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The intersection numbers of the p-spin curves from random matrix theory

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Abstract: The intersection numbers of p-spin curves are computed through correlation functions of Gaussian ensembles of random matrices in an external matrix source. The p-dependence of intersection numbers is determined as polynomial in p; the large p behavior is also considered. The analytic continuation of intersection numbers to negative values of p is discussed in relation to SL(2,R)/U(1) black hole sigma model.

Keywords: Matrix Models, Random Systems, Topological Field Theories

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1 Introduction

The intersection numbers for p-spin curves appear in the generalized Kontsevich matrix model [1–3]. The generating function for p-spin intersection number obeys the p-th KdV equation or Gelfand-Dikii equation. In a random matrix theory, the correlation functions at the edges of the spectrum, where one can tune a degeneracy of order p, are expressed through intersection numbers [4–7]. In conformal field theory, the p-spin curve intersection theory is related to $\mathcal{N}$=2 superconformal minimal theory for Lie algebra A_{p-1} type. It has been pointed out that it corresponds to a gauged Wess-Zumino-Witten (WZW) model of $\mathrm{SU}(2)_k/\mathrm{U}(1)$, where $k = p - 2$ is the level of the Kac-Moody algebra of Lie group SU(2) [8, 9], and is related to SL(2,R)/U(1) black hole sigma model when k becomes negative [10].

The free energy for the p-spin curve satisfies interesting universal equations, such as a string equation, dilaton equation, and WDVV equation, so called tautological equations or universal equations, and has been studied in the connection to Gromov-Witten theory [11–13]. Although the p-spin curve intersection numbers can be obtained through tautological equations in a recursive way, the actual computation for higher genuses is limitted [14–16].

In previous papers, we have derived explicit integral formula for the p-spin curve intersection numbers of the moduli space $\overline{M}_{g,n}$ valid for all order of genus g. We have shown that they are obtained analytically for a fixed number n of marked points. Our formulation starts from simple Gaussian matrix models with an external matrix source and based upon a duality relation [5–7], from which one recovers a generalized Kontsevich matrix model.

The intersection numbers for the spin moduli spaces with n-marked points are obtained from the n n-point correlation functions $\mathrm{U}(s_1, \ldots, s_n)$ of Gaussian random matrices in a scaling limit at critical edges [17, 18]. In a previous article [19], we have computed explicitly the intersection numbers of moduli space of p-spin curves with one marked point, for arbitrary values of p, as polynomials in p. This allowed us to consider continuations in p; in particular the limit $p \to -1$ exhibits an interesting relation between the intersection numbers, and the orbifold Euler characteristics $\chi(\overline{M}_{g,1}) = \zeta(1-2g)$, where $\zeta(x)$ is the Riemann zeta function) [21, 22].

In this paper, we extend the evaluation of the intersection numbers beyond the one marked point ($n = 1$) for arbitrary p. The obtained intersection numbers are consistent with previously known results [14–16] for small values of p. We pursue the large p behavior, $p \to \infty$ limit. The p-spin curve intersection theory is equivalent to gauged WZW model. For this gauged WZW theory, in which $k = p-2$ appears as overall factor, the large k limit may give a semi-classical solution [10, 24]. In the negative k, the gauged WZW model on SU(2)/U(1) is changed to WZW model on non-compact SL$(2,R)$/U(1), which is relevant to a black hole σ model [10]. We will discuss the relation between the intersection numbers and the density of state of SL$(2,R)$/U(1) black hole sigma model [25, 26, 28].

2 Generating function for p-spin intersection numbers

The mathematical definition of the intersection numbers of the moduli space of p-spin curves with s-marked points is given by [3]

$$< \tau_{n_1}(U_{j_1}) \cdots \tau_{n_s}(U_{j_s}) >= \frac{1}{p^g} \int_{\overline{M}_{g,s}} C_T(\nu) \prod_{i=1}^{s} (c_1(\mathcal{L}_i))^{n_i} \tag{2.1}$$

where U_j is an operator for the primary matter field (tachyon), related to top Chern class $C_T(\nu)$, and τ_n is a gravitational operator, related to the first Chern class c_1 of the line bundle $\mathcal{L}_i$ at the ith-marked point. We denote $\tau_n(U_j)$ by $\tau_{n,j}$, and j represents the spin index (j=0,. . . ,p-1). The problem of definition (2.1) has been discussed extensively [12].

In a previous paper [19], we have shown that those intersection numbers (2.1) are expressed through the correlation functions $\mathrm{U}(s_1,\ldots,s_n)$ as coefficients of powers of s_j,

$$\begin{aligned} \mathrm{U}(s_1, s_2, \ldots, s_n) &= < \mathrm{tr} e^{s_1 M} \mathrm{tr} e^{s_2 M} \cdots \mathrm{tr} e^{s_n M} > \\ &= \int \prod_{l=1}^{m} d\lambda_l e^{\sum i t_l \lambda_l} < \prod_{1}^{m} \mathrm{tr}\delta(\lambda_j - M) > \end{aligned} \tag{2.2}$$

where $s_l = it_l$; M is an $N \times N$ Hermitian random matrix. The bracket stands for averages with the Gaussian probability measure

$$< X >= \frac{1}{Z} \int dM e^{-\frac{N}{2}\mathrm{tr}M^2 + N\mathrm{tr}MA} X(M), \tag{2.3}$$

in which A is an $N \times N$ external Hermitian matrix source. By an appropriate tuning of the external source matrix A, we may obtain the desired singularity, which generates the p-spin curves. The relation to the generalized Kontsevich model is discussed in § 3,4 of [19].

