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Existence and multiplicity of positive solutions to a fourth-order impulsive integral boundary value problem with deviating argument

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Abstract

In this paper, we study the existence of multiple positive solutions for fourth-order impulsive differential equation with integral boundary conditions and deviating argument. The main tool is based on the Avery and Peterson fixed point theorem. Meanwhile, an example to demonstrate the main results is given.

Keywords: fourth-order; impulsive differential equation; integral boundary condition; fixed point theorem

1 Introduction

As an important area of investigation, the theory and applications of the fourth-order ordinary differential equations are emerging. The study concerns mainly the description of the deformations of an elastic beam by means of a fourth-order differential equation boundary value problem (BVP for short). Owing to its various applications in physics, engineering, and material mechanics, a lot of attention has been received by fourth-order differential equation BVPs. For more information, see [1–8].

In the meantime, many very interesting and significant cases of BVPs are constituted by integral boundary conditions. They include two-, three-, multi-point, and nonlocal BVPs as special cases. Therefore, in recent years, increasing attention has been given to integral boundary conditions [9–15] of fourth-order BVPs. Especially, we intend to mention some recent results.

In [12], Ma studied the following fourth-order BVP:

$$\begin{cases} u^{(4)}(t) = h(t)f(t, u), & 0 < t < 1, \\ u(0) = u(1) = \int_0^1 p(s)u(s) ds, \\ u''(0) = u''(1) = \int_0^1 q(s)u(s) ds, \end{cases}$$

where $p, q \in L^1[0, 1]$, h and f are continuous. From an application of the fixed point index in cones, the existence of at least one symmetric positive solution could be obtained.

In 2015, the authors [13] investigated the fourth-order differential equation with integral boundary conditions,

$$\begin{cases} y^{(4)}(t) = \omega(t)F(t, y(t), y''(t)), & 0 < t < 1, \\ y(0) = y(1) = \int_0^1 h(s)y(s) ds, \\ ay''(0) - by'''(0) = \int_0^1 g(s)y''(s) ds, \\ ay''(1) + by'''(1) = \int_0^1 g(s)y''(s) ds. \end{cases}$$

By using a novel technique and fixed point theories, they showed the existence and multiplicity of positive solutions.

Unlike [12] and [13], a class of fourth-order differential equations with advanced or delayed argument were considered by the author of [14],

$$x^4(t) = h(t)f(t, x(t), x(\alpha(t))), \quad t \in (0, 1),$$

subject to the boundary conditions

$$\begin{cases} x(0) = \gamma x'(0) - \int_0^1 g(s)x(s) ds, \\ x(1) = \beta x(\eta), \quad x''(0) = x''(1) = 0, \end{cases}$$

or

$$\begin{cases} x(0) = \beta x(\eta), \quad x''(0) = x''(1) = 0, \\ x(1) = \gamma x'(0) - \int_0^1 g(s)x(s) ds. \end{cases}$$

The existence of multiple positive solutions is obtained by using a fixed point theorem due to Avery and Peterson.

In addition, to deviating arguments, the authors of [15] studied a fourth-order impulsive BVP as follows:

$$\begin{cases} (\phi_p(y''(t)))' = \lambda \omega(t)f(t, y(\alpha(t))), & t \in (0, 1) \setminus (t_1, t_2, \dots, t_n), \\ \Delta y'_{t_k} = -\mu I_k(t_k, y(t_k)), & k = 1, 2, \dots, n, \\ ay(0) - by'(0) = \int_0^1 g(s)y(s) ds, \\ ay(1) + by'(1) = \int_0^1 g(s)y(s) ds, \\ \phi_p(y''(0)) = \phi_p(y''(1)) = \int_0^1 h(t)\phi_p(y''(t)) dt. \end{cases}$$

The boundary conditions above are special Sturm-Liouville integral boundary conditions, since $ay(0) - by'(0) = ay(1) + by'(1) = \int_0^1 g(s)y(s) ds$. Several existence and multiplicity results were derived by using inequality techniques and fixed point theories. For most research papers on impulsive differential equation BVPs, see [16–19] and the references therein.

Motivated by the mentioned results, we investigate a fourth-order impulsive differential equation Sturm-Liouville integral BVP with deviating argument,

$$x^{(4)}(t) = h(t)f(t, x(t), x(\alpha(t))), \quad t \in J_0, \tag{1.1}$$

subject to

$$\begin{cases} x(0) = x(1) = \int_0^1 g_0(s)x(s) ds, \\ \Delta x'_{t_k} = -I_k(t_k, x(t_k)), \quad k = 1, 2, \dots, m, \\ x''(0) - \xi x'''(0) = \int_0^1 g_1(s)x''(s) ds, \\ x''(1) + \eta x'''(1) = \int_0^1 g_2(s)x''(s) ds, \end{cases} \tag{1.2}$$

where $\xi, \eta > 0$. Compared with [13–15], in this paper, (1.2) contains the general Sturm-Liouville integral boundary conditions, where $g_1(s), g_2(s)$ could be two different functions in $L^1[0, 1]$. In this case, we have to establish a more complicated expression of operator T and to find the proper lower and upper bounds of Green’s functions (Lemma 2.2 and Lemma 2.4). Further, by using the fixed point theorem due to Avery and Peterson, the existence and multiplicity of positive solutions are obtained.

