,t_4^2 - t_1^3\,t_2\,t_3^4\,t_4^2 + 8\,t_1\,t_2^2\,t_3^4\,t_4^2 + 5\,t_1^2\,t_2^4\,t_3^4\,t_4^2 + t_1^2\,t_2^5\,t_4^3 - t_2\,t_3^5\,t_4^3 + t_1^3\,t_2^2\,t_3^5\,t_4^3 + t_1\,t_2^3\,t_3^5\,t_4^3 + 4\,t_1^2\,t_2^5\,t_3^5\,t_4^3 - t_2^2\,t_3^6\,t_4^3 + t_1^3\,t_2^3\,t_3^6\,t_4^3 - 2\,t_1\,t_2^4\,t_3^6\,t_4^3 - t_1^2\,t_2^2\,t_3^3\,t_4^4 + t_2\,t_3^3\,t_4^4 - t_1^3\,t_2^2\,t_3^3\,t_4^4 + 3\,t_1\,t_2^3\,t_3^3\,t_4^4 + t_1^2\,t_2^5\,t_3^3\,t_4^4 + 2\,t_1^2\,t_2\,t_3^4\,t_4^4 - t_2^2\,t_3^4\,t_4^4 + t_1^3\,t_2^4\,t_3^4\,t_4^4 - t_1\,t_3^5\,t_4^4 - t_2^3\,t_3^5\,t_4^4 + t_1^3\,t_2^5\,t_3^5\,t_4^4 - t_1^2\,t_2^6\,t_3^5\,t_4^4 + t_1\,t_2^2\,t_3^6\,t_4^4 - t_1\,t_2^5\,t_3^4\,t_4^5 - 3\,t_1^2\,t_2^3\,t_3^5\,t_4^5 + t_2^4\,t_3^5\,t_4^5 - t_1^3\,t_2^5\,t_3^5\,t_4^5 + t_1\,t_2^2\,t_3^6\,t_4^5\big)$	{0, 1, 1, 0}
$-t_2\,t_3\,\big(t_1\,t_2^4 + t_1\,t_3 + t_1^2\,t_2^2\,t_3 - t_2^3\,t_3 + t_1^3\,t_2^4\,t_3 - t_1\,t_2\,t_3^2 - t_2^4\,t_3^2 + t_1^3\,t_2^5\,t_3^2 - t_1\,t_2^2\,t_3^3 - 2\,t_1^2\,t_2^4\,t_3^3 + t_1\,t_2^3\,t_3^4 - t_1\,t_2^2\,t_4 - t_1^2\,t_2^4\,t_4 - t_1^2\,t_3\,t_4 - t_1\,t_2^3\,t_3\,t_4 - 2\,t_1^2\,t_2^5\,t_3\,t_4 - t_1^2\,t_2\,t_3^2\,t_4 + 2\,t_2^2\,t_3^2\,t_4 - 2\,t_1^3\,t_2^3\,t_3^2\,t_4 + 4\,t_1\,t_2^4\,t_3^2\,t_4 - t_1^2\,t_2^6\,t_3^2\,t_4 + t_1\,t_3^3\,t_4 + 2\,t_1^2\,t_2^2\,t_3^3\,t_4 + 2\,t_1\,t_2^5\,t_3^3\,t_4 + 2\,t_1^2\,t_2^3\,t_3^4\,t_4 - t_2^4\,t_3^4\,t_4 + t_1^3\,t_2^5\,t_3^4\,t_4 - 2\,t_1\,t_2^2\,t_3^5\,t_4 - t_1^2\,t_2^4\,t_3^5\,t_4 - t_1^2\,t_2^5\,t_3^6\,t_4 + 2\,t_1\,t_2\,t_3\,t_4^2 + 2\,t_1^2\,t_2^3\,t_3\,t_4^2 - t_1\,t_2^2\,t_3^2\,t_4^2 + 4\,t_1^2\,t_2^4\,t_3^2\,t_4^2 - t_2^5\,t_3^2\,t_4^2 + t_1^3\,t_2^6\,t_3^2\,t_4^2 - t_2\,t_3^3\,t_4^2 + t_1^3\,t_2^2\,t_3^3\,t_4^2 - 6\,t_1\,t_2^3\,t_3^3\,t_4^2 - t_1^2\,t_2^5\,t_3^3\,t_4^2 - 2\,t_1^2\,t_2\,t_3^4\,t_4^2 + t_2^2\,t_3^4\,t_4^2 - t_1^3\,t_2^3\,t_3^4\,t_4^2 - t_1\,t_2^4\,t_3^4\,t_4^2 - t_1^2\,t_2^6\,t_3^4\,t_4^2 + t_1\,t_3^5\,t_4^2 + t_2^3\,t_3^5\,t_4^2 - t_1^3\,t_2^4\,t_3^5\,t_4^2 + 2\,t_1\,t_2^5\,t_3^5\,t_4^2 + t_1\,t_2\,t_3^6\,t_4^2 + t_1^2\,t_2^3\,t_3^6\,t_4^2 - t_2^4\,t_3^6\,t_4^2 + t_1^3\,t_2^5\,t_3^6\,t_4^2 - t_1\,t_2^2\,t_3^3\,t_4^3 - t_1\,t_2^5\,t_3^3\,t_4^3 + t_2^3\,t_3^3\,t_4^3 - t_1^3\,t_2^4\,t_3^3\,t_4^3 + t_1\,t_2^6\,t_3^3\,t_4^3 + t_2^4\,t_3^4\,t_4^3 - t_1^3\,t_2^5\,t_3^4\,t_4^3\big)$	{0, 2, 0, 0}
$t_1\,t_2^2 + t_1^2\,t_2^4 - t_2\,t_3 + t_1^3\,t_2^2\,t_3 - t_1\,t_2^3\,t_3 + t_1^2\,t_2^5\,t_3 - t_1^2\,t_2\,t_3^2 - t_2^2\,t_3^2 + t_1^3\,t_2^3\,t_3^2 - 4\,t_1\,t_2^4\,t_3^2 - t_1^2\,t_2^6\,t_3^2 - 3\,t_1^2\,t_2^2\,t_3^3 + 2\,t_2^3\,t_3^3 - 2\,t_1^3\,t_2^4\,t_3^3 - t_1\,t_2^5\,t_3^3 - 2\,t_1^2\,t_2^7\,t_3^3 + 2\,t_1\,t_2\,t_3^4 + t_1^2\,t_2^3\,t_3^4 + 2\,t_2^4\,t_3^4 - 2\,t_1^3\,t_2^5\,t_3^4 + 3\,t_1\,t_2^6\,t_3^4 + t_1\,t_2^2\,t_3^5 + 4\,t_1^2\,t_2^4\,t_3^5 - t_2^5\,t_3^5 + t_1^3\,t_2^6\,t_3^5 + t_1\,t_2^7\,t_3^5 - t_1\,t_2^3\,t_3^6 + t_1^2\,t_2^5\,t_3^6 - t_2^6\,t_3^6 + t_1^3\,t_2^7\,t_3^6 - t_1\,t_2^4\,t_3^7 - t_1^2\,t_2^6\,t_3^7 - t_1^2\,t_2^2\,t_4 - t_1\,t_2^5\,t_4 - 4\,t_1^2\,t_2^3\,t_3\,t_4 + 2\,t_2^4\,t_3\,t_4 - 2\,t_1^3\,t_2^5\,t_3\,t_4 + t_2^3\,t_4 - t_1^3\,t_2\,t_3^2\,t_4 + 7\,t_1\,t_2^2\,t_3^2\,t_4 + 3\,t_1^2\,t_2^4\,t_3^2\,t_4 + t_2^5\,t_3^2\,t_4 - t_1^3\,t_2^6\,t_3^2\,t_4 + t_1\,t_2^7\,t_3^2\,t_4 + t_1^2\,t_2^3\,t_3^3\,t_4 + 5\,t_1\,t_2^3\,t_3^3\,t_4 + 10\,t_1^2\,t_2^5\,t_3^3\,t_4 - t_2^6\,t_3^3\,t_4 + t_1^3\,t_2^7\,t_3^3\,t_4 + t_1\,t_2^8\,t_3^3\,t_4 - 4\,t_2^2\,t_3^4\,t_4 + 4\,t_1^3\,t_2^4\,t_3^4\,t_4 - 8\,t_1\,t_2^6\,t_3^4\,t_4 + 3\,t_1^2\,t_2^6\,t_3^4\,t_4 - t_2^7\,t_3^4\,t_4 + t_1^3\,t_2^8\,t_3^4\,t_4 - t_1\,t_3^5\,t_4 - 4\,t_1^2\,t_2^2\,t_3^5\,t_4 - t_2^3\,t_3^5\,t_4 + t_1^3\,t_2^4\,t_3^5\,t_4 - 8\,t_1\,t_2^5\,t_3^5\,t_4 - 4\,t_1^2\,t_2^7\,t_3^5\,t_4 - 4\,t_1^2\,t_2^3\,t_3^6\,t_4 + 3\,t_2^4\,t_3^6\,t_4 - 3\,t_1^3\,t_2^5\,t_3^6\,t_4 + t_1\,t_2^6\,t_3^6\,t_4 - t_1^2\,t_2^8\,t_3^6\,t_4 + t_1\,t_2^2\,t_3^7\,t_4 + t_2^5\,t_3^7\,t_4 - t_1^3\,t_2^6\,t_3^7\,t_4 + t_1\,t_2^7\,t_3^7\,t_4 + t_1\,t_2^3\,t_3^8\,t_4 + t_1^2\,t_2^5\,t_3^8\,t_4 + t_1\,t_2^3\,t_4^2 + t_1^2\,t_2^5\,t_4^2 + t_1^2\,t_2\,t_3\,t_4^2 - t_2^2\,t_3\,t_4^2 + t_1^3\,t_2^3\,t_3\,t_4^2 + t_1^2\,t_2^6\,t_3\,t_4^2 - t_1\,t_2^3\,t_3^2\,t_4^2 - t_1^2\,t_2^5\,t_3^2\,t_4^2 + 3\,t_2^3\,t_3^2\,t_4^2 + 3\,t_1^3\,t_2^4\,t_3^2\,t_4^2 - 4\,t_1\,t_2^5\,t_3^2\,t_4^2 - 4\,t_1\,t_2\,t_3^3\,t_4^2 - 8\,t_1^2\,t_2^3\,t_3^3\,t_4^2 + t_2^4\,t_3^3\,t_4^2 - t_1^3\,t_2^5\,t_3^3\,t_4^2 - 4\,t_1\,t_2^6\,t_3^3\,t_4^2 - t_1^2\,t_2^8\,t_3^3\,t_4^2 + t_3^4\,t_4^2 - t_1^3\,t_2\,t_3^4\,t_4^2 + 3\,t_1\,t_2^2\,t_3^4\,t_4^2 - 8\,t_1^2\,t_2^4\,t_3^4\,t_4^2 + 4\,t_2^5\,t_3^4\,t_4^2 - 4\,t_1^3\,t_2^6\,t_3^4\,t_4^2 + t_1^2\,t_3^5\,t_4^2 + t_2\,t_3^5\,t_4^2 - t_1^3\,t_2^2\,t_3^5\,t_4^2 + 10\,t_1\,t_2^3\,t_3^5\,t_4^2 + 5\,t_1^2\,t_2^5\,t_3^5\,t_4^2 + t_2^6\,t_3^5\,t_4^2 - t_1^3\,t_2^7\,t_3^5\,t_4^2 + t_1\,t_2^8\,t_3^5\,t_4^2 + t_1^2\,t_2\,t_3^6\,t_4^2 - t_2^2\,t_3^6\,t_4^2 + t_1^3\,t_2^3\,t_3^6\,t_4^2 + 3\,t_1\,t_2^4\,t_3^6\,t_4^2 + 7\,t_1^2\,t_2^6\,t_3^6\,t_4^2 - t_2^7\,t_3^6\,t_4^2 + t_1^3\,t_2^8\,t_3^6\,t_4^2 - 2\,t_2^3\,t_3^7\,t_4^2 + 2\,t_1^3\,t_2^4\,t_3^7\,t_4^2 - 4\,t_1\,t_2^5\,t_3^7\,t_4^2 - t_1^2\,t_2^3\,t_3^8\,t_4^2 - t_1\,t_2^6\,t_3^8\,t_4^2 - t_1\,t_2^2\,t_3\,t_4^3 - t_1^2\,t_2^4\,t_3\,t_4^3 + t_2\,t_3^2\,t_4^3 - t_1^3\,t_2^2\,t_3^2\,t_4^3 + t_1\,t_2^3\,t_3^2\,t_4^3 - t_1^2\,t_2^5\,t_3^2\,t_4^3 + t_1^2\,t_2\,t_3^3\,t_4^3 + t_2^2\,t_3^3\,t_4^3 - t_1^3\,t_2^3\,t_3^3\,t_4^3 + 4\,t_1\,t_2^4\,t_3^3\,t_4^3 + t_1^2\,t_2^6\,t_3^3\,t_4^3 + 3\,t_1^2\,t_2^2\,t_3^4\,t_4^3 - 2\,t_2^3\,t_3^4\,t_4^3 + 2\,t_1^3\,t_2^4\,t_3^4\,t_4^3 + t_1\,t_2^5\,t_3^4\,t_4^3 - t_1^2\,t_2^3\,t_3^5\,t_4^3 - 2\,t_2^4\,t_3^5\,t_4^3 + 2\,t_1^3\,t_2^5\,t_3^5\,t_4^3 - 3\,t_1\,t_2^6\,t_3^5\,t_4^3 - t_1\,t_2^2\,t_3^6\,t_4^3 - 4\,t_1^2\,t_2^4\,t_3^6\,t_4^3 + t_2^5\,t_3^6\,t_4^3 - t_1^3\,t_2^6\,t_3^6\,t_4^3 - t_1\,t_2^7\,t_3^6\,t_4^3 + t_1\,t_2^3\,t_3^7\,t_4^3 - t_1^2\,t_2^5\,t_3^7\,t_4^3 + t_2^6\,t_3^7\,t_4^3 - t_1^3\,t_2^7\,t_3^7\,t_4^3 + t_1\,t_2^4\,t_3^8\,t_4^3 + t_1^2\,t_2^6\,t_3^8\,t_4^3$	{1, 0, 0, 1}

$P_{A4}(t_i)$	Character