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An exact and useful integral representation for $\mathrm{U}(s_1, \ldots, s_n)$ is known in presence of an arbitrary external matrix source A with eigenvalues a_α [20]:

$$\begin{aligned}\mathrm{U}(s_1, \cdots, s_n) &= \frac{1}{N}\langle \mathrm{tr} e^{s_1 \mathrm{M}} \cdots \mathrm{tr} e^{s_n \mathrm{M}} \rangle \\ &= e^{\sum_1^n s_i^2} \oint \prod_1^n \frac{du_i}{2\pi i} e^{\sum_1^n u_i s_i} \prod_{\alpha=1}^N \prod_{i=1}^n \left(1 - \frac{s_i}{a_\alpha - u_i}\right) \det \frac{1}{u_i - u_j + s_i}\end{aligned} \tag{2.4}$$

This representation involves contour integrals around $u_i = a_\alpha$. In the large N limit, it is convenient to express the factors in the determinant as additional integrals. For instance, in the case of the two point correlation (n=2), after the shift $u_i \to u_i - \frac{s_i}{2}$, $s_i \to \frac{s_i}{N}$, in the two point function, we have

$$\begin{aligned}&\frac{1}{u_1 - u_2 + \frac{1}{2N}(s_1 + s_2)} \frac{1}{u_1 - u_2 - \frac{1}{2N}(s_1 + s_2)} \\ &\qquad = \frac{N}{s_1 + s_2} \int_0^\infty dx e^{-x(u_1 - u_2)} \mathrm{sh}\left(\frac{x}{2N}(s_1 + s_2)\right)\end{aligned} \tag{2.5}$$

Tuning now the a_α's, and taking the large N limit, we obtain

$$\begin{aligned}\mathrm{U}(s_1, s_2) = &\frac{2N}{s_1 + s_2} \frac{1}{(2\pi i)^2} \int_0^\infty dx \int du_1 du_2 \mathrm{sh}\left(\frac{1}{2N} x(s_1 + s_2)\right) e^{-(u_1 - u_2)x} \\ &\times \exp\left[-\frac{N}{p^2 - 1} \sum \frac{1}{a_\alpha^{p+1}} \left(\sum_i \left(u_i + \frac{1}{2N} s_i\right)^{p+1} - \sum_i \left(u_i - \frac{1}{2N} s_i\right)^{p+1}\right)\right]\end{aligned} \tag{2.6}$$

For the three and four point correlations, similar useful formulae for the determinant part of (2.4) may be found in the appendices A and B of [19].

3 Intersection numbers for $p = 3$ with two marked points

The intersection numbers are obtained as coefficients of the power series in s_1, s_2 of $\mathrm{U}(s_1, s_2)$. In a previous paper [19], for p=3, we have computed the intersection numbers with two marked points or genus one case (g=1) starting from (2.6). As an example, we compute the $p = 3$ case up to genus 3. The general expansion

$$\mathrm{U}(s_1, s_2) = \sum_{g,m,j} < \tau_{m_1, j_1} \tau_{m_2, j_2} >_g \Gamma\left(1 - \frac{1 + j_1}{p}\right) \Gamma\left(1 - \frac{1 + j_2}{p}\right) {s_1}^{m_1'} {s_2}^{m_2'} \tag{3.1}$$

with the condition,

$$(p + 1)(2g - 2 + n) = \sum_{i=1}^s (pm_i + j_i + 1), \;\; m_k' = m_k + \frac{1 + j_k}{p} \;\; (k = 1, 2) \tag{3.2}$$

is applied to the special case $n = 2, p = 3$ The gamma functions in (3.1) represent the spin factors.

After rescaling of the parameters,

$$\mathrm{U}(s_1,s_2) = \frac{2}{(s_1+s_2)(3s_2)^{1/3}}\int_0^\infty dy \sinh\left(\frac{s_1+s_2}{2}(3s_1)^{1/3}y\right)A_i\left(y-\frac{1}{4\cdot 3^{1/3}}{s_1}^{8/3}\right) \times A_i\left(-ay-\frac{1}{4\cdot 3^{1/3}}{s_2}^{8/3}\right) \tag{3.3}$$

in which $a=(s_1/s_2)^{1/3}$, and the Airy function is

$$A_i(y) = \int_{-\infty}^{+\infty}\frac{du}{2\pi}e^{\frac{i}{3}u^3+iuy} \tag{3.4}$$

The Airy function satisfies the differential equation

$$A_i''(y) = yA_i(y), \quad A_i''(-ay) = -a^3yA_i(-ay) \tag{3.5}$$

The genus one case (g=1) has been studied in [7].

If one expands the hyperbolic sine function and the Airy functions in (3.3) up to relevant orders, we find a sum of six terms which, for $g=2$, involve the following integrals:

$$\begin{aligned} I_1 &= \int_0^\infty dy y^5 A_i(y)A_i(-ay), & I_2 &= \int_0^\infty dy y A_i''(y)A_i(-ay),\\ I_3 &= \int_0^\infty dy y A_i(y)A_i''(-ay), & I_4 &= \int_0^\infty dy y A_i'(y)A_i'(-ay),\\ I_5 &= \int_0^\infty dy y^3 A_i'(y)A_i(-ay), & I_6 &= \int_0^\infty dy y^3 A_i(y)A_i'(-ay) \end{aligned} \tag{3.6}$$

A repeated use of (3.5) plus integrations by parts allows us to write all these integrals in terms of

$$A_i(0) = \frac{3^{-2/3}}{\Gamma(2/3)} = \frac{1}{2\pi 3^{1/3}}\Gamma\left(\frac{1}{3}\right), \qquad A_i'(0) = -\frac{3^{-1/3}}{\Gamma(1/3)} = -\frac{1}{2\pi}\Gamma\left(\frac{2}{3}\right) \tag{3.7}$$

plus the integral

$$T = \int_0^\infty dy A_i(y)A_i'(-ay) \tag{3.8}$$

which cannot be reduced to $A_i(0)$ or $A_i'(0)$. For instance one finds

$$(1+a^3)I_2 = A_i(0)^2 - 2T \tag{3.9}$$

and so on. However, all the T-dependence cancels when we sum up all the terms relevant to $g=2$ in $\mathrm{U}(s_1,s_2)$. For instance the sum of all terms of order $s_2^{\frac{16}{3}}$ is given by