In (1.2), t_k ($k = 1, 2, \dots, m$) are fixed points with $0 = t_0 < t_1 < t_2 < \dots < t_m < t_{m+1} = 1$, $\Delta x'_{t_k} = x'(t_k^+) - x'(t_k^-)$, $x'(t_k^+)$ and $x'(t_k^-)$ represent the right-hand limit and the left-hand limit of $x'(t_k)$ at $t = t_k$, respectively.

Throughout the paper, we always assume that $J = [0, 1]$, $J_0 = (0, 1) \setminus \{t_1, t_2, \dots, t_m\}$, $\mathbb{R}^+ = [0, +\infty)$, $J_k = (t_k, t_{k+1}]$, $k = 0, 1, \dots, m - 1$. $\alpha : J \rightarrow J$ is continuous and:

- (H₁) $f \in C(J \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$, with $f(t, u, v) > 0$ for $t \in J$, $u > 0$, and $v > 0$;
- (H₂) h is a nonnegative continuous function defined on $(0, 1)$; h is not identically zero on any subinterval on J_0 ;
- (H₃) $I_k : J \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is continuous with $I_k(t, u) > 0$ ($k = 1, 2, \dots, m$) for all $t \in J$ and $u > 0$;
- (H₄) $g_0, g_1, g_2 \in L^1[0, 1]$ are nonnegative and $\gamma = \int_0^1 g_0(s) ds \in (0, 1)$.

2 Expression and properties of Green’s function

For $v(t) \in C(J)$, we consider the equation

$$x^4(t) = v(t), \quad 0 < t < 1, \tag{2.1}$$

with boundary conditions (1.2).

We shall reduce BVP (2.1) and (1.2) to two second-order problems. To this goal, first, by means of the transformation

$$x''(t) = -y(t), \tag{2.2}$$

we convert problem (2.1) and (1.2) into

$$\begin{cases} y''(t) = -v(t), \\ y(0) - \xi y'(0) = \int_0^1 g_1(s)y(s) ds, \\ y(1) + \eta y'(1) = \int_0^1 g_2(s)y(s) ds, \end{cases} \tag{2.3}$$

and

$$\begin{cases} x''(t) = -y(t), \\ x(0) = x(1) = \int_0^1 g_0(s)x(s) ds, \\ \Delta x'_{t_k} = -I_k, \quad k = 1, 2, \dots, m. \end{cases} \tag{2.4}$$

Lemma 2.1 $\lambda(t)$ is the solution of $x''(t) = 0, x(0) = \xi, x'(0) = 1$. $\mu(t)$ is the solution of $y''(t) = 0, y(1) = \eta, y'(1) = -1$. Then $\lambda(t)$ is strictly increasing on $J, \lambda(t) > 0$ on $(0, 1]$; $\mu(t)$ is strictly decreasing on J , and $\mu(t) > 0$ on $[0, 1)$. For any $v(t) \in C(J)$, then the BVP (2.3) has a unique solution as follows:

$$y(t) = (Fv)(t) + A(v)\lambda(t) + B(v)\mu(t), \tag{2.5}$$

where

$$(Fv)(t) = \int_0^1 G_1(t, s)v(s) ds, \tag{2.6}$$

$$G_1(t, s) = \begin{cases} \frac{1}{\rho}(\eta + 1 - t)(s + \xi), & 0 \leq s \leq t \leq 1, \\ \frac{1}{\rho}(\eta + 1 - s)(t + \xi), & 0 \leq t \leq s \leq 1, \end{cases} \tag{2.7}$$

$$\rho = 1 + \xi + \eta, \quad \lambda(t) = t + \xi, \quad \mu(t) = \eta + 1 - t, \tag{2.8}$$

and

$$A(v) = \frac{1}{\Delta} \begin{vmatrix} \alpha[Fv] & \rho - \alpha[\mu] \\ \beta[Fv] & -\beta[\mu] \end{vmatrix}, \quad B(v) = \frac{1}{\Delta} \begin{vmatrix} -\alpha[\lambda] & \alpha[Fv] \\ \rho - \beta[\lambda] & \beta[Fv] \end{vmatrix}, \tag{2.9}$$

where

$$\Delta = \begin{vmatrix} -\alpha[\lambda] & \rho - \alpha[\mu] \\ \rho - \beta[\lambda] & -\beta[\mu] \end{vmatrix}, \tag{2.10}$$

$$\alpha[v] = \int_0^1 g_1(s)v(s) ds, \quad \beta[v] = \int_0^1 g_2(s)v(s) ds.$$

Proof Since λ and μ are two linearly independent solutions of the equation $y''(t) = 0$, we know that any solution of $y'' = v(t)$ can be represented by (2.5).

It is easy to check that the function defined by (2.5) is a solution of (2.3) if A and B are as in (2.9), respectively.

Now we show that the function defined by (2.5) is a solution of (2.3) only if A and B are as in (2.9), respectively.