$-t_2^3 t_3 + t_1^3 t_2^4 t_3 - t_1 t_2^5 t_3 - 4 t_1^2 t_2^3 t_3^2 + t_2^4 t_3^2 - t_1^3 t_2^5 t_3^2 - t_1 t_2^6 t_3^2 + 3 t_1 t_2^2 t_3^3 + t_2^5 t_3^3 -$ $t_1^3 t_2^6 t_3^3 + t_1 t_2^7 t_3^3 + 4 t_1 t_2^3 t_3^4 + 5 t_1^2 t_2^5 t_3^4 - t_2^6 t_3^4 + t_1^3 t_2^7 t_3^4 - 2 t_2^2 t_3^5 + 2 t_1^3 t_2^3 t_3^5 -$ $3 t_1 t_2^4 t_3^5 + t_1^2 t_2^6 t_3^5 - t_1^2 t_2^2 t_3^6 - 2 t_1 t_2^5 t_3^6 - t_1^2 t_2^7 t_3^6 - t_1^2 t_2^4 t_4 + t_2 t_3 t_4 - t_1^3 t_2^2 t_3 t_4 +$ $4 t_1 t_2^3 t_3 t_4 + t_1^2 t_2^5 t_3 t_4 + 2 t_1^2 t_2 t_3^2 t_4 - t_2^2 t_3^2 t_4 + t_1^3 t_2^3 t_3^2 t_4 + 4 t_1 t_2^4 t_3^2 t_4 +$ $4 t_1^2 t_2^6 t_3^2 t_4 - t_1 t_3^3 t_4 + 2 t_1^2 t_2^2 t_3^3 t_4 - 5 t_2^3 t_3^3 t_4 + 5 t_1^3 t_2^4 t_3^3 t_4 - 5 t_1 t_2^5 t_3^3 t_4 +$ $t_1^2 t_2^7 t_3^3 t_4 - 3 t_1 t_2 t_3^4 t_4 - 9 t_1^2 t_2^3 t_3^4 t_4 - 6 t_1 t_2^6 t_3^4 t_4 - t_1^2 t_2^8 t_3^4 t_4 + t_1 t_2^2 t_3^5 t_4 -$ $9 t_1^2 t_2^4 t_3^5 t_4 + 4 t_2^5 t_3^5 t_4 - 4 t_1^3 t_2^6 t_3^5 t_4 + t_1 t_2^7 t_3^5 t_4 + t_2 t_3^6 t_4 - t_1^3 t_2^2 t_3^6 t_4 +$ $6 t_1 t_2^3 t_3^6 t_4 + 2 t_1^2 t_2^5 t_3^6 t_4 + 2 t_1 t_2^4 t_3^7 t_4 + 2 t_1^2 t_2^6 t_3^7 t_4 - t_1 t_2 t_3 t_4^2 + t_1^2 t_2^3 t_3 t_4^2 -$ $t_2^4 t_3 t_4^2 + t_1^3 t_2^5 t_3 t_4^2 - t_1 t_2^6 t_3 t_4^2 - 4 t_1 t_2^2 t_3^2 t_4^2 - 6 t_1^2 t_2^4 t_3^2 t_4^2 + t_2^5 t_3^2 t_4^2 -$ $t_1^3 t_2^6 t_3^2 t_4^2 - t_1 t_2^7 t_3^2 t_4^2 + 2 t_2 t_3^3 t_4^2 - 2 t_1^3 t_2^2 t_3^3 t_4^2 + 2 t_1 t_2^3 t_3^3 t_4^2 - 5 t_1^2 t_2^5 t_3^3 t_4^2 +$ $t_2^6 t_3^3 t_4^2 - t_1^3 t_2^7 t_3^3 t_4^2 + 2 t_1^2 t_2 t_3^4 t_4^2 + 3 t_2^2 t_3^4 t_4^2 - 3 t_1^3 t_2^3 t_3^4 t_4^2 + 12 t_1 t_2^4 t_3^4 t_4^2 +$ $4 t_1^2 t_2^6 t_3^4 t_4^2 + 6 t_1^2 t_2^2 t_3^5 t_4^2 - 4 t_2^3 t_3^5 t_4^2 + 4 t_1^3 t_2^4 t_3^5 t_4^2 + 2 t_1 t_2^5 t_3^5 t_4^2 + 4 t_1^2 t_2^7 t_3^5 t_4^2 -$ $2 t_1 t_2 t_3^6 t_4^2 + t_1^2 t_2^3 t_3^6 t_4^2 - 3 t_2^4 t_3^6 t_4^2 + 3 t_1^3 t_2^5 t_3^6 t_4^2 - 4 t_1 t_2^6 t_3^6 t_4^2 - 2 t_1 t_2^2 t_3^7 t_4^2 -$ $3 t_1^2 t_2^4 t_3^7 t_4^2 - t_1 t_2^7 t_3^7 t_4^2 - t_1^2 t_2^5 t_3^8 t_4^2 + t_2^3 t_3^3 t_4^3 - t_1^3 t_2^4 t_3^3 t_4^3 + t_1 t_2^5 t_3^3 t_4^3 +$ $4 t_1^2 t_2^3 t_3^3 t_4^3 - t_2^4 t_3^3 t_4^3 + t_1^3 t_2^5 t_3^3 t_4^3 + t_1 t_2^6 t_3^3 t_4^3 - 3 t_1 t_2^2 t_3^4 t_4^3 - t_2^5 t_3^4 t_4^3 + t_1^3 t_2^6 t_3^4 t_4^3 -$ $t_1 t_2^7 t_3^4 t_4^3 - 4 t_1 t_2^3 t_3^5 t_4^3 - 5 t_1^2 t_2^5 t_3^5 t_4^3 + t_2^6 t_3^5 t_4^3 - t_1^3 t_2^7 t_3^5 t_4^3 + 2 t_2^2 t_3^6 t_4^3 -$ $2 t_1^3 t_2^3 t_3^6 t_4^3 + 3 t_1 t_2^4 t_3^6 t_4^3 - t_1^2 t_2^6 t_3^6 t_4^3 + t_1^2 t_2^2 t_3^7 t_4^3 + 2 t_1 t_2^5 t_3^7 t_4^3 + t_1^2 t_2^7 t_3^7 t_4^3$	{1, 0, 1, 0}
$-t_1 t_2^2 t_3 - 2 t_1^2 t_2^4 t_3 + t_2 t_3^2 - t_1^3 t_2^2 t_3^2 + 3 t_1 t_2^3 t_3^2 + t_1^2 t_2 t_3^3 + 3 t_1 t_2^4 t_3^3 + 2 t_1^2 t_2^6 t_3^3 +$ $t_1^2 t_2^2 t_3^4 - 2 t_2^3 t_3^4 + 2 t_1^3 t_2^4 t_3^4 - 2 t_1 t_2^5 t_3^4 + t_1^2 t_2^7 t_3^4 - t_1 t_2 t_3^5 - 2 t_1^2 t_2^3 t_3^5 - 2 t_1 t_2^6 t_3^5 -$ $t_1^2 t_2^4 t_3^6 + t_2^5 t_3^6 - t_1^3 t_2^6 t_3^6 + t_1 t_2^4 t_4 + 2 t_1^2 t_2^2 t_3 t_4 - t_2^3 t_3 t_4 + t_1^3 t_2^4 t_3 t_4 + t_1 t_2^5 t_3 t_4 -$ $2 t_1 t_2 t_3^2 t_4 + 2 t_1^2 t_2^3 t_3^2 t_4 - 2 t_2^4 t_3^2 t_4 + 2 t_1^3 t_2^5 t_3^2 t_4 - t_1 t_2^6 t_3^2 t_4 - 6 t_1 t_2^2 t_3^3 t_4 -$ $7 t_1^2 t_2^4 t_3^3 t_4 - 2 t_1 t_2^7 t_3^3 t_4 + t_2 t_3^4 t_4 - t_1^3 t_2^2 t_3^4 t_4 + t_1 t_2^3 t_3^4 t_4 - 7 t_1^2 t_2^5 t_3^4 t_4 +$ $2 t_2^6 t_3^4 t_4 - 2 t_1^3 t_2^7 t_3^4 t_4 + t_1^2 t_2 t_3^5 t_4 + 2 t_2^2 t_3^5 t_4 - 2 t_1^3 t_2^3 t_3^5 t_4 + 8 t_1 t_2^4 t_3^5 t_4 +$ $2 t_1^2 t_2^6 t_3^5 t_4 + 2 t_1^2 t_2^2 t_3^6 t_4 - t_2^3 t_3^6 t_4 + t_1^3 t_2^4 t_3^6 t_4 + 2 t_1 t_2^5 t_3^6 t_4 + 2 t_1^2 t_2^7 t_3^6 t_4 +$ $t_1^2 t_2^3 t_3^7 t_4 - t_2^4 t_3^7 t_4 + t_1^3 t_2^5 t_3^7 t_4 - t_1 t_2^6 t_3^7 t_4 - 2 t_1 t_2^3 t_3 t_4^2 - 2 t_1^2 t_2^5 t_3 t_4^2 -$ $t_1^2 t_2 t_3^2 t_4^2 + 2 t_2^2 t_3^2 t_4^2 - 2 t_1^3 t_2^3 t_3^2 t_4^2 + 2 t_1 t_2^4 t_3^2 t_4^2 - t_1^2 t_2^6 t_3^2 t_4^2 + t_1 t_3^3 t_4^2 +$ $2 t_1^2 t_2^2 t_3^3 t_4^2 + 2 t_2^3 t_3^3 t_4^2 - 2 t_1^3 t_2^4 t_3^3 t_4^2 + 6 t_1 t_2^5 t_3^3 t_4^2 + t_1^2 t_2^7 t_3^3 t_4^2 + t_1 t_2 t_3^4 t_4^2 +$ $8 t_1^2 t_2^3 t_3^4 t_4^2 - 3 t_2^4 t_3^4 t_4^2 + 3 t_1^3 t_2^5 t_3^4 t_4^2 + 2 t_1 t_2^6 t_3^4 t_4^2 + t_1^2 t_2^8 t_3^4 t_4^2 - 5 t_1 t_2^2 t_3^5 t_4^2 +$ $t_1^2 t_2^4 t_3^5 t_4^2 - 3 t_2^5 t_3^5 t_4^2 + 3 t_1^3 t_2^6 t_3^5 t_4^2 - 2 t_1 t_2^7 t_3^5 t_4^2 - 5 t_1 t_2^3 t_3^6 t_4^2 - 6 t_1^2 t_2^5 t_3^6 t_4^2 +$ $t_2^6 t_3^6 t_4^2 - t_1^3 t_2^7 t_3^6 t_4^2 + t_2^2 t_3^7 t_4^2 - t_1^3 t_2^3 t_3^7 t_4^2 + t_1 t_2^4 t_3^7 t_4^2 + t_1^2 t_2^6 t_3^7 t_4^2 - t_1 t_2^2 t_3^3 t_4^3 +$ $2 t_1^2 t_2^2 t_3^4 t_4^3 + 2 t_2^3 t_3^4 t_4^3 - 2 t_1^3 t_2^4 t_3^4 t_4^3 + 6 t_1 t_2^5 t_3^4 t_4^3 + t_1^2 t_2^7 t_3^4 t_4^3 + t_1 t_2 t_3^5 t_4^3 +$ $8 t_1^2 t_2^3 t_3^5 t_4^3 - 3 t_2^4 t_3^5 t_4^3 + 3 t_1^3 t_2^5 t_3^5 t_4^3 + 2 t_1 t_2^6 t_3^5 t_4^3 + t_1^2 t_2^8 t_3^5 t_4^3 - 5 t_1 t_2^2 t_3^6 t_4^3 +$ $t_1^2 t_2^4 t_3^6 t_4^3 - 3 t_2^5 t_3^6 t_4^3 + 3 t_1^3 t_2^6 t_3^6 t_4^3 - 2 t_1 t_2^7 t_3^6 t_4^3 - 5 t_1 t_2^3 t_3^7 t_4^3 - 6 t_1^2 t_2^5 t_3^7 t_4^3 +$ $t_2^6 t_3^7 t_4^3 - t_1^3 t_2^8 t_3^7 t_4^3 + t_1^2 t_2^2 t_3^8 t_4^3 - t_1^3 t_2^3 t_3^8 t_4^3 + t_1 t_2^4 t_3^8 t_4^3 - t_1^2 t_2^6 t_3^8 t_4^3 -$ $t_1 t_2^2 t_3^4 t_4^4 + 2 t_1^2 t_2^4 t_3^4 t_4^4 - t_2^5 t_3^4 t_4^4 + t_1^3 t_2^6 t_3^4 t_4^4 - t_1 t_2^7 t_3^4 t_4^4 + t_1^2 t_2^3 t_3^5 t_4^4 - t_2^4 t_3^5 t_4^4 +$ $t_1^3 t_2^5 t_3^5 t_4^4 + t_1 t_2^6 t_3^5 t_4^4 - 2 t_1^2 t_2^2 t_3^6 t_4^4 - t_2^3 t_3^6 t_4^4 + t_1^3 t_2^4 t_3^6 t_4^4 + 2 t_1 t_2^5 t_3^6 t_4^4 +$ $t_1^2 t_2^7 t_3^6 t_4^4 - t_2^2 t_3^7 t_4^4 + t_1^3 t_2^3 t_3^7 t_4^4 - t_1 t_2^4 t_3^7 t_4^4 - t_1^2 t_2^6 t_3^7 t_4^4 + t_1 t_2^2 t_3^5 t_4^5 -$ $t_1^2 t_2^4 t_3^5 t_4^5 + t_2^5 t_3^5 t_4^5 - t_1^3 t_2^6 t_3^5 t_4^5 + t_1 t_2^7 t_3^5 t_4^5 - t_1^2 t_2^3 t_3^6 t_4^5 + t_2^4 t_3^6 t_4^5 -$ $t_1^3 t_2^5 t_3^6 t_4^5 - t_1 t_2^6 t_3^6 t_4^5 + 2 t_1^2 t_2^2 t_3^7 t_4^5 + t_2^3 t_3^7 t_4^5 - t_1^3 t_2^4 t_3^7 t_4^5 + t_1 t_2^5 t_3^7 t_4^5 -$ $t_1^2 t_2^7 t_3^7 t_4^5 + t_1^3 t_2^2 t_3^8 t_4^5 - t_1 t_2^3 t_3^8 t_4^5 + t_1^2 t_2^5 t_3^8 t_4^5 - t_1^2 t_2^2 t_3^6 t_4^6 + t_1^3 t_2^3 t_3^6 t_4^6 -$ $t_1 t_2^4 t_3^6 t_4^6 + t_1^2 t_2^6 t_3^6 t_4^6 - t_1^3 t_2^2 t_3^7 t_4^6 + t_1 t_2^3 t_3^7 t_4^6 - t_1^2 t_2^5 t_3^7 t_4^6 + t_1^2 t_2^2 t_3^8 t_4^6 -$ $t_1^3 t_2^3 t_3^8 t_4^6 + t_1 t_2^4 t_3^8 t_4^6 - t_1^2 t_2^6 t_3^8 t_4^6 + t_1^3 t_2^2 t_3^6 t_4^7 - t_1 t_2^3 t_3^6 t_4^7 + t_1^2 t_2^5 t_3^6 t_4^7 -$ $t_1^2 t_2^2 t_3^7 t_4^7 + t_1^3 t_2^3 t_3^7 t_4^7 - t_1 t_2^4 t_3^7 t_4^7 + t_1^2 t_2^6 t_3^7 t_4^7 - t_1^3 t_2^2 t_3^8 t_4^7 + t_1 t_2^3 t_3^8 t_4^7 -$ $t_1^2 t_2^5 t_3^8 t_4^7 + t_1^2 t_2^2 t_3^6 t_4^8 - t_1^3 t_2^3 t_3^6 t_4^8 + t_1 t_2^4 t_3^6 t_4^8 - t_1^2 t_2^6 t_3^6 t_4^8 + t_1^3 t_2^2 t_3^7 t_4^8 -$ $t_1 t_2^3 t_3^7 t_4^8 + t_1^2 t_2^5 t_3^7 t_4^8 - t_1^2 t_2^2 t_3^8 t_4^8 + t_1^3 t_2^3 t_3^8 t_4^8 - t_1 t_2^4 t_3^8 t_4^8 + t_1^2 t_2^6 t_3^8 t_4^8$	{1, 1, 0, 0}