$$\begin{aligned} &\frac{1}{5!}\frac{1}{16}3^{4/3}(1+a^3)^4a^5s_2^{\frac{16}{3}}I_1 + \frac{1}{2}\left(\frac{1}{4\cdot 3^{1/3}}\right)^2 a^{17}s_2^{\frac{16}{3}}I_2\\ &+\frac{1}{2}\left(\frac{1}{4\cdot 3^{1/3}}\right)^2 a^{-1}s_2^{\frac{16}{3}}I_3 - \left(\frac{1}{4\cdot 3^{1/3}}\right)^2 a^8 s_2^{\frac{16}{3}}I_4\\ &-\frac{1}{3!}\frac{1}{16}3^{1/3}(1+a^3)^2a^{11}s_2^{\frac{16}{3}}I_5 + \frac{1}{3!}\frac{1}{16}3^{1/3}a^2(1+a^3)^2s_2^{\frac{16}{3}}I_6 \end{aligned} \tag{3.10}$$

and we add up the six terms and expand in powers of s_1 to the relevant orders we find

$$\mathrm{U}(s_1,s_2)|_{g=2} = \frac{(A_i(0))^2}{32\cdot 3^{2/3}}\Big(-s_1^{14/3}s_2^{2/3} - \frac{11}{5}s_1^{11/3}s_2^{5/3} - \frac{17}{5}s_1^{8/3}s_2^{8/3} - \frac{11}{5}s_1^{5/3}s_2^{11/3} - s_1^{2/3}s_2^{14/3}\Big). \quad (3.11)$$

From these results, we obtain the intersection numbers

$$< \tau_{0,1}\tau_{4,1} >_{g=2} = \frac{1}{864}$$
$$< \tau_{1,1}\tau_{3,1} >_{g=2} = \frac{11}{4320}$$
$$< \tau_{2,1}\tau_{2,1} >_{g=2} = \frac{17}{4320} \quad (3.12)$$

Rather than computing the exact dependence in a of the terms proportional to $s_2^{16/3}$ and then re-expand in a to obtain the various terms of (3.11), we may proceed in a simpler way by expanding $A_i(-ay), A_i'(-ay), A_i''(-ay)$ for small a:

$$A_i(-ay) = A_i(0) - ayA_i'(0) + \frac{a^2}{2}y^2A_i''(0) + \cdots \quad (3.13)$$

and we then recover (3.12).

In the genus- three case (g=3), we have again ten distinct integrals $J_1 - J_{10}$ for the terms of order $s_2^8 a^m$, in the small s_1, s_2 expansion of (3.3).

$$J_1 = \int_0^\infty dyy^7A_i(y)A_i(-ay), \qquad J_2 = \int_0^\infty dyy^5A_i'(y)A_i(-ay)$$
$$J_3 = \int_0^\infty dyy^5A_i(y)A_i'(-ay), \qquad J_4 = \int_0^\infty dyy^3A_i'(y)A_i'(-ay)$$
$$J_5 = \int_0^\infty dyy^3A_i''(y)A_i(-ay), \qquad J_6 = \int_0^\infty dyy^3A_i(y)A_i''(-ay)$$
$$J_7 = \int_0^\infty dyyA_i'''(y)A_i(-ay), \qquad J_8 = \int_0^\infty dyyA_i(y)A_i'''(-ay)$$
$$J_9 = \int_0^\infty dyyA_i''(y)A_i'(-ay), \qquad J_{10} = \int_0^\infty dyyA_i'(y)A_i''(-ay) \quad (3.14)$$

The genus 3 contribution for $\mathrm{U}(s_1,s_2)$ is then expressed as the sum of four terms, $U^{(1)}$ - $U^{(4)}$. The term $U^{(1)}$, which is related to J_1, is

$$U^{(1)} = \frac{9}{7!\cdot 64}s_1^{7/3}s_2^{17/3}(1+a^3)^6J_1$$
$$= \frac{3}{8960}s_2^8(-15a^7 - 42a^8 + 90a^{10} + 63a^{11} + 63a^{13} + 90a^{14} - 42a^{16} - 15a^{17})A_i(0)A_i'(0). \quad (3.15)$$

The term $U^{(2)}$, which is related to J_2, is

$$U^{(2)} = -\frac{1}{2560}s_2^8a^{13}(1+a^3)^4J_2$$
$$= -\frac{1}{2560}s_2^8a^{13}(30 + 72a - 120a^3 - 90a^4 + 12a^6)A_i(0)A_i'(0). \quad (3.16)$$

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The term $U^{(3)}$, which is related to J_3, is

$$\begin{aligned} U^{(3)} &= \frac{1}{2560} s_2^8 a^4 (1+a^3)^4 J_3 \\ &= \frac{1}{2560} s_2^8 a^5 (-12 + 90a^2 + 120a^3 - 72a^5 - 30a^6) A_i(0) A_i'(0). \end{aligned} \tag{3.17}$$

The term $U^{(4)}$ from the sum of the contributions of J_4 to J_{10}. We have

$$\begin{aligned} J_{10} &= \frac{1}{1+a^3}(2a^3 K_1 - a^3 K_2) \\ J_9 &= -K_2 - J_{10} \\ J_8 &= \frac{a^3 + 2a^4 - 2a^6 - a^7}{(1+a^3)^2} L \\ J_7 &= \frac{1}{a^3} J_8 \\ J_6 &= \frac{6a^3}{1+a^3} J_7 \\ J_5 &= -\frac{1}{a^3} J_6 \\ J_4 &= a^3 J_5 - 3J_9 \end{aligned} \tag{3.18}$$

with $L = A_i(0)A_i'(0)$, and K_1,K_2 are given below. We have also the following relations between J_1, J_2 and J_3,

$$\begin{aligned} J_1 &= \frac{1}{1+a^3}(30J_5 + 12J_3) \\ J_2 &= \frac{1}{1+a^3}(-5J_5 + 4J_4) \\ J_3 &= \frac{1}{1+a^3}(-5a^3 J_5 - 4J_4) \end{aligned} \tag{3.19}$$

Thus $U^{(4)}$ becomes

$$\begin{aligned} U^{(4)} = \frac{1}{1152} s_2^8 (a &+ 5a^4 - 20a^7 + 23a^{10} + 16a^{13} \\ & -19a^{16} + 13a^{19} + 2a^{22} + (2a^2 + 13a^5 - 19a^8 \\ & +16a^{11} + 23a^{14} - 20a^{17} + 5a^{20} + a^{23})) A_i(0) A_i'(0). \end{aligned} \tag{3.20}$$