Let $y(t) = (Fv)(t) + A\lambda(t) + B\mu(t)$ be a solution of (2.3), then we have

$$y(t) = \int_0^t \frac{1}{\rho}(\eta + 1 - t)(s + \xi)v(s) ds + \int_t^1 \frac{1}{\rho}(\eta + 1 - s)(t + \xi)v(s) ds + A\lambda(t) + B\mu(t),$$

$$y'(t) = - \int_0^t \frac{1}{\rho}(s + \xi)v(s) ds + \int_t^1 \frac{1}{\rho}(\eta + 1 - s)v(s) ds + A\lambda'(t) + B\mu'(t),$$

and

$$y''(t) = -\frac{1}{\rho}(t + \xi)v(t) - \frac{1}{\rho}(\eta + 1 - t)v(t) + A\lambda''(t) + B\mu''(t).$$

Thus, by (2.8), we can obtain

$$y'' = -v(t).$$

Since

$$y(0) = \int_0^1 \frac{\xi}{\rho} (\eta + 1 - s)v(s) ds + A\lambda(0) + B\mu(0),$$

$$y'(0) = \int_0^1 \frac{1}{\rho} (\eta + 1 - s)v(s) ds + A\lambda'(0) + B\mu'(0),$$

we have

$$A \left(- \int_0^1 g_1(s)\lambda(s) ds \right) + B \left(\rho - \int_0^1 g_1(s)\mu(s) ds \right) = \int_0^1 g_1(s)(Fv)(s) ds. \tag{2.11}$$

Since

$$y(1) = \int_0^1 \frac{\eta}{\rho} (s + \xi)v(s) ds + A\lambda(1) + B\mu(1),$$

$$y'(1) = - \int_0^1 \frac{1}{\rho} (s + \xi)v(s) ds + A\lambda'(1) + B\mu'(1),$$

we have

$$A \left(\rho - \int_0^1 g_2(s)\lambda(s) ds \right) + B \left(- \int_0^1 g_2(s)\mu(s) ds \right) = \int_0^1 g_2(s)(Fv)(s) ds. \tag{2.12}$$

From (2.11) and (2.12), we get

$$\begin{pmatrix} -\alpha[\lambda] & \rho - \alpha[\mu] \\ \rho - \beta[\lambda] & -\beta[\mu] \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \alpha[Fv] \\ \beta[Fv] \end{pmatrix},$$

which implies that A and B satisfy (2.9), respectively. □

Assume that

$$(H_5) \quad \Delta < 0, \alpha[\mu] < \rho, \beta[\lambda] < \rho.$$

Lemma 2.2 *Denote $e_1(t) = G_1(t, t)$, $\hat{e}_1(t) = \frac{1}{\rho}(1 - t)(t + \xi)$, for $t \in J$. Let $\kappa[v] = \int_0^1 e_1(s)v(s) ds$ and $\hat{\kappa}[v] = \int_0^1 \hat{e}_1(s)v(s) ds$, for $v \in C(J, \mathbb{R}^+)$. If (H_5) is satisfied, then the following results are true:*

- (1) $\hat{e}_1(s)\hat{e}_1(t) \leq G_1(t, s) \leq e_1(s)$, for $t, s \in J$;
- (2) $0 \leq \underline{A}\hat{\kappa}[v] \leq A(v) \leq \overline{A}\kappa[v]$, for $v \in C(J, \mathbb{R}^+)$;
- (3) $0 \leq \underline{B}\hat{\kappa}[v] \leq B(v) \leq \overline{B}\kappa[v]$, for $v \in C(J, \mathbb{R}^+)$,

where

$$\overline{A} = \frac{1}{\Delta} \begin{vmatrix} \alpha[1] & \rho - \alpha[\mu] \\ \beta[1] & -\beta[\mu] \end{vmatrix}, \quad \overline{B} = \frac{1}{\Delta} \begin{vmatrix} -\alpha[\lambda] & \alpha[1] \\ \rho - \beta[\lambda] & \beta[1] \end{vmatrix},$$

$$\underline{A} = \frac{1}{\Delta} \begin{vmatrix} \alpha[\hat{e}_1] & \rho - \alpha[\mu] \\ \beta[\hat{e}_1] & -\beta[\mu] \end{vmatrix}, \quad \underline{B} = \frac{1}{\Delta} \begin{vmatrix} -\alpha[\lambda] & \alpha[\hat{e}_1] \\ \rho - \beta[\lambda] & \beta[\hat{e}_1] \end{vmatrix}.$$

Proof Now we show that (1) is true. Obviously, $G_1(t, s) \leq e_1(s)$ for $t, s \in J$.

In fact, $\hat{e}_1(s)\hat{e}_1(t) = \frac{1}{\rho^2}(1-s)(s+\xi)(1-t)(t+\xi)$, for $s, t \in J$.

For $0 \leq s \leq t \leq 1$, we notice that

$$\frac{\hat{e}_1(s)\hat{e}_1(t)}{G_1(t, s)} = \frac{(1-s)(1-t)(t+\xi)}{\rho(\eta+1-t)} = \frac{(1-s)(1-t)(t+\xi)}{(1+\xi+\eta)(\eta+1-t)}.$$

It is easy to see that $1-s \leq 1, 1-t \leq \eta+1-t, t+\xi \leq 1+\xi+\eta$, for $s, t \in J, \xi, \eta > 0$, which implies

$$\frac{(1-s)(1-t)(t+\xi)}{(1+\xi+\eta)(\eta+1-t)} \leq 1.$$

Hence, we have

$$\hat{e}_1(s)\hat{e}_1(t) \leq G_1(t, s), \quad \text{for } 0 \leq s \leq t \leq 1.$$

Similarly, we can obtain

$$\hat{e}_1(s)\hat{e}_1(t) \leq G_1(t, s), \quad \text{for } 0 \leq t \leq s \leq 1.$$

In the following we show (2) and (3) hold. In view of (H₅), for $v \in C(J, \mathbb{R}^+)$, we have

$$\begin{aligned} A(v) &= -\frac{1}{\Delta}(\alpha[Fv]\beta[\mu] + \beta[Fv](\rho - \alpha[\mu])) \\ &\leq -\frac{1}{\Delta}(\alpha[1]\beta[\mu] + \beta[1](\rho - \alpha[\mu]))\kappa[v] = \bar{A}\kappa[v], \\ A(v) &= -\frac{1}{\Delta}(\alpha[Fv]\beta[\mu] + \beta[Fv](\rho - \alpha[\mu])) \\ &\geq -\frac{1}{\Delta}(\alpha[\hat{e}_1]\beta[\mu] + \beta[\hat{e}_1](\rho - \alpha[\mu]))\hat{\kappa}[v] = \underline{A}\hat{\kappa}[v]. \end{aligned}$$

In the same way, we have $B(v) \leq \bar{B}\kappa[v], B(v) \geq \underline{B}\hat{\kappa}[v]$, for $v \in C(J, \mathbb{R}^+)$. □

Analogously to Lemma 2.1 in [12], we obtain the following result; we omit the proof.