$-t_1 t_2^3 + t_2^2 t_3 - t_1^3 t_2^3 t_3 + t_1 t_2^4 t_3 + 2 t_1^2 t_2^2 t_3^2 - t_2^3 t_3^2 + t_1^3 t_2^4 t_3^2 - t_1 t_2 t_3^3 + t_1^2 t_2^3 t_3^3 +$ $t_1 t_2^6 t_3^3 - 2 t_1 t_2^2 t_3^4 - t_1^2 t_2^4 t_3^4 - t_2^5 t_3^4 + t_1^3 t_2^6 t_3^4 - t_1 t_2^7 t_3^4 - 3 t_1^2 t_2^5 t_3^5 + t_2^6 t_3^5 -$ $t_1^3 t_2^7 t_3^5 + t_1 t_2^4 t_3^6 + t_1 t_2^5 t_3^7 + t_1^2 t_2^3 t_3 t_4 - t_1 t_2^2 t_3 t_4 + t_1^2 t_2^4 t_3 t_4 - 4 t_1 t_2^3 t_3^2 t_4 -$ $2 t_1^2 t_2^5 t_3^2 t_4 - t_1^2 t_2 t_3^3 t_4 + 3 t_2^2 t_3^3 t_4 - 3 t_1^3 t_2^3 t_3^3 t_4 + t_1 t_2^4 t_3^3 t_4 - 2 t_1^2 t_2^6 t_3^3 t_4 +$ $t_1 t_3^4 t_4 + 2 t_1^2 t_2^2 t_3^4 t_4 + t_2^3 t_3^4 t_4 - t_1^3 t_2^4 t_3^4 t_4 + 4 t_1 t_2^5 t_3^4 t_4 + t_1^2 t_2^7 t_3^4 t_4 +$ $t_1 t_2 t_3^5 t_4 + 4 t_1^2 t_2^3 t_3^5 t_4 - t_2^4 t_3^5 t_4 + t_1^3 t_2^5 t_3^5 t_4 + 2 t_1 t_2^6 t_3^5 t_4 + t_1^2 t_2^8 t_3^5 t_4 -$ $t_1 t_2^2 t_3^6 t_4 + 2 t_1^2 t_2^4 t_3^6 t_4 - 2 t_2^5 t_3^6 t_4 + 2 t_1^3 t_2^6 t_3^6 t_4 - t_1 t_2^7 t_3^6 t_4 - t_1 t_2^3 t_3^7 t_4 -$ $t_1^2 t_2^5 t_3^7 t_4 - t_1 t_2^4 t_3^8 t_4 - t_1 t_2^4 t_4^2 - t_1^2 t_2^2 t_3 t_4^2 + t_2^3 t_3 t_4^2 - t_1^3 t_2^4 t_3 t_4^2 + t_1 t_2^5 t_3 t_4^2 +$ $2 t_1 t_2 t_3^2 t_4^2 + 2 t_1^2 t_2^3 t_3^2 t_4^2 - t_2^4 t_3^2 t_4^2 + t_1^3 t_2^5 t_3^2 t_4^2 - t_3^3 t_4^2 + t_1^3 t_2 t_3^3 t_4^2 + 4 t_1^2 t_2^4 t_3^3 t_4^2 +$ $t_1 t_2^7 t_3^3 t_4^2 - t_1^2 t_2^4 t_3^4 t_4^2 - t_2 t_3^4 t_4^2 + t_1^3 t_2^2 t_3^4 t_4^2 - 5 t_1 t_2^3 t_3^4 t_4^2 + t_1^2 t_2^5 t_3^4 t_4^2 - t_2^6 t_3^4 t_4^2 +$ $t_1^3 t_2^7 t_3^4 t_4^2 - t_1 t_2^8 t_3^4 t_4^2 - t_1^2 t_2 t_3^5 t_4^2 - 5 t_1 t_2^4 t_3^5 t_4^2 - 4 t_1^2 t_2^6 t_3^5 t_4^2 + t_2^7 t_3^5 t_4^2 -$ $t_1^3 t_2^8 t_3^5 t_4^2 - t_1^2 t_2^2 t_3^6 t_4^2 + 2 t_2^3 t_3^6 t_4^2 - 2 t_1^3 t_2^4 t_3^6 t_4^2 - t_1^2 t_2^7 t_3^6 t_4^2 + t_1^2 t_2^3 t_3^7 t_4^2 +$ $t_2^4 t_3^7 t_4^2 - t_1^3 t_2^5 t_3^7 t_4^2 + 2 t_1 t_2^6 t_3^7 t_4^2 + t_1^2 t_2^4 t_3^8 t_4^2 + t_1 t_2^3 t_3 t_4^3 - t_2^2 t_3^3 t_4^3 +$ $t_1^3 t_2^2 t_3^3 t_4^3 - t_1 t_2^4 t_3^3 t_4^3 - 2 t_1^2 t_2^2 t_3^4 t_4^3 + t_2^3 t_3^4 t_4^3 - t_1^3 t_2^4 t_3^4 t_4^3 + t_1 t_2 t_3^5 t_4^3 -$ $t_1^2 t_2^3 t_3^5 t_4^3 - t_1 t_2^6 t_3^5 t_4^3 - 2 t_1^2 t_2^5 t_3^6 t_4^3 + t_2^6 t_3^6 t_4^3 - t_1^3 t_2^7 t_3^6 t_4^3 + t_1^3 t_2^4 t_3^7 t_4^3 -$ $t_1 t_2^5 t_3^7 t_4^3 + t_1^2 t_2^7 t_3^7 t_4^3 - t_1^2 t_2^4 t_3^3 t_4^4 + t_2^5 t_3^3 t_4^4 - t_1^3 t_2^6 t_3^3 t_4^4 + t_1 t_2^7 t_3^3 t_4^4 -$ $t_1^3 t_2^2 t_3^4 t_4^4 + t_1 t_2^3 t_3^4 t_4^4 + t_1^2 t_2^5 t_3^4 t_4^4 - t_2^6 t_3^4 t_4^4 + t_1^3 t_2^7 t_3^4 t_4^4 - t_1 t_2^8 t_3^4 t_4^4 -$ $t_1^2 t_2^2 t_3^5 t_4^4 + t_2^3 t_3^5 t_4^4 - t_1^3 t_2^4 t_3^5 t_4^4 + t_1 t_2^5 t_3^5 t_4^4 + t_1^2 t_2^7 t_3^5 t_4^4 - t_2^8 t_3^5 t_4^4 +$ $t_1^3 t_2^2 t_3^6 t_4^4 - t_1 t_2^3 t_3^6 t_4^4 + t_1^2 t_2^5 t_3^6 t_4^4 - t_2^6 t_3^6 t_4^4 + t_1^3 t_2^7 t_3^6 t_4^4 - t_1 t_2^8 t_3^6 t_4^4 -$ $t_1^2 t_2^2 t_3^7 t_4^4 + t_2^3 t_3^7 t_4^4 - t_1^3 t_2^4 t_3^7 t_4^4 + t_1 t_2^5 t_3^7 t_4^4 + t_1^2 t_2^7 t_3^7 t_4^4 - t_2^8 t_3^7 t_4^4 +$ $t_1^3 t_2^2 t_3^8 t_4^4 - t_1 t_2^3 t_3^8 t_4^4 + t_1^2 t_2^5 t_3^8 t_4^4 - t_2^6 t_3^8 t_4^4 + t_1^3 t_2^7 t_3^8 t_4^4 - t_1 t_2^8 t_3^8 t_4^4 -$ $t_1^2 t_2^2 t_3^6 t_4^5 + t_2^3 t_3^6 t_4^5 - t_1^3 t_2^4 t_3^6 t_4^5 + t_1 t_2^5 t_3^6 t_4^5 + t_1^2 t_2^7 t_3^6 t_4^5 - t_2^8 t_3^6 t_4^5 +$ $t_1^3 t_2^2 t_3^7 t_4^5 - t_1 t_2^3 t_3^7 t_4^5 + t_1^2 t_2^5 t_3^7 t_4^5 - t_2^6 t_3^7 t_4^5 + t_1^3 t_2^7 t_3^7 t_4^5 - t_1 t_2^8 t_3^7 t_4^5 -$ $t_1^2 t_2^2 t_3^8 t_4^5 + t_2^3 t_3^8 t_4^5 - t_1^3 t_2^4 t_3^8 t_4^5 + t_1 t_2^5 t_3^8 t_4^5 + t_1^2 t_2^7 t_3^8 t_4^5 - t_2^8 t_3^8 t_4^5 +$ $t_1^3 t_2^2 t_3^6 t_4^6 - t_1 t_2^3 t_3^6 t_4^6 + t_1^2 t_2^5 t_3^6 t_4^6 - t_2^6 t_3^6 t_4^6 + t_1^3 t_2^7 t_3^6 t_4^6 - t_1 t_2^8 t_3^6 t_4^6 -$ $t_1^2 t_2^2 t_3^7 t_4^6 + t_2^3 t_3^7 t_4^6 - t_1^3 t_2^4 t_3^7 t_4^6 + t_1 t_2^5 t_3^7 t_4^6 + t_1^2 t_2^7 t_3^7 t_4^6 - t_2^8 t_3^7 t_4^6 +$ $t_1^3 t_2^2 t_3^8 t_4^6 - t_1 t_2^3 t_3^8 t_4^6 + t_1^2 t_2^5 t_3^8 t_4^6 - t_2^6 t_3^8 t_4^6 + t_1^3 t_2^7 t_3^8 t_4^6 - t_1 t_2^8 t_3^8 t_4^6 -$ $t_1^2 t_2^2 t_3^7 t_4^7 + t_2^3 t_3^7 t_4^7 - t_1^3 t_2^4 t_3^7 t_4^7 + t_1 t_2^5 t_3^7 t_4^7 + t_1^2 t_2^7 t_3^7 t_4^7 - t_2^8 t_3^7 t_4^7 +$ $t_1^3 t_2^2 t_3^8 t_4^7 - t_1 t_2^3 t_3^8 t_4^7 + t_1^2 t_2^5 t_3^8 t_4^7 - t_2^6 t_3^8 t_4^7 + t_1^3 t_2^7 t_3^8 t_4^7 - t_1 t_2^8 t_3^8 t_4^7 -$ $t_1^2 t_2^2 t_3^8 t_4^8 + t_2^3 t_3^8 t_4^8 - t_1^3 t_2^4 t_3^8 t_4^8 + t_1 t_2^5 t_3^8 t_4^8 + t_1^2 t_2^7 t_3^8 t_4^8 - t_2^8 t_3^8 t_4^8 +$ $t_1^3 t_2^2 t_3^7 t_4^9 - t_1 t_2^3 t_3^7 t_4^9 + t_1^2 t_2^5 t_3^7 t_4^9 - t_2^6 t_3^7 t_4^9 + t_1^3 t_2^7 t_3^7 t_4^9 - t_1 t_2^8 t_3^7 t_4^9 -$ $t_1^2 t_2^2 t_3^8 t_4^9 + t_2^3 t_3^8 t_4^9 - t_1^3 t_2^4 t_3^8 t_4^9 + t_1 t_2^5 t_3^8 t_4^9 + t_1^2 t_2^7 t_3^8 t_4^9 - t_2^8 t_3^8 t_4^9 +$ $t_1^3 t_2^2 t_3^8 t_4^{10} - t_1 t_2^3 t_3^8 t_4^{10} + t_1^2 t_2^5 t_3^8 t_4^{10} - t_2^6 t_3^8 t_4^{10} + t_1^3 t_2^7 t_3^8 t_4^{10} - t_1 t_2^8 t_3^8 t_4^{10}$	{2, 0, 0, 0}

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$P_{A4}(t_i)$	Character
$t_2\,t_3\,\left(t_1\,t_2^3+t_1^2\,t_2\,t_3-t_2^2\,t_3+t_1^3\,t_2^3\,t_3-t_1^2\,t_2^2\,t_3^2-t_2^4\,t_3^3+t_1^3\,t_2^5\,t_3^3-\right.$ $t_1^2\,t_2^4\,t_3^4-t_1^2\,t_2^3\,t_4-t_1\,t_2^2\,t_3\,t_4-t_1^2\,t_2^4\,t_3\,t_4-t_1^2\,t_2^3\,t_4+t_2\,t_3^2\,t_4-t_1^3\,t_2^2\,t_3^2\,t_4+$ $t_1\,t_2^3\,t_3^2\,t_4-t_1^2\,t_2^5\,t_3^2\,t_4+t_1^2\,t_2\,t_3^3\,t_4+2\,t_1\,t_2^4\,t_3^3\,t_4-t_1^2\,t_2^6\,t_3^3\,t_4+$ $t_2^3\,t_3^4\,t_4-t_1^3\,t_2^4\,t_3^4\,t_4+2\,t_1\,t_2^5\,t_3^4\,t_4+t_1^2\,t_2^3\,t_3^5\,t_4-t_2^4\,t_3^5\,t_4+t_1^3\,t_2^5\,t_3^5\,t_4-$ $t_1^2\,t_2^4\,t_3^6\,t_4+2\,t_1^2\,t_2^2\,t_3\,t_4^2+2\,t_1^2\,t_2^3\,t_3^2\,t_4^2-2\,t_1\,t_2^2\,t_3^3\,t_4^2+t_1^2\,t_2^4\,t_3^3\,t_4^2-$ $t_2^5\,t_3^3\,t_4^2+t_1^3\,t_2^6\,t_3^3\,t_4^2-2\,t_1\,t_2^3\,t_3^4\,t_4^2-t_1^2\,t_2^5\,t_3^4\,t_4^2-t_1\,t_2^3\,t_3\,t_4^3-t_1^2\,t_2\,t_3^2\,t_4^3+$ $\left.t_2^2\,t_3^3\,t_4^3-t_1^3\,t_2^3\,t_3^3\,t_4^3+t_1^2\,t_2^2\,t_3^3\,t_4^3+t_2^4\,t_3^4\,t_4^3-t_1^3\,t_2^5\,t_3^4\,t_4^3+t_1^2\,t_2^4\,t_3^5\,t_4^3\right)$	{0, 0, 0, 3}