Since $a = (s_1/s_2)^{1/3}$, the above expression for $\mathrm{U}(s_1, s_2)$ is a symmetric function in s_1 and s_2. Denoting

$$s^{m+\frac{1+j}{3}} = t_{m,j} \tag{3.21}$$

and dividing $\mathrm{U}(s_1, s_2)$ by $1/g^p$, i.e. $1/27$ in this case, we obtain the intersection numbers $< \tau_{m_1,j_1} \tau_{m_2,j_2} >_{g=3}$ as the coefficient of $t_{m_1,j_1} t_{m_2,j_2}$ in $\mathrm{U}(s_1, s_2)$ taking into account the spin factors. The following spin factor appears as a over all factor in $\mathrm{U}(s_1, s_2)$ at genus 3.

$$A_i(0)A_i'(0) = -\frac{1}{(2\pi)^2 3^{1/3}} \Gamma\left(\frac{1}{3}\right) \Gamma\left(\frac{2}{3}\right) \tag{3.22}$$

where $\Gamma(\frac{1}{3})$, $\Gamma(\frac{2}{3})$ are spin 1 and spin 2 factors, respectively, as (3.1). All the integrals $J_1, \ldots, J_{10}$ are expressed by (3.22), and there are no terms like (3.8), which appeared in the integrals for the g=1,g=2 cases. Finally we have to compute the following terms

$$K_1 = \int dy A_i''(y) A_i(-ay) = -A_i(0)A_i'(0) - K_2$$
$$K_2 = \int dy A_i'(y) A_i'(-ay). \tag{3.23}$$

For these integrals, we find

$$K_1 = -\frac{1+a}{1+a^3} A_i(0)A_i'(0), \quad K_2 = \frac{a-a^3}{1+a^3} A_i(0)A_i'(0) \tag{3.24}$$

and all the integrals reduces to the spin factor (3.22). Summing up the results of $U^(1)$ to U(4), we obtain the intersection numbers for $p=3, g=3$,

$$< \tau_{0,0}\tau_{7,1} >_{g=3} = \frac{1}{31104}, \qquad < \tau_{0,1}\tau_{7,0} >_{g=3} = \frac{1}{15552}$$
$$< \tau_{1,0}\tau_{6,1} >_{g=3} = \frac{5}{31104}, \qquad < \tau_{1,1}\tau_{6,0} >_{g=3} = \frac{19}{77760}$$
$$< \tau_{2,0}\tau_{5,1} >_{g=3} = \frac{103}{217728}, \qquad < \tau_{2,1}\tau_{5,0} >_{g=3} = \frac{47}{77760}$$
$$< \tau_{3,0}\tau_{4,1} >_{g=3} = \frac{443}{544320}, \qquad < \tau_{3,1}\tau_{4,0} >_{g=3} = \frac{67}{77760} \tag{3.25}$$

The above results are in complete agreement with the previous results [14, 16], which were obtained by recursion relations.

4 Intersection numbers for $p > 3$

For higher multicritical points the algebra is similar, except that we have to deal with generalized Airy functions. For instance for $p=4$ instead of $A_i(x)$ we have to work with $\phi(x)$ defined as

$$\phi(x) = \int_0^\infty dv e^{-\frac{1}{4}v^4 + vx} \tag{4.1}$$

which satifies

$$\phi'''(x) = x\phi. \tag{4.2}$$

Then, similarly

$$\mathrm{U}(s_1, s_2) = \frac{2}{(s_1+s_2)(4s_2)^{1/4}} \int_0^\infty dx \int_0^\infty dv_1 dv_2 \mathrm{sh}\left(\frac{s_1+s_2}{2}(4s_1)^{1/4}x\right) \tag{4.3}$$
$$\times e^{-\frac{s_1^3}{2}(\frac{1}{4s_1})^{1/2}v_1^2 - \frac{s_2^3}{2}(\frac{1}{4s_2})^{1/2}v_2^2} e^{-\frac{1}{4}v_1^4 + xv_1 - \frac{1}{4}v_2^4 - axv_2}$$

where $a = (s_1/s_2)^{1/4}$. In complete analogy with the $p=3$ case, a repeated use of integration by parts and of (4.2) leads to the expansion of $\mathrm{U}(s_1, s_2)$. In the genus one case,

$$\mathrm{U}(s_1, s_2)|_{g=1} = \frac{1}{4}(\phi''(0))^2 s_1^{1/4} s_2^{1/4}(s_1^2 + s_1 s_2 + s_2^2)$$
$$+\frac{1}{12}(s_1 s_2)^{3/4}(s_1+s_2)(\phi(0))^2 \tag{4.4}$$

with

$$\phi''(0) = 2^{1/2}\Gamma\left(\frac{3}{4}\right), \quad \phi(0) = 2^{-1/2}\Gamma\left(\frac{1}{4}\right) \tag{4.5}$$

which provide the $j = 0, j = 2$ spin factors respectively. Replacing s_1, s_2 by t_m, j, $(s^{m+(1+j)/p} = t_{m,j})$,

$$\begin{aligned} \mathrm{U}(s_1, s_2)|_{g=1} &= \frac{1}{2}(t_{2,0}t_{0,0} + t_{1,0}t_{1,0} + t_{0,0}t_{2,0})\left(\Gamma\left(\frac{3}{4}\right)\right)^2 \\ &\quad + \frac{1}{24}(t_{1,2}t_{0,2} + t_{0,2}t_{1,2})\left(\Gamma\left(\frac{1}{4}\right)\right)^2 \end{aligned} \tag{4.6}$$

Multiplying by a factor $\frac{1}{p^g}$, we obtain the intersection numbers as coefficients of (4.6) for $p = 4$ in the genus one case,

$$< \tau_{0,0}\tau_{2,0} >_{g=1} = \frac{1}{8}, \quad < \tau_{1,0}\tau_{1,0} >_{g=1} = \frac{1}{8}, \quad < \tau_{0,2}\tau_{1,2} >_{g=1} = \frac{1}{96} \tag{4.7}$$