Lemma 2.3 *If (H₄) holds, for any $y \in C(J)$, the problem (2.4) has a unique solution x expressed in the form*

$$x(t) = \int_0^1 H(t, s)y(s) ds + \sum_{k=1}^m H(t, t_k)I_k, \tag{2.13}$$

where

$$H(t, s) = G_2(t, s) + \frac{1}{1-\gamma} \int_0^1 G_2(s, \tau)g_0(\tau) d\tau, \tag{2.14}$$

$$G_2(t, s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases} \tag{2.15}$$

From (2.14) and (2.15), we can prove that $H(t, s)$ and $G_2(t, s)$ have the following properties.

Lemma 2.4 *Let $H(t, s), G_2(t, s)$ be given as in Lemma 2.3. Assume that (H_4) holds, then the following results are true:*

- (1) $e_2(s)e_2(t) \leq G_2(t, s) \leq e_2(s)$, for $t, s \in J$;
- (2) $\frac{\Gamma}{1-\gamma}e_2(s) \leq H(t, s) \leq \frac{1}{1-\gamma}e_2(s)$, for $t, s \in J$,

where

$$\Gamma = \int_0^1 e_2(s)g_0(s) ds, \quad e_2(t) = G_2(t, t), \quad \text{for } t \in J.$$

Proof It is easy to see that (1) holds. In the following, we prove that (2) is satisfied:

$$\begin{aligned} H(t, s) &= G(t, s) + \frac{1}{1-\gamma} \int_0^1 G_2(s, \tau)g_0(\tau) d\tau \\ &\leq e_2(s) + \frac{1}{1-\gamma} \int_0^1 e_2(s)g_0(\tau) d\tau \\ &= \frac{1}{1-\gamma}e_2(s), \quad \text{for } s \in J, \end{aligned}$$

and

$$\begin{aligned} H(t, s) &\geq e_2(s)e_2(t) + \frac{1}{1-\gamma} \int_0^1 e_2(s)e_2(\tau)g_0(\tau) d\tau \\ &= e_2(s) \left[e_2(t) + \frac{1}{1-\gamma} \int_0^1 e_2(\tau)g_0(\tau) d\tau \right] \\ &= \frac{\Gamma}{1-\gamma}e_2(s), \quad \text{for } t, s \in J. \quad \square \end{aligned}$$

Lemma 2.5 *Assume that (H_1) - (H_4) hold. Then problem (2.1) and (1.2) has a unique solution x given by*

$$x(t) = \int_0^1 H(t, s) [(Fv)(s) + A(v)\lambda(s) + B(v)\mu(s)] ds + \sum_{k=1}^m H(t, t_k)I_k. \tag{2.16}$$

Lemma 2.6 *Assume that (H_3) - (H_5) hold, for $v \in C(J, \mathbb{R}^+)$, the unique solution x of problem (2.1) and (1.2) satisfies $x(t) \geq 0$ on J .*

Proof By Lemma 2.4, we can obtain $H(t, s) \geq 0$ for $t, s \in J$. Hence, from Lemma 2.5, combining with Lemma 2.2 and (H_3) , we can obtain

$$x(t) \geq 0, \quad \text{for } t \in J.$$

This completes the proof. □

3 Background materials and definitions

Now, we present the prerequisite definitions in Banach spaces from the theory of cones.

Definition 3.1 Let E be a real Banach space. A nonempty convex closed set $P \subset E$ is said to be a cone if

- (i) $ku \in P$ for all $u \in P$ and all $k \geq 0$, and
- (ii) $u, -u \in P$ implies $u = 0$.

Definition 3.2 On a cone P of a real Banach space E , a map Λ is said to be a nonnegative continuous concave functional if $\Lambda : P \rightarrow \mathbb{R}^+$ is continuous and

$$\Lambda(tx + (1 - t)y) \geq t\Lambda(x) + (1 - t)\Lambda(y),$$

for all $x, y \in P$ and $t \in J$.

At the same time, on a cone P of a real Banach space E , a map φ is said to be a nonnegative continuous convex functional if $\varphi : P \rightarrow \mathbb{R}^+$ is continuous and

$$\varphi(tx + (1 - t)y) \leq t\varphi(x) + (1 - t)\varphi(y),$$

for all $x, y \in P$ and $t \in J$.

Definition 3.3 If it is continuous and maps bounded sets into pre-compact sets, an operator is called completely continuous.