$t_2\,t_3$ $\left(t_1^2\,t_2^2-t_1\,t_2\,t_3-t_1\,t_2^2\,t_3^2-t_1^2\,t_2^4\,t_3^2-t_1^2\,t_2^5\,t_3^3+t_1\,t_2^4\,t_3^4-t_1\,t_2^3\,t_4-t_1^2\,t_2\,t_3\,t_4+t_2^2\,t_3\,t_4-\right.$ $t_1^3\,t_2^3\,t_3\,t_4+t_1\,t_2^3\,t_4+2\,t_1^2\,t_2^2\,t_3^2\,t_4+t_2^3\,t_3^2\,t_4-t_1^3\,t_2^4\,t_3^2\,t_4+t_1\,t_2^5\,t_3^2\,t_4+t_1\,t_2\,t_3^3\,t_4+$ $4\,t_1^2\,t_2^3\,t_3^3\,t_4+t_1\,t_2^6\,t_3^3\,t_4-t_1\,t_2^2\,t_3^4\,t_4+2\,t_1^2\,t_2^4\,t_3^4\,t_4-t_2^5\,t_3^4\,t_4+t_1^3\,t_2^6\,t_3^4\,t_4-$ $2\,t_1\,t_2^3\,t_3^5\,t_4-t_1^2\,t_2^5\,t_3^5\,t_4+t_1^2\,t_2^3\,t_4^2+t_1^2\,t_2^4\,t_3\,t_4^2-t_2\,t_2^3\,t_4^2+t_1^3\,t_2^2\,t_3^2\,t_4^2-$ $3\,t_1\,t_2^3\,t_3^2\,t_4^2-2\,t_1^2\,t_2\,t_3^3\,t_4^2-3\,t_1\,t_2^4\,t_3^3\,t_4^2-t_1^2\,t_2^6\,t_3^3\,t_4^2-t_1^2\,t_2^2\,t_3^4\,t_4^2+$ $t_2^3\,t_3^4\,t_4^2-t_1^3\,t_2^4\,t_3^4\,t_4^2+t_1^2\,t_2^3\,t_3^5\,t_4^2+t_2^4\,t_3^5\,t_4^2-t_1^3\,t_2^5\,t_3^5\,t_4^2+t_1^2\,t_2^4\,t_3^6\,t_4^2-$ $\left.t_1^2\,t_2^2\,t_3\,t_4^3+t_1\,t_2\,t_3^2\,t_4^3+t_1\,t_2^2\,t_3^3\,t_4^3+t_1^2\,t_2^4\,t_3^3\,t_4^3+t_1^2\,t_2^5\,t_3^4\,t_4^3-t_1\,t_2^4\,t_3^5\,t_4^3\right)$	{0, 0, 1, 2}
$t_2^2\,t_3\,\left(t_1^2\,t_2^2\,t_3^4-t_1^2\,t_2\,t_4+t_1\,t_3\,t_4+t_1\,t_2\,t_3^2\,t_4+2\,t_1^2\,t_2^3\,t_3^2\,t_4-t_3^3\,t_4+t_1^3\,t_2\,t_3^3\,t_4-\right.$ $2\,t_1\,t_2^2\,t_3^3\,t_4-t_1^2\,t_3^4\,t_4-t_1\,t_2^3\,t_3^4\,t_4-t_1^2\,t_2^5\,t_3^4\,t_4-t_1^2\,t_2\,t_3^5\,t_4-t_1^2\,t_2\,t_3^2\,t_4^2+t_1\,t_3^3\,t_4^2-$ $\left.2\,t_1^2\,t_2^2\,t_3^3\,t_4^2+2\,t_1\,t_2\,t_3^4\,t_4^2+t_1^2\,t_2^3\,t_3^4\,t_4^2+t_1\,t_2^2\,t_3^5\,t_4^2+t_1^2\,t_2^4\,t_3^5\,t_4^2-t_1^2\,t_2^2\,t_3^5\,t_4^3\right)$	{0, 0, 2, 1}
$-t_1\,t_2^2\,t_3^4\,t_4\,\left(t_1\,t_2-t_3+t_1\,t_2^2\,t_3-t_1\,t_2\,t_3^2\,t_4\right)$	{0, 0, 3, 0}
$-t_2\,t_3\,\left(t_1\,t_2^2+2\,t_1\,t_2^3\,t_3+t_1^2\,t_2^5\,t_3+t_1^2\,t_2\,t_3^2-t_2^2\,t_3^2+t_1^3\,t_2^3\,t_3^2-t_1^2\,t_2^2\,t_3^3-t_2^3\,t_3^3+t_1^3\,t_2^4\,t_3^3-\right.$ $t_1\,t_2^5\,t_3^3-t_1^2\,t_2^3\,t_3^4-2\,t_1\,t_2\,t_3\,t_4-2\,t_1^2\,t_2^3\,t_3\,t_4-t_2^4\,t_3\,t_4+t_1^3\,t_2^5\,t_3\,t_4-$ $2\,t_1\,t_2^2\,t_3^2\,t_4-6\,t_1^2\,t_2^4\,t_3^2\,t_4-t_1^2\,t_3^3\,t_4+t_2\,t_3^3\,t_4-t_1^3\,t_2^2\,t_3^3\,t_4+4\,t_1\,t_2^3\,t_3^3\,t_4-$ $2\,t_1^2\,t_2^5\,t_3^3\,t_4+t_1^2\,t_2\,t_3^4\,t_4+t_2^2\,t_3^4\,t_4-t_1^3\,t_2^3\,t_3^4\,t_4+5\,t_1\,t_2^4\,t_3^4\,t_4+t_1^2\,t_2^2\,t_3^5\,t_4-$ $t_2^3\,t_3^5\,t_4+t_1^3\,t_2^4\,t_3^5\,t_4-t_1^2\,t_2^3\,t_3^6\,t_4+t_2^2\,t_3\,t_4^2-t_1^3\,t_2^3\,t_3\,t_4^2+2\,t_1\,t_2^4\,t_3\,t_4^2+$ $t_1\,t_2^3\,t_4^2+5\,t_1^2\,t_2^2\,t_3^2\,t_4^2+2\,t_1\,t_2^5\,t_3^2\,t_4^2-t_1\,t_2\,t_3^3\,t_4^2+4\,t_1^2\,t_2^3\,t_3^3\,t_4^2-2\,t_2^4\,t_3^3\,t_4^2+$ $2\,t_1^3\,t_2^5\,t_3^3\,t_4^2-4\,t_1\,t_2^2\,t_3^4\,t_4^2-2\,t_1^2\,t_2^4\,t_3^4\,t_4^2-t_2^5\,t_3^4\,t_4^2+t_1^3\,t_2^6\,t_3^4\,t_4^2-t_1\,t_2^3\,t_3^5\,t_4^2-$ $2\,t_1^2\,t_2^5\,t_3^5\,t_4^2+t_1\,t_2^4\,t_3^6\,t_4^2-t_1\,t_2^2\,t_3\,t_4^3-2\,t_1\,t_2^3\,t_3^2\,t_4^3-t_1^2\,t_2^5\,t_3^2\,t_4^3-t_1^2\,t_2\,t_3^3\,t_4^3+$ $\left.t_2^2\,t_3^3\,t_4^3-t_1^3\,t_2^3\,t_3^3\,t_4^3+t_1^2\,t_2^2\,t_3^4\,t_4^3+t_2^3\,t_3^4\,t_4^3-t_1^3\,t_2^4\,t_3^4\,t_4^3+t_1\,t_2^5\,t_3^4\,t_4^3+t_1^2\,t_2^3\,t_3^5\,t_4^3\right)$	{0, 1, 0, 2}
$-t_2\,t_3\,\left(t_1^2\,t_2^2\,t_3-t_1\,t_2\,t_3^2-t_1\,t_2^2\,t_3^3-2\,t_1^2\,t_2^4\,t_3^3+t_1\,t_2^3\,t_3^4-t_1\,t_2^2\,t_4-2\,t_1\,t_2^3\,t_3\,t_4-\right.$ $2\,t_1^2\,t_2^5\,t_3\,t_4-t_1^2\,t_2\,t_3^2\,t_4+2\,t_2^2\,t_3^2\,t_4-2\,t_1^3\,t_2^3\,t_3^2\,t_4+3\,t_1\,t_2^4\,t_3^2\,t_4+$ $t_1\,t_3^3\,t_4+4\,t_1^2\,t_2^2\,t_3^3\,t_4+2\,t_1\,t_2^5\,t_3^3\,t_4+4\,t_1^2\,t_2^3\,t_3^4\,t_4-t_2^4\,t_3^4\,t_4+t_1^3\,t_2^5\,t_3^4\,t_4-$ $2\,t_1\,t_2^2\,t_3^5\,t_4-t_1^2\,t_2^4\,t_3^5\,t_4+t_1\,t_2\,t_3\,t_4^2+2\,t_1^2\,t_2^3\,t_3\,t_4^2-t_1\,t_2^2\,t_3^2\,t_4^2+3\,t_1^2\,t_2^4\,t_3^2\,t_4^2-$ $t_2\,t_3^3\,t_4^2+t_1^3\,t_2^3\,t_3^3\,t_4^2-6\,t_1\,t_2^3\,t_3^3\,t_4^2-2\,t_1^2\,t_2^5\,t_3^3\,t_4^2-2\,t_1^2\,t_2\,t_3^4\,t_4^2+t_2^2\,t_3^4\,t_4^2-$ $t_1^3\,t_2^3\,t_3^4\,t_4^2-t_1\,t_2^4\,t_3^4\,t_4^2-t_1^2\,t_2^6\,t_3^4\,t_4^2+t_2^3\,t_3^5\,t_4^2-t_1^3\,t_2^4\,t_3^5\,t_4^2+t_1\,t_2^5\,t_3^5\,t_4^2+$ $\left.t_1^2\,t_2^3\,t_3^6\,t_4^2-t_1^2\,t_2^2\,t_3^3\,t_4^3+t_1\,t_2\,t_3^3\,t_4^3+t_1\,t_2^2\,t_3^4\,t_4^3+2\,t_1^2\,t_2^4\,t_3^4\,t_4^3-t_1\,t_2^3\,t_3^5\,t_4^3\right)$	{0, 1, 1, 1}
$t_1\,t_2^2\,t_3^2\,t_4\,\left(t_1\,t_2-t_3-t_1\,t_2^2\,t_3+t_2\,t_3^2-t_1\,t_2^3\,t_3^2+t_2^2\,t_3^3+\right.$ $\left.t_1\,t_2^4\,t_3^3+t_1\,t_2\,t_3^2\,t_4-t_3^3\,t_4+t_1\,t_2^2\,t_3^3\,t_4-t_2\,t_3^4\,t_4-t_1\,t_2^3\,t_3^4\,t_4\right)$	{0, 1, 2, 0}
$t_2\,t_3^2\,\left(t_1\,t_2^2+t_1\,t_2^3\,t_3+t_1^2\,t_2^5\,t_3-t_1\,t_2^4\,t_3^2-t_2^3\,t_4+t_1^3\,t_2^4\,t_4-t_1\,t_2^5\,t_4-\right.$ $2\,t_1\,t_2\,t_3\,t_4-3\,t_1^2\,t_2^3\,t_3\,t_4-3\,t_1^2\,t_2^4\,t_3^2\,t_4+3\,t_1\,t_2^3\,t_3^3\,t_4+t_1\,t_2^3\,t_4^2-t_1^2\,t_2^5\,t_4^2+$ $t_2^2\,t_3\,t_4^2-t_1^3\,t_2^3\,t_3\,t_4^2+4\,t_1\,t_2^4\,t_3\,t_4^2+t_1\,t_2^2\,t_3^2\,t_4^2+3\,t_1^2\,t_2^2\,t_3^2\,t_4^2-t_2^3\,t_3^2\,t_4^2+$ $t_1^3\,t_2^4\,t_3^2\,t_4^2+t_1\,t_2^5\,t_3^2\,t_4^2-t_1\,t_2\,t_3^3\,t_4^2-t_2^4\,t_3^3\,t_4^2+t_1^3\,t_2^5\,t_3^3\,t_4^2-t_1\,t_2^2\,t_3^4\,t_4^2-$ $2\,t_1^2\,t_2^4\,t_3^4\,t_4^2+t_1\,t_2^3\,t_3^5\,t_4^2-t_1\,t_2^2\,t_3\,t_4^3-t_1\,t_2^3\,t_3^2\,t_4^3-t_1^2\,t_2^5\,t_3^2\,t_4^3+t_1\,t_2^4\,t_3^3\,t_4^3\right)$	{0, 2, 0, 1}
$-t_1\,t_2^2\,t_3^2\,t_4\,\left(t_2+t_1\,t_2^3-t_2^2\,t_3+t_1\,t_2^4\,t_3-t_2^3\,t_3^2-t_3\,t_4-\right.$ $\left.t_1\,t_2^2\,t_3\,t_4+t_2\,t_3^2\,t_4-t_1\,t_2^3\,t_3^2\,t_4+t_2^2\,t_3^3\,t_4+t_1\,t_2^4\,t_3^3\,t_4-t_2^3\,t_3^4\,t_4\right)$	{0, 2, 1, 0}
$t_1\,t_2^4\,t_3^2\,t_4\,\left(t_2-t_3\,t_4+t_1\,t_2^2\,t_3\,t_4-t_2\,t_3^2\,t_4\right)$	{0, 3, 0, 0}

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$P_{A4}(t_i)$	Character
$-t_2\,t_3\,\left(t_1^2\,t_2+t_1\,t_2^4+2\,t_1^2\,t_2^2\,t_3-t_2^3\,t_3+t_1^3\,t_2^4\,t_3-2\,t_1\,t_2\,t_3^2-t_1^2\,t_2^3\,t_3^2-t_2^4\,t_3^2+t_1^3\,t_2^5\,t_3^2-t_1\,t_2^2\,t_3^3-3\,t_1^2\,t_2^4\,t_3^3+t_1\,t_2^3\,t_3^4-t_1^2\,t_2^5\,t_3^4+t_1\,t_2^4\,t_3^5-2\,t_1\,t_2^2\,t_4-t_1^2\,t_2^4\,t_4-t_1^2\,t_3\,t_4+t_2\,t_3\,t_4-t_1^3\,t_2^2\,t_3\,t_4-2\,t_1\,t_2^3\,t_3\,t_4-2\,t_1^2\,t_2^5\,t_3\,t_4-t_1^2\,t_2\,t_3^2\,t_4+3\,t_2^2\,t_3^2\,t_4-3\,t_1^3\,t_2^3\,t_3^2\,t_4+4\,t_1\,t_2^4\,t_3^2\,t_4-t_1^2\,t_2^6\,t_3^2\,t_4+2\,t_1\,t_2^3\,t_3\,t_4+5\,t_1^2\,t_2^2\,t_3^3\,t_4+t_2^3\,t_3^3\,t_4-t_1^3\,t_2^4\,t_3^3\,t_4+4\,t_1\,t_2^5\,t_3^3\,t_4+5\,t_1^2\,t_2^3\,t_3^4\,t_4-2\,t_2^4\,t_3^4\,t_4+2\,t_1^3\,t_2^5\,t_3^4\,t_4+t_1\,t_2^6\,t_3^4\,t_4-2\,t_1\,t_2^2\,t_3^5\,t_4-t_1^2\,t_2^4\,t_3^5\,t_4-t_2^5\,t_3^5\,t_4+t_1^3\,t_2^6\,t_3^5\,t_4-t_1\,t_2^3\,t_3^6\,t_4-t_1^2\,t_2^5\,t_3^6\,t_4+t_1^2\,t_2^2\,t_4^2+2\,t_1\,t_2\,t_3\,t_4^2+4\,t_1^2\,t_2^3\,t_3\,t_4^2-t_2^3\,t_4^2+t_1^3\,t_2\,t_3^2\,t_4^2-3\,t_1\,t_2^2\,t_3^2\,t_4^2+4\,t_1^2\,t_2^4\,t_3^2\,t_4^2-t_2^5\,t_3^2\,t_4^2+t_1^3\,t_2^6\,t_3^2\,t_4^2-t_1\,t_2^3\,t_3^3\,t_4^2+t_1^3\,t_2^2\,t_3^3\,t_4^2-8\,t_1\,t_2^3\,t_3^3\,t_4^2-2\,t_1^2\,t_2^5\,t_3^3\,t_4^2-2\,t_1^2\,t_2\,t_3^4\,t_4^2+t_2^2\,t_3^4\,t_4^2-t_1^3\,t_2^3\,t_3^4\,t_4^2-3\,t_1\,t_2^4\,t_3^4\,t_4^2-2\,t_1^2\,t_2^6\,t_3^4\,t_4^2+2\,t_2^3\,t_3^5\,t_4^2-2\,t_1^3\,t_2^4\,t_3^5\,t_4^2+2\,t_1\,t_2^5\,t_3^5\,t_4^2+2\,t_1^2\,t_2^3\,t_3^6\,t_4^2-t_1^2\,t_2\,t_3\,t_4^3-t_1\,t_2^4\,t_3\,t_4^3-2\,t_1^2\,t_2^2\,t_3^3\,t_4^3+t_2^3\,t_3^3\,t_4^3-t_1^3\,t_2^4\,t_3^3\,t_4^3+2\,t_1\,t_2\,t_3^4\,t_4^3+t_1^2\,t_2^3\,t_3^3\,t_4^3+t_1^4\,t_2^2\,t_3^3\,t_4^3-t_1^3\,t_2^5\,t_3^3\,t_4^3+t_1\,t_2^2\,t_3^4\,t_4^3+3\,t_1^2\,t_2^4\,t_3^4\,t_4^3-t_1\,t_2^3\,t_3^5\,t_4^3+t_1^2\,t_2^5\,t_3^5\,t_4^3-t_1\,t_2^4\,t_3^6\,t_4^3\right)$	{1, 0, 0, 2}