For $g = 2$, $p = 4$, from the term $s_2^{\frac{18}{4}} s_1^{\frac{2}{4}}$, we have similarly

$$< \tau_{0,1}\tau_{4,1} >_{g=2} = \frac{1}{320} \tag{4.8}$$

For general p we have to deal with the generalized Airy functions $\phi(x)$ for $p > 2$, which satisfy the differential equation,

$$\phi^{(p)}(x) = x\phi(x) \tag{4.9}$$

where $\phi^{(p)}(x)$ means the p-th derivative of $\phi(x)$. The generalized Airy function has an integral representation,

$$\phi(y) = \int_0^\infty du \, e^{-\frac{u^p}{p} + yu}. \tag{4.10}$$

As examples of what the method can provide we give a few results: for the case $p = 5$, we obtain

$$\begin{aligned} < \tau_{1,3}\tau_{0,2} >_{g=1} &= < \tau_{1,2}\tau_{0,3} >_{g=1} = \frac{1}{60} \\ < \tau_{1,0}\tau_{1,0} >_{g=1} &= < \tau_{0,0}\tau_{2,0} >_{g=1} = \frac{1}{6} \\ < \tau_{0,1}\tau_{4,1} >_{g=2} &= \frac{7}{1200}. \end{aligned} \tag{4.11}$$

For the case $p = 6$,

$$< \tau_{0,3}\tau_{1,3} >_{g=1} = \frac{1}{36}, \quad < \tau_{0,2}\tau_{1,4} >_{g=1} = \frac{1}{48}, \quad < \tau_{0,4}\tau_{1,2} >_{g=1} = \frac{1}{48}. \tag{4.12}$$

For the case $p = 7$,

$$\begin{aligned} < \tau_{0,2}\tau_{1,5} >_{g=1} &= < \tau_{1,2}\tau_{0,5} >_{g=1} = \frac{1}{42} \\ < \tau_{0,4}\tau_{1,3} >_{g=1} &= < \tau_{0,3}\tau_{1,4} >_{g=1} = \frac{1}{28} \\ < \tau_{1,0}\tau_{1,0} >_{g=1} &= < \tau_{0,0}\tau_{2,0} >_{g=1} = \frac{1}{4}. \end{aligned} \tag{4.13}$$

5 The p dependence of the intersection numbers

In a previous article [19], we have considered the intersection numbers with one marked point for arbitrary p, and found results such as

$$
\begin{aligned}
<\tau_{1,0}>_{g=1} &= \frac{p-1}{24} \\
<\tau_{n,j}>_{g=2} &= \frac{(p-1)(p-3)(1+2p)}{p\cdot 5!\cdot 4^2\cdot 3}\frac{\Gamma(1-\frac{3}{p})}{\Gamma(1-\frac{1+j}{p})} \\
<\tau_{n,j}>_{g=3} &= \frac{(p-5)(p-1)(1+2p)(8p^2-13p-13)}{p^2\cdot 7!\cdot 4^3\cdot 3^2}\frac{\Gamma(1-\frac{5}{p})}{\Gamma(1-\frac{1+j}{p})} \\
<\tau_{n,j}>_{g=4} &= \frac{(p-7)(p-1)(1+2p)(72p^4-298p^3-17p^2+562p+281)}{p^3\cdot 9!\cdot 4^4\cdot 15} \\
&\quad \times\frac{\Gamma(1-\frac{7}{p})}{\Gamma(1-\frac{1+j}{p})}
\end{aligned}
\tag{5.1}
$$

with $n=2g-1+\frac{2g-2-j}{p}$. In the large p limit, the intersection numbers $<\tau_{n,j}>_g$ behave as

$$
<\tau_{n,j}>_g = \frac{B_g}{(2g)!(2g)}p^g + O(p^{g-1}) \tag{5.2}
$$

with B_g is a Bernouilli number, $B_1=\frac{1}{6}, B_2=\frac{1}{30}, B_3=\frac{1}{42}, B_4=\frac{1}{30}$. Note the well known relation to $\zeta(2g)$ as

$$
\frac{B_g}{(2g)!(2g)} = \frac{1}{(2\pi)^{2g}g}\zeta(2g) \tag{5.3}
$$

We have derived (5.2) from U(s) in the large p limit. The one point function U(s) has the following expression [17],

$$
\mathrm{U}(s) = \frac{1}{Ns}\int\frac{du}{2i\pi}\exp\left(-\frac{c}{p+1}\left(\left(u+\frac{1}{2}s\right)^{p+1}-\left(u-\frac{1}{2}s\right)^{p+1}\right)\right) \tag{5.4}
$$

With $s=\frac{\sigma}{p}$, and $u^{p+1}=x^2$, we have

$$
\mathrm{U}(s) = \frac{2}{N\sigma}\int\frac{dx}{2i\pi}x^{-1+\frac{2}{p}}e^{-\frac{c}{p+1}x^2(e^{\sigma/2}-e^{-\sigma/2})} \tag{5.5}
$$

Thus we obtain

$$
\mathrm{U}(s) = \frac{2}{N\sigma}\Gamma\left(\frac{2}{p}\right)\left(\frac{2c}{p+1}\mathrm{sh}\frac{\sigma}{2}\right)^{-1/p} \tag{5.6}
$$

This may be written as

$$
\mathrm{U}(s) = \frac{2}{N\sigma}\Gamma\left(\frac{2}{p}\right)\left(\frac{2c}{p+1}\right)^{-\frac{1}{2}}\left(\frac{\sigma}{2}\right)^{-\frac{1}{p}}\exp\left(-\frac{1}{p}\log\frac{\mathrm{sh}\frac{\sigma}{2}}{\frac{\sigma}{2}}\right) \tag{5.7}
$$

and expanding the exponent in $\frac{1}{p}$, we find

$$
\mathrm{U}(s) = \frac{2}{N\sigma}\Gamma\left(\frac{2}{p}\right)\left(\frac{2c}{p+1}\right)^{-\frac{1}{2}}\left(\frac{\sigma}{2}\right)^{-\frac{1}{p}}\left(1-\frac{1}{p}\log\left(\frac{\mathrm{sh}(\frac{\sigma}{2})}{(\frac{\sigma}{2})}\right)\right) \tag{5.8}
$$