Let φ and Θ be nonnegative continuous convex functionals on P , Λ be a nonnegative continuous concave functional on P , and Ψ be a nonnegative continuous functional on P . Then for positive numbers a, b, c , and d , we define the following sets:

$$\begin{aligned} P(\varphi, d) &= \{x \in P : \varphi(x) < d\}, \\ P(\varphi, \Lambda, b, d) &= \{x \in P : b \leq \Lambda(x), \varphi(x) \leq d\}, \\ P(\varphi, \Theta, \Lambda, b, c, d) &= \{x \in P : b \leq \Lambda(x), \Theta(x) \leq c, \varphi(x) \leq d\}, \end{aligned}$$

and

$$R(\varphi, \Psi, a, d) = \{x \in P : a \leq \Psi(x), \varphi(x) \leq d\}.$$

We will make use of the following fixed point theorem of Avery and Peterson to establish multiple positive solutions to problem (1.1) and (1.2).

Theorem 3.1 (See [20]) *Let P be a cone in a real Banach space E . Let φ and Θ be nonnegative continuous convex functionals on P , Λ be nonnegative continuous concave functional on P , and Ψ be a nonnegative continuous functional on P satisfying $\Psi(kx) \leq k\Psi(x)$ for $0 \leq k \leq 1$, such that for some positive numbers M and d ,*

$$\Lambda(x) \leq \Psi(x) \quad \text{and} \quad \|x\| \leq M\varphi(x),$$

for all $x \in \overline{P(\varphi, d)}$. Suppose

$$T : \overline{P(\varphi, d)} \rightarrow \overline{P(\varphi, d)},$$

is completely continuous and there exist positive numbers a, b , and c with $a < b$ such that

- (S₁) $\{x \in P(\varphi, \Theta, \Lambda, b, c, d) : \Lambda(x) > b\} \neq \emptyset$ and $\Lambda(Tx) > b$ for $x \in P(\varphi, \Theta, \Lambda, b, c, d)$;
- (S₂) $\Lambda(Tx) > b$ for $x \in P(\varphi, \Lambda, b, d)$ with $\Theta(Tx) > c$;
- (S₃) $0 \notin R(\varphi, \Psi, a, d)$ and $\Psi(Tx) < a$ for $x \in R(\varphi, \Psi, a, d)$ with $\Psi(x) = a$.

Then T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\varphi, d)}$, such that

$$\begin{aligned} \varphi(x_i) &\leq d, \quad \text{for } i = 1, 2, 3, \\ b < \Lambda(x_1), \quad a < \Psi(x_2) \quad \text{with } \Lambda(x_2) < b, \end{aligned}$$

and

$$\Psi(x_3) < a.$$

4 Existence result for the case of $\alpha(t) \geq t$ on J

Function $h(t)$ in (1.1) satisfies (H₂). We introduce the notations

$$l_1 = \kappa[h] = \int_0^1 e_1(s)h(s) \, ds, \quad \hat{l}_1 = \int_0^1 \hat{e}_1(s)h(s) \, ds.$$

Let $X = C(J, \mathbb{R})$ be our Banach space with the maximum norm $\|x\| = \max_{t \in J} |x|$.

Set

$$\begin{aligned} P &= \{x \in X : x \text{ is nonnegative, concave and } x(t) \geq \Gamma \|x\|, t \in J\}, \\ \overline{P}_r &= \{x \in P : \|x\| \leq r\}, \end{aligned} \tag{4.1}$$

where Γ is defined as in Lemma 2.4. We define the nonnegative continuous concave functional $\Lambda = \Lambda_1$ on P by

$$\Lambda_1(x) = \min_{t \in [\delta, 1]} |x(t)|,$$

where $\delta \in (0, 1)$ is such that $0 < \delta < 1 - \delta < 1$. Set $J_{\delta_1} = [\delta, 1]$.

Note that $\Lambda_1(x) \leq \|x\|$. Put $\Psi(x) = \Theta(x) = \varphi(x) = \|x\|$.

Theorem 4.1 *Let assumptions (H₁)-(H₅) hold and $\alpha(t) \geq t$ on J . In addition, we assume that there exist positive constants a, b, c, d, ω, L with $a < b$ such that*

$$\begin{aligned} \omega &> \frac{1}{1-\gamma} \left[\frac{l_1}{6} + \left(\frac{1}{12} + \frac{\xi}{6} \right) l_1 \overline{A} + \left(\frac{1}{12} + \frac{\eta}{6} \right) l_1 \overline{B} + \sum_{k=1}^m t_k(1-t_k) \right], \\ 0 &< L < \frac{\Gamma}{(1-\gamma)} \left[\left(\frac{1}{30} + \frac{\xi}{12} \right) \hat{l}_1 + \left(\frac{1}{12} + \frac{\xi}{6} \right) \hat{l}_1 \underline{A} + \left(\frac{1}{12} + \frac{\eta}{6} \right) \hat{l}_1 \underline{B} + \sum_{k=1}^m t_k(1-t_k) \right], \end{aligned} \tag{4.2}$$

and

- (A₁) $f(t, u, v) \leq \frac{d}{\omega}$, for $(t, u, v) \in J \times [0, d] \times [0, d]$, $I_k(t, u) \leq \frac{d}{\omega}$, for $(t, u) \in J_k \times [0, d]$;
- (A₂) $f(t, u, v) \geq \frac{b}{L}$, for $(t, u, v) \in J_{\delta_1} \times [b, \frac{b}{\Gamma}] \times [b, \frac{b}{\Gamma}]$, $I_k(t, u) \geq \frac{b}{L}$, for $(t, u) \in J_{\delta_1} \cap J_k \times [b, \frac{b}{\Gamma}]$;
- (A₃) $f(t, u, v) \leq \frac{a}{\omega}$, for $(t, u, v) \in J \times [0, a] \times [0, a]$, $I_k(t, u) \leq \frac{a}{\omega}$, for $(t, u) \in J_k \times [0, a]$.