$-t_2\,t_3\,\left(t_1^2\,t_2^3-2\,t_1\,t_2^2\,t_3-3\,t_1\,t_2^3\,t_3^2-2\,t_1^2\,t_2^5\,t_3^2-t_1^2\,t_2\,t_3^3+2\,t_2^2\,t_3^3-2\,t_1^3\,t_2^3\,t_3^3+2\,t_1\,t_2^4\,t_3^3+2\,t_1^2\,t_2^2\,t_3^4+t_1\,t_2^5\,t_3^4+t_1^2\,t_2^3\,t_3^5-t_1^2\,t_2\,t_4-t_1\,t_2^4\,t_4+t_1\,t_3\,t_4-2\,t_1^2\,t_2^2\,t_3\,t_4+2\,t_2^3\,t_3\,t_4-2\,t_1^3\,t_2^4\,t_3\,t_4+t_1\,t_2^5\,t_3\,t_4+5\,t_1\,t_2\,t_3^2\,t_4+8\,t_1^2\,t_2^3\,t_3^2\,t_4+t_1\,t_2^6\,t_3^2\,t_4-t_3^3\,t_4+t_1^3\,t_2\,t_3^3\,t_4-t_1\,t_2^2\,t_3^3\,t_4+8\,t_1^2\,t_2^4\,t_3^3\,t_4-t_2^5\,t_3^3\,t_4+t_1^3\,t_2^6\,t_3^3\,t_4-2\,t_2\,t_3^4\,t_4+2\,t_1^3\,t_2^2\,t_3^4\,t_4-9\,t_1\,t_2^3\,t_3^4\,t_4-2\,t_1^2\,t_2^5\,t_3^4\,t_4-2\,t_1^2\,t_2\,t_3^5\,t_4+t_2^2\,t_3^5\,t_4-t_1^3\,t_2^3\,t_3^5\,t_4-2\,t_1\,t_2^4\,t_3^5\,t_4-t_1^2\,t_2^6\,t_3^5\,t_4+t_1\,t_2^2\,t_4^2+t_1^2\,t_2^4\,t_4^2-t_2\,t_3\,t_4^2+t_1^3\,t_2^2\,t_3\,t_4^2-t_1\,t_2^3\,t_3\,t_4^2-6\,t_1\,t_2^4\,t_3\,t_4^2+t_1^2\,t_2^6\,t_3\,t_4^2-9\,t_1^2\,t_2^2\,t_3^2\,t_4^2+3\,t_2^3\,t_3^2\,t_4^2-3\,t_1^3\,t_2^4\,t_3^2\,t_4^2-t_1\,t_2^5\,t_3^2\,t_4^2+4\,t_1\,t_2\,t_3^3\,t_4^2-t_1^2\,t_2^3\,t_3^3\,t_4^2+2\,t_2^4\,t_3^3\,t_4^2-2\,t_1^3\,t_2^5\,t_3^3\,t_4^2+t_1\,t_2^6\,t_3^3\,t_4^2+4\,t_1\,t_2^2\,t_3^4\,t_4^2+5\,t_1^2\,t_2^4\,t_3^4\,t_4^2+t_1^2\,t_2^5\,t_3^5\,t_4^2-t_1^2\,t_2^2\,t_3\,t_4^3+2\,t_1\,t_2^3\,t_3\,t_4^3+3\,t_1\,t_2^2\,t_3^3\,t_4^3+2\,t_1^2\,t_2^5\,t_3^3\,t_4^3+t_1^2\,t_2\,t_3^4\,t_4^3-2\,t_2^2\,t_3^4\,t_4^3+2\,t_1^3\,t_2^3\,t_3^4\,t_4^3-2\,t_1\,t_2^4\,t_3^4\,t_4^3-2\,t_1^2\,t_2^2\,t_3^5\,t_4^3-t_1\,t_2^5\,t_3^5\,t_4^3-t_1^2\,t_2^3\,t_3^6\,t_4^3\right)$	{1, 0, 1, 1}
$t_2^2\,t_3\,\left(t_1^2\,t_2\,t_3^2-t_1\,t_3^3-t_1^2\,t_2^2\,t_3^3-t_1^2\,t_2^3\,t_3^4+t_1^2\,t_2^2\,t_4-2\,t_1\,t_2\,t_3\,t_4-t_1^2\,t_2^3\,t_3\,t_4+t_2^2\,t_3\,t_4-t_1^3\,t_2\,t_3^2\,t_4-t_1^2\,t_2^4\,t_3^2\,t_4+t_1^2\,t_3^3\,t_4+t_2\,t_3^3\,t_4-t_1^3\,t_2^2\,t_3^3\,t_4+3\,t_1\,t_2^3\,t_3^3\,t_4+4\,t_1^2\,t_2\,t_3^4\,t_4-t_2^2\,t_3^4\,t_4+t_1^3\,t_2^3\,t_3^4\,t_4-t_1\,t_3^5\,t_4+t_1^2\,t_2^2\,t_3^5\,t_4+3\,t_1^2\,t_2^2\,t_3^2\,t_4^2-3\,t_1\,t_2^3\,t_3^2\,t_4^2-2\,t_1^2\,t_2^4\,t_3^2\,t_4^2-2\,t_1^2\,t_2^3\,t_3^3\,t_4^2+t_1\,t_2^5\,t_3^3\,t_4^2+t_1^2\,t_2^4\,t_3^4\,t_4^2+t_1^2\,t_2^3\,t_3^5\,t_4^2-t_2^3\,t_4^3+t_1\,t_2^4\,t_4^3-t_1^2\,t_2^3\,t_3\,t_4^3+2\,t_1\,t_2^2\,t_3^2\,t_4^3+3\,t_1\,t_2^3\,t_3^3\,t_4^3+2\,t_1^2\,t_2^5\,t_3^3\,t_4^3+t_1^2\,t_2\,t_3^4\,t_4^3-2\,t_2^2\,t_3^5\,t_4^3-t_1^2\,t_2^3\,t_3^6\,t_4^3\right)$	{1, 0, 2, 0}
$t_2\,t_3\,\left(t_1\,t_2^3+t_1^2\,t_2\,t_3+2\,t_1\,t_2^4\,t_3+2\,t_1^2\,t_2^2\,t_3^2-2\,t_2^3\,t_3^2+2\,t_1^3\,t_2^4\,t_3^2-t_1\,t_2^5\,t_3^2-2\,t_1\,t_2\,t_3^3-3\,t_1^2\,t_2^3\,t_3^3-2\,t_1^3\,t_2^5\,t_3^3-t_1\,t_2^2\,t_3^4-5\,t_1\,t_2^4\,t_3^4-4\,t_1^2\,t_2^4\,t_3\,t_4-t_1^2\,t_3^2\,t_4+2\,t_2\,t_3^2\,t_4-2\,t_1^3\,t_2^2\,t_3^2\,t_4+t_1\,t_2^3\,t_3^2\,t_4-4\,t_1^2\,t_2^5\,t_3^2\,t_4+t_1^2\,t_2\,t_3^3\,t_4+3\,t_2^2\,t_3^3\,t_4-3\,t_1^3\,t_2^3\,t_3^3\,t_4+9\,t_1\,t_2^4\,t_3^3\,t_4+t_1\,t_3^4\,t_4+6\,t_1^2\,t_2^2\,t_3^4\,t_4-2\,t_2^3\,t_3^4\,t_4+2\,t_1^3\,t_2^4\,t_3^4\,t_4+2\,t_1\,t_2^5\,t_3^4\,t_4-t_1\,t_2\,t_3^5\,t_4+t_1^2\,t_2^3\,t_3^5\,t_4-t_2^4\,t_3^5\,t_4+t_1^3\,t_2^5\,t_3^5\,t_4-t_1\,t_2^2\,t_3^6\,t_4-t_1^2\,t_2^4\,t_3^6\,t_4+t_1\,t_3\,t_4^2+2\,t_1^2\,t_2^2\,t_3\,t_4^2+t_2^3\,t_3\,t_4^2-t_1^3\,t_2^4\,t_3\,t_4^2+2\,t_1\,t_2^5\,t_3\,t_4^2+2\,t_1\,t_2\,t_3^2\,t_4^2+9\,t_1^2\,t_2^3\,t_3^2\,t_4^2-2\,t_2^4\,t_3^2\,t_4^2+2\,t_1^3\,t_2^5\,t_3^2\,t_4^2-t_3^3\,t_4^2+t_1^3\,t_2\,t_3^3\,t_4^2-8\,t_1\,t_2^2\,t_3^3\,t_4^2+t_1^2\,t_2^4\,t_3^3\,t_4^2-t_2^5\,t_3^3\,t_4^2+t_1^3\,t_2^6\,t_3^3\,t_4^2-t_1^2\,t_3^4\,t_4^2-8\,t_1\,t_2^3\,t_3^4\,t_4^2-5\,t_1^2\,t_2^5\,t_3^4\,t_4^2-t_1^2\,t_2\,t_3^5\,t_4^2+2\,t_2^2\,t_3^5\,t_4^2-2\,t_1^3\,t_2^3\,t_3^5\,t_4^2+2\,t_1\,t_2^4\,t_3^5\,t_4^2-t_1^2\,t_2^6\,t_3^5\,t_4^2+t_1^2\,t_2^2\,t_3^6\,t_4^2+t_1\,t_2^5\,t_3^6\,t_4^2-t_1\,t_2^3\,t_3\,t_4^3-t_1^2\,t_2\,t_3^2\,t_4^3-2\,t_1\,t_2^4\,t_3^2\,t_4^3-2\,t_1^2\,t_2^2\,t_3^3\,t_4^3+2\,t_2^3\,t_3^3\,t_4^3-2\,t_1^3\,t_2^4\,t_3^3\,t_4^3+t_1\,t_2^5\,t_3^3\,t_4^3+2\,t_1\,t_2^2\,t_3^4\,t_4^3+3\,t_1^2\,t_2^3\,t_3^4\,t_4^3+2\,t_1^2\,t_2^4\,t_3^5\,t_4^3-t_1\,t_2^3\,t_3^6\,t_4^3\right)$	{1, 1, 0, 1}
$t_2\,t_3\,\left(t_1^2\,t_2^3\,t_3-2\,t_1\,t_2^2\,t_3^2-t_1^2\,t_2^4\,t_3^2-t_1^2\,t_2^5\,t_3^3+t_1\,t_2^4\,t_3^4-t_1\,t_2^3\,t_4-t_1^2\,t_2\,t_3\,t_4+t_2^2\,t_3\,t_4-t_1^3\,t_2^3\,t_3\,t_4+t_1\,t_2^2\,t_3^2\,t_4+t_1^2\,t_2^2\,t_3^3\,t_4+t_2^3\,t_3^3\,t_4-t_1^3\,t_2^4\,t_3^3\,t_4+t_1^2\,t_2^4\,t_3^4\,t_4+2\,t_1\,t_2^3\,t_3^4\,t_4+6\,t_1^2\,t_2^3\,t_3^3\,t_4-t_2^4\,t_3^3\,t_4+t_1^3\,t_2^5\,t_3^3\,t_4-3\,t_1\,t_2^2\,t_3^4\,t_4+t_1^2\,t_2^4\,t_3^4\,t_4-2\,t_1\,t_2^3\,t_3^5\,t_4-t_1^2\,t_2^5\,t_3^5\,t_4+t_1\,t_2^2\,t_3\,t_4^2+2\,t_1^2\,t_2^4\,t_3\,t_4^2-t_2^3\,t_3\,t_4^2+t_1^3\,t_2^2\,t_3^2\,t_4^2-4\,t_1\,t_2^3\,t_3^2\,t_4^2-2\,t_1^2\,t_2^5\,t_3^2\,t_4^2-4\,t_1\,t_2^4\,t_3^3\,t_4^2-t_1^2\,t_2^6\,t_3^3\,t_4^2-3\,t_1^2\,t_2^2\,t_3^4\,t_4^2+2\,t_2^3\,t_3^4\,t_4^2-2\,t_1^3\,t_2^4\,t_3^4\,t_4^2+t_1\,t_2^5\,t_3^4\,t_4^2+2\,t_1\,t_2\,t_3^5\,t_4^2+2\,t_1^2\,t_2^3\,t_3^5\,t_4^2+t_1^2\,t_2^4\,t_3^6\,t_4^2-t_1^2\,t_2^2\,t_3^3\,t_4^3+t_1^2\,t_2^5\,t_3^4\,t_4^3-t_1\,t_2^4\,t_3^5\,t_4^3\right)$	{1, 1, 1, 0}

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$P_{A4}(t_i)$	Character
$-t_2\,t_3^2\left(t_1\,t_2^3-t_1\,t_2\,t_4-t_1^2\,t_2^3\,t_4-t_1\,t_2^2\,t_3\,t_4-2\,t_1^2\,t_2^4\,t_3\,t_4+2\,t_1\,t_2^3\,t_3^2\,t_4-t_1^2\,t_2^5\,t_3^2\,t_4+t_1\,t_2^4\,t_3^3\,t_4+t_1\,t_2^4\,t_4^2+t_1\,t_3\,t_4^2+t_1^2\,t_2^2\,t_3\,t_4^2+t_1\,t_2^5\,t_3\,t_4^2+2\,t_1^2\,t_2^3\,t_3^2\,t_4^2-t_2^4\,t_3^2\,t_4^2+t_1^3\,t_2^5\,t_3^2\,t_4^2-2\,t_1\,t_2^2\,t_3^3\,t_4^2-t_1^2\,t_2^4\,t_3^3\,t_4^2-t_1^2\,t_2^5\,t_3^4\,t_4^2+t_1\,t_2^4\,t_3^5\,t_4^2-t_1\,t_2^3\,t_3\,t_4^3\right)$	{1, 2, 0, 0}