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If we use the expansion

$$\log\left(\frac{\mathrm{sh}\frac{\sigma}{2}}{\frac{\sigma}{2}}\right) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{B_n\sigma^{2n}}{(2n)!2n} \tag{5.9}$$

and drop the factors $(\frac{2c}{p+1})^{-\frac{1}{p}}$, and $(\sigma/2)^{-1/p}$ which are close to one in the large p limit, we obtain

$$\mathrm{U}(s) = \left(1 - \frac{1}{p}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{B_n}{(2n)!2n}\sigma^{2n}\right)\frac{p}{N\sigma}\Gamma\left(1+\frac{2}{p}\right) \tag{5.10}$$

Since the intersection numbers $< \tau_{n,j} >_g$ are related to $\mathrm{U}(s)$ by [17]

$$\mathrm{U}(s) = \sum_g < \tau_{n,j} >_g \frac{1}{N\pi}\Gamma\left(1 - \frac{1+j}{p}\right) s^{(2g-1)(1+\frac{1}{p})}p^{g-1} \tag{5.11}$$

with $(p+1)(2g-1) = pn + j + 1$, we have rederived the large p behavior of (5.2).

From (5.9), taking a derivative with respect to σ, gives,

$$\frac{1}{e^{\sigma}-1} + \frac{1}{2} - \frac{1}{\sigma} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{B_n}{(2n)!}\sigma^{2n-1}, \tag{5.12}$$

Using this relation one obtains

$$\frac{d}{d\sigma}(\sigma\mathrm{U}(\sigma)) = \frac{1}{\sigma} - \frac{1}{2} - \frac{1}{e^{\sigma}-1} \tag{5.13}$$

The di-gamma function $\psi(z)$ has the following expression,

$$\psi(z) = \frac{d}{dz}\log\Gamma(z) = \log z - \frac{1}{2z} - \int_0^{\infty} d\sigma\left(\frac{1}{2} - \frac{1}{\sigma} + \frac{1}{e^{\sigma}-1}\right)e^{-\sigma z}. \tag{5.14}$$

From (5.13) and (5.14) we find thus in the large p limit,

$$\begin{aligned}\frac{d}{dz}\log\Gamma(z) &= \log z - \frac{1}{2z} + \int_0^{\infty} d\sigma\left(\frac{d}{d\sigma}(\sigma\mathrm{U}(\sigma))\right)e^{-\sigma z} \\ &= \log z - \frac{1}{2z} - z\frac{d}{dz}\int d\sigma\mathrm{U}(\sigma)e^{-\sigma z}\end{aligned} \tag{5.15}$$

The last integral is related to the density of states. In (2.2), s is replaced by $s = it$, and if we replace z by iE, and take the imaginary part, we obtain the density of states $\rho(E)$. After integration by parts, we obtain

$$\rho(E) = \frac{d}{dE}\mathrm{Im}\,\log\Gamma(iE) - \frac{\pi}{2} - \frac{1}{2E} \tag{5.16}$$

We will discuss this expression in connection to the density of states of the $\mathrm{SL}(2,R)/\mathrm{U}(1)$ black hole sigma model in the next section.

Next we consider the two point correlation function $\mathrm{U}(s_1, s_2)$. For general p, $\mathrm{U}(s_1, s_2)$ is expressed as

$$\begin{aligned}\mathrm{U}(s_1,s_2) = &\frac{2}{(s_1+s_2)(ps_2)^{1/p}}\int_0^{\infty}dx\int_0^{\infty}dv_1dv_2\mathrm{sh}\left(\frac{s_1+s_2}{2}(ps_1)^{1/p}\right) \\ &\times e^{-\frac{v_1^p}{p}+xv_1-\frac{p(p-1)}{24}s_1^3(ps_1)^{\frac{2-p}{p}}v_1^{p-2}+\dots}e^{-\frac{v_2^p}{p}-axv_2-\frac{p(p-1)}{24}s_2^3(ps_2)^{\frac{2-p}{p}}v_2^{p-2}+\dots}\end{aligned} \tag{5.17}$$

The exponent of (5.17) follows from the binomial expansion,

$$\left(u+\frac{s}{2}\right)^{p+1} = u^{p+1} + (p+1)u^p\left(\frac{s}{2}\right) + (p+1)p\frac{1}{2}u^{p-1}\left(\frac{s}{2}\right)^2 + \dots \tag{5.18}$$

and we use $c(p+1) = 1$, $u^p s = v^p/p$. As in the case of p=3, polynomials in a (3.15) give the intersection numbers. Therefore we expand (5.17) in power series of a. At lowest order in a, we obtain two terms from (5.17),

$$\begin{aligned} U_1 &= \frac{1}{3!4}(s_1+s_2)^2 \frac{(ps_1)^{\frac{3}{p}}}{(ps_2)^{\frac{1}{p}}} \int dx x^3 \phi(x)\phi(-ax) \\ U_2 &= -\frac{p(p-1)}{24}\left(\frac{s_1}{s_2}\right)^{\frac{1}{p}} s_2^3 (ps_2)^{\frac{2-p}{p}} \int dx \phi(x)\phi^{(p-2)}(-ax)(-a)^{2-p} \end{aligned} \tag{5.19}$$

From U_2 we find a term proportional to $as_2^{2+\frac{2}{p}}$, namely

$$\Delta U_2 = \frac{p-1}{24} p^{\frac{2}{p}} a s_2^{2+\frac{2}{p}} (\phi^{(p-2)}(0))^2 \tag{5.20}$$

with

$$\begin{aligned} \phi^{(p-2)}(0) &= \int_0^\infty du u^{p-2} e^{-\frac{u^p}{p}} \\ &= p^{-\frac{1}{p}} \Gamma\left(1-\frac{1}{p}\right). \end{aligned} \tag{5.21}$$

Since $s_2^{2+\frac{1}{p}} s_1^{\frac{1}{p}} = t_{2,0}t_{0,0}$, we obtain

$$< \tau_{0,0}\tau_{2,0} >_{g=1} = \frac{p-1}{24} \tag{5.22}$$

From U_1 and U_2, we collect terms proportional to $a^3 s_2^{2+\frac{2}{p}}$ and obtain

$$< \tau_{0,2}\tau_{1,p-2} >_{g=1} = \frac{p-3}{24p} \tag{5.23}$$

This result agrees with those obtained previously for $p = 4, 5, 6$ and 7 in (4.7), (4.11), (4.12) and (4.13). The intersection number (5.23) can be neglected in the large p limit in comparison with (5.22).