Then problem (1.1) and (1.2) has at least three positive solutions x_1, x_2, x_3 satisfying $\|x_i\| \leq d, i = 1, 2, 3$, and

$$b \leq \Lambda_1(x_1), \quad a < \|x_2\| \quad \text{with } \Lambda_1(x_2) < b$$

and

$$\|x_3\| < a.$$

Proof For any $x \in C(J, \mathbb{R}^+)$, we define operator T by

$$\begin{aligned} (Tx)(t) = & \int_0^1 H(t,s) [F(hf_x(s)) + A(hf_x(s))\lambda(s) + B(hf_x(s))\mu(s)] ds \\ & + \sum_{k=1}^m H(t,t_k) I_k(t_k, x(t_k)), \end{aligned} \tag{4.3}$$

where $hf_x(s) = h(s)f(s, x(s), x(\alpha(s)))$. Indeed, $T : X \rightarrow X$. Problem (1.1) and (1.2) has a solution x if and only if x solves the operator equation $x = Tx$.

We need to prove the existence of at least three fixed points of T by verifying that operator T satisfies the Avery-Peterson fixed point theorem.

From the definition of T , we can obtain

$$(Tx)''(t) = -F(hf_x(s)) - A(hf_x(s))\lambda(t) - B(hf_x(s))\mu(t). \tag{4.4}$$

In view of $(H_1), (H_2)$, Lemma 2.1, and Lemma 2.2, we have

$$(Tx)''(t) \leq 0, \quad \text{for } t \in J.$$

So Tx is concave on J . From (4.3) and (4.4), combining with Lemma 2.4 and (H_3) , we can obtain

$$(Tx)(t) \geq 0, \quad \text{for } t \in J.$$

Noting (2) in Lemma 2.4, it follows that

$$\begin{aligned} \|Tx\| = & \max_{t \in J} \left\{ \int_0^1 H(t,s) [F(hf_x(s)) + A(hf_x(s))\lambda(s) + B(hf_x(s))\mu(s)] ds \right. \\ & \left. + \sum_{k=1}^m H(t,t_k) I_k(t_k, x(t_k)) \right\} \\ \leq & \frac{1}{1-\gamma} \left\{ \int_0^1 e_2(s) [F(hf_x(s)) + A(hf_x(s))\lambda(s) + B(hf_x(s))\mu(s)] ds \right. \\ & \left. + \sum_{k=1}^m e_2(t_k) I_k(t_k, x(t_k)) \right\}. \end{aligned} \tag{4.5}$$

On the other hand, from the properties of $H(t, s)$, we have

$$\begin{aligned}
 (Tx)(t) &= \int_0^1 H(t, s)[F(hf_x(s)) + A(hf_x(s))\lambda(s) + B(hf_x(s))\mu(s)] ds \\
 &\quad + \sum_{k=1}^m H(t, t_k)I_k(t_k, x(t_k)) \\
 &\geq \frac{\Gamma}{1-\gamma} \left\{ \int_0^1 e_2(s)[F(hf_x(s)) + A(hf_x(s))\lambda(s) + B(hf_x(s))\mu(s)] ds \right. \\
 &\quad \left. + \sum_{k=1}^m e_2(t_k)I_k(t_k, x(t_k)) \right\} \\
 &\geq \Gamma \|Tx\|.
 \end{aligned} \tag{4.6}$$

This proves that $TP \subset P$.

Now we prove that the operator $T : P \rightarrow P$ is completely continuous. Let $x \in \overline{P_r}$, then $\|x\| \leq r$. Note that h and f are continuous, so h is bounded on J and f is bounded on $J \times [-r, r]$. It means that there exists a constant $K > 0$ such that $\|Tx\| \leq K$. This proves that $T\overline{P}$ is uniformly bounded. On the other hand, for $t_1, t_2 \in J$ there exists a constant $L_1 > 0$ such that

$$|(Tx)(t_1) - (Tx)(t_2)| \leq L_1|t_1 - t_2|.$$

This shows that $T\overline{P}$ is equicontinuous on J , so T is completely continuous.

Let $x \in \overline{P(\varphi, d)}$, so $0 \leq x(t) \leq d, t \in J$, and $\|x\| \leq d$. Note that also $0 \leq x(\alpha(t)) \leq d, t \in J$ because $0 \leq t \leq \alpha(t) \leq 1$ on J . Hence

$$\varphi(Tx) = \|Tx\| = \max_{t \in J} |(Tx)(t)| = \max_{t \in J} (Tx)(t).$$

By (4.2), Lemma 2.2, Lemma 2.4, and (A_1) , we have

$$\begin{aligned}
 \varphi(Tx) &= \max_{t \in J} \left\{ \int_0^1 H(t, s)[F(hf_x(s)) + A(hf_x(s))\lambda(s) + B(hf_x(s))\mu(s)] ds \right. \\
 &\quad \left. + \sum_{k=1}^m H(t, t_k)I_k(t_k, x(t_k)) \right\} \\
 &\leq \frac{1}{1-\gamma} \left[\int_0^1 \int_0^1 e_2(s)e_1(\tau)h(\tau)f_x(\tau) d\tau ds + \int_0^1 e_2(s)A(hf_x(s))(s + \xi) ds \right. \\
 &\quad \left. + \int_0^1 e_2(s)B(hf_x(s))(\eta + 1 - s) ds + \sum_{k=1}^m e_2(t_k)I_k(t_k, x(t_k)) \right] \\
 &\leq \frac{d}{\omega(1-\gamma)} \left[\int_0^1 \int_0^1 e_2(s)e_1(\tau)h(\tau) d\tau ds + \overline{A} \int_0^1 \kappa[h]e_2(s)(s + \xi) ds \right. \\
 &\quad \left. + \overline{B} \int_0^1 \kappa[h]e_2(s)(\eta + 1 - s) ds + \sum_{k=1}^m t_k(1 - t_k) \right]
 \end{aligned}$$