$t_2\,t_3$ $\left(t_1^2\,t_2^2-t_1\,t_2\,t_3+t_1^2\,t_2^3\,t_3-3\,t_1\,t_2^2\,t_3^2-t_1^2\,t_2^4\,t_3^2+t_2\,t_3^3-t_1^3\,t_2^2\,t_3^3-t_1\,t_2^3\,t_3^3-2\,t_1^2\,t_2^5\,t_3^3+t_2^2\,t_3^4-t_1^3\,t_2^3\,t_3^4+2\,t_1\,t_2^4\,t_3^4+t_1^2\,t_2^2\,t_3^5+t_1\,t_2^5\,t_3^5-2\,t_1\,t_2^3\,t_4-2\,t_1^2\,t_2\,t_3\,t_4+2\,t_2^2\,t_3\,t_4-2\,t_1^3\,t_2^3\,t_3\,t_4+2\,t_1\,t_2^3\,t_4+3\,t_1^2\,t_2^2\,t_3^2\,t_4+t_2^3\,t_3^2\,t_4-t_1^3\,t_2^4\,t_3^2\,t_4+2\,t_1\,t_2^5\,t_3^2\,t_4+2\,t_1\,t_2\,t_3^3\,t_4+8\,t_1^2\,t_2^3\,t_3^3\,t_4-t_2^4\,t_3^3\,t_4+t_1^3\,t_2^5\,t_3^3\,t_4+t_1\,t_2^6\,t_3^3\,t_4-t_3^4\,t_4+t_1^3\,t_2\,t_3^4\,t_4-4\,t_1\,t_2^2\,t_3^4\,t_4+3\,t_1^2\,t_2^4\,t_3^4\,t_4-t_2^5\,t_3^4\,t_4+t_1^3\,t_2^6\,t_3^4\,t_4-4\,t_1\,t_2^3\,t_3^5\,t_4-2\,t_1^2\,t_2^5\,t_3^5\,t_4-t_1\,t_2^4\,t_3^6\,t_4+t_1\,t_2\,t_4^2+t_1^2\,t_2^3\,t_4^2-t_3\,t_4^2+t_1^3\,t_2\,t_3\,t_4^2+t_1\,t_2^2\,t_3\,t_4^2+2\,t_1^2\,t_2^4\,t_3\,t_4^2-t_1^2\,t_2^2\,t_3^2\,t_4^2-2\,t_2^2\,t_3^3\,t_4^2+2\,t_1^3\,t_2^2\,t_3^3\,t_4^2-5\,t_1\,t_2^3\,t_3^2\,t_4^2-4\,t_1^2\,t_2\,t_3^3\,t_4^2+t_2^2\,t_3^3\,t_4^2-t_1^3\,t_2^3\,t_3^3\,t_4^2-5\,t_1\,t_2^4\,t_3^3\,t_4^2-2\,t_1^2\,t_2^6\,t_3^3\,t_4^2+t_1\,t_3^4\,t_4^2-4\,t_1^2\,t_2^2\,t_3^4\,t_4^2+3\,t_2^3\,t_3^4\,t_4^2-3\,t_1^3\,t_2^4\,t_3^4\,t_4^2+t_1\,t_2^5\,t_3^4\,t_4^2+2\,t_1\,t_2\,t_3^5\,t_4^2+2\,t_1^2\,t_2^3\,t_3^5\,t_4^2+t_2^4\,t_3^5\,t_4^2-t_1^3\,t_2^5\,t_3^5\,t_4^2+t_1\,t_2^6\,t_3^5\,t_4^2+t_1\,t_2^2\,t_3^6\,t_4^2+2\,t_1^2\,t_2^4\,t_3^6\,t_4^2-t_1^2\,t_2^2\,t_3\,t_4^3+t_1\,t_2\,t_3^2\,t_4^3-t_1^2\,t_2^3\,t_3^2\,t_4^3+3\,t_1\,t_2^2\,t_3^3\,t_4^3+t_1^2\,t_2^4\,t_3^3\,t_4^3-t_2\,t_3^4\,t_4^3+t_1^3\,t_2^2\,t_3^4\,t_4^3+t_1\,t_2^3\,t_3^4\,t_4^3+2\,t_1^2\,t_2^5\,t_3^4\,t_4^3-t_2^2\,t_3^5\,t_4^3+t_1^3\,t_2^3\,t_3^5\,t_4^3-2\,t_1\,t_2^4\,t_3^5\,t_4^3-t_1^2\,t_2^2\,t_3^6\,t_4^3-t_1\,t_2^5\,t_3^6\,t_4^3\right)$	{2, 0, 0, 1}
$-t_2\,t_3\left(t_1\,t_2^3\,t_3+t_1^2\,t_2\,t_3^2-t_2^2\,t_3^2+t_1^3\,t_2^3\,t_3^2+t_1\,t_2^4\,t_3^2-t_2^3\,t_3^3+t_1^3\,t_2^4\,t_3^3-t_1\,t_2^5\,t_3^3-t_1\,t_2\,t_3^4-2\,t_1^2\,t_2^3\,t_3^4-t_1^2\,t_2^4\,t_3^5+t_1^2\,t_2^2\,t_4-2\,t_1\,t_2\,t_3\,t_4-t_1^2\,t_2^3\,t_3\,t_4+t_3^2\,t_4-t_1^3\,t_2^2\,t_3^2\,t_4-2\,t_1\,t_2^2\,t_3^2\,t_4-4\,t_1^2\,t_2^4\,t_3^2\,t_4+2\,t_2\,t_3^3\,t_4-2\,t_1^3\,t_2^2\,t_3^3\,t_4+4\,t_1\,t_2^3\,t_3^3\,t_4-t_1^2\,t_2^5\,t_3^3\,t_4+2\,t_1^2\,t_2\,t_3^4\,t_4+5\,t_1\,t_2^4\,t_3^4\,t_4+t_1^2\,t_2^6\,t_3^4\,t_4+2\,t_1^2\,t_2^2\,t_3^5\,t_4-t_2^3\,t_3^5\,t_4+t_1^3\,t_2^4\,t_3^5\,t_4-t_1\,t_2^3\,t_4^2+t_2^2\,t_3\,t_4^2-t_1^3\,t_2^3\,t_3\,t_4^2+t_1\,t_2^4\,t_3\,t_4^2+5\,t_1^2\,t_2^2\,t_3^2\,t_4^2-t_2^3\,t_3^2\,t_4^2+t_1^3\,t_2^4\,t_3^2\,t_4^2+t_1\,t_2^5\,t_3^2\,t_4^2-2\,t_1\,t_2\,t_3^3\,t_4^2+4\,t_1^2\,t_2^3\,t_3^3\,t_4^2-t_2^4\,t_3^3\,t_4^2+t_1^3\,t_2^5\,t_3^3\,t_4^2-t_1\,t_2^6\,t_3^3\,t_4^2-6\,t_1\,t_2^2\,t_3^4\,t_4^2-2\,t_1^2\,t_2^4\,t_3^4\,t_4^2+t_2\,t_3^5\,t_4^2-t_1^3\,t_2^2\,t_3^5\,t_4^2-2\,t_1\,t_2^3\,t_3^5\,t_4^2-2\,t_1^2\,t_2^5\,t_3^5\,t_4^2-t_1\,t_2^3\,t_3^2\,t_4^3-t_1^2\,t_2\,t_3^3\,t_4^3+t_2^2\,t_3^3\,t_4^3-t_1^3\,t_2^3\,t_3^3\,t_4^3-t_1\,t_2^4\,t_3^3\,t_4^3+t_2^3\,t_3^4\,t_4^3-t_1^3\,t_2^4\,t_3^4\,t_4^3+t_1\,t_2^5\,t_3^4\,t_4^3+2\,t_1^2\,t_2^3\,t_3^5\,t_4^3+t_1^2\,t_2^4\,t_3^6\,t_4^3\right)$	{2, 0, 1, 0}
$-t_2\,t_3\left(t_1^2\,t_2^2\,t_3-t_1\,t_2\,t_3^2-t_1\,t_2^2\,t_3^2-t_1^2\,t_2^4\,t_3^2-t_1^2\,t_2^5\,t_3^3+t_1\,t_2^4\,t_3^5-t_1\,t_2^2\,t_4+t_2\,t_3\,t_4-t_1^3\,t_2^2\,t_3\,t_4+t_1\,t_2^3\,t_3\,t_4+3\,t_1^2\,t_2^2\,t_3^2\,t_4+2\,t_1\,t_2^5\,t_3^2\,t_4+3\,t_1^2\,t_2^3\,t_3^3\,t_4-t_2^4\,t_3^3\,t_4+t_1^3\,t_2^5\,t_3^3\,t_4-t_1\,t_2^2\,t_3^5\,t_4-t_1\,t_2^3\,t_3^6\,t_4+t_1\,t_2\,t_3\,t_4^2+2\,t_1^2\,t_2^3\,t_3\,t_4^2-t_2^3\,t_4^2+t_1^3\,t_2^2\,t_3^2\,t_4^2-2\,t_1\,t_2^2\,t_3^2\,t_4^2+t_1^2\,t_2^4\,t_3^2\,t_4^2-t_1^2\,t_3^3\,t_4^2-4\,t_1\,t_2^3\,t_3^3\,t_4^2-t_1^2\,t_2^5\,t_3^3\,t_4^2-t_1^2\,t_2\,t_3^4\,t_4^2-t_1\,t_2^3\,t_3^4\,t_4^2-t_1^2\,t_2^4\,t_3^5\,t_4^2-t_1^3\,t_2^5\,t_3^5\,t_4^2-2\,t_1\,t_2^4\,t_3^4\,t_4^2-t_1^2\,t_2^6\,t_3^4\,t_4^2+t_2^3\,t_3^5\,t_4^2-t_1^3\,t_2^4\,t_3^5\,t_4^2+t_1\,t_2^5\,t_3^5\,t_4^2-t_1^2\,t_2^2\,t_3^3\,t_4^3+t_1\,t_2\,t_3^4\,t_4^3+t_1^2\,t_2^3\,t_3^4\,t_4^3+t_2^4\,t_3^4\,t_4^3+t_1^3\,t_2^3\,t_3^4\,t_4^3-t_1\,t_2^4\,t_3^5\,t_4^3+t_1^2\,t_2^2\,t_3^6\,t_4^3\right)$	{2, 1, 0, 0}
$t_2\,t_3$ $\left(t_1\,t_2^2\,t_3-t_2\,t_3^2+t_1^3\,t_2^2\,t_3^2+t_1\,t_2^4\,t_3^3-t_2^3\,t_3^4+t_1^3\,t_2^4\,t_3^4-t_1\,t_2^5\,t_3^4-t_1^2\,t_2^3\,t_3^5-t_1\,t_2\,t_3^2\,t_4-2\,t_1^2\,t_2^3\,t_3^2\,t_4+t_3^3\,t_4-t_1^3\,t_2\,t_3^3\,t_4+t_1\,t_2^2\,t_3^3\,t_4+2\,t_1^2\,t_2^4\,t_3^3\,t_4+2\,t_1\,t_2^3\,t_3^4\,t_4+2\,t_1\,t_2^4\,t_3^5\,t_4-t_1\,t_2^2\,t_4^2+t_2\,t_3\,t_4^2-t_1^3\,t_2^2\,t_3\,t_4^2+t_1\,t_2^3\,t_3\,t_4^2+2\,t_1^2\,t_2^4\,t_3\,t_4^2-t_2^2\,t_3^2\,t_4^2+t_1^3\,t_2^3\,t_3^2\,t_4^2-t_1\,t_3^3\,t_4^2+2\,t_1^2\,t_2^2\,t_3^3\,t_4^2+t_1\,t_2^5\,t_3^3\,t_4^2-t_1\,t_2\,t_3^4\,t_4^2+t_1^2\,t_2^3\,t_3^4\,t_4^2-t_2^4\,t_3^4\,t_4^2+t_1^3\,t_2^5\,t_3^4\,t_4^2-t_1\,t_2^6\,t_3^4\,t_4^2-t_1\,t_2^2\,t_3^5\,t_4^2-t_1^2\,t_2^4\,t_3^5\,t_4^2-t_1\,t_2^3\,t_3^6\,t_4^2-t_1\,t_2^2\,t_3^2\,t_4^3+t_2\,t_3^3\,t_4^3-t_1^3\,t_2^3\,t_3^3\,t_4^3-t_1\,t_2^4\,t_3^3\,t_4^3+t_2^3\,t_3^5\,t_4^3-t_1^3\,t_2^4\,t_3^5\,t_4^3+t_1\,t_2^5\,t_3^5\,t_4^3+t_1^2\,t_2^3\,t_3^6\,t_4^3\right)$	{3, 0, 0, 0}
$-t_2^3\,t_3^2\left(t_1^2\,t_2-t_1^2\,t_3\,t_4-t_2^2\,t_3^2\,t_4+t_1^3\,t_2^3\,t_3^2\,t_4-t_1^2\,t_2^2\,t_3^3\,t_4+t_1\,t_2\,t_3\,t_4^2+t_1\,t_2^2\,t_3^2\,t_4^2-t_1^2\,t_2\,t_3\,t_4^3\right)$	{0, 0, 0, 4}
$t_1\,t_2^3\,t_3^2\,t_4\left(t_1\,t_2+t_1\,t_2^3\,t_3^2-t_2^2\,t_3^3-t_1\,t_3\,t_4\right)$	{0, 0, 1, 3}
$t_2^3\,t_3^2\left(t_1^2\,t_2\,t_3-t_1^2\,t_2^3\,t_4-t_2\,t_3^2\,t_4+t_1^3\,t_2^2\,t_3^2\,t_4-t_1\,t_2^3\,t_3^2\,t_4^2+t_1^2\,t_2\,t_3^3\,t_4-t_1^2\,t_2\,t_4^2+t_1\,t_3\,t_4^2+2\,t_1\,t_2\,t_3^2\,t_4^2+t_1\,t_2^2\,t_3^3\,t_4^2-t_1^2\,t_2\,t_3^2\,t_4^3\right)$	{0, 1, 0, 3}

$P_{A4}(t_i)$	Character
$-t_1 t_2^3 t_3^3 t_4 \left(t_1 t_2 + t_1 t_2^2 t_3 - t_2 t_3^2 - t_1 t_3 t_4\right)$	{0, 1, 1, 2}
$t_1 t_2^3 t_3^3 t_4 \left(t_2^2 t_3 + t_1 t_2 t_4 - t_3 t_4 - t_2 t_3^2 t_4\right)$	{0, 2, 0, 2}
$t_2^2 t_3^2 \left(t_1 t_2 + t_1^2 t_2^3 - t_1^2 t_2 t_4 - t_1 t_3 t_4 - 2 t_1^2 t_2^2 t_3 t_4 - t_2^3 t_3 t_4 + t_1^3 t_2^4 t_3 t_4 - 2 t_1^2 t_2^3 t_3^2 t_4 + t_1 t_2^2 t_3^3 t_4 - t_1^2 t_2^4 t_3^3 t_4 + t_1 t_2^3 t_3^4 t_4 + t_1 t_2^2 t_4^2 + t_1^2 t_3 t_4^2 + 2 t_1 t_2^3 t_3 t_4^2 + t_1^2 t_2 t_3^2 t_4^2 + t_1 t_2^4 t_3^2 t_4^2 - t_2^3 t_3^3 t_4^2 + t_1^3 t_2^4 t_3^3 t_4^2 - t_1^2 t_2^3 t_3^4 t_4^2 - t_1 t_2 t_3 t_4^3 - t_1^2 t_2^3 t_3 t_4^3\right)$	{1, 0, 0, 3}