Similarly one obtains the g=2 terms from the coefficients of $a^m s_2^{4+\frac{4}{p}}$ (m=1,2,3),

$$\begin{aligned} < \tau_{0,0}\tau_{4,2} >_{g=2} &= \frac{(p-1)(p-3)(2p+1)}{5760p} \\ < \tau_{0,1}\tau_{4,1} >_{g=2} &= \frac{(p-1)(p-2)(p+2)}{2880p} \\ < \tau_{0,2}\tau_{4,0} >_{g=2} &= \frac{(p-1)(p-3)(2p+11)}{5760p} \end{aligned} \tag{5.24}$$

For the particular values of $p = 3, 4, 5$, the above expressions agree with the previous results (3.12), (4.8) and (4.11) for the genus two case.

From the $a^5 s_2^{4+\frac{4}{p}}$ term, one finds

$$< \tau_{0,4}\tau_{3,p-2} >_{g=2}= \frac{2p^3 + 13p^2 - 158p + 215}{5760p^2} \tag{5.25}$$

which is valid for $p \geq 6$.

In the large p limit, the three terms of (5.24) become equal, and coincide with the result for the one point intersection number (5.2).

$$< \tau_{0,m}\tau_{4,2-m} >_{g=2}= \frac{B_2 p^2}{4! \cdot 4} \quad (p \to \infty) \tag{5.26}$$

Note that (5.25) is order p, and is negligible compared to (5.24).

From the terms $a^m s_2^{6+\frac{6}{p}}$ in the small a expansion of U(s_1, s_2), we obtain the g=3 (genus 3) terms. In the case m=1, we have

$$< \tau_{0,0}\tau_{6,4} >_{g=3}= \frac{(p-1)(p-5)(2p+1)(8p^2 - 13p - 13)}{p^2 \cdot 7! 4^3 3^2} \quad (p > 5) \tag{5.27}$$

This is identical to $< \tau_{5,4} >_{g=3}$ in (5.1). The identity follows from the string equation, in which the insertion of $\tau_{0,0}$ reduces the intersection number from s to $s-1$ marked points:

$$< \tau_{0,0} \prod_{i=1}^{s} \tau_{n_i,j_i} >_g = \sum_{l=1}^{s} < \tau_{n_l-1,j_l} \prod_{i=1, i\neq l}^{s} \tau_{n_i,j_i} >_g \tag{5.28}$$

In our formulation, this string equation follows from the integral representation for the intersection numbers, when one collects the terms proportional to a. By explicit calculation of two marked points, we verified this string equation. It might be possible to verify this string equation for n-marked points by the taking account of the term of a.

From $a^2 s_2^{6+\frac{6}{p}}$, we have for $p > 5$,

$$< \tau_{0,1}\tau_{6,3} >_{g=3}= \frac{(p-1)(p-2)(p-4)(p+2)(2p+1)}{p^2 \cdot 7! \cdot 8 \cdot 3^2} \tag{5.29}$$

From $a^3 s_2^{6+\frac{6}{p}}$,

$$< \tau_{0,2}\tau_{6,2} >_{g=3}= \frac{(p-1)(p-3)(16p^3 + 34p^2 - 155p - 129)}{p^2 \cdot 7! \cdot 64 \cdot 3^2} \tag{5.30}$$

In the large p limit, these g=3 terms exhibit same behavior as in (5.2),

$$< \tau_{0,m}\tau_{6,4-m} >_{g=3}= \frac{B_3}{6! \cdot 6} p^3 + O(p^2) \quad (p \to \infty) \tag{5.31}$$

6 Analytic continuation to negative p

One may analytically continue the integral representations of the correlation functions to negative values of p. This continuation was already examined in [19], and we recall some of the results here:

$$\mathrm{U}(s) = \frac{1}{Ns}\int \frac{du}{2i\pi} e^{-c[(u+\frac{1}{2}s)^{p+1}-(u-\frac{1}{2}s)^{p+1}]} \tag{6.1}$$

where $c = \frac{N}{p^2-1}\sum \frac{1}{a_\alpha^{p+1}}$.

Expanding the exponent, we obtain

$$\mathrm{U}(s) = \int \frac{du}{2i\pi}\exp\left[-c\left(su^p + \frac{p(p-1)}{3!4}s^3u^{p-2} + \frac{p(p-1)(p-2)(p-3)}{5!4^2}s^5u^{p-4} + \cdots\right)\right]. \tag{6.2}$$

This integrals yield Gamma functions after the replacement $u = (\frac{t}{cs})^{1/p}$,

$$\begin{aligned}
\mathrm{U}(s) &= \frac{1}{Nsp}\cdot\frac{1}{(cs)^{1/p}}\int_0^\infty dt t^{\frac{1}{p}-1} e^{-(t+\frac{p(p-1)}{3!4}s^{2+\frac{2}{p}}c^{\frac{1}{p}}t^{1-\frac{2}{p}}+\frac{p(p-1)(p-2)(p-3)}{5!4^2}s^{4+\frac{4}{p}}c^{\frac{4}{p}}t^{1-\frac{4}{p}}+\cdots)}\\
&= \frac{1}{Nsp}\cdot\frac{1}{(cs)^{1/p}}\left[-\frac{p-1}{24}c^{\frac{2}{p}}y\Gamma\left(1-\frac{1}{p}\right)+\frac{(p-1)(p-3)(1+2p)}{5!\cdot 4^2\cdot 3}y^2\Gamma\left(1-\frac{3}{p}\right)\right.\\
&\quad -\frac{(p-5)(p-1)(1+2p)(8p^2-13p-13)}{7!4^33^2}y^3\Gamma\left(1-\frac{5}{p}\right)\\
&\quad +(p-7)(p-1)(1+2p)(72p^4-298p^3-17p^2+562p+281)\\
&\quad \left.\times\frac{1}{9!4^415}y^4\Gamma\left(1-\frac{7}{p}\right)\cdots\right]
\end{aligned} \tag{6.3}$$

with $y = c^{\frac{2}{p}}s^{2+\frac{2}{p}}$.