$$= \frac{d}{\omega(1-\gamma)} \left[\frac{l_1}{6} + \left(\frac{1}{12} + \frac{\xi}{6} \right) l_1 \bar{A} + \left(\frac{1}{12} + \frac{\eta}{6} \right) l_1 \bar{B} + \sum_{k=1}^m t_k(1-t_k) \right] < d.$$

This shows that $T : \overline{P(\varphi, d)} \rightarrow \overline{P(\varphi, d)}$.

To check condition (S₁) we choose

$$x(t) = \frac{1}{2} \left(b + \frac{b}{\Gamma} \right), \quad t \in J.$$

Then

$$\|x\| = \frac{b(\Gamma + 1)}{2\Gamma} < \frac{b}{\Gamma},$$

so

$$\Lambda_1(x) = \min_{t \in [\delta, 1]} x(t) = \frac{b(\Gamma + 1)}{2\Gamma} > b = \frac{b}{\Gamma} \Gamma \geq \Gamma \|x\|.$$

It proves that

$$\left\{ x \in P \left(\varphi, \Theta, \Lambda_1, b, \frac{b}{\Gamma}, d \right) : b < \Lambda_1(x) \right\} \neq \emptyset.$$

Let $b \leq x(t) \leq \frac{b}{\Gamma}$ for $t \in [\delta, 1]$, then $\delta \leq t \leq \alpha(t) \leq 1$ on $[\delta, 1]$. It yields $b \leq x(\alpha(t)) \leq \frac{b}{\Gamma}$ on $[\delta, 1]$. It gives

$$\Lambda_1(Tx) = \min_{t \in [\delta, 1]} (Tx)(t).$$

By (4.2), Lemma 2.2, Lemma 2.4, and (A₂), we have

$$\begin{aligned} \Lambda_1(Tx) &= \min_{t \in [\delta, 1]} \left\{ \int_0^1 H(t, s) [F(hf_x(s)) + A(hf_x(s))\lambda(s) + B(hf_x(s))\mu(s)] ds \right. \\ &\quad \left. + \sum_{k=1}^m H(t, t_k) I_k(t_k, x(t_k)) \right\} \\ &\geq \frac{\Gamma}{1-\gamma} \left[\int_0^1 \int_0^1 e_2(s) \hat{e}_1(s) \hat{e}_1(\tau) h(\tau) f_x(\tau) d\tau ds \right. \\ &\quad + \int_0^1 e_2(s) A(hf_x(s))(s + \xi) ds \\ &\quad \left. + \int_0^1 e_2(s) B(hf_x(s))(\eta + 1 - s) ds + \sum_{k=1}^m e_2(t_k) I_k(t_k, x(t_k)) \right] \\ &\geq \frac{b\Gamma}{L(1-\gamma)} \left[\int_0^1 \int_0^1 e_2(s) \hat{e}_1(s) \hat{e}_1(\tau) h(\tau) d\tau ds + \underline{A} \int_0^1 \hat{\kappa}[h] e_2(s)(s + \xi) ds \right. \\ &\quad \left. + \underline{B} \int_0^1 \hat{\kappa}[h] e_2(s)(\eta + 1 - s) ds + \sum_{k=1}^m t_k(1-t_k) \right] \end{aligned}$$

$$\begin{aligned}
 &= \frac{b\Gamma}{L(1-\gamma)} \left[\left(\frac{1}{30} + \frac{\xi}{12} \right) \hat{l}_1 + \left(\frac{1}{12} + \frac{\xi}{6} \right) \hat{l}_1 \underline{A} \right. \\
 &\quad \left. + \left(\frac{1}{12} + \frac{\eta}{6} \right) \hat{l}_1 \underline{B} + \sum_{k=1}^m t_k(1-t_k) \right] \\
 &> b.
 \end{aligned}$$

It proves condition (S₁) holds.

Now we need to prove that condition (S₂) is satisfied. Take

$$x \in P(\varphi, \Lambda_1, b, d) \quad \text{with } \|Tx\| > \frac{b}{\Gamma} = c.$$

Then

$$\Lambda_1(Tx) = \min_{t \in [\delta, 1]} (Tx)(t) \geq \Gamma \|Tx\| > \Gamma \frac{b}{\Gamma} = b.$$

So condition (S₂) holds.

We finally show that condition (S₃) also holds. Clearly, as $\Psi(0) = 0 < a$, so $0 \notin R(\varphi, \Psi, a, d)$. Suppose that $x \in R(\varphi, \Psi, a, d)$ with $\Psi(x) = \|x\| = a$.