$-t_2^3 t_3^2 \left(t_1^2 t_2 t_3^2 + t_1 t_4 + t_1^2 t_2^2 t_4 + t_1^2 t_2^3 t_3 t_4 - t_3^2 t_4 + t_1^3 t_2 t_3^2 t_4 - 2 t_1 t_2^2 t_3^2 t_4 - 2 t_1^2 t_3^3 t_4 - t_1 t_2^3 t_3^3 t_4 - t_1^2 t_4^2 - 2 t_1^2 t_2 t_3 t_4^2 + 2 t_1 t_3^2 t_4^2 + 2 t_1 t_2 t_3^3 t_4^2 + t_1^2 t_2^3 t_3^3 t_4^2 - t_1^2 t_2 t_3^3 t_4^3\right)$	{1, 0, 1, 2}
$t_1 t_2^3 t_3^4 t_4 \left(2 t_1 t_2 - t_3 - t_1 t_3 t_4\right)$	{1, 0, 2, 1}
$-t_2^2 t_3^2 \left(t_1 t_2 t_3 + t_1^2 t_2^3 t_3 - t_1^2 t_2 t_3 t_4 - t_2^2 t_3 t_4 + t_1^3 t_2^3 t_3 t_4 - t_1 t_2^4 t_3 t_4 - t_1 t_3^2 t_4 - 4 t_1^2 t_2^2 t_3^2 t_4 + t_1 t_2 t_3^3 t_4 - t_1^2 t_2^3 t_4^2 + 3 t_1 t_2^2 t_3 t_4^2 + t_1^2 t_3^2 t_4^2 + 3 t_1 t_2^3 t_3^2 t_4^2 + t_1^2 t_2 t_3^3 t_4^2 - t_2^2 t_3^3 t_4^2 + t_1^3 t_2^3 t_3^3 t_4^2 - t_1^2 t_2^2 t_3^4 t_4^2 - t_1 t_2 t_3^2 t_4^3 - t_1^2 t_2^3 t_3^2 t_4^3\right)$	{1, 1, 0, 2}
$t_1 t_2^3 t_3^3 t_4 \left(1 + 2 t_1 t_2^2 - 2 t_2 t_3 - t_2^2 t_3^2 - t_1 t_4 - 2 t_1 t_2 t_3 t_4 + 2 t_3^2 t_4 + t_1 t_2^2 t_3^2 t_4\right)$	{1, 1, 1, 1}
$-t_1 t_2^4 t_3^3 t_4 \left(t_2 + t_1 t_2 t_4 - 2 t_3 t_4\right)$	{1, 2, 0, 1}
$-t_2^2 t_3^2 \left(t_1 t_2^2 - t_1^2 t_2^2 t_3^3 - t_1 t_4 - t_1^2 t_2^2 t_4 - 2 t_1 t_2 t_3 t_4 - 2 t_1^2 t_2^3 t_3 t_4 + t_3^2 t_4 - t_1^3 t_2 t_3^2 t_4 + 2 t_1 t_2^2 t_3^2 t_4 - t_1^2 t_2^4 t_3^2 t_4 + t_1^2 t_3^3 t_4 + 2 t_1 t_2^3 t_3^3 t_4 + t_1^2 t_2 t_3^4 t_4 + t_1 t_2^4 t_3^4 t_4 + t_1^2 t_4^2 + t_1 t_2^3 t_4^2 + 2 t_1^2 t_2 t_3 t_4^2 + t_1 t_2^4 t_3 t_4^2 - t_1 t_3^2 t_4^2 + 2 t_1^2 t_2^2 t_3^2 t_4^2 - t_2^3 t_3^2 t_4^2 + t_1^3 t_2^4 t_3^2 t_4^2 - 2 t_1 t_2 t_3^3 t_4^2 - 2 t_1^2 t_2^3 t_3^3 t_4^2 - t_1 t_2^2 t_4^3 t_4^2 - t_1^2 t_2^4 t_3^4 t_4^2 - t_1 t_2^2 t_3 t_4^3 + t_1^2 t_2^2 t_3^4 t_4^3\right)$	{2, 0, 0, 2}
$t_2^2 t_3^2 \left(t_1 t_2 t_3^2 + t_1^2 t_2^3 t_3^2 + t_1 t_2^2 t_4 - t_2 t_3 t_4 + t_1^3 t_2^2 t_3 t_4 - t_1 t_2^3 t_3 t_4 - 3 t_1^2 t_2 t_3^2 t_4 - t_1 t_2^4 t_3^2 t_4 - 3 t_1^2 t_2^2 t_3 t_4 + t_1 t_2 t_3^4 t_4 - t_1^2 t_2^2 t_4^2 + t_1 t_2 t_3 t_4^2 - t_1^2 t_2^3 t_3 t_4^2 + 4 t_1 t_2^2 t_3^2 t_4^2 + t_1^2 t_2^4 t_3^2 t_4^2 + t_1^2 t_3^3 t_4^2 - t_2 t_3^3 t_4^2 + t_1^3 t_2^2 t_3^3 t_4^2 + t_1 t_2^3 t_3^3 t_4^2 - t_1 t_2 t_3^4 t_4^3 - t_1^2 t_2^3 t_3^3 t_4^3\right)$	{2, 0, 1, 1}
$-t_1 t_2^3 t_3^3 t_4 \left(t_1 t_2 + t_1 t_2^2 t_3 - t_2 t_3^2 - t_1 t_3 t_4\right)$	{2, 0, 2, 0}
$t_2^2 t_3^3 \left(t_1 t_2^2 - t_1 t_4 - 2 t_1^2 t_2^2 t_4 - 2 t_1^2 t_2^3 t_3 t_4 + 2 t_1 t_2^2 t_3^2 t_4 + t_1 t_2^3 t_3^3 t_4 + t_1^2 t_4^2 + 2 t_1 t_2^3 t_4^2 + 2 t_1^2 t_2 t_3 t_4^2 - t_2^2 t_3 t_4^2 + t_1^3 t_2 t_3^2 t_4^2 - t_1 t_2^2 t_3^3 t_4^2 - t_1^2 t_2^3 t_3^3 t_4^2 - t_1 t_2^3 t_3 t_4^3 - t_1^2 t_2^2 t_3^2 t_4^3\right)$	{2, 1, 0, 1}
$t_1 t_2^3 t_3^3 t_4 \left(t_2^2 t_3 + t_1 t_2 t_4 - t_3 t_4 - t_2 t_3^2 t_4\right)$	{2, 1, 1, 0}
$-t_2^2 t_3^2 \left(t_1 t_2 t_3^3 + t_1^2 t_2^3 t_3^3 + t_1 t_2 t_4 - t_3 t_4 + t_1^3 t_2 t_3 t_4 - t_1^2 t_2^2 t_3 t_4 - t_1 t_2^3 t_3^2 t_4 - 2 t_1^2 t_2 t_3^3 t_4 - t_1 t_2^4 t_3^3 t_4 - t_1^2 t_2^2 t_3^4 t_4 - t_1 t_3 t_4^2 - t_1^2 t_2^2 t_3 t_4^2 + 2 t_1 t_2 t_3^2 t_4^2 - t_3^3 t_4^2 + t_1^3 t_2 t_3^3 t_4^2 + 2 t_1 t_2^2 t_3^3 t_4^2 + t_1^2 t_2^4 t_3^3 t_4^2 + t_1 t_2^3 t_3^4 t_4^2 - t_1 t_2 t_3^4 t_4^3 - t_1^2 t_2^3 t_3^4 t_4^3\right)$	{3, 0, 0, 1}
$-t_2^2 t_3^3 \left(t_1 t_2^2 t_3 - t_1^2 t_2 t_4 - 2 t_1^2 t_2^2 t_3 t_4 - t_1^2 t_2^3 t_3 t_4 + t_1 t_2^2 t_3^2 t_4 + t_1 t_2^2 t_4^2 + t_1^2 t_3 t_4^2 - t_2 t_3 t_4^2 + t_1^3 t_2 t_3 t_4^2 + t_1 t_2^3 t_3 t_4^2 - t_1 t_2^2 t_3^3 t_4^2\right)$	{3, 0, 1, 0}
$-t_1 t_2^2 t_3^3 t_4 \left(t_2^3 t_3^2 + t_1 t_2 t_4 - t_3 t_4 - t_2^2 t_3^3 t_4\right)$	{3, 1, 0, 0}
$t_2^2 t_3^3 \left(t_1 t_2^2 t_3^2 - t_1^2 t_2 t_3 t_4 - t_1^2 t_2^2 t_3^2 t_4 + t_1 t_2 t_4^2 - t_3 t_4^2 + t_1^3 t_2 t_3 t_4^2 + t_1 t_2^3 t_3^2 t_4^2 - t_1 t_2^2 t_3^3 t_4^3\right)$	{4, 0, 0, 0}

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E Generating function for dimensions of A_4 irreps

$$
\begin{aligned}
&\big(1 + 3\,t_2 - 10\,t_1\,t_2 + 5\,t_1^2\,t_2 - t_1^3\,t_2 + t_2^2 - 5\,t_1\,t_2^2 + 10\,t_1^2\,t_2^2 - 3\,t_1^3\,t_2^2 - t_1^3\,t_2^3 + 3\,t_3 - 5\,t_1\,t_3 + t_1^2\,t_3 - 16\,t_2\,t_3 + \\
&\quad 10\,t_1\,t_2\,t_3 + 8\,t_1^2\,t_2\,t_3 - 3\,t_1^3\,t_2\,t_3 - 7\,t_2^2\,t_3 + 70\,t_1\,t_2^2\,t_3 - 69\,t_1^2\,t_2^2\,t_3 + 16\,t_1^3\,t_2^2\,t_3 - 21\,t_1\,t_2^3\,t_3 + 7\,t_1^3\,t_2^3\,t_3 + \\
&\quad 7\,t_1\,t_2^4\,t_3 - t_1\,t_2^5\,t_3 + t_3^2 - 5\,t_1\,t_3^2 + 3\,t_1^2\,t_3^2 - 7\,t_2\,t_3^2 + 35\,t_1\,t_2\,t_3^2 - 16\,t_1^2\,t_2\,t_3^2 - t_1^3\,t_2\,t_3^2 + 42\,t_2^2\,t_3^2 - 105\,t_1\,t_2^2\,t_3^2 + \\
&\quad 28\,t_1^2\,t_2^2\,t_3^2 + 7\,t_1^3\,t_2^2\,t_3^2 - 7\,t_2^3\,t_3^2 - 28\,t_1\,t_2^3\,t_3^2 + 105\,t_1^2\,t_2^3\,t_3^2 - 42\,t_1^3\,t_2^3\,t_3^2 + t_2^4\,t_3^2 + 16\,t_1\,t_2^4\,t_3^2 - 35\,t_1^2\,t_2^4\,t_3^2 + \\
&\quad 7\,t_1^3\,t_2^4\,t_3^2 - 3\,t_1\,t_2^5\,t_3^2 + 5\,t_1^2\,t_2^5\,t_3^2 - t_1^3\,t_2^5\,t_3^2 + t_1^2\,t_3^3 - 7\,t_1^2\,t_2\,t_3^3 - 7\,t_2^2\,t_3^3 + 21\,t_1^2\,t_2^2\,t_3^3 - 16\,t_2^3\,t_3^3 + 69\,t_1\,t_2^3\,t_3^3 - \\
&\quad 70\,t_1^2\,t_2^3\,t_3^3 + 7\,t_1^3\,t_2^3\,t_3^3 + 3\,t_2^4\,t_3^3 - 8\,t_1\,t_2^4\,t_3^3 - 10\,t_1^2\,t_2^4\,t_3^3 + 16\,t_1^3\,t_2^4\,t_3^3 - t_1\,t_2^5\,t_3^3 + 5\,t_1^2\,t_2^5\,t_3^3 - 3\,t_1^3\,t_2^5\,t_3^3 + \\
&\quad t_2^2\,t_3^4 + 3\,t_2^3\,t_3^4 - 10\,t_1\,t_2^3\,t_3^4 + 5\,t_1^2\,t_2^3\,t_3^4 - t_1^3\,t_2^3\,t_3^4 + t_2^4\,t_3^4 - 5\,t_1\,t_2^4\,t_3^4 + 10\,t_1^2\,t_2^4\,t_3^4 - 3\,t_1^3\,t_2^4\,t_3^4 - t_1^3\,t_2^5\,t_3^4 - \\
&\quad t_1\,t_4 - 5\,t_2\,t_4 + 7\,t_1\,t_2\,t_4 - 5\,t_2^2\,t_4 + 24\,t_1\,t_2^2\,t_4 - 24\,t_1^2\,t_2^2\,t_4 + 5\,t_1^3\,t_2^2\,t_4 - 7\,t_1^2\,t_2^3\,t_4 + 5\,t_1^3\,t_2^3\,t_4 + t_1^2\,t_2^4\,t_4 - \\
&\quad 10\,t_3\,t_4 + 7\,t_1\,t_3\,t_4 + 10\,t_2\,t_3\,t_4 + 50\,t_1\,t_2\,t_3\,t_4 - 50\,t_1^2\,t_2\,t_3\,t_4 + 10\,t_1^3\,t_2\,t_3\,t_4 + 35\,t_2^2\,t_3\,t_4 - 161\,t_1\,t_2^2\,t_3\,t_4 + \\
&\quad 93\,t_1^2\,t_2^2\,t_3\,t_4 - 10\,t_1^3\,t_2^2\,t_3\,t_4 - t_1\,t_2^3\,t_3\,t_4 + 75\,t_1^2\,t_2^3\,t_3\,t_4 - 35\,t_1^3\,t_2^3\,t_3\,t_4 - 14\,t_1^2\,t_2^4\,t_3\,t_4 + t_1^2\,t_2^5\,t_3\,t_4 - \\
&\quad 5\,t_3^2\,t_4 + 24\,t_1\,t_3^2\,t_4 - 10\,t_1^2\,t_3^2\,t_4 + 70\,t_2\,t_3^2\,t_4 - 161\,t_1\,t_2\,t_3^2\,t_4 + 45\,t_1^2\,t_2\,t_3^2\,t_4 + 5\,t_1^3\,t_2\,t_3^2\,t_4 - 105\,t_2^2\,t_3^2\,t_4 + \\