From this expansion, we obtain the intersection numbers for one marked point as (5.1). The intersection number $<\tau_{n,j}>_g$ is obtained from the term $y^g\Gamma(1-\frac{1}{p}-\frac{j}{p})$ in (6.3).

The continuation to $p<0$ is straightforward. The t-integral in (6.3) can be changed to v by $t=\frac{1}{v}$,$(0<v<\infty)$, and one obtains the small s expansion for negative p. Therefore the expression for the intersection numbers (5.1) can be analytically continued to negative p. This analytic continuation can also be done for two marked points, since we have computed them in the previous sections for general p. For instance, from (5.1), we have the intersection numbers for $p=-3$,

$$\begin{aligned}
&<\tau_{1,0}>_{g=1} = -\frac{1}{6}, \qquad <\tau_{3,2}>_{g=2} = \frac{1}{144}\\
&<\tau_{6,1}>_{g=3} = -\frac{35}{34992}
\end{aligned} \tag{6.4}$$

In a previous article [19], we have computed the intersection numbers $<\tau_{1,0}>_g$ for the case of $p=-1$ from $\mathrm{U}(s)$, which provides the orbifold Euler characteristics $\chi(\mathcal{M}_{g,1})$ with one marked point,

$$<\tau_{1,0}>_g = \chi(\mathcal{M}_{g,1}) = \zeta(1-2g) = -\frac{B_g}{2g} \tag{6.5}$$

with the Bernoulli number B_g, $(B_1 = \frac{1}{6}, B_2 = \frac{1}{30}, B_3 = \frac{1}{42}, \ldots)$. The s-point orbifold Euler characteristics $\chi(\mathcal{M}_{g,s})$ may be obtained from the dilaton equation:

$$< \tau_{1,0}\tau_{n_1,j_1} \cdots \tau_{n_k,j_k} >_g = (2g - 2 + k) < \tau_{n_1,j_1} \cdots \tau_{n_k,j_k} >_g \tag{6.6}$$

Since the Euler characteristics with s marked points is $< \tau_{1,0} \cdots \tau_{1,0} >_g$, the dilaton equation yields from (6.5),

$$\chi(\mathcal{M}_{g,s}) =< (\tau_{1,0})^s >_g = -\frac{2g-1}{(2g)!}(2g + s - 3)!B_g \tag{6.7}$$

This agrees with previous results obtained in [21–23].

For $p = -2$, we have considered previously the equivalence with the unitary matrix model in a matrix source [27].

The central charge of the gauged Wess-Zumino-Witten model with symmetry $\mathrm{SU}(2)_k/\mathrm{U}(1)$ is

$$C = 2 - \frac{6}{k+2} \tag{6.8}$$

Changing p to $p = -p'$, k to $k = -k'$ ($p < 0$,$k < 0$), we have $p' = k' - 2$, and the central charge C is given by

$$C = 2 + \frac{6}{k'-2} \tag{6.9}$$

The analytic continuation to negative p yields a gauged WZW model for $\mathrm{SL}(2, R)_{k'}/\mathrm{U}(1)$. It is known that this model represents a black hole σ model [10], in particular for the value $k' = \frac{9}{4}$ $(p = -\frac{1}{4})$, for which the central charge C becomes 26.

The density of states for the $\mathrm{SL}(2, R)/\mathrm{U}(1)$ black hole has been studied in [25, 26, 28],

$$\rho(E) = \frac{1}{\pi}\mathrm{log}\epsilon + \frac{1}{4\pi i}\frac{d}{dE}\mathrm{log}\frac{\Gamma(-iE + \frac{1}{2} - m)\Gamma(-iE + \frac{1}{2} + \tilde{m})}{\Gamma(+iE + \frac{1}{2} + \tilde{m})\Gamma(+iE + \frac{1}{2} - m)} \tag{6.10}$$

in which ϵ is a regularization factor, and $m = \frac{1}{2}(n - kw)$, $\tilde{m} = -\frac{1}{2}(kw + n)$ are eigenvalues of J_0^3 and $\bar{J}_0^3$ in CFT ($J_0^3 - \bar{J}_o^3 = n$, $J_0^3 + \bar{J}_0^3 = -kw$). If we neglect m , $\tilde{m}$, and the $\frac{1}{2}$ terms in the large E limit, we obtain

$$\rho(E) = \frac{1}{\pi}\mathrm{log}\epsilon + \frac{1}{2\pi i}\frac{d}{dE}\mathrm{log}\frac{\Gamma(-iE)}{\Gamma(+iE)} \tag{6.11}$$

or

$$\rho(E) = \frac{2}{\pi}\frac{d}{dE}\mathrm{Im}\ \mathrm{log}\ \Gamma(-iE) \tag{6.12}$$

This expression agrees with (5.16), obtained from the intersection numbers for large p. We have scaled $s = \sigma/p$, and the expression (5.16) is valid for small s. Therefore, the Fourier transform of $\mathrm{U}(s)$ gives the large E behavior, in which the terms m,$\tilde{m}$ and 1/2 in (6.10) can be neglected.

7 Discussion

In this article, we have shown that the correlation functions $\mathrm{U}(s_1, s_2, \cdots, s_n)$ of a Gaussian matrix model in a tuned external source, provide the intersection numbers for p-spin curves. For instance, from the two point function $\mathrm{U}(s_1, s_2)$, in the case of p=3, the intersection numbers are computed up to genus 3,

We have also computed the intersection numbers for general p. They are given by power series in a, $a = (\frac{s_1}{s_2})^{\frac{1}{p}}$. Then we have considered the large p behavior for the two point functions. The density of states $\rho(E)$ becomes a di-gamma function in the large p limit, and this expression agrees with the density of states of a $\mathrm{SL}(2, R)_k/\mathrm{U}(1)$ WZW model, which has been studied in the context of two dimensional black hole solutions. The n-point correlation functions $\mathrm{U}(s_1, \cdots, s_n)$ are known through the determinant of a kernel for the p-spin curve case. It will be interesting to investigate further the detailed comparison of those correlation functions, between $\mathrm{SL}(2, R)_k/\mathrm{U}(1)$ WZW theory and the intersection numbers for negative p-spin curves.

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