Similarly, by (4.2), Lemma 2.2, Lemma 2.4, and (A₃), we have

$$\begin{aligned}
 \Psi(Tx) &= \|Tx\| = \max_{t \in J} (Tx)(t) \\
 &= \max_{t \in J} \left\{ \int_0^1 H(t, s) [F(hf_x(s)) + A(hf_x(s))\lambda(s) + B(hf_x(s))\mu(s)] ds \right. \\
 &\quad \left. + \sum_{k=1}^m H(t, t_k) I_k(t_k, x(t_k)) \right\} \\
 &\leq \frac{a}{\omega} \left\{ \frac{1}{1-\gamma} \left[\frac{l_1}{6} + \left(\frac{1}{12} + \frac{\xi}{6} \right) l_1 \bar{A} + \left(\frac{1}{12} + \frac{\eta}{6} \right) l_1 \bar{B} + \sum_{k=1}^m t_k(1-t_k) \right] \right\} \\
 &< a.
 \end{aligned}$$

It proves that condition (S₃) is satisfied.

By Theorem 4.1, there exist at least three positive solutions x_1, x_2, x_3 of problem (1.1) and (1.2) such that $\|x_i\| \leq d$ for $i = 1, 2, 3$,

$$b \leq \min_{t \in [\delta, 1]} x_1(t), \quad a < \|x_2\| \quad \text{with } \min_{t \in [\delta, 1]} x_2(t) < b,$$

and $\|x_3\| < a$. This ends the proof. □

Example We consider the following BVP:

$$\begin{cases}
 x^{(4)}(t) = h(t)f(x(\alpha(t))), & t \in J_0, \\
 x(0) = x(1) = \int_0^1 \frac{s}{2} x(s) ds, & \Delta x'_{t_1} = -I_1(t_1, x(t_1)), \\
 x''(0) - \frac{1}{2} x'''(0) = \int_0^1 s x''(s) ds, & x''(1) + \frac{1}{2} x'''(1) = \int_0^1 s^2 x''(s) ds,
 \end{cases} \tag{4.7}$$

with $\alpha \in C(J, J)$, $h(t) = Dt$, $\alpha(t) \geq t$, $\xi = \eta = \frac{1}{2}$, $\rho = 2$, and $t_1 = \frac{1}{12}$. It follows that $\mu(t) = \frac{3}{2} - t$, $\lambda(t) = t + \frac{1}{2}$, for $t \in J$, and

$$f(v) = I_1(t, v) = \begin{cases} \frac{v^2}{250}, & 0 \leq v \leq 1, \\ \frac{1}{250}(748v - 747), & 1 \leq v \leq \frac{3}{2}, \\ \frac{13}{69}v + \frac{28}{23}, & \frac{3}{2} \leq v \leq 36, \\ 8, & v \geq 36. \end{cases}$$

Note that $f \in C(\mathbb{R}^+, \mathbb{R}^+)$. As a function α we can take, for example, $\alpha(t) = \sqrt{t}$.

In this case we have $\gamma = \frac{1}{4}$, $\Delta = -\frac{85}{36}$, $\bar{A} = \frac{47}{170}$, $\bar{B} = \frac{71}{170}$, $\underline{A} = \frac{269}{6,800}$, $\underline{B} = \frac{457}{6,800}$, $l_1 = \frac{11}{48}D$, $\hat{l}_1 = \frac{1}{12}D$, $\Gamma = \frac{1}{24}$. Let $a = 1$, $b = \frac{3}{2}$, $c = 36$, $d \geq 2,000$, $D = 2,784$. In this case we can take $\omega = 250$, $L = 1$. We see that all assumptions of Theorem 4.1 hold, so BVP (4.7) has at least three positive solutions.

5 Existence result for the case of $\alpha(t) \leq t$ on J

The cone P is defined as in (4.1). We define the nonnegative continuous concave functional $\Lambda = \Lambda_2$ on P by

$$\Lambda_2(x) = \min_{t \in [0, 1-\delta]} |x(t)|,$$

where $\delta \in (0, 1)$ satisfying $0 < \delta < 1 - \delta < 1$. Set $J_{\delta_2} = [0, 1 - \delta]$. Similar to the proof of Theorem 4.1, we have the following result.

Theorem 5.1 *Let assumptions (H₁)-(H₅) hold and $\alpha(t) \leq t$ on J . In addition, we assume that there exist positive constants a, b, c, d, ω, L with $a < b$ such that (4.2) holds and*

- (A₁) $f(t, u, v) \leq \frac{d}{\omega}$, for $(t, u, v) \in J \times [0, d] \times [0, d]$, $I_k(t, u) \leq \frac{d}{\omega}$, for $(t, u) \in J_k \times [0, d]$;
- (A₂) $f(t, u, v) \geq \frac{b}{L}$, for $(t, u, v) \in J_{\delta_2} \times [b, \frac{b}{\Gamma}] \times [b, \frac{b}{\Gamma}]$, $I_k(t, u) \geq \frac{b}{L}$, for $(t, u) \in J_{\delta_2} \cap J_k \times [b, \frac{b}{\Gamma}]$;
- (A₃) $f(t, u, v) \leq \frac{a}{\omega}$, for $(t, u, v) \in J \times [0, a] \times [0, a]$, $I_k(t, u) \leq \frac{a}{\omega}$, for $(t, u) \in J_k \times [0, a]$.

Then problem (1.1) and (1.2) has at least three positive solutions x_1, x_2, x_3 satisfying $\|x_i\| \leq d$, $i = 1, 2, 3$, and

$$b \leq \Lambda_2(x_1), \quad a < \|x_2\| \quad \text{with} \quad \Lambda_2(x_2) < b,$$

and

$$\|x_3\| < a.$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgements

The work is supported by Chinese Universities Scientific Fund (Project No. 2016 LX002). The authors would like to thank the anonymous referees very much for helpful comments and suggestions, which lead to the improvement of presentation and quality of the work.

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