&\quad 100\,t_1\,t_2^2\,t_3^2\,t_4 + 141\,t_1^2\,t_2^2\,t_3^2\,t_4 - 70\,t_1^3\,t_2^2\,t_3^2\,t_4 + 217\,t_1\,t_2^3\,t_3^2\,t_4 - 364\,t_1^2\,t_2^3\,t_3^2\,t_4 + 105\,t_1^3\,t_2^3\,t_3^2\,t_4 - \\
&\quad 70\,t_1\,t_2^4\,t_3^2\,t_4 + 75\,t_1^2\,t_2^4\,t_3^2\,t_4 + 10\,t_1\,t_2^5\,t_3^2\,t_4 - 7\,t_1^2\,t_2^5\,t_3^2\,t_4 - 5\,t_1^2\,t_3^3\,t_4 - 21\,t_2^2\,t_3^3\,t_4 - t_1\,t_2\,t_3^3\,t_4 + \\
&\quad 35\,t_1^2\,t_2\,t_3^3\,t_4 - 28\,t_2^2\,t_3^3\,t_4 + 217\,t_1\,t_2^2\,t_3^3\,t_4 - 210\,t_1^2\,t_2^2\,t_3^3\,t_4 + 21\,t_1^3\,t_2^2\,t_3^3\,t_4 + 69\,t_2^3\,t_3^3\,t_4 - 266\,t_1\,t_2^3\,t_3^3\,t_4 + \\
&\quad 141\,t_1^2\,t_2^3\,t_3^3\,t_4 + 28\,t_1^3\,t_2^3\,t_3^3\,t_4 - 10\,t_2^4\,t_3^3\,t_4 + 15\,t_1\,t_2^4\,t_3^3\,t_4 + 93\,t_1^2\,t_2^4\,t_3^3\,t_4 - 69\,t_1^3\,t_2^4\,t_3^3\,t_4 + 5\,t_1\,t_2^5\,t_3^3\,t_4 - \\
&\quad 24\,t_1^2\,t_2^5\,t_3^3\,t_4 + 10\,t_1^3\,t_2^5\,t_3^3\,t_4 + 7\,t_2\,t_3^4\,t_4 + 16\,t_2^2\,t_3^4\,t_4 - 70\,t_1\,t_2^2\,t_3^4\,t_4 + 35\,t_1^2\,t_2^2\,t_3^4\,t_4 - 7\,t_1^3\,t_2^2\,t_3^4\,t_4 - \\
&\quad 8\,t_2^3\,t_3^4\,t_4 + 15\,t_1\,t_2^3\,t_3^4\,t_4 + 45\,t_1^2\,t_2^3\,t_3^4\,t_4 - 16\,t_1^3\,t_2^3\,t_3^4\,t_4 - 5\,t_2^4\,t_3^4\,t_4 + 25\,t_1\,t_2^4\,t_3^4\,t_4 - 50\,t_1^2\,t_2^4\,t_3^4\,t_4 + \\
&\quad 8\,t_1^3\,t_2^4\,t_3^4\,t_4 + 5\,t_1^3\,t_2^5\,t_3^4\,t_4 - t_2\,t_3^5\,t_4 - 3\,t_2^2\,t_3^5\,t_4 + 10\,t_1\,t_2^2\,t_3^5\,t_4 - 5\,t_1^2\,t_2^2\,t_3^5\,t_4 + t_1^3\,t_2^2\,t_3^5\,t_4 - t_2^3\,t_3^5\,t_4 + \\
&\quad 5\,t_1\,t_2^3\,t_3^5\,t_4 - 10\,t_1^2\,t_2^3\,t_3^5\,t_4 + 3\,t_1^3\,t_2^3\,t_3^5\,t_4 + t_1^3\,t_2^4\,t_3^5\,t_4 + t_2\,t_4^2 + 3\,t_2^2\,t_4^2 - 10\,t_1\,t_2^2\,t_4^2 + 5\,t_1^2\,t_2^2\,t_4^2 - t_1^3\,t_2^2\,t_4^2 + \\
&\quad t_2^3\,t_4^2 - 5\,t_1\,t_2^3\,t_4^2 + 10\,t_1^2\,t_2^3\,t_4^2 - 3\,t_1^3\,t_2^3\,t_4^2 - t_1^3\,t_2^4\,t_4^2 + 5\,t_3\,t_4^2 + 8\,t_2\,t_3\,t_4^2 - 50\,t_1\,t_2\,t_3\,t_4^2 + 25\,t_1^2\,t_2\,t_3\,t_4^2 - \\
&\quad 5\,t_1^3\,t_2\,t_3\,t_4^2 - 16\,t_2^2\,t_3\,t_4^2 + 45\,t_1\,t_2^2\,t_3\,t_4^2 + 15\,t_1^2\,t_2^2\,t_3\,t_4^2 - 8\,t_1^3\,t_2^2\,t_3\,t_4^2 - 7\,t_2^3\,t_3\,t_4^2 + 35\,t_1\,t_2^3\,t_3\,t_4^2 - \\
&\quad 70\,t_1^2\,t_2^3\,t_3\,t_4^2 + 16\,t_1^3\,t_2^3\,t_3\,t_4^2 + 7\,t_1^3\,t_2^4\,t_3\,t_4^2 + 10\,t_2^3\,t_4^2 - 24\,t_1\,t_2^3\,t_4^2 + 5\,t_1^2\,t_2^3\,t_4^2 - 69\,t_2\,t_3^2\,t_4^2 + 93\,t_1\,t_2\,t_3^2\,t_4^2 + \\
&\quad 15\,t_1^2\,t_2\,t_3^2\,t_4^2 - 10\,t_1^3\,t_2\,t_3^2\,t_4^2 + 28\,t_2^2\,t_3^2\,t_4^2 + 141\,t_1\,t_2^2\,t_3^2\,t_4^2 - 266\,t_1^2\,t_2^2\,t_3^2\,t_4^2 + 69\,t_1^3\,t_2^2\,t_3^2\,t_4^2 + 21\,t_2^3\,t_3^2\,t_4^2 - \\
&\quad 210\,t_1\,t_2^3\,t_3^2\,t_4^2 + 217\,t_1^2\,t_2^3\,t_3^2\,t_4^2 - 28\,t_1^3\,t_2^3\,t_3^2\,t_4^2 + 35\,t_1\,t_2^4\,t_3^2\,t_4^2 - t_1^2\,t_2^4\,t_3^2\,t_4^2 - 21\,t_1^3\,t_2^4\,t_3^2\,t_4^2 - 5\,t_1\,t_2^5\,t_3^2\,t_4^2 - \\
&\quad 7\,t_1\,t_3^3\,t_4^2 + 10\,t_1^2\,t_3^3\,t_4^2 + 75\,t_1\,t_2\,t_3^3\,t_4^2 - 70\,t_1^2\,t_2\,t_3^3\,t_4^2 + 105\,t_2^2\,t_3^3\,t_4^2 - 364\,t_1\,t_2^2\,t_3^3\,t_4^2 + 217\,t_1^2\,t_2^2\,t_3^3\,t_4^2 - \\
&\quad 70\,t_2^3\,t_3^3\,t_4^2 + 141\,t_1\,t_2^3\,t_3^3\,t_4^2 + 100\,t_1^2\,t_2^3\,t_3^3\,t_4^2 - 105\,t_1^3\,t_2^3\,t_3^3\,t_4^2 + 5\,t_2^4\,t_3^3\,t_4^2 + 45\,t_1\,t_2^4\,t_3^3\,t_4^2 - 161\,t_1^2\,t_2^4\,t_3^3\,t_4^2 + \\
&\quad 70\,t_1^3\,t_2^4\,t_3^3\,t_4^2 - 10\,t_1\,t_2^5\,t_3^3\,t_4^2 + 24\,t_1^2\,t_2^5\,t_3^3\,t_4^2 - 5\,t_1^3\,t_2^5\,t_3^3\,t_4^2 + t_1\,t_3^4\,t_4^2 - 14\,t_1\,t_2\,t_3^4\,t_4^2 - 35\,t_2^2\,t_3^4\,t_4^2 + \\
&\quad 75\,t_1\,t_2^2\,t_3^4\,t_4^2 - t_1^2\,t_2^2\,t_3^4\,t_4^2 - 10\,t_2^3\,t_3^4\,t_4^2 + 93\,t_1\,t_2^3\,t_3^4\,t_4^2 - 161\,t_1^2\,t_2^3\,t_3^4\,t_4^2 + 35\,t_1^3\,t_2^3\,t_3^4\,t_4^2 + 10\,t_2^4\,t_3^4\,t_4^2 - \\
&\quad 50\,t_1\,t_2^4\,t_3^4\,t_4^2 + 50\,t_1^2\,t_2^4\,t_3^4\,t_4^2 + 10\,t_1^3\,t_2^4\,t_3^4\,t_4^2 + 7\,t_1^2\,t_2^5\,t_3^4\,t_4^2 - 10\,t_1^3\,t_2^5\,t_3^4\,t_4^2 + t_1\,t_2\,t_3^5\,t_4^2 + 5\,t_2^2\,t_3^5\,t_4^2 - \\
&\quad 7\,t_1\,t_2^2\,t_3^5\,t_4^2 + 5\,t_2^3\,t_3^5\,t_4^2 - 24\,t_1\,t_2^3\,t_3^5\,t_4^2 + 24\,t_1^2\,t_2^3\,t_3^5\,t_4^2 - 5\,t_1^3\,t_2^3\,t_3^5\,t_4^2 + 7\,t_1^2\,t_2^4\,t_3^5\,t_4^2 - 5\,t_1^3\,t_2^4\,t_3^5\,t_4^2 - t_1^2\,t_2^5\,t_3^5\,t_4^2 - \\
&\quad t_3\,t_4^3 - 3\,t_2\,t_3\,t_4^3 + 10\,t_1\,t_2\,t_3\,t_4^3 - 5\,t_1^2\,t_2\,t_3\,t_4^3 + t_1^3\,t_2\,t_3\,t_4^3 - t_2^2\,t_3\,t_4^3 + 5\,t_1\,t_2^2\,t_3\,t_4^3 - 10\,t_1^2\,t_2^2\,t_3\,t_4^3 + \\
&\quad 3\,t_1^3\,t_2^2\,t_3\,t_4^3 + t_1^3\,t_2^3\,t_3\,t_4^3 - 3\,t_2^3\,t_4^3 + 5\,t_1\,t_2^3\,t_4^3 - t_1^2\,t_2^3\,t_4^3 + 16\,t_2\,t_3^2\,t_4^3 - 10\,t_1\,t_2\,t_3^2\,t_4^3 - 8\,t_1^2\,t_2\,t_3^2\,t_4^3 + \\
&\quad 3\,t_1^3\,t_2\,t_3^2\,t_4^3 + 7\,t_2^2\,t_3^2\,t_4^3 - 70\,t_1\,t_2^2\,t_3^2\,t_4^3 + 69\,t_1^2\,t_2^2\,t_3^2\,t_4^3 - 16\,t_1^3\,t_2^2\,t_3^2\,t_4^3 + 21\,t_1\,t_2^3\,t_3^2\,t_4^3 - 7\,t_1^3\,t_2^3\,t_3^2\,t_4^3 - \\
&\quad 7\,t_1\,t_2^4\,t_3^2\,t_4^3 + t_1\,t_2^5\,t_3^2\,t_4^3 - t_3^3\,t_4^3 + 5\,t_1\,t_3^3\,t_4^3 - 3\,t_1^2\,t_3^3\,t_4^3 + 7\,t_2\,t_3^3\,t_4^3 - 35\,t_1\,t_2\,t_3^3\,t_4^3 + 16\,t_1^2\,t_2\,t_3^3\,t_4^3 + \\
&\quad t_1^3\,t_2\,t_3^3\,t_4^3 - 42\,t_2^2\,t_3^3\,t_4^3 + 105\,t_1\,t_2^2\,t_3^3\,t_4^3 - 28\,t_1^2\,t_2^2\,t_3^3\,t_4^3 - 7\,t_1^3\,t_2^2\,t_3^3\,t_4^3 + 7\,t_2^3\,t_3^3\,t_4^3 + 28\,t_1\,t_2^3\,t_3^3\,t_4^3 - \\
&\quad 105\,t_1^2\,t_2^3\,t_3^3\,t_4^3 + 42\,t_1^3\,t_2^3\,t_3^3\,t_4^3 - t_2^4\,t_3^3\,t_4^3 - 16\,t_1\,t_2^4\,t_3^3\,t_4^3 + 35\,t_1^2\,t_2^4\,t_3^3\,t_4^3 - 7\,t_1^3\,t_2^4\,t_3^3\,t_4^3 + 3\,t_1\,t_2^5\,t_3^3\,t_4^3 - \\
&\quad 5\,t_1^2\,t_2^5\,t_3^3\,t_4^3 + t_1^3\,t_2^5\,t_3^3\,t_4^3 - t_1^2\,t_4^3\,t_4^3 + 7\,t_1^2\,t_2\,t_4^3\,t_4^3 + 7\,t_2^2\,t_4^3\,t_4^3 - 21\,t_1^2\,t_2^2\,t_4^3\,t_4^3 + 16\,t_2^3\,t_4^3\,t_4^3 - 69\,t_1\,t_2^3\,t_4^3\,t_4^3 + \\
&\quad 70\,t_1^2\,t_2^3\,t_4^3\,t_4^3 - 7\,t_1^3\,t_2^3\,t_4^3\,t_4^3 - 3\,t_2^4\,t_4^3\,t_4^3 + 8\,t_1\,t_2^4\,t_4^3\,t_4^3 + 10\,t_1^2\,t_2^4\,t_4^3\,t_4^3 - 16\,t_1^3\,t_2^4\,t_4^3\,t_4^3 + t_1\,t_2^5\,t_4^3\,t_4^3 - \\
&\quad 5\,t_1^2\,t_2^5\,t_4^3\,t_4^3 + 3\,t_1^3\,t_2^5\,t_4^3\,t_4^3 - t_2^2\,t_3^5\,t_4^3 - 3\,t_2^3\,t_3^5\,t_4^3 + 10\,t_1\,t_2^3\,t_3^5\,t_4^3 - 5\,t_1^2\,t_2^3\,t_3^5\,t_4^3 + t_1^3\,t_2^3\,t_3^5\,t_4^3 - t_2^4\,t_3^5\,t_4^3 + \\
&\quad 5\,t_1\,t_2^4\,t_3^5\,t_4^3 - 10\,t_1^2\,t_2^4\,t_3^5\,t_4^3 + 3\,t_1^3\,t_2^4\,t_3^5\,t_4^3 + t_1^3\,t_2^5\,t_3^5\,t_4^3\big) \Big/ \big((-1+t_1)^5\,(-1+t_2)^7\,(-1+t_3)^7\,(-1+t_4)^5\big)
\end{aligned}
$$

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